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Nonlinear problems related to the Thomas-Fermi equation

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Preface by Haïm Brezis

Most of the results in this work were obtained over the period 1975–77 and were announced at various meetings (see e.g. items [3], [4], [5] under Brezis [16]). This paper has a rather unusual history. Around 1972 I became interested in nonlinear elliptic equations of the form

$$-\Delta u + |u|^{p-1}u = f \quad \text{in a domain } \Omega \subset \mathbb{R}^N, \tag{P.1}$$

with zero Dirichlet condition, where $0 < p < \infty$ and $f \in L^1$. The motivation came from the study of the porous medium equation

$$\frac{\partial v}{\partial t} - \Delta(|v|^{m-1}v) = 0, \tag{P.2}$$

with $0 < m < \infty$.

The space L^1 is a natural functional space associated with (P.2) since (P.2) generates (at least formally) a contraction semi-group in L^1. When trying to apply the Crandall-Liggett theory in L^1 to (P.2) one is led to the question whether the nonlinear operator $Av = -\Delta(|v|^{m-1}v)$ is m-accretive in L^1, and in particular whether the equation

$$v - \lambda\Delta(|v|^{m-1}v) = f$$

admits a solution for every $f \in L^1$ and every $\lambda > 0$. Setting $u = |v|^{m-1}v$ and scaling out λ yields (P.1) with $p = 1/m$. In the sixties, equations of the type (P.1) had been extensively studied by F. Browder (see e.g. Browder [27]) and by J. L. Lions (see e.g. Leray-Lions [46]) using energy estimates and monotonicity methods which are suitable when $f \in H^{-1}$, but *not* when $f \in L^1$. No one in my circles was concerned with L^1 data for (P.1). The only result I had seen was stated in Stampacchia [56] and dealt with the *linear* elliptic equation in divergence form

$$Lu = -\sum \frac{\partial}{\partial x_j}\left(a_{ij}\frac{\partial u}{\partial x_i}\right) = \mu. \tag{P.3}$$

Stampacchia asserted that, given any $\mu \in L^1$ (or even measure), equation (P.3) admits a solution $u \in L^q$, $\forall q < N/(N-2)$; this was an easy consequence, via *duality*, of the DeGiorgi-Stampacchia estimate

$$\|v\|_{L^\infty} \leqslant C_p \|f\|_{L^p} \quad \forall p > N/2,$$

for the solution of $Lv = f$.

In 1972, Walter Strauss and I tackled (P.1) for $f \in L^1$. We proved, in Brezis-Strauss [25], that, for *every* $f \in L^1$ and *every* $0 < p < \infty$, equation (P.1) admits a unique solution $u \in L^p$. More generally, if $\beta : \mathbb{R} \to \mathbb{R}$ is any continuous nondecreasing function (such that $\beta(0) = 0$), we established that, given any $f \in L^1$, there exists a unique solution of

$$-\Delta u + \beta(u) = f \tag{P.4}$$

with $\beta(u) \in L^1$. We even dealt with maximal monotone graphs β in $\mathbb{R} \times \mathbb{R}$ and obtained the same conclusion for the multivalued equation.

$$-\Delta u + \beta(u) \ni f. \tag{P.5}$$

Later, we considered, in Bénilan-Brezis-Crandall [10], similar problems in all of $\mathbb{R}^N$ (instead of domains).

At the International Congress of 1974, I heard a lecture by E. Lieb reporting on the paper Lieb-Simon [48]. One of their results asserts that for some values of λ, $\lambda \geqslant 0$, the Thomas-Fermi equation

$$-\Delta u + [(u-\lambda)^+]^{3/2} = \sum_{i=1}^{\ell} m_i \delta_{a_i} \text{ in } \mathbb{R}^3, \tag{P.6}$$

with $m_i > 0$ and δ_{a_i} = Dirac mass at a_i, admits a solution. Of course, the function $\beta(t) = [(t-\lambda)^+]^{3/2}$ is nondecreasing, continuous and $\beta(0) = 0$ (since $\lambda \geqslant 0$). I became intrigued and decided that it would be interesting to study (P.1)(or(P.4)) for measures instead of L^1 functions. My initial intuition was that measures and L^1 functions are the same "creatures" from the point of view of estimates, and therefore the Brezis-Strauss theorem should extend easily to measures. On the other hand, the method of Lieb-Simon was totally different from ours. In their variational approach, equation (P.6) appears as the Euler equation of a "dual" convex minimization problem. Their technique could be adapted to solve (P.4) for a limited class of nonlinearities β and a limited class of measures f.

I mentioned the problem to Philippe Bénilan in the Spring of 1975 and he liked the idea of working together on this topic. Philippe had been my first Ph.D student, even though he was about four years older than me (he defended his Ph.D in 1972). He had been sent to me in 1970 by his mentor, Jacques Deny, who was one of the leaders of the French school in Potential Theory, jointly with M. Brelot and G. Choquet. He knew much better than me the fine properties of harmonic functions and of measure theory. He was the ideal partner on this project. We had been both invited the following summer to Madison, Wisconsin, by Mike Crandall. I have nostalgic memories from the long days we spent together working on the big tables outside the Memorial Union, facing the inspiring view of Lake Mendota. Philippe, who was an addicted smoker, felt free to finish pack after pack in this open-air environment. We managed rather quickly to prove that (P.1) has a solution for every measure f in the case where $p < N/(N-2)$ for $N > 2$ and no restriction on p for $N = 1, 2$ (see Theorem A.1 in Appendix A). Of course, this was sufficient to handle the Thomas-Fermi model since $N = 3$ and $3/2 < 3$. Still, we were puzzled and tried hard to remove the restriction $p < N/(N-2)$. For a few weeks we had no success, even on the simple equation

$$-\Delta u + u^3 = \delta_a \text{ in } \mathbb{R}^3. \tag{P.7}$$

I remember vividly the shiny day when we discovered, sitting at "our" table next to the lake, that (P.7) has *no* solution: this is the elementary computation in Remark A.4. We were stunned! There was indeed an unexpected difference between measures and L^1 and it was due to the nonlinear nature of the problem. Later, we decided to read carefully the paper of Lieb-Simon [48]. We thought about some of their open problems and succeeded in solving two of them (see Section 5 and 6 below). Then came the painful task of writing up our results. Philippe was a powerful and creative mathematician, able to analyze a concrete differential equation in its most minute details. However, when the time arrived to write a paper, he prefered to "hide" the simple illuminating examples and to present instead a grand abstract framework. He was still strongly influenced by the French school of Potential Theory whose program was to axiomatize Potential Theory "à la Bourbaki"–carefully hiding the Laplace operator! Philippe was a perfectionist, always eager to state a theorem in the most general setting, with minimal assumptions. He wrote a first, partial, draft of our paper (basically, Sections 1, 2, 3 below). I made drastic changes which he did not like, etc. After several divergent iterations we stopped and the paper was "buried" unfinished and unorganized. In the meantime we advertised some of the results through lectures, and some hand-written partial versions were circulated "under the coat" as "samizdats". In fact, our unpublished results gave an impetus to beautiful developments in numerous directions:

a) Solving nonlinear PDE's with L^1, or measures, as data became very fashionable. There is a vast and flourishing literature starting in the eighties. I have listed some of the references in Appendix A. Important connections with Probability Theory (E. Dynkin, J. M. LeGall and their students) have reinvigorated the whole subject in recent years.

b) The nonexistence aspect (e.g. for Dirac masses) has given rise to striking new results about *removable singularities* (e.g. point singularities). On the other hand, singular solutions have also been analyzed and classified; see some references in Appendix A.

c) Our approach turned out to be useful in other models arising in the density-functional theory of atoms and molecules; see e.g. Bénilan - G. Goldstein - J. Goldstein [11],

J. Goldstein - G. Rieder [38], G. Rieder [55], G. Goldstein - J. Goldstein - W. Jia [37], Breazna-G. Goldstein - J. Goldstein [15] and related references.

d) The need for new versions of the strong maximum principle in the case of “bad” coefficients stimulated new research in that direction; see Appendix C and the references therein.

e) The solution u of the Thomas-Fermi equation (P.6) tends to zero at infinity. The set where the density $\rho = [(u-\lambda)^+]^{3/2}$ is positive plays an important role. When $\lambda > 0$, this set is bounded. The regularity of its boundary has been studied by Caffarelli-Friedman [28].

After the tragic death of Philippe I decided that our work should not remain in a drawer. Out of respect for the memory of Philippe I have kept his style of presentation. Our notes were incomplete and the last time we touched them was in 1985. I have tried my best to put them in good order and fill in missing arguments. My apologies to the reader if there are still some inconsistencies. I have also added an extensive list of references published in recent years and which bear some relation to our work.

Haïm Brezis

0. Introduction

The principal motivation of this work comes from the important paper of Lieb-Simon [48]. One of their main results is the following. Given $I > 0$, let

$$K_I = \left\{ \rho \in L^1(\mathbb{R}^3);\ \rho \geqslant 0 \text{ a.e. and } \int \rho = I \right\}.$$

Consider the function

$$V(x) = \sum_{i=1}^{\ell} \frac{m_i}{|x-a_i|},\ m_i > 0,\ a_i \in \mathbb{R}^3, \tag{0.1}$$

and set for $\rho \in L^1 \cap L^{5/3}$, $\rho \geqslant 0$ a.e.,

$$\mathcal{E}(\rho) = \frac{3}{5}\int \rho^{5/3} - \int V\rho + \frac{1}{2}\iint \frac{\rho(x)\rho(y)}{|x-y|}dxdy. \tag{0.2}$$

It is not difficult to check that $\mathcal{E}(\rho)$ is well defined and bounded below. Consider the minimization problem

$$E(I) = \inf\{\mathcal{E}(\rho);\ \rho \in K_I \cap L^{5/3}\}. \tag{0.3}$$

THEOREM 0.1 (Lieb-Simon). *Set*

$$I_0 = \sum_{i=1}^{\ell} m_i \tag{0.4}$$

If $I \leqslant I_0$, *the minimum in* (0.3) *is uniquely achieved by some* ρ. *Moreover there is a constant* $\lambda \geqslant 0$ *such that*

$$\rho^{2/3} - V + B\rho = -\lambda \quad \text{in } [\rho > 0], \tag{0.5}$$

$$-V + B\rho \geqslant -\lambda \quad \text{in } [\rho = 0], \tag{0.6}$$

where

$$B\rho(x) = \int \frac{\rho(y)}{|x-y|} dy. \tag{0.7}$$

In the neutral case, $I = I_0$, *one has* $\rho > 0$ *a.e. and* $\lambda = 0$, *so that* ρ *satisfies*

$$\rho^{2/3} - V + B\rho = 0 \quad \text{a.e. on } \mathbb{R}^3. \tag{0.8}$$

The constant λ plays an important role; $-\lambda$ is called the chemical potential. It appears in the Euler "equation" (0.5)–(0.6), corresponding to the minimization problem (0.3), as a Lagrange multiplier associated with the constraint $\int \rho = I$. The dichotomy (0.5)–(0.6) is standard in variational inequalities involving a constraint of the type $\rho \geqslant 0$.

It is convenient to introduce

$$u = V - B\rho, \tag{0.9}$$

and then (0.5)–(0.6) may be rewritten as

$$-\frac{1}{4\pi}\Delta u = \sum m_i \delta_{a_i} - \rho, \tag{0.10}$$

$$\rho = [(u-\lambda)^+]^{3/2}, \tag{0.11}$$

where $r^+ = \max\{r, 0\}$ and $\delta_a =$ Dirac mass at a.

Hence we are led to the nonlinear PDE

$$-\Delta u + 4\pi[(u-\lambda)^+]^{3/2} = 4\pi \sum m_i \delta_{a_i} \tag{0.12}$$

coupled with a condition at infinity coming from (0.9),

$$u(x) \to 0 \quad \text{as } |x| \to \infty \tag{0.13}$$

(possibly to be understood in a weak sense). Note that here the constant $\lambda \geqslant 0$ is *not* given; it is part of the unknown. But we have instead the additional information

$$\int [(u-\lambda)^+]^{3/2} = I,$$

where $I \geqslant 0$ is *given*.

REMARK 0.1. When $I > I_0$, $E(I) = E(I_0)$ and the infimum in (0.3) is not achieved.

Our work goes in several directions. First, we replace the function $\frac{3}{5}\rho^{5/3}$ by a general convex function $j : \mathbb{R} \to [0, +\infty]$ such that $j(0) = 0$ and we incorporate the constraint $\rho \geqslant 0$ into j by assuming

$$j(r) = +\infty \quad \text{for } r < 0. \tag{0.14}$$

Next, we consider a general measurable function $V(x)$ instead of (0.1). We replace $\mathbb{R}^3$ by $\mathbb{R}^N$, $N \geqslant 3$, and we replace the Coulomb potential by the fundamental solution k of $(-\Delta)$, $k(x) = c_N/|x|^{N-2}$ with $c_N = 1/(N-2)\sigma_N$ and σ_N is the area of the unit space in $\mathbb{R}^N$.

The energy $\mathcal{E}$ takes the form

$$\mathcal{E}(\rho) = \int j(\rho) - \int V\rho + \frac{c_N}{2} \iint \frac{\rho(x)\rho(y)}{|x-y|^{N-2}}\, dxdy, \tag{0.15}$$

whenever it makes sense.

The minimization problem we tackle is

$$E(I) = \inf \left\{ \mathcal{E}(\rho); \int \rho = I \right\}. \tag{M$_I$}$$

The Euler equation (0.5)–(0.6) is replaced, at least formally, by a multivalued equation

$$\partial j(\rho) - V + B\rho \ni -\lambda \quad \text{a.e. on } R^N, \tag{0.16}$$

for some constant λ, where ∂j is the subdifferential of j.

Note that in the special case where j is C^1 on $(0, +\infty)$ we have

$$\partial j(r) = \begin{cases} j'(r) & \text{for } r > 0, \\ (-\infty, j'(0+)] & \text{for } r = 0, \\ \emptyset & \text{for } r < 0, \end{cases}$$

and thus (0.16) is equivalent to

$$j'(\rho) - V + B\rho = -\lambda \quad \text{in } [\rho > 0], \tag{0.17}$$

$$j'(0+) - V + B\rho \geqslant -\lambda \quad \text{in } [\rho = 0], \tag{0.18}$$

which is precisely (0.5)–(0.6) when

$$j(r) = \begin{cases} \frac{1}{p} r^p & \text{for } r \geqslant 0, \\ +\infty & \text{for } r < 0, \end{cases} \tag{0.19}$$

and $p = 5/3$.

Usually we will asume that $V(x) \to 0$ as $|x| \to \infty$ (at least in some weak sense-for example, meas $[|V| > \delta]$ is finite for every $\delta > 0$); we will also assume that $j'(0+) = 0$, and then (0.17)–(0.18) implies.

$$\lambda \geqslant 0. \tag{0.20}$$

As above, we introduce

$$u = V - B\rho, \tag{0.21}$$

so that we obtain

$$-\Delta u + \rho = -\Delta V \tag{0.22}$$

and

$$\partial j(\rho) \ni u - \lambda. \tag{0.23}$$

We now introduce the inverse maximal monotone graph, $\gamma = (\partial j)^{-1}$, which is also equal to ∂j^*, where j^* is the conjugate convex function of j (see e.g. item [2] under Brezis [16]). In the most important examples (see Section 4), γ is *singlevalued.*

Finally we arrive at the nonlinear multivalued PDE

$$-\Delta u + \gamma(u - \lambda) \ni -\Delta V, \tag{0.24}$$

$$u(x) \to 0 \text{ as } |x| \to \infty. \tag{0.25}$$

Again, λ is unknown, but we have the additional information

$$\int \Delta(u - V) = I, \tag{0.26}$$

with $I \geqslant 0$ given, or equivalently when γ is singlevalued

$$\int \gamma(u - \lambda) = I. \tag{0.27}$$

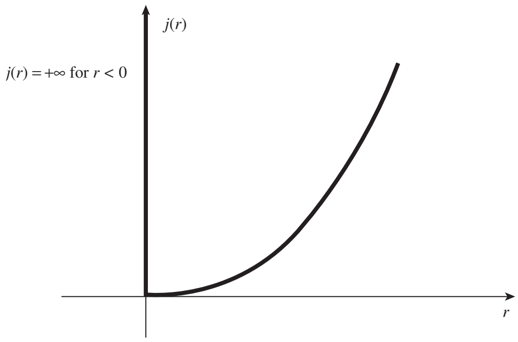

Figure 1 The function j.

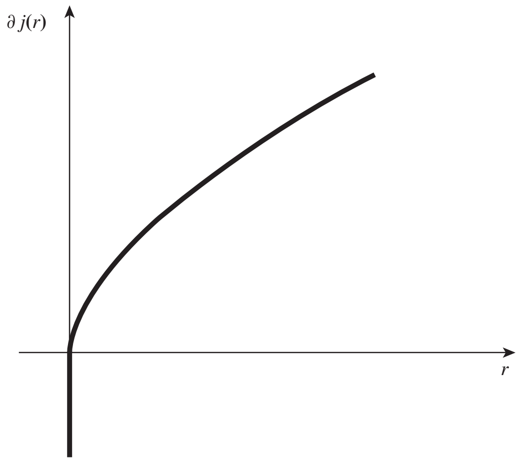

Figure 2 The graph of ∂j.

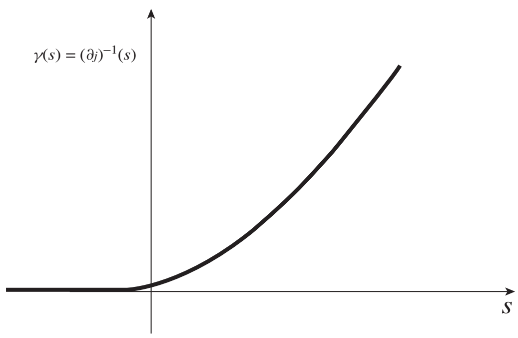

Figure 3 The graph of $\gamma = (\partial j)^{-1}$.

In Section 1 we study the relationship between the variational formulation (M_I) and the Euler equation (0.16). We prove in great generality (see Theorem 1) that if ρ is a minimizer for (M_I), then ρ satisfies the Euler equation (0.16). We establish the converse (0.16) $\Rightarrow$ (M_I) under the additional condition

$$j^*(V - M) \in L^1 \quad \text{for some constant } M, \tag{H}$$

which guarantees that $\mathcal{E}$ is bounded below. For example, when $N = 3$, $V(x) = \sum \frac{m_i}{|x-a_i|}$, and $j(r) = \frac{1}{p}r^p$ for $r \geqslant 0$, condition (H) corresponds to the restriction

$$p > 3/2. \tag{0.28}$$

In fact, when $1 < p \leqslant 3/2$, an easy computation (see Section 4) shows that $E(I) = -\infty$ for every $I > 0$. Despite this fact, we are going to see in Section 4 that the Euler equation (0.16) *does have* a solution when $p > 4/3$.

Therefore, we have a range of p's,

$$\frac{4}{3} < p \leqslant \frac{3}{2}, \tag{0.29}$$

where the *variational formation is meaningless while the PDE approach makes sense*. This is the reason why we have taken, in Sections 3, 4 and in Appendix A, a direct PDE route.

In Section 2 we make basically the following assumptions on j:

$$j \text{ is } C^1 \text{ on } (0, +\infty),\ j'(0+) = 0, \tag{0.30}$$

$$\lim_{r\to+\infty} \frac{j(r)}{r} = +\infty \tag{0.31}$$

On the function V we assume $V(x) \to 0$ as $|x| \to \infty$ in the weak sense that

$$\text{meas}[|V| > \delta] < \infty \quad \text{for every } \delta > 0. \tag{0.32}$$

and that

$$\omega = j^*((1+\theta)(V - M)) \in L^1, \tag{H$^+$}$$

for some constants $\theta > 0$ and $M > 0$. It follows from (H^+) that

$$-V\rho \geqslant -j(\rho) - \omega - M\rho$$

and consequently $\mathcal{E}(\rho)$ is well defined in $(-\infty, +\infty]$ for every $\rho \in L^1, \rho \geqslant 0$ and $j(\rho) \in L^1$.

We then consider the auxilary problem, for every $\lambda \in \mathbb{R}$,

$$\inf \left\{ \mathcal{E}(\rho) + \lambda \int \rho;\ \rho \geqslant 0,\ \rho \in L^1 \text{ and } j(\rho) \in L^1 \right\}. \tag{P$_\lambda$}$$

We will also make the assumption

$$\operatorname*{ess\,sup}_{\mathbb{R}^N} V > 0, \tag{0.33}$$

which is quite natural. If it is not satisfied, then $V \leqslant 0$ a.e. on $\mathbb{R}^N$ and the unique minimizer in (P_λ) is $\rho = 0$.

In Section 2 we prove the following

THEOREM 0.2. *Assume* (H^+), (0.30), (0.31), (0.32) *and* (0.33). *Then, for every* $\lambda > 0$, (P_λ) *admits a unique minimizer* ρ_λ, *and* ρ_λ *satisfies* (0.16). *Set*

$$I(\lambda) = \int \rho_\lambda, \quad \lambda > 0.$$

Then the function $\lambda \longmapsto I(\lambda)$ *is nonincreasing, and continuous from* $(0, \infty)$ *into* $[0, \infty)$ *More precisely,*

$$I(\lambda) \text{ is decreasing on } (0, \operatorname*{ess\,sup}_{\mathbb{R}^N} V), \tag{0.34}$$

$$\begin{cases} I(\lambda) = 0 \quad \forall \lambda \geqslant \operatorname{ess\,sup}_{\mathbb{R}^N} V & \text{if } \operatorname{ess\,sup}_{\mathbb{R}^N} V < \infty, \\ \lim_{\lambda \to \infty} I(\lambda) = 0 & \text{if } \operatorname{ess\,sup}_{\mathbb{R}^N} V = \infty, \end{cases} \tag{0.35}$$

$$\begin{cases} I_0 = \lim_{\lambda \downarrow 0} I(\lambda) = \sup_{\lambda > 0} I(\lambda) < \infty & \text{if and only if} \\ (\mathrm{P}_0) \text{ admits a minimizer } \rho_0, & \text{and then } I_0 = \int \rho_0. \end{cases} \tag{0.36}$$

As a consequence, we easily derive,

COROLLARY 0.3. *Under the assumptions of Theorem* 0.2, *we have*

$$\begin{cases} \text{for every } I \in (0, I_0) \text{ problem } (\mathrm{M}_I) \text{ admits a unique minimizer} \\ \rho^I = \rho_\lambda, \text{ where } \lambda > 0 \text{ is the unique solution of } I(\lambda) = I; \end{cases} \tag{0.37}$$

$$\text{if } I_0 < \infty, \text{ problem } (\mathrm{M}_{I_0}) \text{ admits } \rho_0 \text{ as its unique minimizer;} \tag{0.38}$$

$$\text{if } I_0 < \infty \text{ and } I > I_0, \text{ problem } (\mathrm{M}_I) \text{ admits no minimizer.} \tag{0.39}$$

In Section 3 we investigate situations where assumption (H) is *not* satisfied. For example $N = 3$, V of the form (0.1) and $j(r) = \frac{1}{p} r^p$, $r \geqslant 0$, with p in the range (0.29). We tackle *directly* the Euler equation (0.16), first with $\lambda \geqslant 0$ prescribed and then with λ free and $I = \int \rho$ prescribed.

The main result in Section 3 is

THEOREM 0.4. *Assume* (0.30). *Let* V *be any measurable function satisfying* (0.32) *and* (0.33). *Then, there exists* $\lambda_1 \in [0, +\infty]$ *such that*

$$\text{for every } \lambda > \lambda_1 (\text{and } \lambda < +\infty) \text{ there is a unique solution } \rho_\lambda \text{ of } (0.16) \tag{0.40}$$

$$\text{for } \lambda < \lambda_1 \text{ there is no solution of } (0.16). \tag{0.41}$$

Moreover the function $I(\lambda) = \int \rho_\lambda$ *is nonincreasing continuous on* $(\lambda_1, +\infty)$, *and*

$$I_1 = \sup_{\lambda > \lambda_1} \int \rho_\lambda = \lim_{\lambda \downarrow \lambda_1} \int \rho_\lambda$$

is finite if and only if (0.16) *admits a solution for* $\lambda = \lambda_1$.

It may well happen that λ_1 in Theorem 0.4 is $+\infty$, meaning that there exists *no* λ for which (0.16) has a solution. Consider the case $N = 3$, $V(x)$ of the form (0.1) and $j(r) = \frac{1}{p} r^p$, $r > 0$, with $1 < p \leqslant \frac{4}{3}$. Then (0.16) is equivalent to

$$-\Delta u + [(u - \lambda)^+]^{1/(p-1)} = \sum m_i \delta_{a_i} \tag{0.42}$$

and we know from the nonexistence result (Remark A. 4) in Appendix A that for any $\lambda \geqslant 0$, (0.42) has no solution.

The numbers λ_1 and I_1 in Theorem 0.4 play a central role and it is important to determine their value in concrete situations. This is the content of Section 4. Here are some typical results

THEOREM 0.5. *Assume* (0.30). *Let* V *be any measurable function satisfying* (0.33) *and*

$$V = k * f \text{ for some } f \in L^1, \text{ i.e., } f = -\Delta V \in L^1. \tag{0.43}$$

Then, $\lambda_1 = 0$ *and*

$$0 < I_1 \leqslant \int f^+. \tag{0.44}$$

Under the additional assumption

$$j(r) \sim r^p \text{ near } r = 0 \text{ with } p \geqslant 2(N-1)/N, \tag{0.45}$$

then

$$\int f \leqslant I_1 \leqslant \int f^+ \tag{0.46}$$

and in particular

$$I_1 = \int f \quad \text{if } f \geqslant 0. \tag{0.47}$$

The proof of (0.46) relies heavily on the following ingredient established in Appendix B.

LEMMA 0.6. *Assume*

$$v = k * \mu, \text{ for some bounded measure in } \mathbb{R}^N \tag{0.48}$$

and

$$v^+ \in L^q(\mathbb{R}^N) \text{ for some } q \leqslant N/(N-2) \tag{0.49}$$

then

$$\int_{\mathbb{R}^N} \mu \leqslant 0. \tag{0.50}$$

Assumption (0.43) does not hold e.g. when $N = 3$ and V is given by (0.1) since ΔV is a measure and not an L^1 function. In this case we have

THEOREM 0.7. *Same assumptions as in Theorem* 0.5 *except that we replace* (0.43) *by*

$$V = k * f, \text{ for some bounded measure } f. \tag{0.43'}$$

Assume in addition that

$$j(r) \sim r^p \text{ as } r \to \infty, \text{ with } p > 2(N-1)/N. \tag{0.51}$$

Then all the conclusions of Theorem 0.5 *hold.*

Putting together all the above results, consider now the case where

$$j(r) = r^p, r \geqslant 0, \text{ with } p > 2(N-1)/N, \tag{0.52}$$

$$V = k * f \text{ for some bounded measure } f \geqslant 0,\ f \not\equiv 0. \tag{0.53}$$

Set

$$I_1 = \int f.$$

COROLLARY 0.8. *Assume* (0.52) *and* (0.53). *Given any* $I \in (0, I_1]$ *there exists a unique pair* $\rho \in L^1$, $\rho \geqslant 0$, *and* $\lambda \geqslant 0$, *denoted* ρ_I, λ_I, *satisfying*

$$\rho^{p-1} - V + k * \rho = -\lambda \quad \text{in } [\rho > 0], \tag{0.54}$$

$$-V + k * \rho \geqslant -\lambda \quad \text{in } [\rho = 0], \tag{0.55}$$

$$\int \rho = I \tag{0.56}$$

When $I = I_1$, *then* $\rho_I > 0$ *a.e. and* $\lambda_I = 0$. *When* $I > I_1$, *problem* (0.54)–(0.55)–(0.56) *has no solution. Under the stronger assumption* $p > N/2$, ρ_I *is also the unique minimizer of* $\mathcal{E}(\rho)$ *subject to the constraint* $\{\rho \in L^1 \cap L^p, \rho \geqslant 0 \text{ and } \int \rho = I\}$.

In Sections 5 and 6 we solve two problems raised by Lieb-Simon [48]. The first one concerns the uniqueness of the extremal in some minmax principle.

THEOREM 0.9. *Consider for simplicity the setting of Corollary* 0.8 *with*

$$2(N-1)/N < p \leqslant 2. \tag{0.57}$$

Let $I \in (0, I_1]$*, then* λ_I *given by Corollary* 0.8 *satisfies*

$$\lambda_I = \max_{\substack{\rho\in L^1,\rho\geqslant 0\\ \int\rho=I}} \operatorname*{ess\,inf}_{[\rho>0]}\{V-\rho^{p-1}-k*\rho\} \tag{0.58}$$

and

$$\lambda_I = \min_{\substack{\rho\in L^1,\rho\geqslant 0\\ \int\rho=I}} \operatorname*{ess\,sup}_{\mathbb{R}^N}\{V-\rho^{p-1}-k*\rho\}. \tag{0.59}$$

If $I < I_1$*, the max (resp. min) in* (0.58) *(resp.* (0.59)*) is uniquely achieved by the solution* ρ_I *obtained in Corollary* 0.8*.*

When $N = 3$ and $p = 4/3$, assertions (0.58) and (0.59) are due to Lieb-Simon [48]. They asked whether the max in (0.58) and the min in (0.59) are uniquely achieved. The answer is indeed positive when $I < I_1$. As we shall see in Section 5, the answer is *negative* when $I = I_1$. The proof of uniqueness in Theorem 0.9 involves a new form of the strong maximum principle with "bad" coefficients described in Appendix C.

Our last result concerns the asymptotic behavior of λ_I as $I \uparrow I_1$.

THEOREM 0.10. *Consider for simplicity the setting for Corollary* 0.8*, with*

$$2(N-1)/N < p < 2. \tag{0.60}$$

Then

$$\alpha = \lim_{I\uparrow I_1} \frac{\lambda_I}{(I_1-I)^\tau} \text{ exists}, \tag{0.61}$$

where $\tau = 2(p-1)/(2-2N+pN)$ *and the positive constant* α *can be computed explicitly via the solution of an elementary ODE.*

The exact value of α is given in Theorem 9 (Section 6).

1. The variational problem and its Euler equation; conditions for equivalence

Let Ω be a σ-finite measure space with measure dx. Let $j : \Omega \times \mathbb{R} \to [0, +\infty]$ be a normal convex integrand, i.e., $j(x, r)$ is measurable and, for a.e. $x \in \Omega$, $j(x, \cdot)$ is convex l.s.c. (= lower semi-continuous). We assume that

$$j(x,0) = 0 \text{ for a.e. } x \in \Omega \text{ and } j(x,r) = +\infty \text{ for a.e. } x \in \Omega \text{ and for all } r < 0. \tag{1.1}$$

Set

$$a(x) = \sup\{r \geqslant 0;\ \ j(x,r) < \infty\}. \tag{1.2}$$

Let $j^*(x,s)$ denote the conjugate convex function, that is,

$$j^*(x,s) = \sup_{r\in\mathbb{R}}\{sr - j(x,r)\} \text{ for a.e. } x \in \Omega,\ \text{ for all } s \in \mathbb{R}.$$

Note that

$$\begin{aligned} j^*(x,s) &= 0 \quad \text{for a.e. } x \in \Omega, \quad \text{for all } s \leqslant 0,\\ j^*(x,s) &\geqslant 0 \quad \text{for a.e. } x \in \Omega, \quad \text{for all } s \geqslant 0. \end{aligned}$$

Let $V : \Omega \to \mathbb{R}$ be a measurable function (so that $|V(x)| < \infty$ for a.e. $x \in \Omega$). The following assumption will sometimes play an important role:

$$\text{there exists a constant } M \text{ such that } j^*(\cdot, V(\cdot) - M) \in L^1(\Omega). \tag{H}$$

Note that assumption (H) holds for example if $V^+ \in L^\infty(\Omega)$.

Define the functional $J : L^1(\Omega) \to (-\infty, +\infty]$ to be

$$J(\rho) = \begin{cases} \int_\Omega \{j(x,\rho(x)) - V(x)\rho(x)\}\,dx & \text{if } j(\cdot,\rho) - V\rho \in L^1(\Omega),\\ +\infty & \text{otherwise.} \end{cases}$$

with

$$D(J) = \{\rho \in L^1(\Omega); \quad J(\rho) < +\infty\}.$$

In particular, if $\rho \in D(J)$, then $\rho(x) \geqslant 0$ for a.e. $x \in \Omega$.

REMARK 1. If (H) holds, then J is convex l.s.c. on $L^1(\Omega)$ and bounded below on bounded sets of $L^1(\Omega)$. This is a straightforward consequence of the fact that for every $\rho \in L^1(\Omega)$ we have $j(x,\rho(x)) - V(x)\rho(x) \geqslant -j^*(x, V(x) - M) - M\rho(x)$ for a.e. $x \in \Omega$ (so that we may use Fatou's lemma to check the lower semi-continuity of J). Note that if (H) does not hold, it may happen that $D(J)$ is not convex. Consider, for example $j(x,r) = r^2$ and a function V such that $V \geqslant 0$ a.e. and $V \notin L^2(\Omega)$; then $V \in D(J)$ while $\frac{1}{2}V \notin D(J)$.

Set

$$L_0^\infty(\Omega) = \{\rho \in L^\infty(\Omega);\ \rho = 0 \text{ outside a set of finite measure}\}.$$

Throughout the paper we shall assume that $k : \Omega \times \Omega \to \mathbb{R}$ is a measurable function satisfying

$$k(x,y) = k(y,x) \text{ and } k(x,y) \geqslant 0 \text{ for a.e. } x, y \in \Omega, \tag{1.3}$$

$$\begin{cases} \text{for every } \rho \in L_0^\infty(\Omega), \text{ then } k(x,y)\rho(x)\rho(y) \in L^1(\Omega\times\Omega) \\ \text{and } \iint_{\Omega\times\Omega} k(x,y)\rho(x)\rho(y)dxdy \geqslant 0 \end{cases} \tag{1.4}$$

(i.e., k is a nonnegative kernel).

Define the functional $K : L^1(\Omega) \to [0, +\infty]$ by

$$K(\rho) = \begin{cases} \frac{1}{2}\iint_{\Omega\times\Omega} k(x,y)\rho(x)\rho(y)dxdy & \text{if } \rho \in L^1(\Omega) \text{ and } \rho \geqslant 0 \text{ a.e. on } \Omega \\ +\infty & \text{otherwise,} \end{cases}$$

with

$$D(K) = \{\rho \in L^1(\Omega); \quad K(\rho) < +\infty\}.$$

Set

$$\mathcal{E}(\rho) = J(\rho) + K(\rho) \quad \text{for } \rho \in L^1(\Omega)$$

and

$$D(\mathcal{E}) = D(J) \cap D(K);$$

$\mathcal{E}$ is called the Thomas-Fermi energy functional.

Finally we introduce the mapping B defined for $\rho \in L^1(\Omega)$ with $\rho \geqslant 0$ a.e. on Ω, by

$$B\rho(x) = \int_\Omega k(x,y)\rho(y)dy.$$

The main result in this section is the following:

THEOREM 1. *Let $\rho_0 \in L^1(\Omega)$ with $\rho_0 \geqslant 0$ a.e. on Ω be such that*

$$0 < \int \rho_0(x)\,dx < \int a(x)\,dx, \tag{1.5}$$

and

$$\rho_0 \in D(\mathcal{E}) \text{ and } \mathcal{E}(\rho_0) \leqslant \mathcal{E}(\rho) \quad \forall \rho \in D(\mathcal{E}) \text{ with } \int \rho(x)\,dx = \int \rho_0(x)\,dx. \tag{M}$$

Then[1]

$$\begin{cases} \text{there exists a constant } \lambda \in \mathbb{R} \text{ such that} \\ \partial j(x, \rho_0(x)) + B\rho_0(x) \ni V(x) - \lambda \text{ for a.e. } x \in \Omega. \end{cases} \tag{E}$$

Conversely, when (H) *holds, then* (E) *implies* (M).

[1] $\partial j(x,r)$ denotes the subdifferential of $j(x,r)$ with respect to r.

REMARK 2. λ appears in (E) as a Lagrange multiplier corresponding to the constraint $\int \rho(x)dx = \int \rho_0(x)dx$ in the Euler equation (E) associated to the minimization problem (M).

In proving Theorem 1 we shall make use of the following:

LEMMA 1. *The functional K is convex l.s.c. on $L^1(\Omega)$. In addition*

$$\iint_{\Omega\times\Omega} k(x,y)\varphi(x)\psi(y)\,dxdy \leqslant K(\varphi)+K(\psi) \quad \forall\varphi,\psi\in D(K) \tag{1.6}$$

and equality in (1.6) *holds if and only if $B\varphi = B\psi$. Moreover we have*[2]

$$\text{if } \rho\in D(K) \text{ and } A\subset\Omega \text{ with } |A|<\infty, \text{ then } \chi_A B\rho\in L^1(\Omega). \tag{1.7}$$

Proof of Lemma 1. Let (Ω_n) be a nondecreasing sequence of measurable sets in Ω such that $|\Omega_n| < \infty\,\forall n$ and $\cup_n\Omega_n = \Omega$. Given $\varphi,\psi\in D(K)\cap L^\infty(\Omega)$ set

$$\varphi_n = \chi_{\Omega_n}\varphi \quad \text{and} \quad \psi_n = \chi_{\Omega_n}\psi.$$

By (1.4) we have

$$\iint_{\Omega\times\Omega} k(x,y)[\varphi_n(x)-\psi_n(x)][\varphi_n(y)-\psi_n(y)]\,dxdy \geqslant 0,$$

i.e.

$$\iint_{\Omega\times\Omega} k(x,y)\varphi_n(x)\psi_n(y)\,dxdy \leqslant K(\varphi_n)+K(\psi_n).$$

Using the monotone convergence theorem we obtain (1.6) for $\varphi,\psi\in D(K)\cap L^\infty(\Omega)$. The general case follows by truncation.

The function K is convex since for $\varphi,\psi\in D(K)$ and $t\in(0,1)$ we have

$$\begin{aligned} K((1-t)\varphi+t\psi) &= \frac{1}{2}\iint k(x,y)[(1-t)\varphi(x)+t\psi(x)][(1-t)\varphi(y)+t\psi(y)]\,dx\,dy \\ &= (1-t)^2K(\varphi)+t^2K(\psi)+t(1-t)\iint k(x,y)\varphi(x)\psi(x)\,dx\,dy \\ &\leqslant (1-t)^2K(\varphi)+t^2K(\psi)+t(1-t)\,[K(\varphi)+K(\psi)] \\ &= (1-t)K(\varphi)+tK(\psi). \end{aligned}$$

The lower semi-continuity of K follows from Fatou's lemma.

[2] $|A| = \text{meas}(A)$ denotes the measure of A and χ_A denotes the characteristic function of A.

Next, let $\rho \in D(K)$ and $A \subset \Omega$ with $|A| < \infty$. We have

$$\int_A (B\rho)(x)dx = \iint_{\Omega\times\Omega} k(x,y)\rho(y)\chi_A(x)\,dxdy \leqslant K(\rho) + K(\chi_A).$$

Finally we show that equality in (1.6) holds if and only if $B\varphi = B\psi$.
First, suppose that $B\varphi = B\psi$; then we have

$$\int \varphi B\psi = \frac{1}{2}\int \varphi B\psi + \frac{1}{2}\int \psi B\varphi = K(\varphi) + K(\psi).$$

Conversely, assume that equality in (1.6) holds. Note that

$$\int (\psi + \zeta)B\varphi \leqslant K(\varphi) + K(\psi) + K(\zeta) + \int \zeta B\psi, \quad \forall\varphi, \psi, \zeta \in D(K),$$

and since (1.6) holds we obtain $\int \zeta B\varphi \leqslant K(\zeta) + \int \zeta B\psi$. Replacing ζ by $\lambda\zeta, \lambda > 0$, we see that $\int \zeta B\varphi \leqslant \int \zeta B\psi \forall\zeta \in D(K)$. Reversing φ and ψ we find $\int \zeta B\varphi = \int \zeta B\psi$ $\forall\zeta \in D(K)$ and consequently $\int \zeta B\varphi = \int \zeta B\psi \forall\zeta \in L^\infty_0(\Omega)$. Therefore we have $B\varphi = B\psi$.

REMARK 3. The argument above shows that K is a strictly convex function on $D(K)$ if and only if B is injective.

Proof of Theorem 1. (E) $\Rightarrow$ (M) (*under assumption* (H)).
Indeed, by (E) and the definition of the subdifferential, we have for $\rho \in L^1(\Omega)$

$$j(\cdot,\rho) \geqslant j(\cdot,\rho_0) + (V - B\rho_0 - \lambda)(\rho - \rho_0) \text{ a.e. on } \Omega. \tag{1.8}$$

In particular, for $\rho \equiv 0$, we find

$$j(\cdot,\rho_0) - V\rho_0 + (B\rho_0)\rho_0 \leqslant -\lambda\rho_0 \text{ a.e. on } \Omega.$$

From (H) it follows that $j(\cdot,\rho_0) - V\rho_0$ is bounded below by some L^1 function; thus $\rho_0 \in D(\mathcal{E})$. Now let $\rho \in D(\mathcal{E})$ with $\int \rho(x)dx = \int \rho_0(x)dx$; integrating (1.8) and using (1.6) we obtain (M).
(M) $\Rightarrow$ (E) (*without assumption* (H), but with (1.5)).
First let $\zeta \in D(\mathcal{E})$ with $\int \zeta(x)\,dx = \int \rho_0(x)\,dx$ and $V(\zeta - \rho_0) \in L^1(\Omega)$.
Let

$$\rho_t = (1-t)\rho_0 + t\zeta \quad \text{ with } 0 < t < 1.$$

We claim that

$$\rho_t \in D(J) \text{ and } J(\rho_t) \leqslant (1-t)J(\rho_0) + tJ(\zeta). \tag{1.9}$$

Indeed we have a.e. on Ω,

$$j(\cdot,\rho_t) - V\rho_t \leqslant (1-t)(j(\cdot,\rho_0) - V\rho_0) + t(j(\cdot,\zeta) - V\zeta). \tag{1.10}$$

On the other hand we have, a.e. on Ω,

$$j(\cdot,\rho_t) \geqslant \min\{j(\cdot,\rho_0), j(\cdot,\zeta)\}$$

– this follows from the monotonicity of $j(x,\cdot)$ on $[0,+\infty)$. Therefore we obtain a.e. on Ω,

$$j(\cdot,\rho_t) - V\rho_t \geqslant \min\{j(\cdot,\rho_0) - V\rho_0, j(\cdot,\zeta) - V\zeta\} - 2|V(\rho_0-\zeta)|. \tag{1.11}$$

Combining (1.10) and (1.11) we see that $\rho_t \in D(J)$ and integrating (1.10) we find

$$J(\rho_t) \leqslant (1-t)J(\rho_0) + tJ(\zeta).$$

It follows that

$$\mathcal{E}(\rho_t) \leqslant (1-t)J(\rho_0) + tJ(\zeta) + (1-t)^2K(\rho_0) + t^2K(\zeta) + t(1-t)\int (B\rho_0)\zeta.$$

By assumption (M) we have

$$\mathcal{E}(\rho_0) = J(\rho_0) + K(\rho_0) \leqslant \mathcal{E}(\rho_t)$$

and thus

$$tJ(\rho_0) + (2t-t^2)K(\rho_0) \leqslant tJ(\zeta) + t^2K(\zeta) + t(1-t)\int (B\rho_0)\zeta.$$

Dividing by t and letting $t \to 0$ we find

$$\begin{cases} J(\rho_0) + \int (B\rho_0)\rho_0 \leqslant J(\zeta) + \int (B\rho_0)\zeta \\ \forall \zeta \in D(\mathcal{E}) \text{ with } \int \zeta(x)\,dx = \int \rho_0(x)\,dx \text{ and } V(\zeta-\rho_0) \in L^1(\Omega). \end{cases} \tag{1.12}$$

Set $\widetilde{V} = V - B\rho_0$ and define the functional $\widetilde{J} : L^1(\Omega) \to (-\infty,+\infty]$ by

$$\widetilde{J}(u) = \begin{cases} \int \{j(x,u(x)) - \widetilde{V}(x)u(x)\}\,dx & \text{if } j(\cdot,u) - \widetilde{V}u \in L^1(\Omega), \\ +\infty & \text{otherwise.} \end{cases}$$

It is clear that $\rho_0 \in D(\widetilde{J})$ (since $\rho_0 \in D(\mathcal{E})$).

We claim that

$$\begin{cases} \widetilde{J}(\rho_0) \leqslant \widetilde{J}(\zeta) \quad \forall \zeta \in D(\widetilde{J}) \\ \text{with } \int \zeta(x)\,dx = \int \rho_0(x)\,dx,\ (\zeta-\rho_0) \in L_0^\infty(\Omega) \text{ and } \widetilde{V}(\zeta-\rho_0) \in L^1(\Omega). \end{cases} \tag{1.13}$$

Indeed, suppose ζ satisfies the assumptions in (1.13), then ζ also satisfies the assumptions in (1.12). Note that:

a) $\zeta \in D(K)$, since $\zeta \leqslant \rho_0 + |\zeta - \rho_0| \in D(K) + D(K)$;
b) $\zeta \in D(\widetilde{J}) \cap D(K) \Rightarrow \zeta \in D(J)$, since $j(\cdot,\zeta) - V\zeta = j(\cdot,\zeta) - \widetilde{V}\zeta - (B\rho_0)\zeta$;
c) $V(\zeta-\rho_0) = \widetilde{V}(\zeta-\rho_0) + (B\rho_0)(\zeta-\rho_0)$.

We conclude the proof of Theorem 1 with the help of the next lemma applied to $\widetilde{J}$ (instead of J).

LEMMA 2. *Assume* $\rho_0 \in D(J)$ *satisfies* (1.5)*, as well as*

$$\begin{cases} J(\rho_0) \leqslant J(\rho) \quad \forall \rho \in D(J) \\ \text{with } \int \rho(x)\,dx = \int \rho_0(x)\,dx,\ (\rho - \rho_0) \in L_0^\infty(\Omega) \text{ and } V(\rho - \rho_0) \in L^1(\Omega). \end{cases} \tag{1.14}$$

Then there exists a constant $\lambda \in \mathbb{R}$ *such that*

$$V - \lambda \in \partial j(\cdot, \rho_0) \quad \text{a.e. on } \Omega. \tag{1.15}$$

Proof. Set

$$E_- = \{x \in \Omega\,;\, \rho_0(x) > 0\},$$
$$E_+ = \{x \in \Omega\,;\, \rho_0(x) < a(x)\}.$$

It follows from (1.5) that $|E_-| > 0$ and $|E_+| > 0$. Let (Ω_n) be as in the proof of Lemma 1 and set

$$\Omega_n' = \{x \in \Omega_n;\, |V(x)| + \rho_0(x) < n\},$$

so that $|E_- \cap \Omega_n'| \uparrow |E_-|$ and $|E_+ \cap \Omega_n'| \uparrow |E_+|$ as $n \to \infty$. Fix n_0 such that

$$|E_- \cap \Omega_{n_0}'| > 0 \quad \text{and} \quad |E_+ \cap \Omega_{n_0}'| > 0.$$

In what follows we choose $n \geqslant n_0$; for every $\lambda \in \mathbb{R}$ set

$$u_\lambda(x) = (I + \partial j(x, \cdot))^{-1}(V(x) + \rho_0(x) - \lambda) \quad \text{for } x \in \Omega, \tag{1.16}$$

$$I_\lambda = \int_{\Omega_n'} u_\lambda(x)\,dx. \tag{1.17}$$

Note that I_λ makes sense since $|u_\lambda(x)| \leqslant n + |\lambda|$ on Ω_n' and $|\Omega_n'| < \infty$. Clearly we have $u_\lambda(x) \uparrow a(x)$ as $\lambda \downarrow -\infty$ and $u_\lambda(x) \downarrow 0$ as $\lambda \uparrow +\infty$. Therefore

$$\lim_{\lambda \to -\infty} I_\lambda = \int_{\Omega_n'} a(x)\,dx \ \text{ and } \lim_{\lambda \to +\infty} I_\lambda = 0.$$

On the other hand we have

$$0 < \int_{\Omega_n'} \rho_0(x)\,dx < \int_{\Omega_n'} a(x)\,dx$$

since $n \geqslant n_0$. Thus, there exists a constant $\lambda_n \in \mathbb{R}$ such that

$$I_{\lambda_n} = \int_{\Omega_n'} \rho_0(x) dx \tag{1.18}$$

(note that I_λ is a continuous function of λ and, in fact, $|I_\lambda - I_\mu| \leqslant |\lambda - \mu|\,|\Omega'_n|$). It follows from (1.16) that, a.e. on Ω,

$$u_{\lambda_n}(x) + \partial j(x, u_{\lambda_n}(x)) \ni V(x) + \rho_0(x) - \lambda_n \tag{1.19}$$

and so

$$j(x, \rho_0(x)) - j(x, u_{\lambda_n}(x)) \geqslant (V(x) + \rho_0(x) - \lambda_n - u_{\lambda_n}(x))(\rho_0(x) - u_{\lambda_n}(x)).$$

Hence a.e. on Ω we find

$$\begin{aligned} & j(x, u_{\lambda_n}(x)) - V(x)u_{\lambda_n}(x) \\ & \quad \leqslant j(x, \rho_0(x)) - V(x)\rho_0(x) - (\rho_0(x) - u_{\lambda_n}(x))^2 - \lambda_n(u_{\lambda_n}(x) - \rho_0(x)). \end{aligned} \tag{1.20}$$

On the other hand we have, a.e. on Ω'_n,

$$j(x, u_{\lambda_n}(x)) - V(x)u_{\lambda_n}(x) \geqslant -V(x)u_{\lambda_n}(x) \geqslant -n(n + |\lambda_n|). \tag{1.21}$$

Combining (1.20) and (1.21) we see that

$$j(\cdot, u_{\lambda_n}) - Vu_{\lambda_n} \in L^1(\Omega'_n).$$

Set

$$\rho = \begin{cases} u_{\lambda_n} & \text{on } \Omega'_n, \\ \rho_0 & \text{on } \Omega \setminus \Omega'_n. \end{cases}$$

Therefore ρ satisfies all the assumptions in (1.14) and we deduce that

$$\int_{\Omega'_n} \{j(x, \rho_0(x)) - V(x)\rho_0(x)\}\,dx \leqslant \int_{\Omega'_n} \{j(x, u_{\lambda_n}(x)) - V(x)u_{\lambda_n}(x)\}\,dx. \tag{1.22}$$

Combining (1.20) and (1.22) we find

$$\rho_0 = u_{\lambda_n} \quad \text{a.e. on } \Omega'_n.$$

It follows from (1.19) that

$$V(x) - \lambda_n \in \partial j(x, \rho_0(x)) \quad \text{for a.e. } x \in \Omega'_n.$$

For every $n \geqslant n_0$, set

$$\Lambda_n = \{\lambda \in \mathbb{R};\ V(x) - \lambda \in \partial j(x, \rho_0(x)) \quad \text{for a.e. } x \in \Omega'_n\}.$$

We have just established that $\Lambda_n \neq \emptyset$. Clearly Λ_n is a closed interval. Moreover Λ_n is bounded; indeed if, for instance, Λ_n were unbounded below we would have $\rho_0(x) = a(x)$ for a.e. $x \in \Omega'_n$ – a contradiction with $|E_+ \cap \Omega'_n| > 0$. Since Λ_n decreases with n we obtain

$$\bigcap_{n \geqslant n_0} \Lambda_n \neq \emptyset$$

and the conclusion of Lemma 2 follows. □

Our next lemma – which will be used later – is closely related to Theorem 1.

LEMMA 3. *Let* $\rho_0 \in L^1(\Omega)$ *with* $\rho_0 \geqslant 0$ *a.e. on* Ω *be such that*

$$\rho_0 \in D(\mathcal{E}) \text{ and } \mathcal{E}(\rho_0) \leqslant \mathcal{E}(\rho) \quad \forall \rho \in D(\mathcal{E}). \tag{1.23}$$

Then

$$\partial j(x, \rho_0(x)) + B\rho_0(x) \ni V(x) \quad \text{for a.e. } x \in \Omega. \tag{1.24}$$

Conversely, when (H) *holds, then* (1.24) *implies* (1.23).

REMARK 4. Note that assumption (1.5) is not required in Lemma 3.

Proof of Lemma 3. In order to prove that (1.24) $\Rightarrow$ (1.23) under assumption (H) one proceeds exactly as in the proof of (E) $\Rightarrow$ (M).

In order to prove that (1.23) $\Rightarrow$ (1.24) one uses the same $\widetilde{V}$ and $\widetilde{J}$ as in the proof of Theorem 1 and one shows that

$$\widetilde{J}(\rho_0) \leqslant \widetilde{J}(\zeta) \quad \forall \zeta \in D(\widetilde{J}) \text{ with } (\zeta - \rho_0) \in L^\infty_0(\Omega) \text{ and } \widetilde{V}(\zeta - \rho_0) \in L^1(\Omega). \tag{1.25}$$

Next one considers

$$\Omega'_n = \{x \in \Omega_n;\ |\widetilde{V}(x)| + \rho_0(x) < n\}$$

and one uses (1.25) with

$$\zeta = \begin{cases} u & \text{on } \Omega'_n, \\ \rho_0 & \text{on } \Omega \backslash \Omega'_n, \end{cases}$$

where $u(x) = (I + \partial j(x, \cdot))^{-1}(\widetilde{V}(x) + \rho_0(x))$. This leads to $\rho_0 = u$ a.e. on Ω'_n and $\widetilde{V}(x) \in \partial j(x, \rho_0(x))$ for a.e. $x \in \Omega'_n$.

REMARK 5. Suppose that (E) and (H) hold. Then we have, in fact, a stronger conclusion than (M), namely

$$\mathcal{E}(\rho_0) + \lambda \int \rho_0 \leqslant \mathcal{E}(\rho) + \lambda \int \rho \quad \forall \rho \in D(\mathcal{E}). \tag{1.26}$$

This follows from Lemma 3 applied with $(V - \lambda)$ instead of V. In particular, if we happen to know that $\lambda \geqslant 0$ (for example Lemma 8 implies that this holds when $V_\infty - j'(0+) \geqslant 0$, where V_∞ is defined at the beginning of Section 2), then we have

$$\mathcal{E}(\rho_0) \leqslant \mathcal{E}(\rho) \quad \forall \rho \in D(\mathcal{E}) \text{ with } \int \rho(x)dx \leqslant \int \rho_0(x)dx. \tag{1.27}$$

This explains why one can use a "relaxation" method (see Lieb-Simon [48] and also Proposition 3 in Section 2). In other words, the constraint $\int \rho(x)dx = I$ in the minimization problem is "relaxed" to $\int \rho(x)dx \leqslant I$.

REMARK 6. Assume (H) holds. Then we have

$$\overline{D(\mathcal{E})}^{L^1} = \{\rho \in L^1(\Omega);\, 0 \leqslant \rho(x) \leqslant a(x) \text{a.e. on } \Omega\} \tag{1.28}$$

and consequently, for every constant I with $0 \leqslant I < \int a(x)\,dx$, there is some $\rho \in D(\mathcal{E})$ such that $\int \rho(x)dx = I$. For this purpose, it suffices to show that every function $\rho \in L_0^\infty(\Omega)$ such that $0 \leqslant \rho(x) \leqslant a(x)$ a.e. on Ω, belongs to $\overline{D(\mathcal{E})}^{L^1}$. Indeed, set

$$\rho_\varepsilon(x) = \frac{(\rho(x) - \varepsilon)^+}{1 + \varepsilon j(x, (\rho(x) - \varepsilon)^+) + \varepsilon |V(x)|}, \quad \varepsilon > 0,$$

and note that $\rho_\varepsilon \in D(\mathcal{E})$ and $\rho_\varepsilon \to \rho$ in $L^1(\Omega)$ as $\varepsilon \to 0$.

2. Existence via the variational route

Given a constant I with $0 \leqslant I < \infty$ we set

$$K_I = \left\{\rho \in D(\mathcal{E});\, \int \rho(x)dx = I\right\}.$$

In this section we are concerned with the following problem:

$$\text{find } \bar{\rho} \in K_I \text{ such that } \mathcal{E}(\bar{\rho}) \leqslant \mathcal{E}(\rho) \quad \forall \rho \in K_I. \tag{M_I}$$

For simplicity, we shall now assume that $j(x, r) = j(r)$ is independent of x and we set

$$a = \sup\{r \geqslant 0;\ j(r) < \infty\} \leqslant \infty.$$

Of course, we assume that $a > 0$.

We recall (see Remark 6) that $K_I \neq \emptyset$ for every $I < a|\Omega|$. When $I = a|\Omega|$ (assuming $a|\Omega| < \infty$), then either K_I is reduced to a single element $\{a\}$ or $K_I = \emptyset$ – so that problem (M_I) has no interest. Therefore we may always assume that $I < a|\Omega|$.

We shall encounter two different situations:

– in CASE I, a strong assumption (on V or Ω) implies that problem (M_I) has a solution for every $I < a|\Omega|$,
– in CASE II, problem (M_I) has a solution only for a *limited* range of I's, usually smaller than the interval $[0, a|\Omega|)$.

Throughout Section 2 we make an assumption slightly stronger than (H), namely

$$\text{there exist constants } \theta > 0 \text{ and } M \in \mathbb{R} \text{ such that } j^*((1+\theta)(V - M)) \in L^1(\Omega). \tag{H^+}$$

We also assume that j is coercive, i.e.,

$$\lim_{r \to +\infty} \frac{j(r)}{r} = +\infty. \tag{2.1}$$

Finally, we set[3]

$$V_\infty = \inf\{\alpha \in \mathbb{R}; \quad [V > \alpha] \text{ has finite measure}\}.$$

Note that there exist α's such that $[V > \alpha]$ has finite measure (this is so because (H) holds and $j^* \not\equiv 0$ since $a > 0$). Therefore we have either $V_\infty \in \mathbb{R}$ or $V_\infty = -\infty$. Of course if $|\Omega| < \infty$, then we have $V_\infty = -\infty$. In the special case where $\Omega = \mathbb{R}^N$ and $V(\infty) = \lim_{|x|\to\infty} V(x)$ exists, then $V_\infty = V(\infty)$.

CASE I. We assume here that

$$V_\infty = -\infty. \tag{2.2}$$

The main result is the following:

THEOREM 2. *Assume* (H^+), (2.1) *and* (2.2). *Then, for every I with $0 \leqslant I < a|\Omega|$ there exists a solution of* (M_I).

In the proof of Theorem 2 we shall use

LEMMA 4. *Assume* (H^+). *Let (ρ_n) be a sequence in $D(J)$ such that*

$$\int j(\rho_n) - V\rho_n \leqslant C_1 \text{ and } \int \rho_n \leqslant C_1 \quad \forall n, \text{ for some constant } C_1 > 0. \tag{2.3}$$

Then, there exists a constant C_2 such that

$$\int j(\rho_n) \leqslant C_2 \text{ and } \int |V\rho_n| \leqslant C_2 \quad \forall n. \tag{2.4}$$

Proof of Lemma 4. Set $\omega(x) = j^*((1+\theta)(V(x) - M))$ so that $\omega \in L^1(\Omega)$ and $(1+\theta)(V - M)\rho_n \leqslant j(\rho_n) + \omega$. It follows that

$$V\rho_n \leqslant \frac{1}{1+\theta} j(\rho_n) + \omega + M\rho_n \tag{2.5}$$

and, using (2.3), we obtain

$$\int j(\rho_n) \leqslant C_1 + \frac{1}{1+\theta} \int j(\rho_n) + \int \omega + |M|\, C_1.$$

This leads to $\int j(\rho_n) \leqslant C_2$. Next, set

$$f_n = \frac{1}{1+\theta} j(\rho_n) + \omega + M\rho_n - V\rho_n$$

[3] We use the notations $[V > \alpha] = \{x \in \Omega;\ V(x) > \alpha\}$ and $[V \geqslant \alpha] = \{x \in \Omega;\ V(x) \geqslant \alpha\}$ etc...

so that $f_n \geqslant 0$ (by (2.5)). From (2.3) we have

$$\int j(\rho_n) + f_n - \frac{1}{1+\theta} j(\rho_n) - \omega - M\rho_n \leqslant C_1$$

and thus

$$\int |f_n| \leqslant C_1 + \int \omega + |M| C_1.$$

It follows that

$$\int |V\rho_n| \leqslant C_1 + 2\int \omega + 2|M| C_1 + \int j(\rho_n)$$

and therefore we obtain a bound for $\int |V\rho_n|$.

Proof of Theorem 2. From assumption (H) we have

$$j(\rho) - V\rho \geqslant -j^*(V-M) - M\rho \quad \forall \rho \in K_I,$$

so that

$$\mathcal{E}(\rho) \geqslant -\int j^*(V-M) - MI \quad \forall \rho \in K_I,$$

and consequently

$$E(I) = \inf_{\rho \in K_I} \mathcal{E}(\rho) > -\infty. \tag{2.6}$$

Let (ρ_n) be a minimizing sequence for (2.6). From Lemma 4 we deduce that

$$\int j(\rho_n) \leqslant C \tag{2.7}$$

and

$$\int |V|\rho_n \leqslant C, \tag{2.8}$$

for some constant C. We claim that the sequence (ρ_n) is equi-integrable in Ω, that is,

$$\begin{aligned} &\forall \varepsilon > 0 \ \exists \delta > 0 \quad \text{such that } \int_A \rho_n < \varepsilon \ \forall n, \\ &\quad \forall A \subset \Omega \text{ measurable with } |A| < \delta, \end{aligned} \tag{2.9}$$

and

$$\forall \varepsilon > 0 \ \exists \Omega' \subset \Omega \text{ measurable with } |\Omega'| < \infty \text{ such that } \int_{\Omega \setminus \Omega'} \rho_n < \varepsilon \ \forall n. \tag{2.10}$$

Verification of (2.9). Given any $k > 0$, there is a constant C_k such that

$$j(r) \geqslant kr - C_k \quad \forall r \geqslant 0$$

(this follows from (2.1)).
Consequently, we have for every measurable set $A \subset \Omega$

$$k \int_A \rho_n \leqslant \int_\Omega j(\rho_n) + C_k\,|A|$$

so that (by (2.7)) we obtain

$$\int_A \rho_n \leqslant \frac{C}{k} + \frac{C_k}{k}\,|A|.$$

Given $\varepsilon > 0$ we fix k large so that $\frac{C}{k} < \frac{\varepsilon}{2}$ and then we choose $\delta > 0$ so small that $\frac{C_k}{k}\,\delta < \frac{\varepsilon}{2}$.

Verification of (2.10). We recall that

$$\int |V\rho_n| \leqslant C.$$

Choose $k > 0$ so large that $\frac{C}{k} < \varepsilon$ and set $\Omega' = [V > -k]$. It follows from assumption (2.2) that $|\Omega'| < \infty$. Clearly, we have

$$k \int_{\Omega\setminus\Omega'} \rho_n \leqslant \int_{\Omega\setminus\Omega'} |V\rho_n| \leqslant C$$

and thus

$$\int_{\Omega\setminus\Omega'} \rho_n \leqslant \frac{C}{k} < \varepsilon \quad \forall n.$$

We may therefore apply the Dunford-Pettis theorem (see e.g. Dunford-Schwartz [30], Corollary IV.8.11) and conclude that there exists a subsequence (ρ_{n_k}) such that $\rho_{n_k} \rightharpoonup \bar{\rho}$ weakly in $L^1(\Omega)$. It follows that $\int \bar{\rho} = I$ and $\mathcal{E}(\bar{\rho}) \leqslant \inf_{\rho \in K_I} \mathcal{E}(\rho)$ (since $\mathcal{E}$ is convex and l.s.c. on $L^1(\Omega)$).

We now turn to Case II, which is the most important from the point of view of applications.

CASE II. We assume here that

$$V_\infty > -\infty.$$

This implies in particular that $|\Omega| = \infty$. For simplicity we will assume throughout the rest of this section that

$$V_\infty = 0. \tag{2.11}$$

This is just a normalization condition since in the problems of interest we may always add a constant to V. Note that (2.11) implies in particular that $\operatorname{ess\,sup}_\Omega V \geqslant 0$.

Concerning j we will assume that $j : \mathbb{R} \to [0, +\infty]$ is convex l.s.c.,

$$j(r) = +\infty \quad \text{for } r < 0 \text{ and } j(0) = 0, \tag{2.12}$$

$$j \text{ is finite and } C^1 \text{ on } (0, \infty), \tag{2.13}$$

$$j'(0+) = \lim_{r\downarrow 0} \frac{j(r)}{r} = 0. \tag{2.14}$$

In addition, we assume that

$$\mathcal{E} \text{ is strictly convex on } D(\mathcal{E}) \tag{2.15}$$

and

$$\text{for every } \rho \in D(\mathcal{E}) \text{ and every } \delta > 0, \text{ the set } [B\rho > \delta] \text{ has finite measure.} \tag{2.16}$$

Condition (2.16) says that, in some weak sense, $B\rho \to 0$ at "infinity".

In order to study problem (M_I), it will be extremely useful to introduce an auxiliary problem. For every $\lambda \in \mathbb{R}$, consider

$$\inf\left\{\mathcal{E}(\rho) + \lambda \int \rho; \quad \rho \in D(\mathcal{E})\right\}. \tag{P$_\lambda$}$$

The main result is the following:

THEOREM 3. *Assume* (H^+), (2.1), (2.11), (2.12), (2.13), (2.14), (2.15), *and* (2.16). *Then,*

$$\textit{for every } \lambda > 0, \textit{ problem } (P_\lambda) \textit{ admits a unique minimizer } \rho_\lambda, \tag{2.17}$$

$$\textit{for every } \lambda < 0, \textit{ the infimum in } (P_\lambda) \textit{ is } -\infty. \tag{2.18}$$

Set

$$I(\lambda) = \int \rho_\lambda, \quad \lambda > 0.$$

Then the function $\lambda \longmapsto I(\lambda)$ *is nonincreasing, and continuous from* $(0, \infty)$ *into* $[0, \infty)$ *More precisely,*

$$I(\lambda) \textit{is decreasing on} (0, \operatorname*{ess\,sup}_\Omega V), \tag{2.19}$$

$$\begin{cases} I(\lambda) = 0 \quad \forall\lambda \geqslant \operatorname{ess\,sup}_\Omega V & \textit{if } \operatorname{ess\,sup}_\Omega V < \infty, \\ \lim_{\lambda\to\infty} I(\lambda) = 0 & \textit{if } \operatorname{ess\,sup}_\Omega V = \infty, \end{cases} \tag{2.20}$$

$$\begin{cases} I_0 = \lim_{\lambda\downarrow 0} I(\lambda) = \sup_{\lambda>0} I(\lambda) < \infty \quad \textit{if and only if} \\ (P_0) \textit{ admits a minimizer } \rho_0 \in D(\mathcal{E}), \textit{ and then } I_0 = \int \rho_0. \end{cases} \tag{2.21}$$

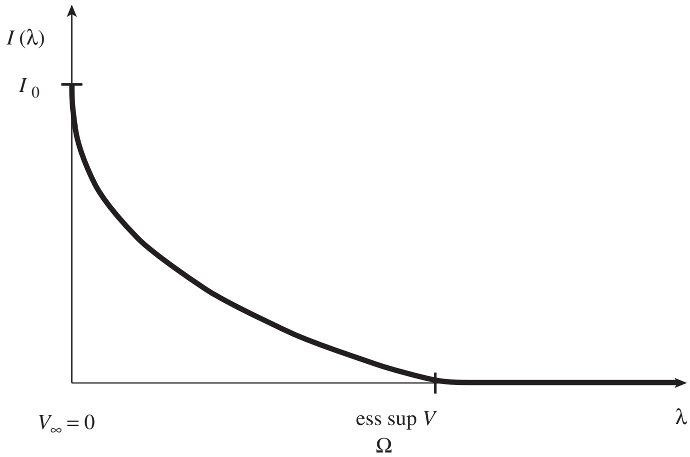

Figure 4 Typical shape of $I(\lambda)$.

The proof of Theorem 3 is based on several lemmas.

LEMMA 5. *Assume* (H^+), (2.1), (2.11), *and* (2.12). *Then, for every* $\varepsilon > 0$ *there exists a function* $\omega_\varepsilon \in L^1(\Omega)$ *such that*

$$j(r) - V(x)r + \varepsilon r \geqslant \omega_\varepsilon(x) \quad \text{for a.e. } x \in \Omega, \quad \forall r \geqslant 0. \tag{2.22}$$

Proof. Set $A = [V > \varepsilon]$ and so $|A| < \infty$ (since $V_\infty = 0$). For $x \in {}^cA = [V \leqslant \varepsilon]$ we have

$$j(r) - V(x)r + \varepsilon r \geqslant j(r) \geqslant 0$$

and thus we choose $\omega_\varepsilon(x) = 0$ on ${}^cA$.

Given any $k > 0$ (to be fixed later) there is a constant C_k such that

$$j(r) \geqslant kr - C_k \quad \forall r \geqslant 0.$$

(Here we have used (2.1)). For $x \in [\varepsilon < V \leqslant k]$ we have

$$j(r) - V(x)r + \varepsilon r \geqslant j(r) - kr \geqslant -C_k$$

and so we choose $\omega_\varepsilon(x) = -C_k$ on $[\varepsilon < V \leqslant k]$.

Finally, we consider the case where $x \in [V > k]$. We now use assumption (H^+) to write

$$(1+\theta)(V(x) - M)\, r \leqslant j(r) + \omega(x)$$

where $\omega(x) = j^*((1+\theta)(V(x)-M))$. Therefore we have

$$\begin{aligned} j(r) - V(x)r + \varepsilon r &\geqslant (1+\theta)(V(x)-M)r - \omega(x) - V(x)r + \varepsilon r \\ &\geqslant r[-M + \theta V(x) - \theta M] - \omega(x) \geqslant -\omega(x) \end{aligned}$$

provided we fix k so large that $-M + \theta k - \theta M \geqslant 0$. Hence we may choose $\omega_\varepsilon(x) = -\omega(x)$ on $[V > k]$. □

LEMMA 6. *Same assumptions as in Lemma* 5. *Let* (ρ_n) *be a sequence in* $D(\mathcal{E})$ *such that* $\int \rho_n \leqslant C$ *and* $\rho_n \rightharpoonup \bar{\rho}$ *weakly in* $L^1(\Omega_j)$ *for each* j.[4] *Then*

$$\mathcal{E}(\bar{\rho}) \leqslant \liminf_{n\to\infty} \mathcal{E}(\rho_n).$$

Proof. For every $\varepsilon > 0$, let $\omega_\varepsilon(x)$ be as in Lemma 5. We have

$$\mathcal{E}(\rho_n) \geqslant \mathcal{E}(\rho_n \chi_{\Omega_j}) + \int_{\Omega\setminus\Omega_j} \omega_\varepsilon(x)\,dx - \varepsilon \int_\Omega \rho_n(x)\,dx.$$

For each j, $\rho_n \chi_{\Omega_j} \rightharpoonup \bar{\rho}\,\chi_{\Omega_j}$ weakly in $L^1(\Omega)$.

Hence we obtain, for each j,

$$\liminf_{n\to\infty} \mathcal{E}(\rho_n) \geqslant \mathcal{E}(\bar{\rho}\,\chi_{\Omega_j}) + \int_{\Omega\setminus\Omega_j} \omega_\varepsilon(x)dx - \varepsilon C.$$

We conclude by letting $j \to \infty$ and then $\varepsilon \to 0$. □

LEMMA 7. *Same assumptions as in Lemma* 5. *Then for every* $\lambda > 0$ *there is some* $\bar{\rho} \in D(\mathcal{E})$ *such that*

$$\mathcal{E}(\bar{\rho}) + \lambda \int \bar{\rho} \leqslant \mathcal{E}(\rho) + \lambda \int \rho \quad \forall \rho \in D(\mathcal{E}). \tag{2.23}$$

Proof. Applying Lemma 5 with $\varepsilon = \lambda/2$, we obtain some function $\omega \in L^1(\Omega)$ such that

$$j(r) - V(x)r + \frac{\lambda}{2} r \geqslant \omega(x) \text{ a.e. in } \Omega,\ \forall r \geqslant 0,$$

and so

$$j(r) - V(x)r + \lambda r \geqslant \frac{\lambda}{2} r + \omega(x) \text{ a.e. in } \Omega,\ \forall r \geqslant 0.$$

Therefore, for every $\rho \in D(\mathcal{E})$, we have

$$\mathcal{E}(\rho) + \lambda \int \rho \geqslant \frac{\lambda}{2} \int \rho - C.$$

[4] We recall that (Ω_j) is a nondecreasing sequence of measurable sets in Ω such that $|\Omega_j| < \infty \quad \forall j$ and $\cup_j \Omega_j = \Omega$

Thus if (ρ_n) is a minimizing sequence for (2.23), then $\int \rho_n \leqslant C$ and also $\int j(\rho_n) - V\rho_n \leqslant C$. We deduce from Lemma 4 that $\int j(\rho_n) \leqslant C$. Therefore, the sequence (ρ_n) is equi-integrable on each Ω_j and we may extract a subsequence still denoted (ρ_n) such that $\rho_n \rightharpoonup \bar{\rho}$ weakly in $L^1(\Omega_j)$ for each j. We conclude with the help of Lemma 6 that (2.23) holds. □

A final lemma,

LEMMA 8. *Assume* (2.12), (2.13), (2.14), (2.16), *and suppose* $\rho \in D(\mathcal{E})$. *Let W be any measurable function satisfying*

$$\partial j(\rho) + B\rho \ni W \quad \text{a.e. on } \Omega. \tag{2.24}$$

Then

$$W_\infty \leqslant 0. \tag{2.25}$$

Proof. Let $\alpha > 0$; we shall prove that $W_\infty \leqslant \alpha$. Indeed fix ε such that $0 < \varepsilon < \alpha$. By assumption (2.16) the set $\Omega_1 = [B\rho > \varepsilon]$ has finite measure. Since $\alpha - \varepsilon > 0$ there exists $\delta > 0$ such that $\partial j(r) \subset (-\infty, \alpha - \varepsilon]$ for $r \in [0, \delta]$. (Here we have used (2.14)). The set $\Omega_2 = [\rho > \delta]$ has also finite measure (since $\rho \in L^1(\Omega)$). Using (2.24) we see that

$$[W > \alpha] \subset \Omega_1 \cup \Omega_2$$

and thus the set $[W > \alpha]$ has finite measure. □

Proof of Theorem 3. We split the proof into 5 steps.

STEP 1. The existence of a minimizer ρ_λ for (P_λ) when $\lambda > 0$ has been established in Lemma 7. We prove that $I(\lambda) = \int \rho_\lambda$ is nonincreasing and continuous on $(0, \infty)$.

Proof. Let $\lambda, \mu > 0$. We have

$$\begin{cases} \mathcal{E}(\rho_\lambda) + \lambda I(\lambda) \leqslant \mathcal{E}(\rho_\mu) + \lambda I(\mu) \\ \mathcal{E}(\rho_\mu) + \mu I(\mu) \leqslant \mathcal{E}(\rho_\lambda) + \mu I(\lambda) \end{cases}$$

and thus

$$(\lambda - \mu)(I(\lambda) - I(\mu)) \leqslant 0,$$

so that the function $\lambda \longmapsto I(\lambda)$ is nonincreasing. □

We now prove that $I(\lambda)$ is continous on $(0, +\infty)$. Let $\lambda_n \to \bar{\lambda}$ with $\bar{\lambda} > 0$ and set $\rho_n = \rho_{\lambda_n}$. It is easy to see (as in the proof of Lemma 7) that $\int \rho_n \leqslant C$ and $\int j(\rho_n) \leqslant C$. Therefore we may extract a subsequence (ρ_{n_k}) such that $\rho_{n_k} \rightharpoonup \bar{\rho}$ weakly in $L^1(\Omega_j)$ for each j. We have

$$\mathcal{E}(\rho_{n_k}) + \lambda_{n_k} \int \rho_{n_k} \leqslant \mathcal{E}(\rho) + \lambda_{n_k} \int \rho \quad \forall \rho \in D(\mathcal{E}); \tag{2.26}$$

passing to the limit as $k \to \infty$ we find

$$\mathcal{E}(\bar{\rho}) + \bar{\lambda} \int \bar{\rho} \leqslant \mathcal{E}(\rho) + \bar{\lambda} \int \rho \quad \forall \rho \in D(\mathcal{E}),$$

so that $\bar{\rho}$ and $\rho_{\bar{\lambda}}$ are both minimizers for the problem $(P_{\bar{\lambda}})$. By (2.15) it follows that $\bar{\rho} = \rho_{\bar{\lambda}}$, $\int \bar{\rho} = I(\bar{\lambda})$. And also $\liminf_{k\to\infty} \int \rho_{n_k} \geqslant \int \bar{\rho} = I(\bar{\lambda})$. Next we have, from (2.26) (choosing $\rho = \bar{\rho}$)

$$\limsup_{k\to\infty} \lambda_{n_k} \int \rho_{n_k} \leqslant \mathcal{E}(\bar{\rho}) + \bar{\lambda} \int \bar{\rho} - \liminf_{k\to\infty} \mathcal{E}(\rho_{n_k}) \leqslant \bar{\lambda} \int \bar{\rho}.$$

We conclude that

$$\limsup_{k\to\infty} \int \rho_{n_k} \leqslant \int \bar{\rho}$$

and so $\lim_{k\to\infty} \int \rho_{n_k} = \int \bar{\rho} = I(\bar{\lambda})$. The uniqueness of the limit shows that, in fact, $\lim_{n\to\infty} I(\lambda_n) = I(\bar{\lambda})$.

STEP 2. Proof of (2.19).

Proof. Indeed let $\lambda, \mu \in (0, \operatorname{ess\,sup}_{\Omega} V)$ be such that $I(\lambda) = I(\mu)$. We have

$$\mathcal{E}(\rho_\lambda) + \lambda \int \rho_\lambda \leqslant \mathcal{E}(\rho_\mu) + \lambda \int \rho_\mu,$$
$$\mathcal{E}(\rho_\mu) + \mu \int \rho_\mu \leqslant \mathcal{E}(\rho_\lambda) + \mu \int \rho_\lambda,$$

and therefore $\mathcal{E}(\rho_\lambda) = \mathcal{E}(\rho_\mu)$. We deduce from the strict convexity of $\mathcal{E}$ that $\rho_\lambda = \rho_\mu$. □

On the other hand we have

$$\partial j(\rho_\lambda) + B\rho_\lambda \ni V - \lambda \quad \text{a.e.},$$
$$\partial j(\rho_\mu) + B\rho_\mu \ni V - \mu \quad \text{a.e.},$$

which means (since j is C^1 on $(0, \infty)$)

$$\begin{cases} j'(\rho_\lambda) + B\rho_\lambda = V - \lambda & \text{a.e. on } [\rho_\lambda > 0] \\ B\rho_\lambda \geqslant V - \lambda & \text{a.e. on } [\rho_\lambda = 0] \end{cases}$$

and similarly for ρ_μ.

If $\rho_\lambda = \rho_\mu = \rho$ is positive on a set of positive measure, then we have

$$V - \lambda - B\rho = V - \mu - B\rho,$$

and thus $\lambda = \mu$. Otherwise, $\rho_\lambda = \rho_\mu = \rho = 0$, and then $V - \lambda \leqslant 0$, $V - \mu \leqslant 0$, i.e., $\lambda \geqslant \operatorname{ess\,sup}_\Omega V$ and $\mu \geqslant \operatorname{ess\,sup}_\Omega V$, but this contradicts the assumption $\lambda, \mu \in (0, \operatorname{ess\,sup}_\Omega V)$.

STEP 3. Proof of (2.20).

Proof. By Lemma 3 we have

$$\partial j(\rho_\lambda) + B\rho_\lambda \ni V - \lambda \quad \text{a.e. on } \Omega$$

and thus

$$\rho_\lambda \in \partial j^*(V - \lambda - B\rho_\lambda).$$

It follows that

$$j^*(V - M) - j^*(V - \lambda - B\rho_\lambda) \geqslant \rho_\lambda(\lambda - M + B\rho_\lambda)$$

and therefore

$$\rho_\lambda \leqslant \frac{j^*(V - M)}{\lambda - M} \quad \text{for } \lambda > M.$$

Using assumption (H) we obtain $\lim_{\lambda \to +\infty} I(\lambda) = 0$. □

From the relation $\partial j(\rho_\lambda) + B\rho_\lambda \ni V - \lambda$ a.e. onΩ we see that

$$[I(\lambda) = 0] \ \Leftrightarrow \ [\rho_\lambda = 0] \ \Leftrightarrow \ [V - \lambda \leqslant 0 \text{ a.e.}] \ \Leftrightarrow \ [\lambda \geqslant \operatorname*{ess\,sup}_\Omega V].$$

STEP 4. Proof of (2.21).

Proof. Suppose that (P_0) admits a minimizer $\rho_0 \in D(\mathcal{E})$. We have

$$\mathcal{E}(\rho_\lambda) + \lambda I(\lambda) \ \leqslant \ \mathcal{E}(\rho_0) + \lambda \int \rho_0 \quad \forall \lambda > 0,$$

and also

$$\mathcal{E}(\rho_0) \leqslant \mathcal{E}(\rho_\lambda)$$

so that $I(\lambda) \leqslant \int \rho_0$ and $I_0 \leqslant \int \rho_0 < \infty$. □

Conversely, suppose $I_0 < \infty$, so that $\int \rho_\lambda \leqslant C \, \forall \lambda > 0$. It follows from Lemma 4 that $\int j(\rho_\lambda) \leqslant C \, \forall \lambda > 0$. Therefore, we may find, as in the proof of Theorem 2, a sequence $\lambda_n \to 0$ such that $\rho_{\lambda_n} \rightharpoonup \rho_0$ weakly in $L^1(\Omega_j)$ for each j. From Lemma 6 we easily see that ρ_0 is a minimizer for (P_0). Moreover, we have

$$\int \rho_0 \leqslant \liminf_{n \to +\infty} \int \rho_{\lambda_n} = \lim_{\lambda \downarrow 0} I(\lambda) = I_0.$$

Combining this with the above argument we find $\int \rho_0 = I_0$.

STEP 5. Proof of (2.18).

Proof. Suppose by contradiction that, for some $\lambda_0 < 0$, $\mathcal{E}(\rho) + \lambda_0 \int \rho$ is bounded below on $D(\mathcal{E})$. We deduce from Ekeland's principle (see Ekeland [32]) that for every $\varepsilon > 0$ there is some $\rho_\varepsilon \in D(\mathcal{E})$ such that

$$\mathcal{E}(\rho) + \lambda_0 \int \rho - \mathcal{E}(\rho_\varepsilon) - \lambda_0 \int \rho_\varepsilon + \varepsilon \int |\rho - \rho_\varepsilon| \geqslant 0 \quad \forall \rho \in D(\mathcal{E}).$$

Applying Lemma 3 and standard convex analysis we see that

$$\partial j(\rho_\varepsilon) + B\rho_\varepsilon \ni V - \lambda_0 - f_\varepsilon \quad \text{ a.e. on } \Omega$$

for some function $f_\varepsilon \in L^\infty(\Omega)$ with $\|f_\varepsilon\|_{L^\infty} \leqslant \varepsilon$. We deduce from Lemma 8 that $(V - \lambda_0 - f_\varepsilon)_\infty \leqslant 0$ and consequently $V_\infty - \lambda_0 \leqslant \varepsilon$, so that $-\lambda_0 \leqslant \varepsilon$. Choosing $\varepsilon < -\lambda_0$ yields a contradiction. □

We may now return to problem (M_I) described at the beginning of this section and state, using the notation introduced in Theorem 3, the following:

COROLLARY 1. *Under the assumptions of Theorem* 3*, we have*

$$\begin{cases} \textit{for every } I \in (0, I_0) \textit{ problem } (\mathrm{M}_I) \textit{ admits a unique minimizer} \\ \rho^I = \rho_\lambda, \textit{ where } \lambda > 0 \textit{ is the unique solution of } I(\lambda) = I; \end{cases} \tag{2.27}$$

$$\textit{if } I_0 < \infty, \textit{ problem } (\mathrm{M}_{I_0}) \textit{ admits } \rho_0 \textit{ as its unique minimizer;} \tag{2.28}$$

$$\textit{if } I_0 < \infty \textit{ and } I > I_0, \textit{ problem } (\mathrm{M}_I) \textit{ admits no minimizer.} \tag{2.29}$$

REMARK 7. If $I_0 < \infty$ and $I > I_0$, any minimizing sequence converges to ρ_0 weakly in $L^1(\Omega_j)$ for every j (this is proved at the end of the section). Note that the constraint $\int \rho = I$ is "lost" in the limit.

Proof of Corollary 1.

Proof of (2.27). We have, by construction,

$$\mathcal{E}(\rho_\lambda) + \lambda \int \rho_\lambda \leqslant \mathcal{E}(\rho) + \lambda \int \rho \quad \forall \rho \in D(\mathcal{E})$$

and thus

$$\mathcal{E}(\rho^I) + \lambda I \leqslant \mathcal{E}(\rho) + \lambda I \quad \forall \rho \in D(\mathcal{E}) \text{ with } \int \rho = I,$$

so that ρ_I is a minimizer for (M_I).

Proof of (2.28). If $I_0 < \infty$, we have

$$\mathcal{E}(\rho_0) \leqslant \mathcal{E}(\rho) \quad \forall \rho \in D(\mathcal{E})$$

and in particular

$$\mathcal{E}(\rho_0) \leqslant \mathcal{E}(\rho) \quad \forall \rho \in D(\mathcal{E}) \text{ with } \int \rho = I_0.$$

Therefore ρ_0 is a minimizer for (M_{I_0}).

Proof of (2.29). Indeed, suppose that problem (M_I) has a solution $\bar{\rho}$ for some $\bar{I} > I_0$. We deduce from Theorem 1 that there is a constant $\bar{\lambda} \in \mathbb{R}$ such that

$$\partial j(\bar{\rho}) + B\bar{\rho} \ni V - \bar{\lambda} \quad \text{a.e. on } \Omega.$$

Lemma 8 implies $V_\infty - \bar{\lambda} \leqslant 0$, i.e., $\bar{\lambda} \geqslant 0$ (since $V_\infty = 0$). From Lemma 3 we see that

$$\mathcal{E}(\bar{\rho}) + \bar{\lambda} \int \bar{\rho} \leqslant \mathcal{E}(\rho) + \bar{\lambda} \int \rho \quad \forall \rho \in D(\mathcal{E}).$$

Since $\mathcal{E}$ is strictly convex we must have $\bar{\rho} = \rho_{\bar{\lambda}}$ and thus $\int \bar{\rho} = \int \rho_{\bar{\lambda}} \leqslant I_0$. But, on the other hand, $\int \bar{\rho} = \bar{I} > I_0$ – a contradiction.

We gather some additional facts in the next propositions.

PROPOSITION 1. *Same assumptions as in Theorem* 3. *Then for every $I \in (0, I_0)$ we have*

$$[\rho^I > 0] \text{ has finite measure,} \tag{2.30}$$

$$\rho^I \leqslant \gamma^0(V) \quad \text{a.e.,} \tag{2.31}$$

where

$$\gamma^0(s) = (j^*)'(s-0) = \lim_{t \uparrow s} \frac{j^*(t) - j^*(s)}{t - s}.$$

If $I_0 < \infty$, we have

$$\rho_0 \leqslant \gamma^0(V) \quad \text{a.e.} \tag{2.32}$$

and in particular

$$I_0 = \int \rho_0 \leqslant \int \gamma^0(V) \leqslant \infty. \tag{2.33}$$

Proof. Since $0 < I < I_0$ there is some $\bar{\lambda} > 0$ such that $\rho^I = \rho_{\bar{\lambda}}$ and thus we have

$$\partial j(\rho^I) + B\rho^I \ni V - \bar{\lambda} \quad \text{a.e. on } \Omega. \tag{2.34}$$

It follows from (2.34) and (2.14) that

$$[\rho^I > 0] \subset [V \geqslant \bar{\lambda}],$$

and so $[\rho^I > 0]$ has finite measure (since $V_\infty = 0$ and $\bar{\lambda} > 0$).

We write (2.34) as

$$\rho^I \in \gamma(V - \bar{\lambda} - B\rho^I)$$

where $\gamma = \partial j^* = (\partial j)^{-1}$; (2.31) follows from the monotonicity of γ.

When $I_0 < \infty$, the proof Theorem 3 (Step 4) shows that

$$\rho^I \underset{I \uparrow I_0}{\rightharpoonup} \rho_0 \quad \text{weakly in } L^1(\Omega_j) \forall j.$$

We deduce from (2.31) that

$$\rho_0 \leqslant \gamma^0(V) \quad \text{a.e. on } \Omega.$$

We now introduce two natural expressions

$$E(I) = \begin{cases} \inf\{\mathcal{E}(\rho);\, \rho \in D(\mathcal{E}) \text{ and } \int \rho = I\} & \text{if } I \geqslant 0 \\ +\infty & \text{if } I < 0, \end{cases} \tag{2.35}$$

and, for every $\lambda \in \mathbb{R}$,

$$\Phi(\lambda) = -\inf\left\{\mathcal{E}(\rho) + \lambda \int \rho;\, \rho \in D(\mathcal{E})\right\}. \tag{2.36}$$

□

PROPOSITION 2. *Same assumptions as in Theorem 3. We have*

E is convex, l.s.c. on $\mathbb{R}$, $E(0) = 0$, (2.37)

E is strictly convex and decreasing on $(0, I_0)$, (2.38)

if $I_0 < \infty$, then $E(I) = E(I_0) = \mathcal{E}(\rho_0)$ for $I \geqslant I_0$, (2.39)

Φ is convex, l.s.c. on $\mathbb{R}$, (2.40)

$$\Phi(\lambda) = \infty \quad \forall \lambda < 0, \tag{2.41}$$

$$\Phi(\lambda) \geqslant 0 \quad \forall \lambda \in \mathbb{R}, \tag{2.42}$$

Φ is finite, C^1, nonincreasing on $(0, \infty)$, (2.43)

$$\Phi'(\lambda) = -I(\lambda) \quad \forall \lambda > 0, \tag{2.44}$$

$$\begin{cases} \Phi(\lambda) = 0 \quad \textit{for } \lambda \geqslant \operatorname{ess\,sup}_\Omega V, & \textit{if } \operatorname{ess\,sup}_\Omega V < \infty \\ \lim_{\lambda\to\infty} \Phi(\lambda) = 0 & \textit{if } \operatorname{ess\,sup}_\Omega V = \infty, \end{cases} \tag{2.45}$$

$$\Phi(\lambda) = E^*(-\lambda) \quad \forall \lambda \in \mathbb{R} \quad \textit{and} \quad E(I) = \Phi^*(-I) \quad \forall I \in \mathbb{R}. \tag{2.46}$$

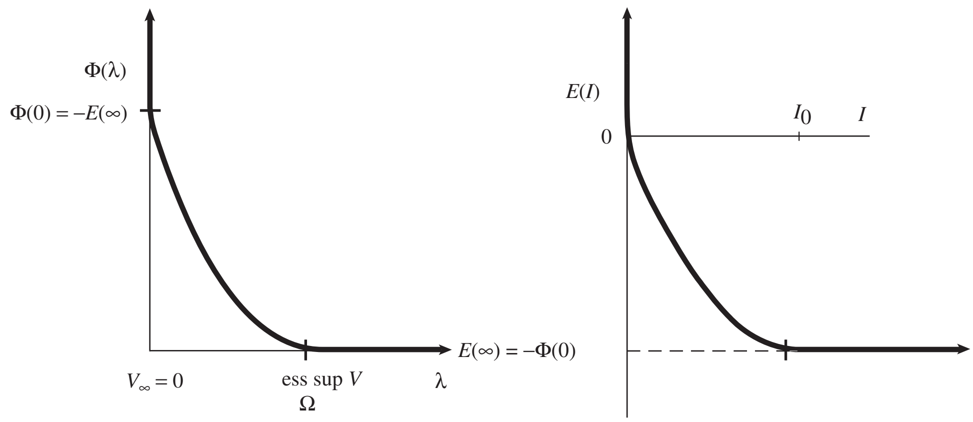

Figure 5 Typical shape of $\Phi(\lambda)$ and $E(I)$.

Proof.

Verification of (2.37). It follows from assumption (H) that $E(I) \geqslant -\int j^*(V-M) - MI$ and thus $E(I) \in \mathbb{R}$ for $I \geqslant 0$. Let $I_1, I_2 \geqslant 0$ and $t \in (0, 1)$. Given $\varepsilon > 0$ there is some $\rho_1 \in D(\mathcal{E})$ such that $\int \rho_1 = I_1$ and $\mathcal{E}(\rho_1) \leqslant E(I_1) + \varepsilon$ and there is some $\rho_2 \in D(\mathcal{E})$ such that $\int \rho_2 = I_2$ and $\mathcal{E}(\rho_2) \leqslant E(I_2)+\varepsilon$. Set $\bar{\rho} = t\rho_1 + (1-t)\rho_2$ so that $\int \bar{\rho} = tI_1 + (1-t)I_2$ and $\mathcal{E}(\bar{\rho}) \leqslant tE(I_1) + (1-t)E(I_2) + \varepsilon$. Therefore we obtain

$$E(tI_1 + (1-t)I_2) \leqslant tE(I_1) + (1-t)E(I_2) + \varepsilon,$$

and so E is convex.

In view of the convexity of E we already know that E is continuous on $(0, +\infty)$ and that $\limsup_{I\downarrow 0} E(I) \leqslant E(0) = 0$. On the other hand if $I_n \to 0$ with $I_n \geqslant 0$, there is a sequence (ρ_n) in $D(\mathcal{E})$ such that $\int \rho_n = I_n$ and $\mathcal{E}(\rho_n) \leqslant E(I_n) + 1/n$. Since $\mathcal{E}$ is l.s.c. on L^1 we conclude that $\liminf_{n\to\infty} E(I_n) \geqslant 0$. Therefore $\lim_{I\downarrow 0} E(I) = E(0) = 0$.

Verification of (2.40), (2.41) **and** (2.42). It is clear that Φ is convex and l.s.c. since it is a sup of affine functions. (2.41) corresponds to assertion (2.18) in Theorem 3. (2.42) is obvious by choosing $\rho = 0$ as testing function in the definition of Φ.

Verification of (2.43) **and** (2.44). We have $\forall \lambda, \mu > 0$,

$$\mathcal{E}(\rho_\mu) + \mu \int \rho_\mu \leqslant \mathcal{E}(\rho_\lambda) + \mu \int \rho_\lambda$$

and thus,

$$-\Phi(\mu) \leqslant \mathcal{E}(\rho_\lambda) + \lambda \int \rho_\lambda + (\mu - \lambda) \int \rho_\lambda = -\Phi(\lambda) + (\mu - \lambda) I(\lambda).$$

Hence

$$\Phi(\mu) - \Phi(\lambda) + I(\lambda)(\mu - \lambda) \geqslant 0 \quad \forall \lambda, \mu > 0.$$

Changing λ and μ yields

$$|\Phi(\mu) - \Phi(\lambda) + I(\lambda)(\mu - \lambda)| \leqslant |I(\mu) - I(\lambda)|\, |\mu - \lambda| \quad \forall \lambda, \mu > 0.$$

Assertions (2.43) and (2.44) follow.

Verification of (2.45). We have $\forall \lambda \in \mathbb{R}$,

$$j(\rho) - (V - \lambda)\rho \geqslant -j^*(V - \lambda)$$

and thus, for $\rho \in D(\mathcal{E})$,

$$\mathcal{E}(\rho) + \lambda \int \rho \geqslant -\int j^*(V - \lambda),$$

so that

$$\Phi(\lambda) \leqslant \int j^*(V - \lambda).$$

If $\operatorname{ess\,sup}_\Omega V < \infty$, we see immediately that $\Phi(\lambda) \leqslant 0$ for $\lambda \geqslant \operatorname*{ess\,sup}_{\Omega} V$ (since $j^*(s) = 0$ for $s \leqslant 0$).

If $\operatorname{ess\,sup}_\Omega V = \infty$, we observe that $j^*(V-\lambda) \leqslant j^*(V-M)$ for $\lambda \geqslant M$ and $j^*(V-\lambda) \to 0$ a.e. as $\lambda \to +\infty$. It follows, by dominated convergence that $\int j^*(V - \lambda) \to 0$ as $\lambda \to +\infty$.

Verification of (2.46). It is clear that, for every $\lambda \in \mathbb{R}$,

$$\inf_{\rho \in D(\mathcal{E})} \left\{ \mathcal{E}(\rho) + \lambda \int \rho \right\} = \inf_{I \geqslant 0} \{E(I) + \lambda I\},$$

i.e., $\Phi(\lambda) = \sup_{I \geqslant 0} \{-\lambda I - E(I)\} = E^*(-\lambda)$. It follows that $E^{**}(I) = \Phi^*(-I)\, \forall I \in \mathbb{R}$. However $E^{**} = E$ since E is convex and l.s.c. on $\mathbb{R}$.

Verification of (2.38) **and** (2.39). Let $I_1, I_2 \in (0, I_0)$ with $I_1 \neq I_2$. We know from Theorem 3 that there exist $\rho_1, \rho_2 \in D(\mathcal{E})$ with $\int \rho_1 = I_1$ and $\int \rho_2 = I_2$, $E(I_1) = \mathcal{E}(\rho_1)$ and $E(I_2) = \mathcal{E}(\rho_2)$. Since $\mathcal{E}$ is strictly convex we have, for $t \in (0, 1)$,

$$\begin{aligned} E(tI_1 + (1-t)I_2) \leqslant \mathcal{E}(t\rho_1 + (1-t)\rho_2) &< t\mathcal{E}(\rho_1) + (1-t)\mathcal{E}(\rho_2) \\ &= tE(I_1) + (1-t)E(I_2). \end{aligned}$$

On the other hand, we have from (2.46) and (2.41)

$$E(I) = \sup_{\lambda \in \mathbb{R}} \{-I\lambda - \Phi(\lambda)\} = \sup_{\lambda \geqslant 0} \{-I\lambda - \Phi(\lambda)\}.$$

For $\lambda \geqslant 0$ the function $I \mapsto (-I\lambda - \Phi(\lambda))$ is nonincreasing and thus the function $I \mapsto E(I)$ is also nonincreasing on $\mathbb{R}$. It follows that E is decreasing on $(0, I_0)$ since it is strictly convex on $(0, I_0)$. Finally, E is constant on $(I_0, +\infty)$. Indeed, if $I_0 < \infty$, there exists (by Theorem 3) some $\rho_0 \in D(\mathcal{E})$ with $\int \rho_0 = I_0$ and $\mathcal{E}(\rho_0) \leqslant \mathcal{E}(\rho) \quad \forall \rho \in D(\mathcal{E})$, so that

$$E(I_0) = \mathcal{E}(\rho_0) \leqslant \mathcal{E}(\rho) \quad \forall \rho \in D(\mathcal{E}).$$

In particular, $E(I_0) \leqslant E(I)\, \forall I$. Since E is nonincreasing on $\mathbb{R}$ we conclude that $E(I) = E(I_0)$ for $I > I_0$. □

REMARK 8. In all the examples related to Thomas-Fermi $I_0 < \infty$ (see Section 4). It would be illuminating to construct examples satisfying all the conditions of Theorem 3 such that $I_0 = \infty$. From the definitions of Φ and (2.46) we have

$$\Phi(0) = -\inf_{\rho \in D(\mathcal{E})} \mathcal{E}(\rho) = -\inf_{I \geqslant 0} E(I).$$

It would be useful to construct some examples where $I_0 = \infty$ and $\Phi(0) < \infty$, and other examples where $I_0 = \infty$ and $\Phi(0) = \infty$. From (2.44) we see that $\Phi(0) < \infty$ if and only if $\int_0^1 I(\lambda)\, d\lambda < \infty$.

The approach via relaxation

Another approach for proving Corollary 1 (without passing through Theorem 3) is the *relaxation* method used by Lieb-Simon [48].

Given a constant $0 \leqslant I < \infty$ set

$$\widehat{K}_I = \left\{ \rho \in D(\mathcal{E});\ \int \rho(x)\, dx \leqslant I \right\}$$

and consider the relaxed minimization problem

$$\text{find } \widehat{\rho} \in \widehat{K}_I \text{ such that } \mathcal{E}(\widehat{\rho}) \leqslant \mathcal{E}(\rho) \quad \forall \rho \in \widehat{K}_I. \tag{$\widehat{\mathrm{M}}_I$}$$

Set, for every $I \geqslant 0$,

$$\widehat{E}(I) = \inf \left\{ \mathcal{E}(\rho);\ \rho \in D(\mathcal{E}) \text{ and } \int \rho \leqslant I \right\}. \tag{2.47}$$

We keep the assumptions of Theorem 3. Clearly, the function $I \mapsto \widehat{E}(I)$ is convex, nonincreasing and continuous on $[0, \infty)$ (the argument is similar to the proof of (2.37) in Proposition 2). It is easy to see that for *every* $I \geqslant 0$ the infimum in (2.47) is *achieved* by some unique element, denoted $\widehat{\rho}_I$ (the argument is similar to the one used in the proof of Lemma 7). A simple consideration about convex functions shows that there exists some $\widehat{I}_0 \in [0, \infty]$ such that:

a) $\widehat{E}$ is *decreasing* on $[0, \widehat{I_0})$,
b) $\widehat{E}$ is constant on $[\widehat{I_0}, \infty]$(assuming $\widehat{I_0} < \infty$).

PROPOSITION 3. *Under the assumptions of Theorem* 3, *this* $\widehat{I_0}$ *satisfies all the properties of* I_0 *described in Corollary* 1. *Moreover* $\widehat{E}(I) = E(I)\ \forall I \geqslant 0$.

Proof. a) If $I \leqslant \widehat{I_0}$ we must have $\int \widehat{\rho}^I = I$, so that $\widehat{\rho}^I$ is a solution of (M_I). Otherwise, set $I' = \int \widehat{\rho}^I < I$. We have

$$\widehat{E}(I') = \mathcal{E}(\widehat{\rho}^{I'}) \leqslant \mathcal{E}(\rho) \quad \forall \rho \in \widehat{K}_{I'}.$$

Choosing $\rho = \widehat{\rho}^I$ we obtain $\widehat{E}(I') \leqslant \widehat{E}(I)$ – absurd.

b) If $I > \widehat{I_0}$, problem (M_I) has no solution. Indeed, suppose, by contradiction, that there is a solution $\bar{\rho}$ of (M_I) with $I > \widehat{I_0}$. We know, by Theorem 1, that there is a constant $\lambda \in \mathbb{R}$ such that

$$\partial j(\bar{\rho}) + B\bar{\rho} \ni V - \lambda \quad \text{a.e.}$$

and by Lemma 8 we find that $\lambda \geqslant 0$. Therefore we have

$$\mathcal{E}(\bar{\rho}) + \lambda \int \bar{\rho} \leqslant \mathcal{E}(\rho) + \lambda \int \rho \quad \forall \rho \in D(\mathcal{E}). \tag{2.48}$$

Choosing $\rho = \widehat{\rho}^{\hat{I}_0}$ in (2.48) we obtain

$$\mathcal{E}(\bar{\rho}) + \lambda I \leqslant \widehat{E}(\widehat{I_0}) + \lambda \widehat{I_0}.$$

However we have

$$\widehat{E}(\widehat{I_0}) = \inf_{\rho \in D(\mathcal{E})} \mathcal{E}(\rho)$$

and this infimum is achieved only when $\rho = \widehat{\rho}^{\hat{I}_0}$ so that $\mathcal{E}(\bar{\rho}) > \widehat{E}(\widehat{I_0})$. It follows that $\lambda < 0$ – absurd. □

Proof of Remark 7. Let $I > I_0$ $(I_0 < \infty)$; we have $E(I) = E(I_0)$. Thus if (ρ_n) is a minimizing sequence for (M_I) we have $\mathcal{E}(\rho_n) \to E(I_0)$. As in Lemma 7 we may extract a subsequence still denoted ρ_n such that $\rho_n \rightharpoonup \bar{\rho}$ weakly in $L^1(\Omega_j)$ for each j. By Lemma 6 we have $\mathcal{E}(\bar{\rho}) \leqslant E(I_0)$. Hence $\bar{\rho}$ is a minimizer for $\mathcal{E}$ on $D(\mathcal{E})$. By uniqueness we have $\bar{\rho} = \rho_0$.

REMARK 9. Throughout this section we have made assumption (2.1), i.e. j is coercive, and it played an essential role in applying the Dunford-Pettis theorem about weak convergence in L^1. When (2.1) does *not* hold it may be natural to extend the setting of problem (M_I) and to allow solutions ρ which are measures. This is an interesting direction of research.

3. A direct approach for solving the Euler equation

As in Section 2, let $j : \mathbb{R} \to [0, +\infty]$ be a convex l.s.c. function such that

$$j(0) = 0 \text{ and } j(r) = +\infty \quad \text{for all } r < 0,$$
$$j \text{ is finite and } C^1 \text{ on } (0, \infty),$$
$$j'(0+) = \lim_{r \downarrow 0} \frac{j(r)}{r} = 0.$$

Let $V : \Omega \to \mathbb{R}$ be a measurable function. Assume $k : \Omega \times \Omega \to \mathbb{R}$ is a measurable function satisfying (1.3) and (1.4). For every $\rho \in L^1(\Omega)$ with $\rho \geqslant 0$ a.e. we set

$$(B\rho)(x) = \int k(x, y)\, \rho(y) dy \ \leqslant \ +\infty.$$

We shall make further assumptions on B:

$$\int (B\rho - 1)^+ \ < \ \infty \quad \forall \rho \in L^1(\Omega) \text{ with } \rho \geqslant 0 \text{ a.e..} \tag{3.1}$$

It is equivalent to assume that for every $\rho \in L^1(\Omega)$ with $\rho \geqslant 0$ a.e. we have

$$\begin{cases} \int_A B\rho < \infty \quad \forall A \subset \Omega \text{ measurable with finite measure} \\ \text{and for every } \delta > 0 \text{ the set } [B\rho > \delta] \text{ has finite measure.} \end{cases} \tag{3.1$'$}$$

$$j(0) \text{ and } j(r) = -\infty \text{ for all } r < 0,$$
$$j \text{ is finite and } C^1 \text{ on } (0, \infty),$$
$$j'(0+) = \lim_{r \downarrow 0} \frac{j(r)}{r} = 0.$$

We may thus extend B as a linear operator from $L^1(\Omega)$ into $L^1(\Omega) + L^\infty(\Omega)$. Sometimes we shall use an assumption slightly stronger than (3.1):

$$\text{for every } M \geqslant 0, \sup \left\{ \int (B\rho - 1)^+; \rho \in L^1(\Omega), \rho \geqslant 0 \text{ a.e.}, \int \rho \leqslant M \right\} < \infty. \tag{3.2}$$

We shall also make an assumption related to the maximum principle:

$$\begin{cases} \int \rho\, p(B\rho) \geqslant 0 \quad \forall \rho \in L^1(\Omega), \quad \forall p \in \mathcal{P} \\ \text{and} \\ \int \rho\, p(B\rho) = 0 \quad \text{if and only if} \quad p(B\rho) = 0, \end{cases} \tag{3.3}$$

where

$$\mathcal{P} = \{p \in C^\infty(\mathbb{R}; \mathbb{R}); 0 \leqslant p \leqslant 1, p' \geqslant 0 \text{ on } \mathbb{R}, p' \in L^\infty(\mathbb{R}), \text{ and } p(t) = 0 \text{ for } t \leqslant 1\}.$$

Finally we suppose that

B is injective. (3.4)

We are concerned with the following problem:

$$\begin{cases} \text{Given a constant } I, \text{ with } 0 < I < \infty, \text{ find a function } \rho \in L^1(\Omega) \text{ and a} \\ \text{constant } \lambda \in \mathbb{R} \text{ such that } \rho \geqslant 0 \text{ a.e., } \int \rho = I \text{ and } \partial j(\rho) + B\rho \ni V - \lambda \text{ a.e..} \end{cases} \tag{E^I}$$

When assumption (H) holds, problem (E^I) is equivalent to problem (M_I) - which has been solved in Section 2. We emphasize that throughout Section 3 we do *not* assume (H) and we solve (E^I) by a direct method. Our main results are the following.

THEOREM 4. *Assume* (3.1), (3.3) *and* (3.4). *Then, there exists* I_1 *with* $0 \leqslant I_1 \leqslant \infty$ *such that:*

a) *for every* $0 < I \leqslant I_1$ *(and* $I < \infty$*) there is a unique solution* ρ^I *of problem* (E^I),
b) *for* $I_1 < I < \infty$ *problem* (E^I) *has no solution.*

REMARK 10. It may well happen that there is *no* $I > 0$ whatsoever for which problem (E^I) admits a solution (see an elementary example in Section 4, Remark 15). In this case we say that $I_1 = 0$. In contrast with the situation of Theorem 3 (where the assumption (H^+) plays a central role), this may happen even if $\operatorname{ess\,sup}_\Omega (V - V_\infty) > 0$. (Again, in the example of Section 4, Remark 15 one has $V_\infty = 0$, $\operatorname{ess\,sup}_\Omega V = +\infty$ but assumption (H^+) fails).

In order to solve (E^I) we proceed as in Section 2 and introduce the auxiliary problem:

$$\begin{cases} \text{Given a constant } \lambda \in \mathbb{R}, \text{ find } \rho \in L^1(\Omega) \text{ with } \rho \geqslant 0 \text{ a.e. such that} \\ \partial j(\rho) + B\rho \ni V - \lambda \text{ a.e..} \end{cases} \tag{E_λ}$$

Theorem 4 is a direct consequence of

THEOREM 5. *Assume* (3.1), (3.3) *and* (3.4). *Let V be any measurable function. Then, there exists* $\lambda_0 \in [V_\infty, +\infty]$ *such that:*

a) *for every* $\lambda > \lambda_0$ *(and* $\lambda < +\infty$*) there is a unique solution* ρ_λ *of* (E_λ),
b) *for* $\lambda < \lambda_0$ *there is no solution of* (E_λ).

The mapping $\lambda \mapsto \rho_\lambda$ *defined for* $\lambda \in (\lambda_0, +\infty)$ *is nonincreasing and continuous with values into* $L^1(\Omega)$*; moreover* $\rho_\lambda \to 0$ *in* $L^1(\Omega)$ *as* $\lambda \to +\infty$. *Set*

$$I_1 = \sup_{\lambda > \lambda_0} \int \rho_\lambda = \lim_{\lambda \downarrow \lambda_0} \int \rho_\lambda.$$

If $\lambda_0 \in \mathbb{R}$ *the following are equivalent:*

(i) $I_1 < \infty$
(ii) (E_{λ_0}) *has a unique solution* ρ_{λ_0},
(iii) *there exist functions* $f \in L^1(\Omega)$, $f \geqslant 0$ *a.e., and* $U : \Omega \to \mathbb{R}$ *measurable with* $\gamma^0(U) \in L^1(\Omega)$ *such that* $V - \lambda_0 = U + Bf$.

where γ^0 *has been defined in Section* 2, *Proposition* 1.
In this case $\rho_\lambda \to \rho_{\lambda_0}$ *in* $L^1(\Omega)$ *as* $\lambda \downarrow \lambda_0$ *and*

$$I_1 \leqslant \int (\gamma^0(U) + f) \tag{3.5}$$

REMARK 11. Very often we will find that $\lambda_0 = V_\infty$ (see e.g. Theorem 6). However it may also happen sometimes that there is *no* $\lambda \in \mathbb{R}$ for which (E_λ) admits a solution (see Section 4, Remark 15).

We start with some lemmas:

LEMMA 9. *Assume* (3.1). *Let* (ρ_n) *be a sequence in* $L^1(\Omega)$ *such that* $\rho_n \rightharpoonup \rho$ *weakly in* $L^1(\Omega)$ *and* $|\rho_n| \leqslant f$ *for some* $f \in L^1(\Omega)$. *Then* $B\rho_n \to B\rho$ *a.e. and in* $L^1(\Omega) + L^\infty(\Omega)$.

Proof. We may always assume that $\rho = 0$. We recall that for a.e. $x \in \Omega$ the function $y \mapsto k(x, y)f(y)$ is integrable. We write, for $M > 0$

$$(B\rho_n)(x) = \int_{[k(x,\cdot) \leqslant M]} k(x, y)\rho_n(y)dy + \int_{[k(x,\cdot)>M]} k(x, y)\rho_n(y)dy \tag{3.6}$$

It follows that

$$\limsup_{n\to\infty} |B\rho_n(x)| \leqslant \int_{[k(x,\cdot)>M]} k(x, y)f(y)dy \quad \forall M > 0.$$

As $M \to \infty$ we see that $B\rho_n \to 0$ a.e. By dominated convergence we have

$$\int (|B\rho_n| - k)^+ \to 0 \quad \forall k > 0.$$

Finally we note that

$$\|B\rho_n\|_{L^1+L^\infty} \leqslant k + \int (|B\rho_n| - k)^+ \quad \forall k > 0.$$

and thus

$$\limsup_{n\to\infty} \|B\rho_n\|_{L^1+L^\infty} \leqslant k \quad \forall k > 0.$$

□

LEMMA 10. *Assume* (3.1). *Then B is a bounded operator from $L^1(\Omega)$ into $L^1(\Omega) + L^\infty(\Omega)$ and from $L^1(\Omega) \cap L^\infty(\Omega)$ into $L^\infty(\Omega)$.*

Proof. Let (ρ_n) be a sequence in $L^1(\Omega)$ such that $\rho_n \to 0$ in $L^1(\Omega)$. We may extract a subsequence still denoted (ρ_n) such that $|\rho_n| \leqslant f$ a.e. with $f \in L^1(\Omega)$. We deduce from Lemma 9 that $B\rho_n \to 0$ in $L^1(\Omega) + L^\infty(\Omega)$. Thus B is a bounded operator from $L^1(\Omega)$ into $L^1(\Omega) + L^\infty(\Omega)$. It follows, by duality, that B is a bounded operator from $L^1(\Omega) \cap L^\infty(\Omega)$ into $L^\infty(\Omega)$. □

LEMMA 11. *Assume* (3.1) *and* (3.3). *Let $\rho \in L^1(\Omega)$ and let $k \geqslant 0$ be a constant. Then we have*

$$\int_{[B\rho > k]} \rho \geqslant 0 \tag{3.7}$$

and

$$[B\rho \leqslant k \text{ a.e. on } [\rho > 0]] \quad \Rightarrow \quad [B\rho \leqslant k \text{ a.e. on } \Omega]. \tag{3.8}$$

Proof. It suffices to consider the case $k = 1$. We have

$$\int \rho\, p(B\rho) \geqslant 0 \quad \forall p \in \mathcal{P}$$

and we obtain (3.7) by choosing a sequence (p_n) in $\mathcal{P}$ such that $p_n(t) \to 1\ \forall t > 1$. If $B\rho \leqslant 1$ on $[\rho > 0]$ we have for $p \in \mathcal{P}$

$$\int \rho\, p(B\rho) = \int_{[\rho \leqslant 0]} \rho\, p(B\rho) + \int_{[\rho > 0]} \rho\, p(B\rho) \leqslant 0$$

since $p(B\rho) = 0$ a.e. on $[\rho > 0]$. It follows that $p(B\rho) = 0$ a.e. on Ω, for every $p \in \mathcal{P}$ and thus $B\rho \leqslant 1$ a.e. on Ω. □

LEMMA 12 (A comparison principle via L^∞). *Assume* (3.1) *and* (3.3). *Let V_1 and V_2 be two measurable functions. Let $\rho_1, \rho_2 \in L^1(\Omega)$ be such that $\rho_1 \geqslant 0,\ \rho_2 \geqslant 0$ and*

$$\begin{cases} \partial j(\rho_1) + B\rho_1 \ni V_1 \\ \partial j(\rho_2) + B\rho_2 \ni V_2. \end{cases} \tag{3.9}$$

Then

$$\|(B\rho_1 - B\rho_2)^+\|_{L^\infty} \leqslant \|(V_1 - V_2)^+\|_{L^\infty}. \tag{3.10}$$

In particular

$$[V_1 \leqslant V_2 \text{ a.e.}] \quad \Rightarrow \quad [B\rho_1 \leqslant B\rho_2 \text{ a.e.}]$$

and if B is injective

$$[V_1 = V_2 \text{ a.e.}] \quad \Rightarrow \quad [\rho_1 = \rho_2 \text{ a.e.}].$$

Proof. Set $k = \|(V_1 - V_2)^+\|_{L^\infty}$. On the set $[\rho_1 - \rho_2 > 0]$ we have, using (3.9), $V_1 - B\rho_1 \geqslant V_2 - B\rho_2$ and so $B(\rho_1 - \rho_2) \leqslant k$. It follows from (3.8) that $B(\rho_1 - \rho_2) \leqslant k$ a.e. on Ω. □

LEMMA 13. *Assume* (3.1) *and* (3.3). *Suppose that there is some* $\bar{\rho} \in L^1(\Omega)$ *with* $\bar{\rho} \geqslant 0$ *such that*

$$\partial j(\bar{\rho}) + B\bar{\rho} \ni V \quad \textit{a.e.}. \tag{3.11}$$

Then, for every $\lambda > 0$ *there is some* $\rho_\lambda \in L^1$ *with* $\rho_\lambda \geqslant 0$ *such that*

$$\partial j(\rho_\lambda) + B\rho_\lambda \ni V - \lambda \quad \textit{a.e.}$$

and

$$\rho_\lambda \leqslant \bar{\rho} \quad \textit{a.e.}.$$

Proof. We divide the proof into 2 steps:

STEP 1. We claim that for every $\varepsilon > 0$ there is some $\rho^\varepsilon \in L^1(\Omega)$ with $\rho^\varepsilon \geqslant 0$ a.e. such that

$$\partial j(\rho^\varepsilon) + \varepsilon\rho^\varepsilon + B\rho^\varepsilon \ni V + \varepsilon\bar{\rho} - \lambda \quad \text{a.e.} \tag{3.12}$$

and

$$\rho^\varepsilon \leqslant \bar{\rho} \quad \text{a.e.} \tag{3.13}$$

Proof. In what follows $\varepsilon > 0$ is *fixed* and we set $V_n = \inf\{V + \varepsilon\bar{\rho}, n\}$. For every n there is a (unique) solution ρ_n of the problem

$$\partial j(\rho_n) + \varepsilon\rho_n + B\rho_n \ni V_n - \lambda \text{ a.e.}; \tag{3.14}$$

this is a consequence of Lemma 7. (Note that $V_\infty \leqslant 0$ – by (3.11) and Lemma 8 – and thus $(V_n)_\infty \leqslant 0$. An easy inspection of the proof shows that Lemma 7 still holds if one assumes $V_\infty \leqslant 0$ instead of $V_\infty = 0$). We have

$$\begin{aligned}
\rho_n &= (\partial j + \varepsilon I)^{-1}(V_n - \lambda - B\rho_n)\\
&\leqslant (\partial j + \varepsilon I)^{-1}(V + \varepsilon\bar{\rho} - \lambda)\\
&= (\partial j + \varepsilon I)^{-1}(V + \varepsilon\bar{\rho} - B\bar{\rho} + B\bar{\rho} - \lambda)\\
&\leqslant \bar{\rho} + \frac{1}{\varepsilon}(B\bar{\rho} - \lambda)^+ \in L^1(\Omega).
\end{aligned}$$

since $(\partial j + \varepsilon I)^{-1}$ is Lipschitz with constant $1/\varepsilon$. We may thus assume (for a subsequence) that

$$\rho_n \rightharpoonup \rho \quad \text{weakly in } L^1(\Omega)$$

and then, by Lemma 9,

$$B\rho_n \to B\rho \quad \text{a.e. and in } L^1(\Omega) + L^\infty(\Omega).$$

Using standard monotone analysis (see e.g. item [1], Lemma 3, under Brezis [16]) we can pass to the limit in (3.14) and conclude that ρ satisfies (3.12). Applying Lemma 12 to $(\partial j + \varepsilon I)$ we deduce from (3.11) and (3.12) that $0 \leqslant B\bar{\rho} - B\rho^\varepsilon \leqslant \lambda$. Therefore we obtain

$$\rho^\varepsilon = (\partial j + \varepsilon I)^{-1}(V + \varepsilon\bar{\rho} - B\rho^\varepsilon - \lambda) \leqslant (\partial j + \varepsilon I)^{-1}(V + \varepsilon\bar{\rho} - B\bar{\rho}) = \bar{\rho}.$$

□

STEP 2. We let $\varepsilon \to 0$. It is easy to see that (for a subsequence $\varepsilon_n \to 0$)

$$\rho^\varepsilon \rightharpoonup \rho \quad \text{weakly in } L^1(\Omega)$$
$$B\rho^{\varepsilon_n} \to B\rho \quad \text{a.e. and in} L^1(\Omega) + L^\infty(\Omega)$$

and ρ satisfies

$$\partial j(\rho) + B\rho \ni V - \lambda \quad \text{a.e.}$$
$$\rho \leqslant \bar{\rho} \quad \text{a.e..}$$

□

Proof of Theorem 5. Uniqueness follows from Lemma 12 since B is assumed to be injective. Let

$$\Lambda = \{\lambda \in \mathbb{R}; (\mathrm{E}_\lambda) \text{ has a solution}\} \text{ and } \lambda_0 = \inf \Lambda$$

($\lambda_0 = +\infty$ if $\Lambda = \emptyset$). It follows from Lemma 13 that λ_0 has all the required properties; moreover the mapping $\lambda \mapsto \rho_\lambda$ is nonincreasing.

In order to check its continuity let $\lambda_n \to \lambda \in (\lambda_0, +\infty)$ be a monotone sequence so that $\rho_{\lambda_n} \to \rho$ in $L^1(\Omega)$ (by monotone convergence). It follows that $B\rho_{\lambda_n} \to B\rho$ in $L^1(\Omega) + L^\infty(\Omega)$ and thus ρ satisfies $\partial j(\rho) + B\rho \ni V - \lambda$ a.e., i.e., $\rho = \rho_\lambda$. As $\lambda \uparrow \infty$, $\rho_\lambda \downarrow \rho$ in $L^1(\Omega)$; since $\rho_\lambda = 0$ a.e. on the set $[V - j'(0+) < \lambda]$, we conclude that $\rho = 0$ a.e. on Ω.

For the last assertion in Theorem 5, we note that (i) $\Rightarrow$ (ii) and (ii) $\Rightarrow$ (iii) are straight forward (choose $f = \rho_{\lambda_0}$ and $U = V - \lambda_0 - B\rho_{\lambda_0}$). It remains to show that (iii) $\Rightarrow$ (i). For $\lambda > \lambda_0$ we have

$$\partial j(\rho_\lambda) + B\rho_\lambda \ni U + Bf + \lambda_0 - \lambda$$

so that $\rho_\lambda \leqslant \gamma^0(U + Bf - B\rho_\lambda)$ and therefore

$$\int_{[Bf - B\rho_\lambda \leqslant 0]} \rho_\lambda \leqslant \int \gamma^0(U). \tag{3.15}$$

On the other hand, we have, by Lemma 11,

$$\int_{[Bf - B\rho_\lambda > 0]} (f - \rho_\lambda) \geqslant 0. \tag{3.16}$$

Combining (3.15) and (3.16) we see that

$$\int \rho_\lambda \leqslant \int (\gamma^0(U) + f).$$

We conclude with a rather general and useful result.

THEOREM 6. *Assume* (2.1), (2.12), (2.13), (2.14), (3.2), (3.3) *and* (3.4). *Assume in addition that there exist a function* $f \in L^1(\Omega)$, *and a measurable function* $U : \Omega \to \mathbb{R}$ *such that*

$$V = U + Bf \tag{3.17}$$

and

$$\int_\omega \gamma^0(U + t) < \infty \quad \forall t > 0, \quad \forall \omega \subset \Omega \text{ with } |\omega| < \infty. \tag{3.18}$$

Then, for every $\lambda > V_\infty$, *problem* (E_λ) *admits a solution, i.e., there exists a* $\rho_\lambda \in L^1$, $\rho_\lambda \geq 0$, *satisfying*

$$\partial j(\rho_\lambda) + B\rho_\lambda \ni V - \lambda.$$

In particular, if $\operatorname{ess\,sup}_\Omega V > V_\infty$, *problem* (E^I) *admits a solution for every* $I \in (0, I_1)$ *where*

$$0 < I_1 = \lim_{\lambda \downarrow V_\infty} \int \rho_\lambda \leqslant \infty.$$

Proof. Let $\lambda > V_\infty$ be fixed and let $V_n = \min\{V, n\}$. Let ρ_n be the solution of (3.19)

$$\partial j(\rho_n) + B\rho_n \ni V_n - \lambda. \tag{3.19}$$

The existence of ρ_n follows from Lemma 7. We claim that

$$\int \rho_n \leqslant C. \tag{3.20}$$

Indeed let μ be such that $\lambda > \mu > V_\infty$. We have

$$\rho_n \leqslant \gamma^0(V_n - \mu - B\rho_n) \leqslant \gamma^0(V - \mu - B\rho_n) = \gamma^0(U - \mu + Bf - B\rho_n)$$

and therefore

$$\int_{[Bf \leqslant B\rho_n]} \rho_n \leqslant \int \gamma^0(U - \mu) < \infty$$

since $[U > \mu]$ has finite measure (note that by (3.17), $U_\infty = V_\infty = 0$). On the other hand we have, by Lemma 12,

$$\int_{[Bf > B\rho_n]} (f - \rho_n) \geqslant 0.$$

It follows that

$$\int \rho_n \leqslant \int (\gamma^0(U - \mu) + f).$$

Clearly $\rho_n \leqslant \gamma^0(V - \mu)$, so that

$$\text{Supp } \rho_n \subset \omega = [V > \mu] \tag{3.21}$$

and $|\omega| < \infty$.

Next, we claim that the sequence (ρ_n) is equi-integrable on ω. Indeed let $t > 0$ and let $A \subset \omega$ be measurable. We write

$$\int_A \rho_n \leqslant \int_{A \cap [Bf - B\rho_n \leqslant t]} \rho_n \quad + \int_{[Bf - B\rho_n > t]} \rho_n.$$

As above we have

$$\int_{A \cap [Bf - B\rho_n \leqslant t]} \rho_n \leqslant \int_A \gamma^0(U - \mu + t)$$

and

$$\int_{[Bf - B\rho_n > t]} \rho_n \leqslant \int_{[Bf - B\rho_n > t]} f \leqslant \int_{[Bf > t]} f.$$

Consequently

$$\int_A \rho_n \leqslant \int_A \gamma^0(U - \mu + t) + \int_{[Bf > t]} f.$$

Given $\varepsilon > 0$ we first choose t large enough so that $\int_{[Bf > t]} f < \varepsilon$. Then we choose $\delta > 0$ small enough so that $|A| < \delta$ implies $\int_A \gamma^0(U - \mu + t) < \varepsilon$.

It follows from Lemma 12 that the sequence $(B\rho_n)$ is a nondecreasing. From (3.20) and assumption (3.2) we have

$$\int (B\rho_n - k)^+ \leqslant C(k) \quad \forall\, k > 0,\ \forall n.$$

Therefore $B\rho_n \uparrow u$ a.e. as $n \uparrow \infty$ and $\int (u - k)^+ < \infty \quad \forall k > 0$.

From (3.21) we deduce that (up to a subsequence)

$$\rho_n \rightharpoonup \rho \quad \text{weakly in } L^1(\Omega).$$

By Lemma 10 we have

$$B\rho_n \rightharpoonup B\rho \quad \text{weakly in } L^1(\Omega) + L^\infty(\Omega).$$

It follows that $B\rho_n \rightharpoonup B\rho$ weakly in $L^1(\Omega')$ for any $\Omega' \subset \Omega$ of finite measure. Since $B\rho_n \to u$ a.e. on Ω we deduce that $u = B\rho$ a.e. on Ω. Using Egorov's lemma and standard monotone analysis, we may now pass to the limit in (3.19) and conclude that

$$\partial j(\rho) + B\rho \ni V - \lambda \quad \text{a.e..}$$

□

REMARK 12. Part of the argument used in the proof of Theorem 6 (e.g. the equi-integrability of ρ_n) is inspired by the papers quoted as items [2] [3] under Gallouët-Morel [35].

REMARK 13. If j is coercive, i.e., γ^0 is everywhere defined, then assumption (3.18) is weaker than (H^+). Indeed we write

$$j^*((1+\theta)(V-M)) - j^*(V+t) \geqslant \gamma^0(V+t)[\theta V - M - \theta M - t].$$

so that

$$\gamma^0(V+t) \leqslant j^*((1+\theta)(V-M)) \quad \text{on } [\theta V - M - \theta M - t \geqslant 1]$$

while

$$\gamma^0(V+t) \leqslant \gamma^0\left(\frac{1+M+\theta M+t}{\theta} + t\right) \quad \text{on } [\theta V - M - \theta M - t < 1].$$

4. Some examples. Further properties of I_0 and I_1

In what follows and throughout the rest of the paper we assume that $\Omega = \mathbb{R}^N$ (with the Lebesgue measure dx) and $N \geqslant 3$.

We take $k(x, y) = k(x - y)$ where $k(x) = c_N/|x|^{N-2}$ with $c_N = 1/[(N-2)\sigma_N]$ and σ_N is the area of the unit sphere in $\mathbb{R}^N$, so that

$$k \in M^{N/(N-2)}(\mathbb{R}^N)$$

and

$$-\Delta k = \delta \quad \text{in the sense of } \mathcal{D}'(\mathbb{R}^N).$$

Here $M^p(\mathbb{R}^N)(1 < p < \infty)$ denotes the Marcinkiewicz (or weak L^p) space, i.e.,

$$M^p(\mathbb{R}^N) = \{u : \mathbb{R}^N \to \mathbb{R};\ u \text{ is measurable and } \|u\|_{M^p} < \infty\}$$

where the norm $\|u\|_{M^p}$ is defined by

$$\|u\|_{M^p} = \sup_{\substack{A\subset\mathbb{R}^N\\ |A|<\infty}} \frac{1}{|A|^{1/p'}} \int_A |u(x)|\, dx.$$

Some elementary properties of the spaces M^p are discussed in the Appendix of Bénilan-Brezis-Crandall [10]. In particular we recall that

$$a_p \|u\|^p_{M^p} \leqslant \sup_{\lambda>0} \lambda^p \text{ meas } [|u| > \lambda] \leqslant \|u\|^p_{M^p} \quad (a_p > 0).$$

We also recall that, for every $f \in L^1(\mathbb{R}^N)$,

$$Bf = k * f \in M^{N/(N-2)}$$

and

$$\|Bf\|_{M^{N/(N-2)}} \leqslant \|k\|_{M^{N/(N-2)}} \|f\|_{L^1}.$$

Moreover we have

$$-\Delta(Bf) = f \quad \text{in the sense of } \mathcal{D}'(\mathbb{R}^N)$$

and, in particular, B is *injective*. Therefore K defined in Section 1 is *strictly convex* (see Remark 3).

We claim that the kernel k satisfies properties (1.4), (3.2) and (3.3).

Verification of (3.2). Let $\rho \in L^1(\mathbb{R}^N)$ with $\rho \geqslant 0$ and $\|\rho\|_{L^1} \leqslant M$. We have

$$\int_{\mathbb{R}^N} (B\rho - 1)^+ \leqslant \int_{[B\rho>1]} B\rho \leqslant \|B\rho\|_{M^p} |A|^{1/p'}$$

where $p = N/(N-2)$ and $A = [B\rho > 1]$. But $|A| \leqslant \|B\rho\|^p_{M^p}$ and therefore

$$\int_{\mathbb{R}^N} (B\rho - 1)^+ \leqslant \|B\rho\|^p_{M^p} \leqslant CM^p.$$

In order to check (1.4) and (3.3) it is convenient to use

LEMMA 14. *Let $p \in C^1(\mathbb{R})$ with $p' \geqslant 0$ and $p(0) = 0$. Let $\rho \in L^1(\mathbb{R}^N)$ be such that $\rho\, p(B\rho) \in L^1(\mathbb{R}^N)$. Then*

$$\int p'(B\rho)|\nabla(B\rho)|^2 \leqslant \int \rho\, p(B\rho).$$

Proof. We already know (by Lemma A.10 in Bénilan-Brezis-Crandall [10]) that the conclusion holds if, in addition, $p \in L^\infty(\mathbb{R})$. In the general case, let (p_n) be a sequence such that $p_n \in C^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$, $p_n' \geqslant 0$, $p_n(0) = 0$, $|p_n(t)| \leqslant |p(t)|\ \forall t \in \mathbb{R}$, $p_n(t) \to p(t)$ $\forall t \in \mathbb{R}$ and $p_n'(t) \to p'(t)\ \forall t \in \mathbb{R}$.

We have

$$\int p_n'(B\rho)|\nabla(B\rho)|^2 \leqslant \int \rho\, p_n(B\rho)$$

and since $|\rho\, p_n(B\rho)| \leqslant |\rho\, p(B\rho)| \in L^1(\mathbb{R}^N)$ we conclude easily, using Fatou's Lemma and dominated convergence. □

Verification of (1.4) **and** (3.3). Applying Lemma 14 with $p(t) = t$ we obtain (1.4) (note that $\int_A |B\rho| < \infty$ for every A with $|A| < \infty$). Suppose now $p \in C^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ with $p' \in L^\infty(\mathbb{R})$, $p' \geqslant 0$ and $p(0) = 0$. Let $\rho \in L^1(\mathbb{R}^N)$ be such that $\int \rho\ p(B\rho) = 0$. It follows from Lemma 14 that $p'(B\rho)|\nabla(B\rho)|^2 = 0$ and thus $\nabla p(B\rho) = p'(B\rho)\,\nabla(B\rho) = 0$. Therefore, $p(B\rho)$ is a constant. On the other hand, $B\rho \to 0$ as $|x| \to \infty$ in a weak sense (i.e., for every $\alpha > 0$ the set $[|B\rho| > \alpha]$ has finite measure) and so does $p(B\rho)$. It follows that $p(B\rho) = 0$.

We recall the main result of Section 3. Let $j : \mathbb{R} \to [0, +\infty]$ be any convex l.s.c. function such that

$$j(0) = 0 \text{and} j(r) = +\infty \quad \text{for all } r < 0.$$

As above we set $\gamma = \partial j^* = (\partial j)^{-1}$.

Let $V : \mathbb{R}^N \to \mathbb{R}$ be any measurable function. We are concerned with the two problems

$$\begin{cases} \text{Given a constant } I \text{ with } 0 < I < \infty,\ \text{find a function } \rho \in L^1(\mathbb{R}^N) \text{ and a} \\ \text{constant } \lambda \in \mathbb{R} \text{ such that } \rho \geqslant 0 \text{ a.e.}, \int \rho = I \text{ and } \partial j(\rho) + B\rho \ni V - \lambda \text{ a.e.} \end{cases} \tag{E^I}$$

and

$$\begin{cases} \text{Given a constant } I \text{ with } 0 < I < \infty \text{ find a function} \\ \rho \in K_I = \{\rho \in D(\mathcal{E});\ \int \rho = I\} \text{ which minimizes } \mathcal{E} \text{ on } K_I. \end{cases} \tag{M$_I$}$$

Corollary 1 says that, under some assumptions, there exists $0 \leqslant I_0 \leqslant \infty$ such that

a) for every $0 < I \leqslant I_0$ (and $I < \infty$) there is a unique solution ρ^I of problem (M$_I$),
b) if $I_0 < \infty$ and $I > I_0$ problem (M$_I$) admits no solution.

Theorem 4 asserts that there exists I_1 with $0 \leqslant I_1 \leqslant \infty$ such that:

a) for every $0 < I \leqslant I_1$ (and $I < \infty$), there is a unique solution ρ^I of problem (E^I),
b) if $I_1 < \infty$ and $I > I_1$, problem (E^I) has no solution.

In what follows we shall examine various examples of functions j and V, discuss the relation between problems (E^I) and (M_I) and describe some additional properties of I_0 and I_1.

Some specific examples of functions j are the following:

EXAMPLE 1. Let $1 < p < \infty$ and let

$$j(r) = \begin{cases} \frac{1}{p} r^p & \text{for } r \geqslant 0 \\ +\infty & \text{for } r < 0 \end{cases}$$

so that, with $\frac{1}{p} + \frac{1}{p'} = 1$,

$$j^*(s) = \begin{cases} \frac{1}{p'} s^{p'} & \text{for } s \geqslant 0 \\ 0 & \text{for } s < 0 \end{cases}$$

$$\partial j(r) = \begin{cases} r^{p-1} & \text{for } r > 0 \\ (-\infty, 0] & \text{for } r = 0 \\ \emptyset & \text{for } r < 0 \end{cases}$$

$$\gamma(s) = \partial j^*(s) = (\partial j)^{-1}(s) = \begin{cases} s^{p'-1} & \text{for } s \geqslant 0 \\ 0 & \text{for } s < 0. \end{cases}$$

The usual Thomas-Fermi problem (see e.g. Lieb-Simon [48], Lieb [47]) corresponds to the case $p = 5/3$.

EXAMPLE 2. Let $1 < p < \infty$ and let

$$j(r) = \begin{cases} \frac{1}{p}[(1+r)^p - 1 - pr] & \text{for } r \geqslant 0 \\ +\infty & \text{for } r < 0 \end{cases}$$

so that

$$j^*(s) = \begin{cases} \frac{1}{p'}(1+s)^{p'} - 1 - p's & \text{for } s \geqslant 0 \\ 0 & \text{for } s \leqslant 0 \end{cases}$$

$$\partial j(r) = \begin{cases} (1+r)^{p-1} - 1 & \text{for } r > 0 \\ (-\infty, 0] & \text{for } r = 0 \\ \emptyset & \text{for } r < 0 \end{cases}$$

$$\gamma(s) = \partial j^*(s) = (\partial j)^{-1}(s) = \begin{cases} (1+s)^{p'-1} - 1 & \text{for } s \geqslant 0 \\ 0 & \text{for } s < 0. \end{cases}$$

Such a j(with $p = 5/3$) occurs in the Thomas-Fermi theory of screening (see Lieb-Simon [48], Section VII). Note that $j(r) \sim r^p$ as $r \to +\infty$ while $j(r) \sim r^2$ as $r \to 0+$.

EXAMPLE 3. Let

$$j(r) = \begin{cases} 3\int_0^{r^{1/3}} t^2(\sqrt{1+t^2}-1)\,dt & \text{for } r \geqslant 0 \\ +\infty & \text{for } r < 0 \end{cases}$$

so that

$$j^*(s) = \begin{cases} \int_0^s (2t+t^2)^{3/2}\,dt & \text{for } s \geqslant 0 \\ 0 & \text{for } s < 0 \end{cases}$$

$$\partial j(r) = \begin{cases} \sqrt{1+r^{2/3}}-1 & \text{for } r \geqslant 0 \\ (-\infty, 0] & \text{for } r = 0 \\ \emptyset & \text{for } r < 0 \end{cases}$$

$$\gamma(s) = \partial j^*(s) = (\partial j)^{-1}(s) = \begin{cases} (2s+s^2)^{3/2} & \text{for } s \geqslant 0 \\ 0 & \text{for } s < 0. \end{cases}$$

Such a j occurs in some relativistic Thomas-Fermi model (E. Lieb, personal communication). Note that $j(r) \sim r^{4/3}$ as $r \to +\infty$ while $j(r) \sim r^{5/3}$ as $r \to 0+$.

EXAMPLE 4. Let $1 < q < p < \infty$ and let

$$j(r) = \begin{cases} \frac{1}{p}r^p - \frac{a}{q}r^q + br & \text{for } r > 1 \\ 0 & \text{for } 0 \leqslant r \leqslant 1 \\ +\infty & \text{for } r < 0 \end{cases}$$

where $a = q(p-1)/p(q-1)$ and $b = (p-q)/p(q-1)$, so that

$$\partial j(r) = \begin{cases} r^{p-1} - ar^{q-1} + b & \text{for } r > 1 \\ 0 & \text{for } 0 < r \leqslant 1 \\ (-\infty, 0] & \text{for } r = 0 \\ \emptyset & \text{for } r < 0 \end{cases}$$

$$\gamma(s) = \begin{cases} 0 & \text{for } s < 0 \\ [0, 1] & \text{for } s = 0 \\ \text{singlevalued} & \text{for } s > 0. \end{cases}$$

Note that $j(r) \sim r^p$ as $r \to +\infty$ while $j(r) = 0$ for $0 < r < 1$ and $\gamma(s) \sim 1 + cs$, for $s > 0, s \sim 0$ with $c = \frac{p}{(p-1)(p-q)}$. Such a j occurs in Thomas-Fermi model with an "exchange correction" (see Benguria [7], Chapter 3).

In what follows we will assume that $N = 3$, but there are similar results for $N > 3$. Throughout the rest of this section we will assume (this is satisfied in all the examples above) that

$$j \text{ is } C^1 \text{ on } (0, \infty) \text{ with } j'(0+) = 0. \tag{4.1}$$

We will consider various types of functions V. In all cases we have $V_\infty = 0$.

TYPE I. $V = k * f$ for some $f \in L^1$.

Thus $V \in M^3$ and $-\Delta V = f$. In particular, we know that for every $\delta > 0$ the set $[|V| > \delta]$ has finite measure. This case is well adapted to the direct approach of Section 3. Indeed, the equation

$$-\Delta u_0 + \gamma(u_0) \ni f \quad \text{in } \mathbb{R}^3 \tag{4.2}$$

admits a unique solution $u_0 \in M^3$ (by Theorem 2.1 in Bénilan-Brezis-Crandall [10]), with $\gamma(u_0) \in L^1$ (more precisely $f + \Delta u_0 \in L^1$) and

$$\int \gamma(u_0) \leqslant \int f^+. \tag{4.3}$$

(Recall $\gamma(t) = 0$ for $t \leqslant 0$). If we set

$$\rho_0 = f + \Delta u_0 = \Delta(u_0 - V)$$

we see (from (4.2)) that

$$u_0 \in \gamma^{-1}(\rho_0) = \partial j(\rho_0)$$

and therefore

$$\partial j(\rho_0) + B\rho_0 \ni V \quad \text{a.e.}. \tag{4.4}$$

More generally, for every $\lambda \geqslant 0$ there exists a unique solution $u_\lambda \in M^3$ of

$$-\Delta u_\lambda + \gamma(u_\lambda - \lambda) \ni f \quad \text{in } \mathbb{R}^3 \tag{4.5}$$

(since $\beta(t) = \gamma(t - \lambda)$ is a maximal monotone graph such that $0 \in \beta(0)$). Then

$$\rho_\lambda = f + \Delta u_\lambda \in L^1$$

satisfies

$$u_\lambda - \lambda \in \gamma^{-1}(f + \Delta u_\lambda) = \partial j(\rho_\lambda)$$

and therefore we have

$$\partial j(\rho_\lambda) + B\rho_\lambda \ni V - \lambda \quad \text{a.e.}. \tag{E$_\lambda$}$$

Set

$$I_1 = \int f + \Delta u_0 = \int \gamma(u_0) \leqslant \int f^+.$$

Note that $I_1 > 0$ whenever $\operatorname{ess\,sup}_\Omega V > 0$. (Indeed $[I_1 = 0] \Leftrightarrow [\gamma(u_0) = 0] \Leftrightarrow [u_0 \leqslant 0]$ because of assumption (4.1), and then by (4.2) we have $u_0 = V$).

COROLLARY 2. *For every* $I \in (0, I_1]$ *there exists a unique solution of problem* (E^I). *In addition, if we assume*

$$\int_0^1 \frac{\gamma^0(s)}{s^4}\, ds = \infty, \tag{4.6}$$

then

$$\int f \leqslant I_1 \leqslant \int f^+;$$

in particular, if $f \geqslant 0$ *a.e. then*

$$I_1 = \int f. \tag{4.7}$$

Proof. The conditions of Theorem 5 are satisfied with $\lambda_0 = V_\infty = 0$. Note that (E_λ) has no solution for $\lambda < 0$. (Indeed, if (E_λ) has a solution for some $\lambda \in \mathbb{R}$ we deduce from Lemma 8 and (4.1) that $(V - \lambda)_\infty = V_\infty - \lambda = -\lambda \leqslant 0$, i.e., $\lambda \geqslant 0$). Hence we have the first assertion of Corollary 2.

Next we assume (4.6). Applying Lemma B.1 and Theorem B.1 (from Appendix B) to the function u_0 we conclude that

$$\int \Delta u_0 \geqslant 0.$$

Therefore, $I_1 = \int f + \Delta u_0 \geqslant \int f$. □

REMARK 14. We emphasize that the first assertion in Corollary 2 applies to Example 1 *without any restriction* on p. The second assertion holds only under the restriction

$$p \geqslant \frac{4}{3} \tag{4.8}$$

(this is an assumption about j *near zero*). It is clearly satisfied for the standard Thomas-Fermi exponent $p = 5/3$.

On the other hand if (4.8) fails, i.e., if $p < 4/3$, then for $f \geqslant 0$ with compact support, $f \not\equiv 0$, we have $I_1 < \int f$. Indeed in this case $\gamma(s) \sim s^q$ as $s \to 0$ with $q = p' - 1 > 3$. Applying a result of Véron (see item [3], Théorème 4.1, under Véron [59]) we see that $u_0(x) \sim c/|x|$ as $|x| \to \infty$ with $c > 0$. Therefore (by Theorem B.1) we have $\int \Delta u_0 < 0$ and $I_1 = \int f + \Delta u_0 < \int f$.

Alternatively, we could also try to apply the variational route of Section 2. This is indeed possible in Example 1 when

$$p > 3/2 \tag{4.9}$$

((4.9) is now an assumption about j *near infinity*). Note that (4.9) holds for the standard Thomas-Fermi exponent $p = 5/3$. However (4.9) *does not hold in Example* 2 (relativistic Thomas-Fermi).

Indeed, the basic condition (H) (or H$^+$) says that for some constant $C \in \mathbb{R}$

$$(V - C)^+ \in L^{p'}. \tag{4.10}$$

Recall that $V \in M^3$ and thus $V|_\omega \in L^q(\omega)$ for any $q < 3$ and any set ω with finite measure. If we take $C > 0$ and $\omega = [|V| > C]$ we see that (4.10) holds provided $p' < 3$, i.e., $p > 3/2$.

When condition (4.9) fails—for example $j(r) = r^p$ with $p \leqslant 3/2$—the functional

$$\mathcal{E}(\rho) + \lambda \int \rho = \int j(\rho) - V\rho + \lambda\rho + \frac{1}{2}\int \rho B\rho \tag{4.11}$$

is usually *unbounded from below for any* $\lambda > 0$. This means that the variational route used in Section 2 is *not* practicable for a general $V = Bf$, $f \in L^1$.

Here is a sketch of the argument. Suppose that we have a lower bound. Then

$$\int V\rho \leqslant \int \rho^p + \frac{1}{2}\int \rho B\rho + C\int \rho + C. \tag{4.12}$$

It is easy to see from Young's inequality on convolutions or the L^p regularity theory that $\|B\rho\|_{L^6} \leqslant C\|\rho\|_{L^{6/5}}$ and thus

$$\int \rho B\rho \leqslant \|\rho\|_{L^{6/5}}\|B\rho\|_{L^6} \leqslant C\|\rho\|^2_{L^{6/5}}.$$

Since $p < 2$ we deduce from (4.12) that

$$\int V\rho \leqslant C(\|\rho\|^2_{L^p} + \|\rho\|^2_{L^{6/5}} + C)$$

and by scaling we find

$$\int V\rho \leqslant C(\|\rho\|_{L^p} + \|\rho\|_{L^{6/5}}).$$

Hence

$$V \in L^{p'} + L^6$$

so that

$$V \in L^q_{\text{loc}} \quad \text{with } q = \min(p', 6).$$

Since $p \leqslant 3/2$ we have $p' \geqslant 3$ and then $q \geqslant 3$. On the other hand B does not map L^1 into L^3 (only into M^3) [otherwise B would also map $L^{3/2}$ into L^∞ and then $k \in L^3$ – impossible]. Hence there are some f's in L^1 such that $V = Bf \notin L^3$. For such V's the functional (4.11) is unbounded below.

TYPE II. $V = k * \mu$ for some bounded measure μ.

This case is especially important in the Thomas-Fermi setting because it includes functions $V(x)$ of the form

$$V(x) = \sum_{i=1}^{\ell} \frac{m_i}{|x - a_i|}, \; m_i \in \mathbb{R}, \tag{4.13}$$

which play a central role in the analysis of Lieb-Simon [48]. Here we have

$$V = k * \mu \quad \text{and} \quad \mu = 4\pi \sum_{i=1}^{\ell} m_i \delta_{a_i}.$$

Again it is well suited to the direct approach of Section 3 *provided we make the additional assumption*

$$\int_{|x|<1} \gamma^0\left(\frac{1}{|x|}\right) = C \int_1^\infty \frac{\gamma^0(s)}{s^4}\, ds < \infty \tag{4.14}$$

which is required in order to apply Theorem A.1 (in Appendix A). In the framework of Examples 1, 2, 4 this corresponds to the condition

$$p > \frac{4}{3}. \tag{4.15}$$

Assumption (4.14) is an assumption about j near *infinity*. It is satisfied for the standard Thomas-Fermi exponent $p = 5/3$. However (4.14) *fails* in Example 2 (relativistic Thomas-Fermi).

As above we solve the equation

$$-\Delta u_0 + \gamma(u_0) \ni \mu \quad \text{in } \mathbb{R}^3 \tag{4.16}$$

with the help of Theorem A.1 and we set

$$\rho_0 = \mu + \Delta u_0 \in L^1$$

and

$$I_1 = \int \mu + \Delta u_0 = \int \gamma(u_0) \leqslant \int \mu^+.$$

Again $I_1 > 0$ whenever $\operatorname{ess\,sup}_{\mathbb{R}^3} V > 0$.

Using the same strategy as in Corollary 2, we have

COROLLARY 3. *Assume* (4.14). *Then for every* $I \in (0, I_1]$ *there exists a unique solution of problem* (E^I).

In addition, if we assume (4.6)*, then*

$$\int \mu \leqslant I_1 \leqslant \int \mu^+;$$

in particular if $\mu \geqslant 0$*, then*

$$I_1 = \int \mu.$$

REMARK 15. Condition (4.15) is *absolutely essential*. When it is not satisfied there is usually *no* I whatsoever such that problem (E^I) admits a solution. Take, for example, $j(r) = \frac{3}{4}r^{4/3}$ and then $\gamma(s) = (s^+)^3$. Let $V(x) = 1/|x|$ (so that $-\Delta V = 4\pi\delta_0$). If we had a solution of (E^I) for some I, it would satisfy

$$\partial j(\rho) + B\rho \ni V - \lambda.$$

Necessarily $\lambda \geqslant 0$ (by Lemma 8) and $u = \frac{c}{|x|} - B\rho$ satisfies

$$-\Delta u + \left[(u-\lambda)^+\right]^3 = \delta_0$$

with $(u-\lambda)^+ \in L^3$. But this is impossible, even locally near 0; see the discussion in Remark A.4. In particular, for the relativistic Thomas-Fermi model (Example 3 above) with the Coulomb potential $V(x) = 1/|x|$, there is *no* I such that problem (E^I) admits a solution; existence holds provided the potential is slightly more "diffuse".

REMARK 16. As above, we see that the variational route discussed in Section 2 holds in Example 1 when $p > 3/2$. If $p \leqslant 3/2$ and $V(x) = \sum_i \frac{m_i}{|x-a_i|}$, the functional $\mathcal{E}(\rho) + \lambda\int\rho$ is *unbounded below*.

TYPE III. $V \in M^3(\mathbb{R}^3)$.

Clearly this situation is more general than Type II (since $k * \mu \in M^3$). Here we cannot anymore rely on Appendix A to solve

$$-\Delta u_0 + \gamma(u_0) \ni -\Delta V$$

since ΔV need not be a measure. Instead we will rely on Theorem 6. The conclusion is less precise since we have little information about I_1 (we suspect that I_1 might sometimes be infinite).

COROLLARY 4. *Assume again* (4.14)*. Let* $V \in M^3(\mathbb{R}^3)$ *be such that* $\operatorname{ess\,sup}_{\mathbb{R}^3} V > 0$*. Then there exists* $0 < I_1 \leqslant \infty$ *such that*

a) *for every* $I \in (0, I_1)$ *there is a unique solution of problem* (E^I)*,*

b) *if* $I_1 < \infty$*, problem* (E^{I_1}) *admits a solution, and problem* (E^{I}) *has no solution when* $I > I_1$.

Proof. Apply Theorem 6 with the decomposition $V = U + Bf$ and $f = 0$. We have to verify (3.18), i.e.,

$$\int_\omega \gamma^0(V+t) < \infty \quad \forall t > 0,\ \forall \omega \subset \Omega \text{ with } |\omega| < \infty.$$

This follows immediately from assumption (4.14) and Lemma A.1 applied to the function $u_n \equiv V + t \in M^3$ on ω. □

REMARK 17. There are *many* variants of Corollary 4. For instance, in the standard Thomas-Fermi theory (Example 1 with $p = 5/3$), it suffices to assume, for example, that for every $\delta > 0$, the set $[V > \delta]$ is bounded and $V \in L^{3/2}_{\mathrm{loc}}$ (singularities such as $|x|^{-\alpha}$, $\alpha < 2$ are admissible).

In the relativistic Thomas-Fermi (Example 2), it suffices to assume that for every $\delta > 0$, the set $[V > \delta]$ is bounded and that $V = V_1 + V_2$ with $V_1 \in L^3_{\mathrm{loc}}$ and $V_2 \in L^1_{\mathrm{loc}}$ with $\Delta V_2 \in L^1_{\mathrm{loc}}$. Note that the singularity $V(x) = 1/|x|$ is excluded, but this is consistent with the discussion in Remark 15 (see also Remark A. 4).

5. A min-max principle for the Lagrange multiplier λ; uniqueness of the extremals

Throughout this section we take $\Omega = \mathbb{R}^N$, $N \geqslant 3$ and $B\rho = k * \rho$ as in Section 4.

Let $j : \mathbb{R} \to [0, +\infty]$ be a convex l.s.c. function such that

$$j(0) = 0 \text{ and } j(r) = +\infty \text{ for all } r < 0 \tag{5.1}$$

$$j \text{ is } C^1 \text{ on } (0,\infty), \quad \text{and} \quad j'(0+) = 0. \tag{5.2}$$

Let $V : \Omega \to \mathbb{R}$ be a measurable function such that $V_\infty = 0$. Recall (see Theorem 5) that exists $\lambda_0 \in [0, +\infty]$ such that for every $\lambda > \lambda_0$ problem

$$\partial j(\rho) + B\rho \ni V - \lambda \quad \text{a.e.} \tag{E$_\lambda$}$$

admits a unique solution $\rho_\lambda \in L^1$, $\rho_\lambda \geqslant 0$. As in the previous sections we set

$$I(\lambda) = \int \rho_\lambda \text{ and } I_1 = \sup_{\lambda > \lambda_0} I(\lambda) = \lim_{\lambda \downarrow \lambda_0} I(\lambda) \leqslant \infty.$$

Note that $I(\lambda) > 0$ if and only if $\lambda < \operatorname{ess\,sup}_{\mathbb{R}^N} V$. Recall that (E_λ) has no solution for $\lambda < \lambda_0$ and (E_{λ_0}) admits a unique solution if and only if $I_1 < \infty$.

THEOREM 7. *For any* $\lambda_0 < \lambda < \operatorname{ess\,sup}_{\mathbb{R}^N} V$ *we have*

$$\lambda = \max_{\substack{\rho \in L^1, \rho \geqslant 0 \\ \int \rho = I(\lambda)}} \operatorname*{ess\,inf}_{[\rho > 0]} \{V - B\rho - j'(\rho)\} \tag{5.3}$$

and

$$\lambda = \min_{\substack{\rho \in L^1, \rho \geqslant 0 \\ \int \rho = I(\lambda)}} \operatorname*{ess\,sup}_{\mathbb{R}^N} \{V - B\rho - j'(\rho)\}. \tag{5.4}$$

Conclusion (5.3) *holds for* $\lambda = \lambda_0 < \infty$ *provided* $I_1 < \infty$*; conclusion* (5.4) *holds for* $\lambda = \lambda_0$ *provided* $\lambda_0 = 0$ *and* $I_1 < \infty$.

In (5.4) we use the convention that $j'(0) = j'(0+)(= 0)$.

REMARK 18. The conclusion of Theorem 7 were obtained by Lieb-Simon [48] (Theorems II.28 and II.29) in the context of the standard Thomas-Fermi model (see Example 1 in Section 4 with $p = 5/3$, and $V(x)$ given by (4.13) with $m_i > 0\ \forall i$).

Proof. If we take $\rho = \rho_\lambda$ we have on the set $A = [\rho_\lambda > 0]$ (which has positive measure because of the assumption $\lambda < \operatorname{ess\,sup}_{\mathbb{R}^N} V$),

$$j'(\rho_\lambda) + B\rho_\lambda = V - \lambda, \tag{5.5}$$

so that

$$\operatorname*{ess\,inf}_{[\rho_\lambda > 0]} \{V - B\rho_\lambda - j'(\rho_\lambda)\} = \lambda.$$

Moreover, on the set $[\rho_\lambda = 0]$ we have

$$V - \lambda - B\rho_\lambda \leqslant 0.$$

and in particular

$$V - B\rho_\lambda - j'(\rho_\lambda) \leqslant \lambda.$$

Thus

$$\operatorname*{ess\,sup}_{\mathbb{R}^N} \{V - B\rho_\lambda - j'(\rho_\lambda)\} = \lambda.$$

To conclude the proof it remains to show that for every $\rho \in L^1$, $\rho \geqslant 0$, with $\int \rho = I(\lambda)$ we have

$$\operatorname*{ess\,inf}_{[\rho > 0]} \{V - B\rho - j'(\rho)\} \leqslant \lambda \tag{5.6}$$

and

$$\operatorname*{ess\,sup}_{\mathbb{R}^N} \{V - B\rho - j'(\rho)\} \geqslant \lambda. \tag{5.7}$$

□

Proof of (5.6). Suppose, by contradiction, that there is some $\bar{\rho} \in L^1, \bar{\rho} \geqslant 0$, with $\int \bar{\rho} = I(\lambda)$, such that

$$\lambda^* = \operatorname*{ess\,inf}_{[\bar{\rho}>0]} \{V - B\bar{\rho} - j'(\bar{\rho})\} > \lambda. \tag{5.8}$$

Let $\rho^* = \rho_{\lambda^*}$ be the unique solution of (E_{λ^*}), i.e.,

$$\partial j(\rho^*) + B\rho^* \ni V - \lambda^*. \tag{5.9}$$

Set

$$W = \begin{cases} j'(\bar{\rho}) + B\bar{\rho} & \text{on } [\bar{\rho} > 0], \\ \min\{B\bar{\rho}, V - \lambda^*\} & \text{on } [\bar{\rho} = 0]. \end{cases}$$

Clearly we have

$$\partial j(\bar{\rho}) + B\bar{\rho} \ni W \quad \text{a.e. on} \mathbb{R}^N, \tag{5.10}$$

and

$$W \leqslant V - \lambda^* \quad \text{a.e. on } \mathbb{R}^N. \tag{5.11}$$

We deduce from (5.9), (5.10), (5.11) and Lemma 12 that

$$B\bar{\rho} \leqslant B\rho^*. \tag{5.12}$$

Applying Theorem B.1 with $u = B(\bar{\rho} - \rho^*) \leqslant 0$, we see that $\int(\bar{\rho} - \rho^*) \leqslant 0$, i.e.,

$$\int \bar{\rho} = I(\lambda) \leqslant I(\lambda^*) = \int \rho^*. \tag{5.13}$$

Let ρ_λ be the solution of (E_λ). From Theorem 5 we know that

$$\rho_{\lambda^*} \leqslant \rho_\lambda. \tag{5.14}$$

Combining (5.14) with (5.13) we deduce that

$$\rho^* = \rho_\lambda. \tag{5.15}$$

Recall that $A = [\rho_\lambda > 0] = [\rho^* > 0]$ has positive measure. Applying (E_λ) and (E_{λ^*}) on A we find

$$V - \lambda = V - \lambda^* \quad \text{a.e. on } A,$$

and thus $\lambda = \lambda^*$− a contradiction.

Proof of (5.7). Suppose, by contradiction, that there is some $\bar{\rho} \in L^1, \bar{\rho} \geqslant 0$, with $\int \bar{\rho} = I(\lambda)$ such that

$$\mu^* = \operatorname*{ess\,sup}_{\mathbb{R}^N}\{V - B\bar{\rho} - j'(\bar{\rho})\} < \lambda. \tag{5.16}$$

Fix μ such that $\max\{\mu^*, \lambda_1\} < \mu < \lambda$. Set

$$W = j'(\bar{\rho}) + B\bar{\rho},$$

so that

$$\partial j(\bar{\rho}) + B\bar{\rho} \ni W \tag{5.17}$$

and

$$W \geqslant V - \mu^* > V - \mu. \tag{5.18}$$

Let ρ_μ be the solution of

$$\partial j(\rho_\mu) + B\rho_\mu \ni V - \mu \tag{5.19}$$

(which exists since $\mu > \lambda_1$). Combining (5.17), (5.19) and (5.18), we deduce from the comparison principle in Lemma 12 that $B\rho_\mu \leqslant B\bar{\rho}$. Applying Theorem B.1 once more yields $\int(\rho_\mu - \bar{\rho}) \leqslant 0$, i.e.,

$$I(\mu) = \int \rho_\mu \leqslant \int \bar{\rho} = I(\lambda).$$

We conclude that $\rho_\lambda = \lambda_\mu$ and obtain a contradiction as above.

In the limiting case $\lambda = \lambda_0$, the proof of (5.6) is unchanged. But we cannot use the above proof for (5.7). In this case we simply observe that

$$\operatorname*{ess\,sup}_{\mathbb{R}^N}\{V - B\rho - j'(\rho)\} \geqslant V_\infty = 0 = \lambda_0.$$

Lieb and Simon [48] have conjectured the uniqueness of the maximizer in (5.3) and the minimizer in (5.4) (see Problem 4 in the Introduction and the discussion in Section II.7). We will prove that the conjecture is *true* when $\lambda_0 < \lambda < \operatorname{ess\,sup}_{\mathbb{R}^N} V$ for a large class of problems including the standard Thomas-Fermi model: Example 1 in Section 4 with $p = 5/3$. (With the notations of Lieb-Simon [48] this means that the conjecture holds when $N < Z$). A basic ingredient is a sharp form of strong maximum principle described in Appendix C.

However we will see that the conjecture *fails* (even for the standard Thomas-Fermi model) in the "neutral" case $\lambda = 0$ (i.e., $N = Z$ with the notations of Lieb-Simon [48]).

A counter example in the neutral case.

Consider for simplicity the case $N = 3$ and the Example of Section 4 with $p > 4/3$. In the neutral case, the Thomas-Fermi ρ is the unique solution of the equations

$$\rho^{p-1} + B\rho = V = \sum_i \frac{m_i}{|x - a_i|} \tag{5.20}$$

with $m_i > 0\,\forall i$. In other words $u = \rho^{p-1}$ is the unique positive solution of

$$-\Delta u + u^{1/(p-1)} = 4\pi \sum_i m_i \delta_{a_i}. \tag{5.21}$$

Moreover we have, by Corollary 3,

$$\int \rho = 4\pi \sum m_i. \tag{5.22}$$

Clearly the function ρ satisfies $\rho > 0$, $\int \rho = I = 4\pi \sum m_i$, and

$$\operatorname*{ess\,inf}_{\mathbb{R}^N}(V - B\rho - \rho^{p-1}) = 0 \tag{5.23}$$

$$\operatorname*{ess\,sup}_{\mathbb{R}^N}(V - B\rho - \rho^{p-1}) = 0. \tag{5.24}$$

We will now construct two functions ρ_1, ρ_2, distinct from ρ, satisfying $\rho_1 > 0, \rho_2 > 0$, $\int \rho_1 = \int \rho_2 = I$,

$$\operatorname*{ess\,inf}_{\mathbb{R}^N}(V - B\rho_1 - \rho_1^{p-1}) = 0, \tag{5.25}$$

$$\operatorname*{ess\,sup}_{\mathbb{R}^N}(V - B\rho_2 - \rho_2^{p-1}) = 0. \tag{5.26}$$

Given $k > 0$, let $u_k > 0$ be the solution of

$$-\Delta u_k + k u_k^{1/(p-1)} = 4\pi \sum m_i \delta_{a_i}, \tag{5.27}$$

and set

$$\rho_k = k u_k^{1/(p-1)}. \tag{5.28}$$

From the results of Appendix B we deduce that

$$\int \rho_k = I = 4\pi \sum m_i \quad \forall k.$$

On the other hand, we see from (5.27) and (5.28) that

$$(k^{-1}\rho_k)^{p-1} + B\rho_k = V = \sum \frac{m_i}{|x - a_i|}$$

and therefore

$$V - B\rho_k - \rho_k^{p-1} = (k^{-(p-1)} - 1)\rho_k^{p-1}.$$

We obtain the desired ρ_1 and ρ_2 satisfying (5.25) and (5.26) by choosing $\rho_1 = \rho_{k_1}$ and $\rho_2 = \rho_{k_2}$ with $k_1 < 1$ and $k_2 > 1$.

Uniqueness of the extremals in the "ionic" case, $0 < I < I_0$

In addition to the standard assumptions (5.1) and (5.2) on j, we assume here that

$$j' \text{ is concave on } (0, \infty), \tag{5.29}$$

and

$$\lim_{r\to\infty} \frac{j(r)}{r} = +\infty. \tag{5.30}$$

As a result, it is easy to see that $\gamma = (\partial j)^{-1}$ is a continuous nondecreasing function on $\mathbb{R}$ such that

$$\gamma(s) = 0 \text{ for } s \leqslant 0, \tag{5.31}$$

and

$$\gamma \text{ is convex on } \mathbb{R}, \tag{5.32}$$

so that $\gamma'(s-)$ exists at every $s \in \mathbb{R}$, and will be denote simply $\gamma'(s)$.

A typical example is

$$j(r) = \begin{cases} \frac{1}{p} r^p & \text{for } r \geqslant 0, \\ +\infty & \text{for } r < 0, \end{cases} \tag{5.33}$$

with $1 < p < 2$ and then $\gamma(r) = (r^+)^{p'-1}$; recall that the standard Thomas-Fermi model corresponds to $p = 5/3$ and then $\gamma(r) = (r^+)^{3/2}$.

Let $\lambda > 0$ and let V be any measurable function such that, for some $R > 0$,

$$V(x) \leqslant \lambda \text{ for a.e. } x \text{ with } |x| > R. \tag{5.34}$$

We will assume that

$$\gamma'(V - \lambda) \in L^1(\mathbb{R}^N). \tag{5.35}$$

The standard Thomas-Fermi model corresponds to $V(x) = \sum \frac{m_i}{|x-a_i|}$ in $\mathbb{R}^3$ and satisfies all the required assumptions (any $p > 5/4$ would be acceptable).

Let $\rho \in L^1$, $\rho \geqslant 0$, be a solution of the problem

$$\partial j(\rho) + B\rho \ni V - \lambda \quad \text{a.e. on } \mathbb{R}^N. \tag{5.36}$$

Suppose now that ρ_1 is a maximizer for (5.3), i.e., $\rho_1 \in L^1$, $\rho_1 \geqslant 0$, satisfies

$$\int \rho_1 = \int \rho \tag{5.37}$$

and

$$\operatorname*{ess\,inf}_{[\rho_1>0]} \{V - B\rho_1 - j'(\rho_1)\} = \lambda. \tag{5.38}$$

Similarly, suppose that ρ_2 is a minimizer for (5.4), i.e., $\rho_2 \in L^1$, $\rho_2 \geqslant 0$, satisfies

$$\int \rho_2 = \int \rho \tag{5.39}$$

and

$$\operatorname*{ess\,sup}_{\mathbb{R}^N} \{V - B\rho_2 - j'(\rho_2)\} = \lambda \tag{5.40}$$

(with the convention that $j'(0) = 0$).

THEOREM 8. *Assume* (5.1), (5.2), (5.29), (5.30), (5.34)–(5.40). *Then*

$$\rho_1 = \rho_2 = \rho.$$

The key ingredient in the proof is the following:

LEMMA 15. *Assume* (5.1), (5.2), (5.29) *and* (5.30). *Let* $\psi_1, \psi_2 \in L^1$ *with* $\psi_1 \geqslant 0$ *a.e. on* $\mathbb{R}^N$, $\psi_2 \geqslant 0$ *a.e. on* $\mathbb{R}^N$ *be such that*

$$\int \psi_1 = \int \psi_2. \tag{5.41}$$

Let f_1, f_2 *be measurable functions on* $\mathbb{R}^N$ *such that*

$$f_1 \leqslant f_2 \quad \text{a.e. on } \mathbb{R}^N, \tag{5.42}$$

$$f_1(x) \leqslant 0 \quad \text{for a.e. } x, |x| > R, \tag{5.43}$$

$$\gamma'(f_1) \in L^1. \tag{5.44}$$

Assume

$$\partial j(\psi_1) + B\psi_1 \ni f_1 \quad \textit{a.e. on } \mathbb{R}^N, \tag{5.45}$$

and

$$\partial j(\psi_2) + B\psi_2 \ni f_2 \quad \textit{a.e. on } \mathbb{R}^N. \tag{5.46}$$

Then

$$\psi_1 = \psi_2. \tag{5.47}$$

Proof of Lemma 15. From Lemma 12 and (5.42) we already know that

$$B\psi_1 \leqslant B\psi_2. \tag{5.48}$$

Set $u = B(\psi_2 - \psi_1) \geqslant 0$ and

$$a = \begin{cases} \frac{\gamma(f_1 - B\psi_1) - \gamma(f_1 - B\psi_2)}{u} & \text{on } [u > 0], \\ 0 & \text{on } [u = 0], \end{cases}$$

so that $a \geqslant 0$ a.e.

Clearly we have

$$\begin{aligned} -\Delta u + au &= (\psi_2 - \psi_1) + au \\ &= \gamma(f_2 - B\psi_2) - \gamma(f_1 - B\psi_1) + au \\ &\geqslant \gamma(f_1 - B\psi_2) - \gamma(f_1 - B\psi_1) + au \equiv 0. \end{aligned} \tag{5.49}$$

From the convexity of γ we see that

$$\gamma(f_1 - B\psi_2) - \gamma(f_1 - B\psi_1) \geqslant \gamma'(f_1 - B\psi_1)(B\psi_1 - B\psi_2),$$

and thus, by (5.44),

$$a(x) \leqslant \gamma'(f_1) \in L^1. \tag{5.50}$$

On the other hand $u \in M^{N/(N-2)}$, $\Delta u \in L^1$ and $\int \Delta u = 0$ (by (5.41)); moreover

$$-\Delta u = \psi_2 - \psi_1 \geqslant -\gamma(f_1),$$

since $\psi_2 \geqslant 0$ and $\psi_1 = \gamma(f_1 - B\psi_1) \leqslant \gamma(f_1)$. From (5.43) we infer that

$$-\Delta u \geqslant 0 \quad \text{for a.e. } x, |x| > R. \tag{5.51}$$

Applying Corollary B.3 we see that $u \equiv 0$ in $[|x| > R]$. We may then invoke Theorem C. 1 to conclude that $u \equiv 0$, i.e., $\psi_1 = \psi_2$.

We may now go to the

Proof of Theorem 8. Set

$$W = j'(\rho_2) + B\rho_2 \text{ a.e. on } \mathbb{R}^N,$$

so that

$$\partial j(\rho_2) + B\rho_2 \ni W \text{ a.e.}$$

and by (5.40)

$$W \geqslant V - \lambda \text{ a.e.}$$

Applying Lemma 15 to $\psi_1 = \rho$, $f_1 = V - \lambda$, $\psi_2 = \rho_2$ and $f_2 = W$, we find that $\rho = \rho_2$. Next, letting

$$Z = \begin{cases} j'(\rho_1) + B\rho_1 & \text{on } [\rho_1 > 0], \\ \min\{B\rho_1, V - \lambda\} & \text{on } [\rho_1 = 0], \end{cases}$$

we see that

$$\partial j(\rho_1) + B\rho_1 \ni Z \quad \text{a.e. on } \mathbb{R}^N$$

and

$$W \leqslant V - \lambda \quad \text{a.e. on } \mathbb{R}^N.$$

Applying Lemma 15 to $\psi_1 = \rho_1$, $f_1 = W$, $\psi_2 = \rho$ and $f_2 = V - \lambda$ we find that $\rho_1 = \rho$.

6. Asymptotic estimates for $I(\lambda)$ as $\lambda \downarrow 0$; behavior of the chemical potential in the weakly ionized limit

In this section we assume that (where the symbol $\sim$ means, as usual, that the ratio tends to 1),

$$\gamma(s) \sim s^q \quad \text{as } s \downarrow 0, \text{ for some } 1 < q < \frac{N}{N-2}, \tag{6.1}$$

and

$$f = -\Delta V \quad \text{is a nonnegative, nonzero, measure in } \mathbb{R}^N \text{ with compact support,} \tag{6.2}$$

where $V \in M^{N/(N-2)}(\mathbb{R}^n)$. If $f \notin L^1(\mathbb{R}^N)$, we suppose, in addition, that

$$\gamma^0\left(\frac{1}{|x|^{N-2}}\right) \in L^1_{\text{loc}}(\mathbb{R}^N). \tag{6.3}$$

Using Theorem 2.1 in Bénilan-Brezis-Crandall [10] if $f \in L^1(\mathbb{R}^N)$, or Theorem A.1 in Appendix A if $f \notin L^1(\mathbb{R}^N)$, we know that for every $\lambda \geqslant 0$, there exists $(u_\lambda, \rho_\lambda) \in M^{N/(N-2)} \times L^1$ such that

$$\rho_\lambda \in \gamma(u_\lambda - \lambda) \quad \text{a.e.} \quad \text{and} \quad -\Delta u_\lambda + \rho_\lambda = f \quad \text{in } \mathcal{D}'(\mathbb{R}^N). \tag{6.4}$$

We start with a result which is basically known (see e.g. Hille [39], Lieb-Simon [48], item [3] under Véron [59]):

PROPOSITION 4. *We have*

$$u_0(x) \sim \left(\frac{B}{|x|}\right)^k \quad \text{as } |x| \to \infty, \tag{6.5}$$

where

$$k = \frac{2}{q-1} \quad \text{and} \quad B = B(k, N) = (k(k - N + 2))^{1/2}. \tag{6.6}$$

We now set

$$I(\lambda) = \int \rho_\lambda(x)\, dx,$$

$$\overline{R}_\lambda = \inf\{r > 0;\ u_\lambda(x) < \lambda \text{ a.e. on } [|x| > r]\},$$

$$\underline{R}_\lambda = \sup\{r > 0;\ u_\lambda(x) > \lambda \text{ a.e. on } [|x| < r]\}.$$

Clearly, we have $\underline{R}_\lambda \leqslant \overline{R}_\lambda$, $\operatorname{supp} \rho_\lambda \subset [|x| \leqslant \overline{R}_\lambda]$, and $\rho_\lambda(x) > 0$ a.e. on $[|x| < \underline{R}_\lambda]$.

The main result of this section is the following

THEOREM 9. *We have, as* $\lambda \downarrow 0$,

$$\overline{R}_\lambda \sim \underline{R}_\lambda \sim B\left(\frac{A_0}{\lambda}\right)^{1/k}, \ I_0 - I(\lambda) \sim a A_0^\theta \lambda^{1-\theta}, \tag{6.7}$$

with

$$\theta = \frac{N-2}{k}, \quad a = (N-2) B^{N-2} \sigma_N, \tag{6.8}$$

where $\sigma_N = |S^{N-1}|$, *k and B are given by* (6.6), $A_0 = (2k - N + 2) A^{1/2} (N-2)^{-1}$, *and* $A = h(0)$ *is a constant, depending only on q and N, defined via the solution of an ODE described in Lemmas* 17 *and* 18.

In order to prove Proposition 4, we need the following lemma, essentially due to Hille [39, Theorem 4] (see also Lemme 2.2 in item [3] under Véron [59]):

LEMMA 16. *Let* $N \geqslant 3$*,* $1 < q < \frac{N}{N-2}$*,* $R_0 > 0$*,* $\ell > 0$*,* $\phi_0 > 0$*, and* $v_0 \in C^2([R_0, \infty))$*,* $v_0 \geqslant 0$*, be the solution of*

$$\begin{cases} v_0'' + \frac{N-1}{r} v_0' = \ell v_0^q & \text{in } [R_0, \infty), \\ v_0(R_0) = \phi_0. \end{cases} \tag{6.11}$$

Then

$$v_0(r) \sim \left(\frac{B}{\ell^{1/2}\, r}\right)^k \quad \text{as } r \to \infty, \tag{6.12}$$

where k *and* B *are given by* (6.6).

It is well-known (see item [8] under Brezis [16]) that (6.11) has a unique solution, even without prescribing a condition at infinity. Moreover, there exists a constant $C > 0$ (depending on the given data) such that

$$v_0(r) \leqslant \frac{C}{r^k} \quad \forall r \geqslant R_0. \tag{6.13}$$

Proof of Lemma 16. By a simple scaling argument, it suffices to prove the lemma for $\ell = 1$. Set $v_0(r) = (\frac{B}{r})^k w_0(r^n)$, with

$$n = 2k - (N-2), \tag{6.14}$$

so that $w_0 \in C^2([\sigma_0, \infty))$, $w_0 \geqslant 0$, satisfies

$$\begin{cases} \sigma^2 w_0'' = L w_0 (w_0^{q-1} - 1) & \text{in } [\sigma_0, \infty), \\ w_0(\sigma_0) = \psi_0, \end{cases} \tag{6.15}$$

where $\sigma_0 = R_0^n$, $\psi_0 = \phi_0 (\frac{R_0}{B})^k$, and $L = (\frac{B}{n})^2$. Clearly, in order to prove (6.12), it suffices to show that

$$\lim_{\sigma \to \infty} w_0(\sigma) = 1. \tag{6.16}$$

Note that the function $(w_0 - 1)^2$ is convex; indeed,

$$\frac{1}{2} \frac{d^2}{d\sigma^2} (w_0 - 1)^2 \geqslant (w_0 - 1) w_0'' \geqslant 0$$

by (6.15).

Suppose, by contradiction, that (6.16) does not hold. Since $(w_0 - 1)^2$ is convex and bounded (for this last property we just apply (6.13)), there would exist a $\delta > 0$ small enough so that $(w_0 - 1)^2 \geqslant \delta^2$ on $[\sigma_0, \infty)$. We now split the argument into two cases:

CASE 1. $w_0(\sigma_0) > 1$.

In this case, one has $w_0 \geqslant 1 + \delta$ on $[\sigma_0, \infty)$ and

$$\sigma^2 w_0'' \geqslant \overline{\delta} = L(1+\delta)((1+\delta)^{q-1} - 1) > 0 \quad \text{on } [\sigma_0, \infty). \tag{6.17}$$

In particular, w_0 itself is convex and bounded. Thus it is also decreasing. We then conclude that

$$\lim_{\sigma \to \infty} \sigma w_0'(\sigma) = 0. \tag{6.18}$$

In fact, by the convexity of w_0, we can write

$$0 \leqslant -\sigma w_0'(\sigma) \leqslant 2(w_0(\sigma/2) - w_0(\sigma)) \quad \text{for } \sigma \geqslant 2\sigma_0.$$

Since $w_0(\sigma)$ converges as $\sigma \to \infty$, (6.18) follows.

On the other hand, it follows from (6.17) that

$$-w_0'(\sigma) = \int_\sigma^\infty w_0''(\tau)\, d\tau \geqslant \frac{\overline{\delta}}{\sigma} \quad \forall \sigma \geqslant \sigma_0,$$

which contradicts (6.18). This proves (6.16) in Case 1.

CASE 2. $w_0(\sigma_0) < 1$.

We have $0 < w_0 \leqslant 1 - \delta$ on $[\sigma_0, \infty)$, so that w_0 is concave. We deduce that w_0 is increasing,

$$\lim_{\sigma \to \infty} \sigma w_0'(\sigma) = 0 \quad \text{and} \quad \sigma^2 w_0''(\sigma) \leqslant -\overline{\delta}$$

for some $\bar{\delta} > 0$. As before, this gives a contradiction.

Proof of Proposition 4. By the maximum principle, we have $0 \leqslant u_0 \leqslant V$ on $\mathbb{R}^N$. Since V is harmonic outside some large ball, $\lim_{|x| \to \infty} V(x) = 0$. Then for any pair of positive numbers $\overline{\ell}, \underline{\ell}$ with $0 < \overline{\ell} < 1 < \underline{\ell}$, there exists $R_0 > 0$ such that

$$\overline{\ell} u_0^q \leqslant \rho_0 \leqslant \underline{\ell} u_0^q \quad \text{a.e. on } [|x| > R_0].$$

We may also assume that the support of f is contained in $[|x| < R_0/2]$; in particular, u_0 is C^2 on $[|x| \geqslant R_0]$ (see e.g. Theorem 3 in item [8] under Brezis [16]).

Set $\overline{\phi}_0 = \max_{|x|=R_0} u_0(x)$, and consider the solution $\overline{v}_0 \in C^2([R_0, \infty))$, $\overline{v}_0 \geqslant 0$, of (6.11) with ℓ and ϕ_0 replaced by $\overline{\ell}$ and $\overline{\phi}_0$, respectively. By the maximum principle, we have $u_0(x) \leqslant \overline{v}_0(|x|)$ on $[|x| \geqslant R_0]$, so that, by Lemma 16,

$$\limsup_{|x| \to \infty} \left[\left(\frac{|x|}{B} \right)^k u_0(x) \right] \leqslant \left(\frac{1}{\overline{\ell}} \right)^{k/2}. \tag{6.19}$$

We now claim that $u_0 > 0$ on $[|x| \geqslant R_0]$. For a.e. $x \in \mathbb{R}^N$, let

$$a(x) = \begin{cases} \frac{\rho_0(x)}{u_0(x)} & \text{if } u_0(x) \neq 0, \\ 0 & \text{if } u_0(x) = 0, \end{cases}$$

so that u_0 satisfies

$$-\Delta u_0 + a u_0 = f \geqslant 0 \quad \text{in } \mathcal{D}'(\mathbb{R}^N). \tag{6.20}$$

Using (6.1), we deduce that $a \in L^1(\mathbb{R}^N)$; moreover, a is bounded on $[|x| \geqslant R_0]$. By the strong maximum principle, then either $u_0 > 0$ on $[|x| \geqslant R_0]$, or $u_0 \equiv 0$ on $[|x| \geqslant R_0]$. Suppose, by contradiction, that $u_0 \equiv 0$ on $[|x| \geqslant R_0]$; in this case, Theorem C.1 in Appendix C would imply that $u_0 \equiv 0$ in $\mathbb{R}^N$, which is not possible because, by assumption (6.2), f is a nonzero measure. We deduce that $u_0 > 0$ on $[|x| \geqslant R_0]$, as claimed.

Set $\underline{\phi}_0 = \min_{|x|=R_0} u_0(x) > 0$, and consider the solution $\underline{v}_0 \in C^2([R_0, \infty))$, $\underline{v}_0 \geqslant 0$, of (6.11) corresponding to $\underline{\ell}$ and $\underline{\phi}_0$. We have $u_0(x) \geqslant \underline{v}_0(|x|)$ on $[|x| \geqslant R_0]$, and then

$$\liminf_{|x|\to\infty} \left(\frac{|x|}{B}\right)^k u_0(x) \geqslant \left(\frac{1}{\underline{\ell}}\right)^{k/2}. \tag{6.21}$$

Since (6.19) and (6.21) hold for every $0 < \overline{\ell} < 1 < \underline{\ell}$, the proposition follows.

In order to prove Theorem 9, we need the following

LEMMA 17. *Let $K \in C^1([0, 1])$ with $K > 0$ on $(0, 1)$, and $K'(1) < 0$. Then there exists a unique solution $h \in C^1([0, 1])$ of*

$$\begin{cases} \frac{1}{2}h'(\xi) + h(\xi)^{1/2} + K(\xi) = 0 & \textit{in } [0, 1], \\ h(1) = 0, \quad h(\xi) \geqslant 0 & \forall \xi \in [0, 1]. \end{cases} \tag{6.22}$$

Proof. (We present a modification due to M. Crandall of our original proof). Given $\varepsilon > 0$, set

$$F_\varepsilon(s) = \begin{cases} s^{1/2} & \text{if } s \geqslant \varepsilon, \\ \frac{s}{\varepsilon^{1/2}} & \text{if } 0 < s < \varepsilon, \\ 0 & \text{if } s \leqslant 0. \end{cases}$$

Then F_ε is Lipschitz continuous, and there exists a (unique) solution $h_\varepsilon \in C^1([0, 1])$ of

$$\begin{cases} \frac{1}{2}h_\varepsilon'(\xi) + F_\varepsilon(h_\varepsilon(\xi)) + K(\xi) = 0 & \text{in } [0, 1], \\ h_\varepsilon(1) = \varepsilon. \end{cases}$$

Since $h_\varepsilon' \leqslant 0$, we have $h_\varepsilon \geqslant \varepsilon$, and

$$\frac{1}{2}h_\varepsilon'(\xi) + h_\varepsilon(\xi)^{1/2} + K(\xi) = 0 \quad \forall \xi \in [0, 1].$$

Moreover, $\varepsilon \longmapsto h_\varepsilon(\xi)$ is increasing, and the limit h_0 of h_ε as $\varepsilon \downarrow 0$ is a solution of (6.22).

We now turn to uniqueness. Let $\tilde{h}$ be any solution of (6.22). Since $\tilde{h}' < 0$ on $(0, 1)$, we have $\tilde{h} > 0$ on $[0, 1)$; also, $\tilde{h} < h_\varepsilon$ on $[0, 1)$ for every $\varepsilon > 0$, and so $\tilde{h} \leqslant h_0$ on $[0, 1]$.

Take $\xi_0 \in [0, 1)$ so that $K' < 0$ on $[\xi_0, 1]$. For $0 < \delta < 1 - \xi_0$, let h^δ be a function defined on $[\xi_0 + \delta, 1]$ by $h^\delta(\xi) = \tilde{h}(\xi - \delta)$. We have

$$\frac{dh^\delta}{d\xi}(\xi) + h^\delta(\xi)^{1/2} + K(\xi) = K(\xi) - K(\xi - \delta) \leqslant 0,$$
$$h^\delta(1) = \tilde{h}(1 - \delta).$$

Thus if we take $\varepsilon = \tilde{h}(1-\delta) > 0$, then $h^\delta \geqslant h_\varepsilon$ on $[\xi_0+\delta, 1]$. At the limit as $\delta \downarrow 0$, $\tilde{h} \geqslant h_0$ on $[\xi_0, 1]$, and so $\tilde{h} = h_0$ on $[\xi_0, 1]$. In particular, if we now choose $\varepsilon = \tilde{h}(\xi_0) = h_0(\xi_0)$, then both $\tilde{h}$ and h_0 satisfy the initial value problem:

$$\begin{cases} \frac{1}{2}h'(\xi) + F_\varepsilon(h(\xi)) + K(\xi) = 0 & \text{in } [0, \xi_0], \\ h(\xi_0) = \varepsilon, \end{cases}$$

since $\tilde{h}, h_0 \geqslant \varepsilon$ on $[0, \xi_0]$, and $F_\varepsilon(s) = s^{1/2}$ if $s \geqslant \varepsilon$. By uniqueness, we conclude that $\tilde{h} = h_0$ on $[0, \xi_0]$, and hence on the entire interval $[0, 1]$. □

We now prove the following

LEMMA 18. *Let* $N \geqslant 3$, $1 < q < \frac{N}{N-2}$, $R_0 > 0$, $\ell > 0$, $\phi_0 > 0$, $\lambda > 0$, *and* $v_\lambda \in C^2([R_0, \infty))$ *be the solution of*

$$\begin{cases} v_\lambda'' + \frac{N-1}{r} v_\lambda' = \ell[(v_\lambda - \lambda)^+]^q & \textit{in } [R_0, \infty), \\ v_\lambda(R_0) = \phi_0, \quad \lim_{r\to\infty} v_\lambda(r) = 0. \end{cases} \tag{6.23}$$

Then $v_\lambda(r)$ *is decreasing with respect to* r *on* $[R_0, \infty)$.
For every $0 < \lambda \leqslant \phi_0$, *let* $R_\lambda \in [R_0, \infty)$ *be such that* $v_\lambda(R_\lambda) = \lambda$. *We have*

$$-v_\lambda'(R_\lambda) = \frac{(N-2)\lambda}{R_\lambda} \sim \frac{nA^{1/2}}{R_\lambda} \left(\frac{B}{\ell^{1/2} R_\lambda} \right)^k \quad \textit{as } \lambda \downarrow 0, \tag{6.24}$$

with k *and* B *given by* (6.6), n *given by* (6.14), *and* $A = h(0)$, *where* h *is the solution of* (6.22) *corresponding to*

$$K(\xi) = \left(\frac{B}{n}\right)^2 \xi(1 - \xi^{q-1}).$$

Proof. By a simple scaling argument, it suffices to prove the lemma for $\ell = 1$. Firstly, we have

$$\frac{d}{dr}(r^{N-1} v_\lambda'(r)) = r^{N-1}[(v_\lambda(r) - \lambda)^+]^q \geqslant 0 \quad \forall r \geqslant R_0.$$

In particular, since $v_\lambda(R_0) > 0$ and $\lim_{r\to\infty} v_\lambda(r) = 0$, it follows from the maximum principle that $v_\lambda > 0$ in $[R_0, \infty)$. We claim that $v_\lambda' < 0$ in $[R_0, \infty)$. In fact, if $v_\lambda'(r_0) \geqslant 0$ for some $r_0 \geqslant R_0$, then we would have

$$r^{N-1} v_\lambda'(r) \geqslant r_0^{N-1} v_\lambda'(r_0) \geqslant 0 \quad \text{for every } r \geqslant r_0.$$

In other words, $v_\lambda'(r) \geqslant 0$ for $r \geqslant r_0$, and so

$$\liminf_{r\to\infty} v_\lambda(r) \geqslant v_\lambda(r_0) > 0.$$

But this contradicts $\lim_{r\to\infty} v_\lambda(r) = 0$. We then deduce that $v_\lambda' < 0$ in $[R_0, \infty)$.

For each $0 < \lambda \leqslant \phi_0$, it follows that there exists a unique $R_\lambda \in [R_0, \infty)$ such that $v_\lambda(R_\lambda) = \lambda$. Moreover, if $r \geqslant R_\lambda$, then $\frac{d}{dr}\left(r^{N-1} v_\lambda'(r)\right) = 0$. Thus

$$v_\lambda(r) = \lambda \left(\frac{R_\lambda}{r}\right)^{N-2} \quad \text{and} \quad v_\lambda'(r) = -\frac{(N-2)\lambda}{R_\lambda}\left(\frac{R_\lambda}{r}\right)^{N-1}.$$

In particular,

$$v_\lambda'(R_\lambda) = -\frac{(N-2)\lambda}{R_\lambda}. \tag{6.25}$$

Now set, as in Lemma 16, $v_\lambda(r) = (\frac{B}{r})^k w_\lambda(r^n) + \lambda$, so that w_λ satisfies

$$\begin{cases} w_\lambda \in C^2([\sigma_0, \sigma_\lambda]), \quad w_0 \geqslant 0, \\ \sigma^2 w_\lambda'' = L w_\lambda (w_\lambda^{q-1} - 1) \quad \text{in } [\sigma_0, \sigma_\lambda], \\ w_\lambda(\sigma_0) = \psi_\lambda, \quad w_\lambda(\sigma_\lambda) = 0, \end{cases} \tag{6.26}$$

where $\sigma_0 = R_0^n$, $\sigma_\lambda = R_\lambda^n$, $\psi_\lambda = (\phi_0 - \lambda)(\frac{R_0}{B})^k$, and $L = (\frac{B}{n})^2$. Using this notation, we can rewrite (6.25) as

$$v_\lambda'(R_\lambda) = \frac{n}{R_\lambda}\left(\frac{B}{R_\lambda}\right)^k \sigma_\lambda w_\lambda'(\sigma_\lambda).$$

In order to establish (6.24), it suffices to show that

$$\lim_{\lambda\downarrow 0} (\sigma_\lambda w_\lambda'(\sigma_\lambda))^2 = A. \tag{6.27}$$

Before proving (6.27), we first remark that if v_0 and w_0 are the functions introduced in Lemma 16, it follows from the standard maximum principle that

$$v_\lambda \downarrow v_0 \quad \text{and} \quad v_\lambda - \lambda \uparrow v_0 \quad \text{as } \lambda \downarrow 0, \tag{6.28}$$

so that

$$\sigma_\lambda \uparrow \infty \quad \text{and} \quad w_\lambda \uparrow w_0 \quad \text{as } \lambda \downarrow 0. \tag{6.29}$$

As in Lemma 16, we split the proof of (6.27) into two cases:

CASE 1. $w_0(\sigma_0) \leqslant 1$.

Since $(w_0 - 1)^2$ is convex and $\lim_{\sigma\to\infty} w_0(\sigma) = 1$, we have $w_0 \leqslant 1$, and then $w_\lambda < 1$ for $\lambda > 0$. It follows from (6.26) that w_λ is strictly concave.

Let $m_\lambda = \max w_\lambda$, and $\overline{\sigma}_\lambda \in [\sigma_0, \sigma_\lambda]$ be such that $w_\lambda(\overline{\sigma}_\lambda) = m_\lambda$. We have $w'_\lambda < 0$ on $(\overline{\sigma}_\lambda, \sigma_\lambda]$, and, by (6.29), $m_\lambda \uparrow 1$.

Define $\varphi_\lambda : [0, m_\lambda] \to [\overline{\sigma}_\lambda, \sigma_\lambda]$ to be the inverse function of $w_\lambda|_{[\overline{\sigma}_\lambda, \sigma_\lambda]}$.
Set

$$h_\lambda(\xi) = [w'_\lambda(\varphi_\lambda(\xi))\, \varphi_\lambda(\xi)]^2.$$

We have

$$\begin{aligned}
&\varphi'_\lambda(\xi)\, w'_\lambda(\varphi_\lambda(\xi)) = 1,\\
&\varphi_\lambda(\xi)\, w'_\lambda(\varphi_\lambda(\xi)) = -h_\lambda(\xi)^{1/2},\\
&\varphi_\lambda(\xi)^2\, w''_\lambda(\varphi_\lambda(\xi)) = L\xi(\xi^{q-1} - 1),
\end{aligned}$$

so that h_λ satisfies

$$\begin{cases} \frac{1}{2}h'_\lambda(\xi) + h_\lambda(\xi)^{1/2} + L\xi(1 - \xi^{q-1}) = 0 \quad \text{in } [0, m_\lambda],\\ h_\lambda(m_\lambda) = 0. \end{cases}$$

Since $h_\lambda(0) = (\sigma_\lambda w'_\lambda(\sigma_\lambda))^2$ and $m_\lambda \uparrow 1$, the lemma follows in this case.

CASE 2. $w_0(\sigma_0) > 1$.
For $\lambda > 0$ small enough, $w_\lambda(\sigma_0) > 1$ by (6.29). It then follows from the convexity of $(w_\lambda - 1)^2$ that there exists a unique $\overline{\sigma}_\lambda \in (\sigma_0, \sigma_\lambda)$ such that $w_\lambda(\overline{\sigma}_\lambda) = 1$, and $w'_\lambda < 0$ on $[\overline{\sigma}_\lambda, \sigma_\lambda]$.

Define $\varphi_\lambda : [0, 1] \to [\overline{\sigma}_\lambda, \sigma_\lambda]$ to be the inverse function of $w_\lambda|_{[\overline{\sigma}_\lambda, \sigma_\lambda]}$. As before, set

$$h_\lambda(\xi) = [w'_\lambda(\varphi_\lambda(\xi))\, \varphi_\lambda(\xi)]^2,$$

so that h_λ satisfies

$$\frac{1}{2}h'_\lambda(\xi) + h_\lambda(\xi)^{1/2} + L\xi(1 - \xi^{q-1}) = 0 \quad \text{in } [0, 1].$$

Note that

$$h_\lambda(0) = (\sigma_\lambda w'_\lambda(\sigma_\lambda))^2.$$

By Lemma 17, to conclude this second case it suffices to show that $\lim_{\lambda\downarrow 0} h_\lambda(1) = 0$; in other words, we only need to prove that

$$\lim_{\lambda\downarrow 0} \overline{\sigma}_\lambda w'_\lambda(\overline{\sigma}_\lambda) = 0. \tag{6.30}$$

The convexity of w_λ on $[\sigma_0, \overline{\sigma}_\lambda]$ implies that

$$0 \leqslant w'_\lambda(\overline{\sigma}_\lambda)(r - \overline{\sigma}_\lambda) \leqslant w_\lambda(r) - w_\lambda(\overline{\sigma}_\lambda) \quad \forall r \in [\sigma_0, \overline{\sigma}_\lambda];$$

consequently

$$0 \geqslant \overline{\sigma}_\lambda w'_\lambda(\overline{\sigma}_\lambda) \geqslant \frac{\overline{\sigma}_\lambda}{\overline{\sigma}_\lambda - r}[1 - w_\lambda(r)] \quad \forall r \in [\sigma_0, \overline{\sigma}_\lambda].$$

Taking $\lambda \downarrow 0$, and then $r \to \infty$, we get (6.30) as desired. □

Proof of Theorem 9. Let $0 < \overline{\ell} < 1 < \underline{\ell}$. Since $0 \leqslant u_0 \leqslant V$, there exists $R_0 > 0$ such that

$$\overline{\ell}[(u_\lambda - \lambda)]^q \leqslant \rho_\lambda \leqslant \underline{\ell}[(u_\lambda - \lambda)]^q \quad \text{a.e. on } [|x| > R_0].$$

We may also assume that the support of f is contained in $[|x| < R_0/2]$.

Set

$$\underline{\varphi}_0 = \min_{|x|=R_0} u_0(x) \quad \text{and} \quad \overline{\varphi}_0 = \max_{|x|=R_0} u_0(x),$$

and consider $\underline{v}_\lambda, \overline{v}_\lambda \in C^2([R_0, \infty))$ to be the corresponding solutions given by Lemma 18. By the maximum principle, we have

$$\underline{v}_\lambda(|x|) \leqslant u_\lambda(x) \leqslant \overline{v}_\lambda(|x|) \quad \text{on } [|x| \geqslant R_0]. \tag{6.31}$$

It is clear that

$$\underline{R}'_\lambda = \underline{v}_\lambda^{-1}(\lambda) \leqslant \underline{R}_\lambda \leqslant \overline{R}_\lambda \leqslant \overline{v}_\lambda^{-1}(\lambda) = \overline{R}'_\lambda,$$

and then, by Lemma 18,

$$\frac{nA^{1/2}}{N-2}\left(\frac{B}{\underline{\ell}^{1/2}}\right)^k \leqslant \liminf_{\lambda\downarrow 0} \lambda \underline{R}_\lambda^k \leqslant \limsup_{\lambda\downarrow 0} \lambda \overline{R}_\lambda^k \leqslant \frac{nA^{1/2}}{N-2}\left(\frac{B}{\overline{\ell}^{1/2}}\right)^k.$$

Since the estimates above hold for any $0 < \overline{\ell} < 1 < \underline{\ell}$, we conclude that

$$\underline{R}_\lambda \sim \overline{R}_\lambda \sim B\left(\frac{nA^{1/2}}{(N-2)\lambda}\right)^{1/k} \quad \text{as } \lambda \downarrow 0.$$

Take $\underline{u}_\lambda,\ \overline{u}_\lambda \in M^{N/(N-2)}$, with $\Delta\underline{u}_\lambda,\ \Delta\overline{u}_\lambda \in \mathcal{M}$, to be any extensions inside $[|x| < R]$ of $\underline{v}_\lambda(|x|)$ and $\overline{v}_\lambda(|x|)$, respectively. By Corollary B1 in Appendix B, and by (6.31), we have

$$\int_{\mathbb{R}^N} \Delta\underline{u}_\lambda \geqslant \int_{\mathbb{R}^N} \Delta u_\lambda \geqslant \int_{\mathbb{R}^N} \Delta\overline{u}_\lambda. \tag{6.32}$$

Then, for $\lambda > 0$ sufficiently small (so that $\underline{R}'_\lambda > R_0$),

$$\int_{\mathbb{R}^N} \Delta\underline{u}_\lambda = \int_{|x|<\underline{R}'_\lambda} \Delta\underline{u}_\lambda = \sigma_N(\underline{R}'_\lambda)^{N-1}\underline{v}'_\lambda(\underline{R}'_\lambda), \tag{6.33}$$

since $\Delta \underline{u}_\lambda = 0$ on $|x| > \underline{R}'_\lambda$. Similarly,

$$\int_{\mathbb{R}^N} \Delta \overline{u}_\lambda = \int_{|x|<\overline{R}'_\lambda} \Delta \overline{u}_\lambda = \sigma_N (\overline{R}'_\lambda)^{N-1} \overline{v}'_\lambda (\overline{R}'_\lambda). \tag{6.34}$$

Thus, for $\lambda > 0$ small enough, it follows from (6.32), (6.33) and (6.34) that

$$\sigma_N (\underline{R}'_\lambda)^{N-1} \underline{v}'_\lambda (\underline{R}'_\lambda) \geqslant \int_{\mathbb{R}^N} \Delta u_\lambda \geqslant \sigma_N (\overline{R}'_\lambda)^{N-1} \overline{v}'_\lambda (\overline{R}'_\lambda).$$

Using Lemma 18, at the limit as $\underline{\ell} \downarrow 1$ and $\overline{\ell} \uparrow 1$, we obtain

$$-\int_{\mathbb{R}^N} \Delta u_\lambda \sim a A_0^\theta \lambda^{1-\theta} \quad \text{as } \lambda \downarrow 0, \tag{6.35}$$

where a is given by (6.8) and $A_0 = \frac{nA^{1/2}}{N-2}$. We now apply Corollary B.2 and Lemma B.1 (with $p = N/(N-2)$) in Appendix B. By (6.1), we have $\int_{\mathbb{R}^N} \Delta u_0 = 0$, so that

$$I_0 - I(\lambda) = \int_{\mathbb{R}^N} (\Delta u_0 - \Delta u_\lambda) = -\int_{\mathbb{R}^N} \Delta u_\lambda. \tag{6.36}$$

Combining (6.35) and (6.36), we conclude that

$$I_0 - I(\lambda) \sim a A_0^\theta \lambda^{1-\theta} \quad \text{as } \lambda \downarrow 0.$$

Behavior of the chemical potential in the weakly ionized limit

We consider the standard Thomas-Fermi model and we follow now the notations of Lieb-Simon [48] (except that we set $\mu = -\varepsilon_F$, instead of φ_0, where ε_F is the chemical potential). The functions φ_μ and ρ_μ satisfy, with $\mu > 0$,

$$-\Delta \varphi_\mu + 4\pi [(\varphi_\mu - \mu)^+]^{3/2} = 4\pi \sum z_i \delta_{a_i} \tag{6.37}$$

and

$$\rho_\mu = [(\varphi_\mu - \mu)^+]^{3/2}. \tag{6.38}$$

Set

$$J(\mu) = \int \rho_\mu$$

and

$$J(0) = \int \rho_0 = \sum z_i = Z$$

Lieb-Simon [48, Problem 5] raised the following problem: prove that

$$\lim_{\mu \downarrow 0} \frac{\mu}{[Z - J(\mu)]^{4/3}} \text{ exists.} \tag{6.39}$$

The answer is indeed positive and can be easily derived from Theorem 9.

COROLLARY 5. *We have*

$$\lim_{\mu \downarrow 0} \frac{\mu}{[Z - J(\mu)]^{4/3}} = \left(\frac{\pi^2}{63A^{1/2}}\right)^{1/3} \tag{6.40}$$

where $A = h(0)$*, and* h *is the unique solution* $h > 0$ *of the differential equation*

$$\begin{cases} \frac{1}{2}h'(\xi) + h(\xi)^{1/2} + \frac{12}{49}\xi(1 - \xi^{1/2}) = 0 \quad \text{in } (0, 1), \\ h(1) = 0. \end{cases} \tag{6.41}$$

REMARK 19. Solving numerically (6.41) yields $A = h(0) = 1.129359\ldots$ and then

$$\lim_{\mu \downarrow 0} \frac{\mu}{[Z - J(\mu)]^{4/3}} = 0.52826\ldots; \tag{6.42}$$

with the notation of Lieb-Simon [48], (6.42) reads

$$\lim_{N \uparrow Z} \frac{\varepsilon_F(N)}{[Z - N]^{4/3}} = -0.52826\ldots. \tag{6.43}$$

This exact value is consistent with a lower bound for $-\varepsilon_F(N)$ near $N = Z$ obtained by Benguria-Yáñez [8] with the help of a new variational characterization for ε_F; we refer the reader to the paper of Benguria-Yáñez [8] for other comments on this question.

Proof of Corollary 5. Let $M = 1/16\pi^2$ and set

$$u = M^{-1}\varphi_\mu. \tag{6.44}$$

From (6.37) we obtain

$$-\Delta u + \left[\left(u - \frac{\mu}{M}\right)^+\right]^{3/2} = (4\pi/M)\sum z_i\delta_{a_i}. \tag{6.45}$$

We may apply Theorem 9 with $\lambda = \mu/M$, $q = 3/2$, $k = 4$, $B = (12)^{1/2}$, $\theta = 1/4$, $a = 4\pi(12)^{1/2}$, $A_0 = 7A^{1/2}$, and we obtain

$$I_0 - I(\lambda) \sim 4\pi(12)^{1/2}(7A^{1/2})^{1/4}\lambda^{3/4}. \tag{6.46}$$

Here $I_0 = \frac{4\pi}{M}Z = (4\pi)^3 Z$ and

$$I(\lambda) = \int \left[\left(u - \frac{\mu}{M}\right)^+\right]^{3/2}. \tag{6.47}$$

Note that

$$J(\mu) = \int \rho_\mu = \int [(\varphi - \mu)^+]^{3/2} = \int [(Mu - \mu)^+]^{3/2} = M^{3/2}I(\lambda),$$

and

$$J(0) = Z = M^{3/2} I_0.$$

Thus, by (6.46),

$$\begin{aligned} J(0) - J(\mu) &= M^{3/2}[I_0 - I(\lambda)] \\ &\sim \frac{1}{(4\pi)^2}(12)^{1/2}(7A^{1/2})^{1/4}[16\pi^2\mu]^{3/4} \\ &= (4\pi)^{-1/2}(12)^{1/2}(7A^{1/2})^{1/4}\mu^{3/4}, \end{aligned}$$

which is the desired result (6.39).

7. Another dual variational formulation

In this section we assume that (H) holds, and that meas$[V > \delta] < \infty$ for every $\delta > 0$. Set

$$L = \left\{ (u, \lambda) \;\middle|\; \begin{array}{l} u \text{ is a measurable function, } \lambda \geqslant 0, \\ j^*(u - \lambda) \in L^1, \\ u - V \in M^{N/(N-2)} \text{ and } \nabla(u - V) \in L^2 \end{array} \right\}.$$

Fix $I > 0$; consider the following convex functional defined on L:

$$\Phi(u, \lambda) = \frac{1}{2}\int |\nabla(u - V)|^2 + \int j^*(u - V) + \lambda I.$$

THEOREM 10. *Let* (u_0, λ_0) *be such that*

$$\begin{cases} u_0 \textit{ is measurable, } \lambda_0 > 0, \\ u_0 - V \in M^{N/(N-2)}, \quad \Delta(u_0 - V) \in L^1 \textit{ and } \int \Delta(u_0 - V) = I, \\ -\Delta(u_0 - V) + \gamma(u_0 - \lambda_0) \ni 0 \quad a.e. \end{cases}$$

Then

$$(u_0, \lambda_0) \in L \quad \textit{and} \quad \Phi(u_0, \lambda_0) \leqslant \Phi(u, \lambda) \quad \forall (u, \lambda) \in L.$$

Proof. Set $\rho_0 = \Delta(u_0 - V)$, so that $\rho_0 \in L^1$, $\rho_0 \geqslant 0$, $\int \rho_0 = I$, and

$$\partial j(\rho_0) + B\rho_0 \ni V - \lambda \quad \text{a.e.}.$$

By Theorem 1, $\rho_0 \in D(K)$; it follows from Lemma 14 that $\nabla B\rho_0 = \nabla(V - u_0) \in L^2$, and

$$\int \rho_0 B\rho_0 = \int |\nabla(u_0 - V)|^2 = -\int (u_0 - V)\, \Delta(u_0 - V).$$

Using the fact that $\gamma = \partial j^*$, we have, for any $(u, \lambda) \in L$,

$$\begin{aligned} j^*(u-\lambda) - j^*(u_0-\lambda_0) &\geqslant \Delta(u_0 - V)\,[(u-\lambda) - (u_0-\lambda_0)] \\ &= \Delta(u_0 - V)\,[(u-V) - (u_0 - V)] \\ &\quad + (\lambda_0 - \lambda)\,\Delta(u_0 - V). \end{aligned} \tag{7.1}$$

Take first $u = V$ and $\lambda \geqslant 0$ such that $j^*(V-\lambda) \in L^1$ (here we use assumption (H)); we deduce from (7.1) that

$$j^*(u_0-\lambda_0) \leqslant j^*(V-\lambda) + \rho_0\,(B\rho_0 + (\lambda - \lambda_0)) \in L^1,$$

and thus $(u_0, \lambda_0) \in L$.

Now suppose $(u, \lambda) \in L$ is such that $u - V \in L^\infty$. In this case, all the functions in (7.1) are integrable, and we find

$$\begin{aligned} &\int j^*(u-\lambda) - \int j^*(u_0-\lambda_0) \\ &\quad \geqslant \int \Delta(u_0 - V)\,(u-V) + \int |\nabla(u_0 - V)|^2 + (\lambda_0 - \lambda) I \\ &\quad = -\int \nabla(u_0 - V)\cdot\nabla(u-V) + \int |\nabla(u_0 - V)|^2 + (\lambda_0 - \lambda) I \\ &\quad \geqslant -\frac{1}{2}\int |\nabla(u-V)|^2 + \frac{1}{2}\int |\nabla(u_0 - V)|^2 + (\lambda_0 - \lambda) I, \end{aligned}$$

i.e., $\Phi(u_0, \lambda_0) \leqslant \Phi(u, \lambda)$.

[Here we have used the fact that

$$-\int (\Delta\varphi)\psi = \int \nabla\varphi\cdot\nabla\psi$$

$\forall \varphi, \psi$ with $\varphi \in M^{N/(N-2)}$, $\Delta\varphi \in L^1$, $\nabla\varphi \in L^2$, $\psi \in L^\infty \cap M^{N/(N-2)}$, $\nabla\psi \in L^2$; and this may be easily justified by a smoothing argument.]

For a general $(u, \lambda) \in L$, set $u_n = T_n(u - V) + V$, where

$$T_n(r) = \begin{cases} n & \text{if } r > n \\ r & \text{if } |r| \leqslant n \\ -n & \text{if } r < -n \end{cases}$$

so that $u_n - V \in M^{N/(N-2)} \cap L^\infty$, $\nabla(u_n - V) \in L^2$, and $\nabla(u_n - V) \to \nabla(u - V)$ in L^2 as $n \to \infty$. Moreover,

$$\begin{aligned} j^*(u_n - \lambda) &\leqslant j^*(u-\lambda) && \text{on } [u - V \geqslant -n], \\ j^*(u_n - \lambda) &= j^*(V - n - \lambda) && \text{on } [u - V < -n]. \end{aligned}$$

Note that for $n \geqslant M - \lambda$, we have $j^*(V - n - \lambda) \leqslant j^*(V - M) \in L^1$ (by (H)); thus, for n sufficiently large,

$$j^*(u_n - \lambda) \in L^1 \quad \text{and} \quad j^*(u_n - \lambda) \to j^*(u - \lambda) \quad \text{in } L^1 \text{ as } n \to \infty.$$

Therefore, $(u_n, \lambda) \in L$ for n large, and

$$\Phi(u_0, \lambda_0) \leqslant \Phi(u_n, \lambda) \to \Phi(u, \lambda) \quad \text{as } n \to \infty.$$

This completes the proof of the theorem. □

Appendix A

The equation $-\Delta u + \beta(u) \ni \mu$ with μ measure

Let β be a maximal monotone graph on $\mathbb{R}$ with $0 \in \beta(0)$. Let $\mathcal{M}(\mathbb{R}^N)$ denote the space of bounded measures on $\mathbb{R}^N$ with the usual norm:

$$\|\mu\|_{\mathcal{M}} = \sup\left\{\int \varphi \, d\mu;\ \varphi \in C_0(\mathbb{R}^N) \text{ and } \|\varphi\|_{L^\infty} \leqslant 1\right\},$$

where $C_0(\mathbb{R}^N)$ is the space of continuous functions on $\mathbb{R}^N$ tending to zero at infinity. In this Appendix we assume that $N \geqslant 3$.

THEOREM A.1. *Assume β satisfies*

$$D(\beta) = \mathbb{R} \quad \textit{and} \quad \beta^0\Big(\pm \frac{1}{|x|^{N-2}}\Big) \in L^1_{loc}(\mathbb{R}^N). \tag{A.1}$$

Then, for every measure $\mu \in \mathcal{M}(\mathbb{R}^N)$ there exists a unique solution $u \in M^{N/(N-2)}(\mathbb{R}^N)$ of the problem

$$-\Delta u + \beta(u) \ni \mu \quad \textit{a.e. in } \mathbb{R}^N \tag{A.2}$$

such that

$$w \equiv \Delta u + \mu \in L^1(\mathbb{R}^N). \tag{A.3}$$

Moreover if $\widehat{u}$ is the solution corresponding to $\widehat{\mu}$ we have

$$\|u - \widehat{u}\|_{M^{N/(N-2)}} + \|\nabla(u - \widehat{u})\|_{M^{N/(N-1)}} \leqslant C\|\mu - \widehat{\mu}\|_{\mathcal{M}}, \tag{A.4}$$

$$\|(w - \widehat{w})^+\|_{L^1} \leqslant \|(\mu - \widehat{\mu})^+\|_{\mathcal{M}}, \tag{A.5}$$

and

$$[\mu \leqslant \widehat{\mu} \quad \textit{in } \mathcal{M}] \quad \Rightarrow \quad [u \leqslant \widehat{u} \quad \textit{a.e.}]. \tag{A.6}$$

The proof of Theorem A.1 relies on the following

LEMMA A.1. *Let Ω be a measurable space of finite measure. Let (u_n) be a bounded sequence in $M^p(\Omega)$ for some $1 < p < \infty$. Let $\beta : \mathbb{R} \to \mathbb{R}$ be a nondecreasing function such that*

$$\int_{|s|>1} \frac{|\beta(s)|}{|s|^{p+1}}\, ds < \infty. \tag{A.7}$$

Then $(\beta(u_n))$ is bounded in $L^1(\Omega)$ and equi-integrable.

Proof. We may always assume that $\beta(0) = 0$. Set $\gamma(s) = \beta(s+0) - \beta(-s-0)$ for $s \geqslant 0$, so that γ is also nondecreasing and satisfies

$$\int_1^\infty \frac{\gamma(s)}{s^{p+1}}\, ds < \infty.$$

Let

$$\alpha_n(\lambda) = \text{meas}[|u_n| > \lambda].$$

Since (u_n) is bounded in $M^p(\Omega)$ there is a constant C such that $\alpha_n(\lambda) \leqslant C/\lambda^p$, $\forall \lambda > 0$. Let $A \subset \Omega$ be measurable and let $t > 0$. We have

$$\begin{aligned}
\int_A |\beta(u_n)|\, dx &\leqslant \int_A \gamma(|u_n|)\, dx \\
&\leqslant \int_{A\cap[|u_n|\leqslant t]} \gamma(|u_n|) + \int_{[|u_n|>t]} \gamma(|u_n|) \\
&\leqslant |A|\,\gamma(t) + \alpha_n(t)\gamma(t) + \int_t^\infty \alpha_n(\lambda)\, d\gamma(\lambda) \\
&\leqslant |A|\,\gamma(t) + C\left(\frac{\gamma(t)}{t^p} + \int_t^\infty \frac{1}{\lambda^p}\, d\gamma(\lambda)\right) \\
&= |A|\,\gamma(t) + C_p \int_t^\infty \frac{\gamma(\lambda)}{\lambda^{p+1}}\, d\lambda.
\end{aligned}$$

Given $\varepsilon > 0$ we fix t_0 large enough so that

$$C_p \int_{t_0}^\infty \frac{\gamma(\lambda)}{\lambda^{p+1}}\, d\lambda < \varepsilon/2.$$

Then we have $\int_A |\beta(u_n)| \leqslant \varepsilon$ for every A such that $|A| < \delta \equiv \varepsilon/2\ \gamma(t_0)$. □

Proof of Theorem A.1.
Uniqueness. Let $\widehat{u}$ be another solution. We have

$$-\Delta(u - \widehat{u}) + w - \widehat{w} = 0 \quad \text{in } \mathbb{R}^N$$

and thus, by Kato's inequality (see Kato [43]), we obtain

$$-\Delta|u-\widehat{u}| + (w-\widehat{w})\,\mathrm{sign}\,(u-\widehat{u}) \leqslant 0 \quad \text{in } \mathcal{D}'(\mathbb{R}^N).$$

Since $w \in \beta(u)$ and $\widehat{w} \in \beta(\widehat{u})$ it follows that $(w-\widehat{w})\,\mathrm{sign}\,(u-\widehat{u}) \geqslant 0$. Therefore, the function $\varphi = |u-\widehat{u}|$ is subharmonic and, for a.e. x_0, we have

$$\varphi(x_0) \leqslant \frac{1}{|C_n(x_0)|}\int_{C_n(x_0)} \varphi(x)\,dx$$

where $C_n(x_0) = \{x \in \mathbb{R}^N; n < |x-x_0| < 2n\}$. From the fact that $\varphi \in M^{N/N-2}$ we deduce that

$$\int_{C_n(x_0)} \varphi(x)\,dx \leqslant C|C_n(x_0)|^{2/N}.$$

Letting $n \to \infty$ we conclude that $\varphi = 0$ a.e.

Existence. We already know (see Bénilan-Brezis-Crandall [10, Theorem 2.1]) that if $\mu \in L^1(\mathbb{R}^N)$ all the conclusions of Theorem A.1 hold, even without assumption (A.1). If $\mu \in \mathcal{M}(\mathbb{R}^N)$, we let $f_n = \rho_n * \mu$ where (ρ_n) is a sequence of mollifiers. We have

$$f_n \in L^1 \cap C^\infty, \quad \|f_n\|_{L^1} \leqslant \|\mu\|_{\mathcal{M}} \quad \text{and} \quad f_n \rightharpoonup \mu \text{ in the } w^* - \text{topology of } \mathcal{M}.$$

Let $u_n \in M^{N/(N-2)}(\mathbb{R}^N)$ be the (unique) solution of

$$-\Delta u_n + \beta(u_n) \ni f_n$$

with $w_n = \Delta u_n + f_n \in L^1(\mathbb{R}^N)$. We already know that

$$\|u_n\|_{M^{N/(N-2)}} + \|\nabla u_n\|_{M^{N/(N-1)}} \leqslant C\|f_n\|_{L^1}$$

and

$$\|w_n\|_{L^1} \leqslant \|f_n\|_{L^1}.$$

It follows that (u_n) is relatively compact in $L^1_{\mathrm{loc}}(\mathbb{R}^N)$. On the other hand, assumption (A.1) implies (A.7) with $p = N/(N-2)$. We deduce from Lemma A.1 that (w_n) is equi-integrable on every bounded set of $\mathbb{R}^N$. Applying the Dunford–Pettis theorem (see e.g. Dunford-Schwartz [30, Corollary IV.8.11]) we may choose a subsequence such that

$$u_{n_k} \to u \quad \text{in } L^1_{\mathrm{loc}}(\mathbb{R}^N),$$
$$w_{n_k} \rightharpoonup w \quad \text{weakly in } L^1_{\mathrm{loc}}(\mathbb{R}^N).$$

We have $u \in M^{N/(N-2)}(\mathbb{R}^N)$, $w \in L^1(\mathbb{R}^N)$ and (by standard monotone analysis; see e.g. Lemma 3 in item [1] under Brezis [16]) $w \in \beta(u)$ a.e. Therefore u is a solution of (A.2)–(A.3). Properties (A.4), (A.5) and (A.6) follow easily[5] from the corresponding properties for $u_n, \widehat{u}_n$. Indeed $f_n - f_n = \rho_n * (\mu - \widehat{\mu})$ so that

$$\|f_n - \widehat{f_n}\|_{L^1} \leqslant \|\mu - \widehat{\mu}\|_{\mathcal{M}} \quad \text{and} \quad \|(f_n - \widehat{f_n})^+\|_{L^1} \leqslant \|(\mu - \widehat{\mu})^+\|_{\mathcal{M}}.$$

REMARK A.1. The case $N = 2$ has been investigated by J.L. Vázquez item [1] under Vázquez [58].

REMARK A.2. Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with smooth boundary. Under assumption (A.1), the same method as above shows that, for every bounded measure μ on Ω, there exists a unique solution $u \in W_0^{1,1}(\Omega)$ of the problem

$$\begin{cases} -\Delta u + \beta(u) \ni \mu & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

with $w = \Delta u + \mu \in L^1(\Omega)$.

REMARK A.3. Local regularity. Assume ω is an open subset of $\mathbb{R}^N$. Suppose $\mu \in L^q_{\text{loc}}(\omega)$ for some $1 < q < \infty$, then the solution u of (A.2) satisfies $u \in W^{2,q}_{\text{loc}}(\omega)$ (see Brezis [8, Theorem 3]).

REMARK A.4. Non existence without (A.1). Assume $D(\beta) = \mathbb{R}$ but (A.1) does not hold—for example

$$\int_{|x|<1} \beta^0\left(\frac{1}{|x|^{N-2}}\right) dx = \infty. \tag{A.8}$$

Then, for each $c > 0$, problem

$$-\Delta u + \beta(u) \ni c\delta \quad \text{in } \mathbb{R}^N \tag{A.9}$$

has *no* solution (with $w = c\delta + \Delta u \in L^1$).

Indeed, suppose u is a solution of (A.9), then u is radial (by uniqueness) and $u \in C^1(\mathbb{R}^N \backslash \{0\})$ (by Remark A.3). We have

$$\int_{|x|<r} w\,dx = c + \int_{|x|<r} \Delta u\,dx = c + \sigma_N r^{N-1} u'(r)$$

and therefore

$$u'(r) = -\frac{c}{\sigma_N r^{N-1}} + o\left(\frac{1}{r^{N-1}}\right) \quad \text{as } r \to 0.$$

[5] Note that if (v_n) is a bounded sequence in $M^p (1 < p < \infty)$ such that $v_n \rightharpoonup v$ weakly in L^1_{loc}, then $v \in M^p$ and $\|v\|_{M^p} \leqslant \liminf \|v_n\|_{M^p}$. This is clear since $\{v \in M^p;\ \|v\|_{M^p} \leqslant C\}$ is a closed convex set in L^1_{loc} by Fatou's lemma.

It follows that

$$u(r) = \frac{c}{\sigma_N (N-2) r^{N-2}} + o\left(\frac{1}{r^{N-2}}\right) \quad \text{as } r \to 0.$$

This contradicts (A.8) since $\int |\beta^0(u)| < \infty$.

In the special case where $\beta(u) = |u|^{q-1}u$ assumption (A.1) holds if and only if $q < N/(N-2)$. When $q \geqslant N/(N-2)$ the nonexistence of solutions for $\mu = c\delta$ may also be viewed as a consequence of results about removable singularities (see Brezis-Véron [26] and also item [1] under Baras-Pierre [3]). When $q \geqslant N/(N-2)$, the measures μ for which the equation $-\Delta u + |u|^{q-1}u = \mu$ has a solution $u \in L^q$ have been completely characterized; see item [1] under Baras-Pierre [3] (and also item [2] under Gallouët-Morel [35]). The result of Baras-Pierre asserts that, for $1 < q < \infty$, the equation

$$-\Delta u + |u|^{q-1}u = \mu \quad \text{in } \mathbb{R}^N \tag{A.10}$$

has a solution $u \in L^q(\mathbb{R}^N)$ if and only if the bounded measure μ satisfies

$$\mu(E) = 0 \quad \forall E \subset \mathbb{R}^N \text{ such that } \mathrm{cap}_{2,q'}(E) = 0, \tag{A.11}$$

where $\mathrm{cap}_{2,q'}$ is the capacity associated to the Sobolev space $W^{2,q'}$, and $q' = q/(q-1)$.

An equivalent form asserts that (A.10) has a solution if and only if

$$\mu \in L^1 + W^{-2,q}. \tag{A.12}$$

Prior to our study very few authors had considered nonlinear PDE's involving measures as data (see however the pioneering nonexistence result in item [1] under Kamenomostskaia [40] and the paper of Bamberger [2]). Theorem A.1 and the nonexistence result stated above has been the starting point and the motivation for many subsequent works in various directions:

A) Removable singularities. A typical result is the following (see e.g. Brezis-Véron [26], items [6], [7] under Brezis [16]).

Assume $0 \in \Omega \subset \mathbb{R}^N$ and $q \geqslant N/(N-2)$. Let $f \in L^1(\Omega)$, and suppose $u \in L^q_{\mathrm{loc}}(\Omega \backslash \{0\})$ satisfies

$$-\Delta u + |u|^{q-1}u = f \quad \text{in } \mathcal{D}'(\Omega \backslash \{0\}).$$

Then $u \in L^q_{\mathrm{loc}}(\Omega)$, and we have

$$-\Delta u + |u|^{q-1}u = f \quad \text{in } \mathcal{D}'(\Omega).$$

A similar result has been established by Baras-Pierre (see item [1] under Baras-Pierre [3]), when the point 0 is replaced by a closed set $E \subset \Omega$ with $\mathrm{cap}_{2,q'}(E) = 0$ (following earlier works by Loewner-Nirenberg [49] and Véron [1]).

B) Classification of singularities. When the singularities are not removable it is an important task to understand the nature of the singularities and possibly classify them.

A remarkable result of Véron asserts that if $u \in C^2(\Omega\backslash\{0\})$, $u \geqslant 0$, and u satisfies

$$-\Delta u + u^q = 0 \quad \text{in } \Omega\backslash\{0\} \tag{A.13}$$

with $1 < q < N/(N-2)$, then:

a) either u is smooth at 0;
b) or $\lim_{|x|\to 0} |x|^{N-2} u(x) = c$, where c is an arbitrary positive constant;
c) or $\lim_{|x|\to 0} |x|^{\frac{2}{q-1}} u(x) = C(q, N)$, where $C(q, N)$ is an explicit constant such that $C(q, N)|x|^{-\frac{2}{q-1}}$ is an exact solution of (A.13). For example, if $q = 3/2$ and $N = 3$, then $C(q, N) = 144$. Following the terminology introduced in Brezis-Peletier-Terman [22], this type of solution is called *very singular* (VSS).

For the proof we refer to item [2] under Véron [59]; see also Brezis-Oswald [21]. A variety of other results are presented in the book of Véron (item [4] under Véron [59]).

C) Measures as boundary data. Similar questions can be asked for nonlinear equations involving measures as boundary condition. A typical example is the problem

$$-\Delta u + |u|^{q-1}u = 0 \quad \text{in } \Omega, \tag{A.14}$$

$$u = \mu \quad \text{on } \partial\Omega, \tag{A.15}$$

where μ is a positive Borel measure on $\partial\Omega$. The detailed investigation of such questions was initiated by Gmira-Véron [36], and has vastly expanded in recent years; see the papers of Marcus-Véron [50]. Important motivations coming from the theory of probability—and the use of probabilistic methods—have reinvigorated the whole subject; see the pioneering papers of LeGall [45], the recent book of Dynkin [31], and the numerous references therein.

D) Singular solutions and removable singularities for other nonlinear problems. Questions concerning the existence (or nonexistence) of solutions with measure data, removable singularities, and classification of singularities have been investigated for a large variety of nonlinear problems (elliptic and parabolic), such as

$$\frac{\partial u}{\partial t} - \Delta u + |u|^{q-1}u = f,$$

$$\frac{\partial u}{\partial t} - \Delta(|u|^{m-1}u) = f,$$

$$-\operatorname{div}(a(x, \nabla u)) + |u|^{q-1}u = f,$$

$$-\Delta u + u|\nabla u|^2 = f,$$

$$\frac{\partial u}{\partial t} + \gamma\left|\frac{\partial u}{\partial t}\right| - \Delta u = 0, \quad \text{with } 0 < |\gamma| < 1,$$

see Brezis-Friedman [17], Baras-Pierre (item [2] under Baras-Pierre [3]), Brezis-Peletier-Terman [22], Brezis-Nirenberg [20], Boccardo-Gallouët [13], Boccardo-Gallouët-Orsina

[14], Boccardo-Dall' Aglio-Gallouët-Orsina [12], Oswald [52], Pierre [54], and the numerous references in these papers. The study of nonlinear parabolic equations with a Dirac mass as initial data is closely related to the analysis of self-similar solutions; see Barenblatt [4], Barenblatt-Sivashinski [5], Friedman-Kamin [33], Kamenomostskaia (item [2] under [40]), Kamin-Peletier [41], Kamin-Peletier-Vázquez [33], [42], Pattle [53], and Zel'dovich-Kompaneec [60].

E) "Forcing" solutions to exist. Assume $\beta : \mathbb{R} \to \mathbb{R}$ is continuous and nondecreasing, with $\beta(0) = 0$. We make *no* assumption about the behavior of β at infinity, so that (A.1) may fail. Our goal is to solve

$$-\Delta u + \beta(u) = \mu \quad \text{in } \Omega, \tag{A.16}$$

$$u = 0 \quad \text{on } \partial\Omega. \tag{A.17}$$

In general, (A.16)–(A.17) need not have a solution, but we may still consider approximations of (A.16)–(A.17), and try to understand how they fail to converge to a solution of (A.16)–(A.17). There are several natural approximations. For example, we may solve

$$-\Delta u_n + \beta_n(u_n) = \mu \quad \text{in } \Omega, \tag{A.18}$$

$$u_n = 0 \quad \text{on } \partial\Omega, \tag{A.19}$$

where (β_n) is a sequence of continuous nondecreasing functions with $\beta_n(0) = 0$, such that $\beta_n \to \beta$, e.g. uniformly on compact sets. Assume that each β_n has at most a linear growth at infinity, e.g. $\beta_n = \beta$ truncated at $\pm n$, or β_n is the Yosida approximation of β. Then (A.18)–(A.19) admits a unique solution. Another reasonable approximation is

$$-\Delta u_n + \beta(u_n) = \rho_n * \mu 0 \quad \text{in } \Omega, \tag{A.20}$$

$$u_n = 0 \rho_n * \mu \quad \text{on } \partial\Omega, \tag{A.21}$$

where (ρ_n) is a sequence of mollifiers.

Let us start with the case $\beta(u) = |u|^{q-1}u$. It is not difficult to see that if $q < N/(N-2)$, then the solutions (u_n) of (A.18)–(A.19) or (A.20)–(A.21) converge to the solution of (A.16)–(A.17), which exist for every measure μ. The difficulty arises when $q \geqslant N/(N-2)$ and (A.16)–(A.17) has no solution, e.g. when $\mu = \delta_a (a \in \Omega)$. In this case, it has been proved by H. Brezis (see item [7] under reference [16]) that $u_n \to 0$. More generally, if $\mu = f + \delta_a$ with $f \in L^1$, then $u_n \to u^*$, where u^* is the solution of

$$-\Delta u^* + |u^*|^{q-1}u^* = f \quad \text{in } \Omega,$$

$$u^* = 0 \quad \text{on } \partial\Omega.$$

Observe that u^* does *not* satisfy $-\Delta u^* + |u^*|^{q-1}u^* = f + \delta_a$. An interesting aspect to the same phenomenon is that when $\beta(u) = |u|^{q-1}u$ and $q \geqslant N/(N-2)$, the solution of (A.16)–(A.17)—assuming it exists—is "not sensitive" to large perturbation of the data μ, provided these perturbations are localized on sets of small capacity (in the sense of $\mathrm{cap}_{2,q'}$); this is quantified in a recent estimate of Labutin [44] (see also Marcus-Véron, item [4] under reference [50]). For a general measure $\mu \geqslant 0$, it has been proved in Brezis-Marcus-Ponce [19] that $u_n \to u^*$, where u^* is the unique solution of

$$\begin{aligned} &-\Delta u^* + |u^*|^{q-1}u^* = \mu^* \quad \text{in } \Omega, \\ &u^* = 0 \qquad \text{on } \partial\Omega. \end{aligned}$$

Here, μ^* denotes the "regular" part μ_1 of μ in the decomposition

$$\mu = \mu_1 + \mu_2,$$

where $\mu_1(E) = 0$, $\forall E$ with $\mathrm{cap}_{2,q'}(E) = 0$, and μ_2 is concentrated on a set Σ with $\mathrm{cap}_{2,q'}(\Sigma) = 0$; recall that this decomposition exists and is unique—see e.g. Fukushima-Sato-Taniguchi [34].

Returning to a general continuous nondecreasing function $\beta : \mathbb{R} \to \mathbb{R}$, the convergence of the sequences (u_n) has been thoroughly investigated for a general measure $\mu \geqslant 0$ in Brezis-Marcus-Ponce [19]. The sequences (u_n) always converge to a well-defined limit u^* independent of the approximation method. In addition, $\beta(u^*) \in L^1$ and Δu^* is a bounded measure, so that one may define the "reduced" measure

$$\mu^* = -\Delta u^* + \beta(u^*).$$

The measure μ^*, which is a kind of "projection" of μ on the class of "admissible" measures, has a number of remarkable properties. It is the largest measure ν such that $\nu \leqslant \mu$ and

$$\begin{aligned} &-\Delta v + \beta(v) = \nu \quad \text{in } \Omega, \\ &v = 0 \quad \text{on } \partial\Omega, \end{aligned}$$

admits a solution, and therefore u^* is the largest subsolution of (A.16)–(A.17). Moreover, $(\mu - \mu^*)$ is concentrated on a set Σ with $\mathrm{cap}_{1,2}(\Sigma) = 0$.

Applying a result of Vázquez (item [1] under reference [58]), one may identify the measure μ^* when $N = 2$ and $\beta(t) = (e^t - 1)$. The identification of μ^* in more general situations is an interesting direction of research:

OPEN PROBLEM 1. What is μ^* when $\beta(t) = (e^t - 1)$ and $N \geqslant 3$? What is μ^* when $\beta(t) = (e^{t^2} - 1)$ and $N \geqslant 2$?

Similar questions arise when β admits vertical asymptotes. Suppose for example that $\beta : (-1, 1) \to \mathbb{R}$ is a continuous nondecreasing function such that $\beta(0) = 0$ and $\lim_{t\to\pm 1} \beta(t) = \pm\infty$.

OPEN PROBLEM 2. What are the properties of the mapping $\mu \mapsto \mu^*$ in this case ?

Other multivalued graphs β are of interest—for example the graphs

$$\beta(r) = \begin{cases} 0 & \text{if } r < a \\ [0, \infty) & \text{if } r = a \\ \emptyset & \text{if } t > a \end{cases}$$

(for some $a \geqslant 0$), and

$$\beta(r) = \begin{cases} \emptyset & \text{if } r < -1 \text{ and } r > 1 \\ (-\infty, 0] & \text{if } r = -1 \\ 0 & \text{if } -1 < r < 1 \\ [0, \infty) & \text{if } r = 1. \end{cases}$$

They correspond respectively to one-sided and two-sided variational inequalities. The objective is to solve in some natural "weak sense" the equation

$$-\Delta u + \beta(u) \ni \mu,$$

where μ is a given bounded measure. There are some partial results; see e.g. Baxter [6], Dall'Aglio-Dal Maso [29], Orsina-Prignet [51], Brezis-Serfaty [24], and the references therein.

Appendix B

Some properties of $\int \Delta u$

It is clear that if a (smooth) function u decays "very fast" at infinity on $\mathbb{R}^N$ – for example if u has compact support – then $\int \Delta u = 0$; on the other hand, if u decays at infinity like $1/|x|^{N-2}$ then $\int \Delta u \neq 0$. In this paragraph we investigate the relation between $\int \Delta u$ and the behavior of u at infinity. Throughout this Appendix we take $N \geqslant 3$.

THEOREM B.1. *Assume $u \in M^{N/(N-2)}(\mathbb{R}^N)$ with $\Delta u \in \mathcal{M}(\mathbb{R}^N)$. Then*

$$\lim_{\lambda \downarrow 0} (\lambda^{N/(N-2)} \operatorname{meas}[u > \lambda]) \quad \textit{exists and equals} \quad d_N \left[\left(- \int_{\mathbb{R}^N} \Delta u \right)^+ \right]^{N/(N-2)}$$

where d_N is a positive constant depending only on N.

Before proving Theorem B.1 we deduce some corollaries

COROLLARY B.1. *Assume $u \in M^{N/(N-2)}(\mathbb{R}^N)$ with $\Delta u \in \mathcal{M}(\mathbb{R}^N)$. If*

$$\liminf_{\lambda \downarrow 0} (\lambda^{N/(N-2)} \operatorname{meas}[u > \lambda]) = 0,$$

then

$$\int \Delta u \geqslant 0.$$

In particular if $u(x) \leqslant 0$ *for* $|x| > R$, *then* $\int \Delta u \geqslant 0$.

COROLLARY B.2. *Assume* $u \in M^{N/(N-2)}(\mathbb{R}^N)$ *with* $\Delta u \in \mathcal{M}(\mathbb{R}^N)$. *Then*

$$\lim_{\lambda \downarrow 0} (\lambda^{N/(N-2)} \text{ meas}[|u| > \lambda]) \quad \textit{exists and equals} \quad d_N \left| \int \Delta u \right|^{N/(N-2)}.$$

Proof of Corollary B.2. Without loss of generality we may assume that $\int \Delta u \leqslant 0$. By Theorem B.1 we have

$$\lim_{\lambda \downarrow 0} (\lambda^{N/(N-2)} \text{ meas}[u > \lambda]) = d_N \left| \int \Delta u \right|^{N/(N-2)}$$

and

$$\lim_{\lambda \downarrow 0} (\lambda^{N/(N-2)} \text{ meas}[-u > \lambda]) = 0.$$

The conclusion follows since

$$\text{meas}[|u| > \lambda] = \text{meas}[u > \lambda] + \text{meas}[-u > \lambda].$$

It is convenient, in the proof of Theorem B.1, to use the following notations:

$$p = N/(N-2),$$
$$\overline{M}(u) = \limsup_{\lambda \downarrow 0} (\lambda^p \text{ meas}[u > \lambda]),$$
$$\underline{M}(u) = \liminf_{\lambda \downarrow 0} (\lambda^p \text{ meas}[u > \lambda]).$$

Notice that, for any functions u_1, u_2, we have

$$\overline{M}(u_1 + u_2) \leqslant \frac{1}{t^p} \overline{M}(u_1) + \frac{1}{(1-t)^p} \overline{M}(u_2) \quad \forall t \in (0, 1) \tag{B.1}$$

and

$$\underline{M}(u_1 + u_2) \leqslant \frac{1}{t^p} \underline{M}(u_1) + \frac{1}{(1-t)^p} \overline{M}(u_2) \quad \forall t \in (0, 1). \tag{B.2}$$

These relations follow from the fact that

$$[u_1 + u_2 > \lambda] \subset [u_1 > t\lambda] \cup [u_2 > (1-t)\lambda] \quad \forall t \in (0, 1).$$

Proof of Theorem B.1. Set $A = -\int \Delta u$. Given $\varepsilon > 0$, we fix R large enough so that

$$\int_{|x| \geqslant R} |\Delta u| \; < \varepsilon.$$

Let $k(x) = c_N/|x|^{N-2}$ where $c_N = 1/(N-2)\sigma_N$ and σ_N is the area of the unit sphere in $\mathbb{R}^N$ (so that $-\Delta k = \delta_0$). Set

$$f_1 = (-\Delta u)\chi_{B_R} \quad \text{and} \quad f_2 = (-\Delta u)(1 - \chi_{B_R}),$$
$$u_1 = k * f_1 \quad \text{and} \quad u_2 = k * f_2,$$

where χ_{B_R} is the characteristic function of $B_R = \{x \in \mathbb{R}^N \; ; \; |x| < R\}$.

We have

$$u_1 + u_2 = k * (-\Delta u) \; = \; u$$

and

$$\|u_2\|_{M^p} \leqslant \|k\|_{M^p} \|f_2\|_{\mathcal{M}} \leqslant C\varepsilon. \tag{B.3}$$

We claim that there is some $\overline{R} > R$ such that

$$\left| u_1(x) - \frac{Ac_N}{|x|^{N-2}} \right| \leqslant \frac{2\,\varepsilon c_N}{|x|^{N-2}} \quad \text{for } |x| > \overline{R}. \tag{B.4}$$

Indeed we have

$$u_1(x) = \int_{B_R} \frac{c_N}{|x-y|^{N-2}} f_1(y)\, dy$$

and thus

$$u_1(x) - \frac{A\, c_N}{|x|^{N-2}} \; = \; c_N \int_{B_R} \frac{1}{|x-y|^{N-2}} f_1(y)\, dy - \frac{c_N}{|x|^{N-2}} \int_{B_R} f_1(y)\, dy + \frac{c_N}{|x|^{N-2}} \left(\int_{B_R} f_1(y)\, dy - A \right).$$

It follows that

$$\left| u_1(x) - \frac{Ac_N}{|x|^{N-2}} \right| \leqslant c_N \int_{B_R} \left| \frac{1}{|x-y|^{N-2}} - \frac{1}{|x|^{N-2}} \right| |f_1(y)|\, dy + \frac{\varepsilon c_N}{|x|^{N-2}}.$$

On the other hand, we have

$$\left| \frac{1}{|x-y|^{N-2}} - \frac{1}{|x|^{N-2}} \right| \leqslant \frac{(N-2)R}{(|x|-R)^{N-1}} \quad \text{provided } |y| < R < |x|$$

[it suffices to write that $|\varphi(1)-\varphi(0)| \leqslant \int_0^1 |\varphi'(s)|ds$ with $\varphi(t) = 1/|x-ty|^{N-2}$]. Therefore, we obtain

$$\left|u_1(x) - \frac{Ac_N}{|x|^{N-2}}\right| \leqslant \frac{C}{(|x|-R)^{N-1}} + \frac{\varepsilon c_N}{|x|^{N-2}} \quad \text{provided } |x| > R$$

and we deduce (B.4) easily.

We now distinguish two cases:

(i) $\mathbf{A} \leqslant 0$
(ii) $\mathbf{A} > 0$.

CASE (i). It follows easily from (B.4) that

$$\overline{M}(u_1) \leqslant (2\,\varepsilon\, c_N)^p b_N \tag{B.5}$$

where b_N denotes the measure of the unit ball in $\mathbb{R}^N$. Using (B.1), (B.3) and (B.5) we find

$$\overline{M}(u) \leqslant C\varepsilon^p$$

and since ε is arbitrary we conclude that $\overline{M}(u) = 0$.

CASE (ii). It follows easily from (B.4) that

$$[(A-2\varepsilon)c_N]^p b_N \leqslant \underline{M}(u_1) \leqslant \overline{M}(u_1) \leqslant [(A+2\varepsilon)c_N]^p\, b_N \tag{B.6}$$

provided $\varepsilon < A/2$. Using (B.1), (B.3) and (B.6) we find

$$\overline{M}(u) \leqslant \frac{1}{t^p}[(A+2\varepsilon)c_N]^p\, b_N + \frac{1}{(1-t)^p} C\,\varepsilon^p \quad \forall t \in (0,1)$$

Letting $\varepsilon \to 0$ and then $t \to 1$ we are led to

$$\overline{M}(u) \leqslant A^p\, c_N^p\, b_N.$$

On the other hand we have (by (B.2))

$$\underline{M}(u_1) \leqslant \frac{1}{t^p}\underline{M}(u) + \frac{1}{(1-t)^p}\overline{M}(-u_2) \quad \forall t \in (0,1),$$

which implies

$$[(A-2\varepsilon)\, c_N]^p\, b_N \leqslant \frac{1}{t^p}\underline{M}(u) + \frac{1}{(1-t)^p} C\varepsilon^p \quad \forall t \in (0,1).$$

Letting $\varepsilon \to 0$ and then $t \to 1$ we are led to

$$\underline{M}(u) \geqslant A^p c_N^p b_N.$$

We conclude that

$$\underline{M}(u) = \overline{M}(u) = A^p c_N^p b_N.$$

This establishes Theorem B.1 with

$$d_N = c_N^p b_N = \frac{1}{(N-2)^p \sigma_N^p} \cdot \frac{\sigma_N}{N} = \frac{1}{N(N-2)^p \sigma_N^{p-1}}.$$

Here is another useful application.

COROLLARY B.3. *Assume* $u \in M^{N/(N-2)}(\mathbb{R}^N)$, $\Delta u \in \mathcal{M}(\mathbb{R}^N)$ *and*

$$\int_{\mathbb{R}^N} \Delta u = 0. \tag{B.7}$$

Suppose that, for some $R > 0$,

$$u \geqslant 0 \quad \textit{a.e. in } [|x| > R] \tag{B.8}$$

and

$$-\Delta u \geqslant 0 \quad \textit{a.e. in } [|x| > R]. \tag{B.9}$$

Then

$$u \equiv 0 \quad \textit{in } [|x| > R].$$

Proof. From (B.8), (B.9) and the strong maximum principle we know that either

$$u \equiv 0 \quad \text{in } [|x| > R]$$

and the proof is finished, or

$$u > 0 \quad \text{in } [|x| > R]. \tag{B.10}$$

More precisely, for every open set ω with compact closure in $[|x| > R]$ there is a constant $\delta_\omega > 0$ such that

$$u \geqslant \delta_\omega \quad \text{a.e. in } \omega.$$

We will show that (B.10) is impossible. Suppose that (B.10) holds. Fix $R_1 > R$; then for some $\delta > 0$ we have

$$u \geqslant \delta \quad \text{a.e. in } [R_1 < |x| < 2R_1].$$

Fix $\varepsilon > 0$ so that

$$u(x) \geqslant \frac{\varepsilon}{|x|^{N-2}} \quad \text{a.e. in } [R_1 < |x| < 2R_1]. \tag{B.11}$$

Note that by (B.9) we have

$$-\Delta\Big(u - \frac{\varepsilon}{|x|^{N-2}}\Big) \geqslant 0 \quad \text{in } [|x| > R].$$

Applying the maximum principle in the region $[R_1 < |x| < \rho]$ with $\rho > 2R_1$ we see that

$$u(x) - \frac{\varepsilon}{|x|^{N-2}} \geqslant -\frac{\varepsilon}{\rho^{N-2}} \quad \text{in } [R_1 < |x| < \rho].$$

As $\rho \to \infty$ we conclude that

$$u(x) \geqslant \frac{\varepsilon}{|x|^{N-2}} \quad \text{in } [|x| > R_1].$$

From Corollary B.1 applied with $v(x) = \frac{\varepsilon}{|x|^{N-2}} - u(x)$ we obtain $\int \Delta v \geqslant 0$. But $\Delta v = -\varepsilon\delta_0/c_N - \Delta u$, and thus by (B.7), $\int \Delta v = -\varepsilon/c_N < 0$. A contradiction. □

It is sometimes convenient to combine Theorem B.1 with the following:

LEMMA B.1. *Let* Ω *be a measurable space* (*with* $|\Omega| \leqslant \infty$). *Let* $\beta : \mathbb{R} \to \mathbb{R}$ *be a nondecreasing function such that*

$$\beta(0) = 0 \quad \textit{and} \quad \int_0^1 \frac{\beta(s)}{s^{p+1}}\, ds = \infty \quad \textit{for some } 1 < p < \infty. \tag{B.12}$$

Let $u : \Omega \to \mathbb{R}$ *be a measurable function such that*

$$\int_\Omega \beta(u^+(x))\, dx < \infty.$$

Then

$$\liminf_{\lambda \downarrow 0}\, (\lambda^p \operatorname{meas}[u > \lambda]) = 0. \tag{B.13}$$

REMARK B.1. Condition (B.12) is also necessary. More precisely, if u satisfies (B.13) one can show that there exists a function $\beta : \mathbb{R} \to \mathbb{R}$, convex, nondecreasing, Lipschitz continuous, such that $\beta(s) = 0$ for $s \leqslant 0$, $\int_0^1 \frac{\beta(s)}{s^{p+1}}\, ds = \infty$ and $\int_\Omega \beta(u^+(x))\, dx < \infty$.

Proof of Lemma B.1. Assume, by contradiction, that

$$\liminf_{\lambda \downarrow 0}\, (\lambda^p \operatorname{meas}[u > \lambda]) > 0.$$

There exist $\lambda_0 > 0$ and $\varepsilon > 0$ such that

$$\alpha(\lambda) = \operatorname{meas}[u > \lambda] \geqslant \frac{\varepsilon}{\lambda^p} \quad \text{for } 0 < \lambda < \lambda_0.$$

We have, for $0 < \delta < \lambda_0$,

$$\begin{aligned}\int_{[\delta<u<\lambda_0]} \beta(u(x))\,dx &= -\int_\delta^{\lambda_0} \beta(\lambda)\,d\alpha(\lambda)\\ &= -\beta(\lambda_0)\alpha(\lambda_0) + \beta(\delta)\alpha(\delta) + \int_\delta^{\lambda_0} \alpha(\lambda)\,d\beta(\lambda)\\ &\geqslant -\beta(\lambda_0)\alpha(\lambda_0) + \frac{\varepsilon}{\delta^p}\beta(\delta) + \int_\delta^{\lambda_0} \frac{\varepsilon}{\lambda^p}\,d\beta(\lambda)\\ &= -\beta(\lambda_0)\alpha(\lambda_0) + \frac{\varepsilon}{\lambda_0^p}\beta(\lambda_0) + \varepsilon p \int_\delta^{\lambda_0} \frac{1}{\lambda^{p+1}}\beta(\lambda)\,d\lambda.\end{aligned}$$

It follows that $\int_{[\delta<u<\lambda_0]}\beta(u(x))\,dx \to +\infty$ as $\delta \to 0$. A contradiction.

Appendix C

A form of the strong maximum principle for $-\Delta + a(x)$ with $a(x) \in L^1$

The strong maximum principle asserts that if u is smooth, $u \geqslant 0$ and $-\Delta u \geqslant 0$ in a domain $\Omega \subset \mathbb{R}^N$, then either $u \equiv 0$ in Ω or $u > 0$ in Ω. The same conclusion holds when $-\Delta$ is replaced by $-\Delta + a(x)$ with $a \in L^p(\Omega)$, $p > N/2$ (this is a consequence of Harnack's inequality; see e.g. Stampacchia [56], and also Trudinger [57], Corollary 5.3). Another formulation of the same fact says that if $u(x_0) = 0$ for some point $x_0 \in \Omega$, then $u \equiv 0$ in Ω. A similar conclusion fails when $a \notin L^p(\Omega)$, $p > N/2$. For example $u(x) = |x|^2$ satisfies $-\Delta u + a(x)u = 0$ with $a = \frac{2N}{|x|^2} \notin L^{N/2}$.

However if u vanishes on a larger set, not just at one point, one may still hope to conclude that $u \equiv 0$ in Ω. Here is such a result.

THEOREM C.1. *Assume $u \in L^1_{\text{loc}}(\mathbb{R}^N)$ with $u \geqslant 0$ a.e. and $\Delta u \in L^1_{\text{loc}}(\mathbb{R}^N)$. Let $a \in L^1_{\text{loc}}(\mathbb{R}^N)$, $a \geqslant 0$ a.e. Assume u has compact support and satisfies*

$$-\Delta u + au \geqslant 0 \quad \text{a.e. in } \mathbb{R}^N. \tag{C.1}$$

Then $u \equiv 0$.

Proof. (We present a modification due to R. Jensen of our original proof). Set

$$a_n(x) = \min\{a(x), n\}$$

and

$$g_n = -\Delta u + a_n u, \tag{C.2}$$

so that g_n is a nondecreasing sequence of functions in $L^1(\mathbb{R}^N)$ and

$$g_n \uparrow g = -\Delta u + au \quad \text{a.e..}$$

Note that g need not belong to L^1; g is just measurable and $g \geqslant 0$. Fix R sufficiently large, so that $u(x) = 0$ for $|x| > R - 1$. Solve

$$\begin{cases} \Delta b_n = a_n & \text{in } B_R = [|x| < R] \\ b_n = 0 & \text{on } \partial B_R = [|x| = R], \end{cases}$$

so that $b_n \in W^{2,p}(B_R)$ $\forall p < \infty$, $b_n \leqslant 0$ in B_R, $0 \leqslant e^{b_n} \leqslant 1$ in B_R, with

$$\Delta e^{b_n} = e^{b_n}(|\nabla b_n|^2 + \Delta b_n).$$

As $n \to \infty$, $b_n \to b$ in $W^{1,p}(B_R)$ $\forall p < \frac{N}{N-1}$, where b is the solution of

$$\begin{cases} \Delta b = a & \text{in } B_R \\ b = 0 & \text{on } \partial B_R. \end{cases}$$

From (C.2) we have

$$-\int_{B_R} u\Delta\zeta + \int_{B_R} a_n u\zeta = \int_{B_R} g_n\zeta \quad \forall \zeta \in W^{2,q}(B_R) \text{ for some } q > N/2. \tag{C.3}$$

Note that the first integral in (C.3) makes sense since $u \in L^r$ $\forall r < \frac{N}{N-2}$ (recall that $\Delta u \in L^1$). [One may first prove (C.3) for $\zeta \in C^2(\overline{B}_R)$ and then argue by density.]

Choosing $\zeta = e^{b_n}$ in (C.3) yields

$$-\int_{B_R} ue^{b_n}(|\nabla b_n|^2 + \Delta b_n) + \int_{B_R} (\Delta b_n)ue^{b_n} = \int_{B_R} g_n e^{b_n}$$

and, in particular,

$$\int_{B_R} g_n e^{b_n} \leqslant 0.$$

Therefore

$$\int_{B_R} (g_n - g_1)e^{b_n} \leqslant -\int_{B_R} g_1 e^{b_n} \leqslant \int_{B_R} |g_1|. \tag{C.4}$$

Since $g_n - g_1 \geqslant 0$ for $n \geqslant 1$, we conclude by Fatou's lemma that $(g - g_1)e^b \in L^1$ and thus $ge^b \in L^1$. Returning to (C.4) we also have

$$\int_{B_R} (g - g_1)e^b \leqslant -\int_{B_R} g_1 e^b$$

and thus

$$\int_{B_R} ge^b \leqslant 0.$$

Since $g \leqslant 0$ a.e. (by hypothesis (C.1)) we deduce that $g \equiv 0$ and consequently $-\Delta u \leqslant 0$. Therefore $u \leqslant 0$ a.e. By assumption, $u \geqslant 0$ a.e. and thus $u \equiv 0$. □

REMARK C.1. Theorem C.1 is a special case of a much more general result due to Ancona [1]:

THEOREM (Ancona [1]). *Assume* $u \in L^1_{\text{loc}}(\Omega)$, $\Omega \subset \mathbb{R}^N$ *open connected,* $u \geqslant 0$ *a.e.,* $\Delta u \in \mathcal{M}(\Omega)$, $a \in L^1_{\text{loc}}(\Omega)$, $a \geqslant 0$ *a.e., satisfy*

$$\Delta u \leqslant au \quad \textit{in the sense of measures},$$

i.e.,

$$\int_E \Delta u \leqslant \int_E a\, u \quad \textit{for every Borel set } E \subset \Omega. \tag{C.5}$$

(*Note that the integral on the right-hand side is well-defined in* $[0, \infty]$ *since* $au \geqslant 0$ *a.e.*). *Assume that u vanishes on a set* $E \subset \Omega$ *of positive measure, then* $u \equiv 0$.

The proof of Ancona relies on Potential Theory. The interested reader will find another proof based on PDE techniques in Brezis-Ponce [23].

There are several interesting questions related to Theorem C.1:

OPEN PROBLEM 3. Can one replace in Theorem C.1 the assumption $a \in L^1_{\text{loc}}$ by a weaker condition, for example $a^{1/2} \in L^1_{\text{loc}}$ (or $a^{1/2} \in L^p_{\text{loc}}$ for some $p > 1$)?

Note that one cannot hope to go below $L^{1/2}$. For instance the C^2 function u given by

$$u(x) = \begin{cases} (1 - |x|^2)^4 & \text{for } |x| \leqslant 1 \\ 0 & \text{for } |x| > 1 \end{cases}$$

satisfies $-\Delta u + au \geqslant 0$ for some function $a(x)$ such that $a(x) \sim \frac{1}{(1-|x|)^2}$ for $|x| < 1$ and $|x|$ close to 1. Here $a^\alpha \in L^1$, $\forall \alpha < 1/2$, but $a^{1/2} \notin L^1$.

Still one more:

OPEN PROBLEM 4. Assume $u \in C^0$, $\Delta u \in L^1_{\text{loc}}$, $u \geqslant 0$, $a \in L^q_{\text{loc}}$ for some $q \geqslant 1$, $a \geqslant 0$ a.e., satisfy (C.1). Assume that $u = 0$ on a set E with $\text{cap}_{1,2q}(E) > 0$, where $\text{cap}_{1,2q}$ refers to the capacity associated with the Sobolev space $W^{1,2q}$. Can one conclude that $u \equiv 0$?

Ancona [1] (see also Brezis-Ponce [23]) has shown that the answer is positive when $q = 1$. The answer is again positive when $q > \frac{N}{2}$ by the strong maximum principle mentioned above.

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