Progress of Theoretical Physics, Vol. 72, No. 2, August 1984

Non-Linear Realization in Supersymmetric Theories

Masako BANDO, Tetsuji KURAMOTO, Toshihide MASKAWA[†]
and Shozo UEHARA

Department of Physics, Kyoto University, Kyoto 606
[†]*Research Institute for Fundamental Physics, Kyoto University, Kyoto 606*

(Received January 31, 1984)

Non-linear realization is extensively studied in $N=1$ supersymmetric theories with a global symmetry group G breaking down to its subgroup H. A subgroup $\hat{H}$ of G^c, the complexification of the group G, is defined in such a way that it leaves the minimal point of the effective potential invariant. The $\hat{H}$ determines the types of the non-linear realization. The general structure of $\hat{H}$ is fully investigated and summarized in several theorems. General methods are shown how to construct the supersymmetric non-linear lagrangians. Discussions are also given on matter fields.

§ 1. Introduction

About fifteen years ago, Coleman, Wess and Zumino elucidated the general structure of effective lagrangian theories and presented the method to construct invariant lagrangians in which fields are transformed in accordance with non-linear realizations of an internal symmetry group.[1] Recently, Zumino investigated the supersymmetric non-linear σ model with scalar fields taking values in a Kähler manifold and gave a simple explicit formula for the case of the Kähler manifold $U(p+q)/U(p)\otimes U(q)$.[2] In this case, the number of massless scalars is equal to the dimension of the manifold $U(p+q)/U(p)\otimes U(q)$ and the scalars are what is called goldstone bosons. In supersymmetric models, however, it often happens that there appear a number of massless scalars in addition to goldstone bosons.[3]

Several authors studied non-linear realization for more general cases of an original internal symmetry group G breaking down to its subgroup H.[4] They pointed out the importance of complex extension of the group G and showed the origin of massless scalars. It turns out that, even when one fixes G and H, one has a variety of non-linear supersymmetric lagrangians corresponding to the number of massless scalars. There come out, in general, massless scalars called quasi goldstone bosons in addition to goldstone bosons. The masslessness of the quasi goldstone bosons is guaranteed by the masslessness of goldstone bosons as far as the supersymmetry is unbroken. However no definite answer has yet been made as to how many massless scalars appear when an internal symmetry group G breaks down to a group H without breaking the supersymmetry. Furthermore, there has not been clarified the general method for constructing the corresponding non-linear lagrangian for general cases of G/H.

In this paper, we will clarify in detail the non-linear realization in supersymmetric theories. A brief outline of this work has already been presented in Ref. 5). After examining the general properties of the non-linear realization in supersymmetric theories, we show what the number of quasi goldstone bosons is related with, and present the general methods for constructing the correponding lagrangians.

In §2, we elucidate the general properties of non-linear realization in supersymmetric theories. The operation of an internal symmetry group G is extended to that of the complexified group G^c of G.[4] According to that, a symmetry group H can be extended to a group $\hat{H}$ which contains the complexified group H^c of H. This is a special feature of non-linear realization in supersymmetric theories. After we clarify the structure of $\hat{H}$, we can construct the corresponding lagrangian in accordance with the usual method given in Ref. 1). We show explicitly how to construct non-linear lagrangians without quasi goldstone bosons when an internal symmetry group $U(N)$ breaks down to $U(n_1)\otimes U(n_2)\otimes\cdots\otimes U(n_a)$ with $\sum_{i=1}^{a} n_i=N$ in §3 and when $O(N)[Sp(2N)]$ breaks down to $O(m)[Sp(2m)]\otimes U(n_1)\otimes\cdots\otimes U(n_a)$ with $m+2\sum_{i=1}^{a}n_i=N[m+\sum_{i=1}^{a}n_i=N]$ in §4. In Appendices, we give a brief proof of $\hat{H}$-structure theorem (Appendix A) and present general forms of $\hat{H}$ for the case of G and H being classical groups (Appendix B).

§ 2. General properties and methods

In this section we consider models in which a global symmetry group G breaks down spontaneously to its subgroup H preserving ($N=1$) supersymmetry. It is well known that the number of goldstone bosons is equal to the dimension of the coset space G/H. If we assume that the supersymmetry is preserved, each goldstone boson must be a component of a chiral superfield consisting of two scalar bosons and a fermion. In general, all the superfields containing goldstone bosons (goldstone superfields) are classified into two types, which we call "P-type (Pure-type) superfield" and "M-type (Mixed-type) superfield". In a P-type superfield, both of scalar components are goldstone bosons, while, in a M-type superfield, only one of them is a goldstone boson. The supersymmetry gurantees that all the components of those superfields are massless. Hence, a P-type superfield cotains a massless fermion (quasi goldstone fermion) besides two goldstone bosons, while a M-type superfield consists of a goldstone boson, a quasi goldstone boson and a quasi goldstone fermion. The supersymmetric non-linear realization is categorized as follows:

(I) "Maximal realization"; all goldstone superfields of the theory are of M-type.
(II) "Minimal realization"; the theory contains the minimal number of M-type superfields with the rest being P-type superfields.
(III) "Intermediate realization"; the number of M-type superfields is between those of the above two extreme cases.

We should remark that the minimum number of M-type superfields is uniquely determined depending solely on the groups G and H. Certain kinds of G/H allow all the goldstone superfields to be of P-type, which we call "pure realization". Which of these cases is realized depends on the dynamics of the system. In the following we will clarify the general structure of non-linear realization in supersymmetric theories and present some theorems.

A. *Ceneral properties and theorem*

Let $\mathcal{L}_{\rm eff}(\phi^A)$ be the effective lagrangian under consideration. It is a function of chiral superfields ϕ^A, which may be either elementary or composite fields.*) In general the chiral

*) Irrelevant superfields are understood to be integrated out in $\mathcal{L}_{\rm eff}$.

superfields belong to a reducible representation ρ in a compact Lie group G. It is assumed that $\mathcal{L}_{\rm eff}(\phi^A)$ is invariant under G and realizes a spontaneous breakdown preserving a subgroup H of G as well as supersymmetry. The effective potential is obtained by substituting the constant fields ϕ^A for field variables in $\mathcal{L}_{\rm eff}$

$$V_{\rm eff}(\phi^A)=-\mathcal{L}_{\rm eff}(\phi^A). \tag{2·1}$$

The minimal point $\phi_0{}^A$ of $V_{\rm eff}$ is a fixed point of H by the assumption,

$$\rho(H)\phi_0=\phi_0\,. \tag{2·2}$$

(1) Since ϕ^A is a complex field, we can extend the domain of ρ from G to G^c, "complexification of G". It does not mean that G^c is a symmetry of $\mathcal{L}_{\rm eff}$. We define the extension as follows. Since any irreducible representation can be obtained by the irreducible decomposition from direct products of elements of the fundamental system of irreducible representations,[6] choose the natural extension of the fundamental system. If one wants to use complex conjugate representation ρ^* of them, one adopts the contragradient representation $(\rho^T)^{-1}$ as its extension. This extended representation preserves irreducibility and is analytic in the coordinate variables of G^c. We will call it "analytic representation".

(2) Equation (2·2) is rewritten in terms of Lie algebra $\mathfrak{h}$, $\{\Lambda^a{}_H\}$*)

$$\rho(c_a\Lambda^a{}_H)\phi_0=c_a\rho(\Lambda^a{}_H)\phi_0=0\,, \tag{2·3}$$

for any real number c_a. The first equality holds even for a complex number, because we choose the "analytic representation" as an extension of $\rho(G)$ to $\rho(G^c)$. Then ϕ_0 is a fixed point of H^c, "complexification of H". We define $\hat{H}$ as follows:

$$\hat{H}=\{g;\ \rho(g)\phi_0=\phi_0,\ g\in G^c\}. \tag{2·4}$$

By definition,

$$\hat{H}\supset H^c\,. \tag{2·5}$$

(3) Let ξ be "representatives" of $G^c/\hat{H}$.[4b)~d)] Because the operation $\rho(\xi)$ is effective on ϕ_0 (i.e., if $\rho(\xi_1)\phi_0=\rho(\xi_2)\phi_0$, then $\xi_1=\xi_2$), the field variables ϕ can be represented by ξ and the rest freedoms σ,

$$\phi=\phi(\xi,\,\sigma). \tag{2·6}$$

At the vacuum point ξ and σ are chosen to be zero.

(4) The effective potential $V_{\rm eff}$ consists of two parts, D-term V_D and F-term of superpotential W_F. If an analytic function of ϕ^A is G-invariant, then it is also G^c-invariant owing to analytic extension of ρ. Since W_F is an analytic function of ϕ^A, W_F is G^c-invariant.[4a)] Therefore W_F is independent of the variables ξ. On the other hand, V_D is written as

$$V_D=F_\alpha g^{\alpha\beta}(\xi,\,\sigma)\bar{F}_\beta, \qquad (\alpha,\,\beta=\xi,\,\sigma) \tag{2·7}$$

where F_α is the F-component of a chiral superfield ξ or σ and the metric $g^{\alpha\beta}(\xi,\,\sigma)$ is non-singular at the vacuum point. Hence the auxiliary field F_α is given by

*) We use small German letter for Lie algebras and capital Latin letter for the corresponding Lie groups.

$$\bar{F}_\alpha = -(g^{-1})^{\alpha\beta}\delta W_F(\sigma)/\delta\phi^\beta . \tag{2·8}$$

From this equation, together with the condition of preserving supersymmetry,

$$F_\xi|_{\sigma=0}=0\,, \qquad F_\sigma|_{\sigma=0}=0\,. \tag{2·9}$$

Equation (2·9) can be regarded so as to define the minimal point of $V_{\rm eff}$, the condition of which is expressed as

$$\delta W_F/\delta\phi^\beta=0 \tag{2·10}$$

at the minimal point. One should notice that the manifold made up of the whole of the minimal points is invariant under G^c because the minimal point is determined by the G^c invariant condition (2·10). Note that $\phi_0'=\rho(g)\phi_0$, $(g\in G^c)$ with any minimal point ϕ_0, is also a minimal point of $V_{\rm eff}$. This is a remarkable feature particular to the supersymmetric theory, which we would like to emphasize in the following statement.

Statement 1 Let ϕ_0 be a minimal point and $\hat{H}$ be its isotropy group. Then each of

$$\phi_0'=\rho(g)\phi_0(^\forall g\in G^c),$$

is also a minimal point of $V_{\rm eff}$. The corresponding isotropy group is

$$\hat{H}'=\rho(g)\hat{H}\rho(g^{-1}).$$

Equation (2·10) leads to

$$F_a=(\text{const})\sigma+(\text{higher power of } \sigma \text{ and } \xi). \tag{2·11}$$

From Eqs. (2·7) and (2·11), one observes that there is no mass term of ξ,[4d] and only interaction terms can appear. If the supersymmetry is broken, $F_\sigma|_{\sigma=0}\neq 0$, then it is easily seen from Eq. (2·7) that the quasi goldstone bosons get masses. We have the following theorem.[4d]

Theorem 2 Let $\xi=(\xi^i)$ be a representative of $G^c/\hat{H}$, then all ξ^i correspond to massless particles and the number of quasi goldstone bosons N_Q is given by

$$N_Q=\dim[G^c/\hat{H}]-\dim[G/H].^{*)}$$

Note that the proof is valid even in the case of dynamical symmetry breakdown. The structure of $\hat{H}$ is clarified in theorem 3.

(5) Which of the complex subgroups of G^c can be candidates for $\hat{H}$? The following theorem on the structure $\hat{H}$ will answer the question.

Theorem 3 ($\hat{H}$-structure theorem)

Let ρ be an "analytic representation" and

$$\hat{\mathfrak{h}}\equiv\{x;\ \rho(x)\phi_0=0,\ x\in\mathfrak{g}^c\}, \tag{2·12}$$

then

$$\hat{\mathfrak{h}}=\mathfrak{h}^c\oplus\mathfrak{r}\,. \tag{2·13}$$

*) The symbol "dim" counts real dimensions of a manifold.

Here $\mathfrak{h}^c$ is a direct sum of semisimple Lie algebra and $\mathfrak{u}(1)^c$ factors, and $\mathfrak{r}$ is a nilpotent ideal of $\hat{\mathfrak{h}}$ such that all the eigen values of the restriction of $\mathfrak{g}^c$-adjoint representation on $\mathfrak{r}$ vanishes.

Because $\mathfrak{r}$ is an ideal, the group $\hat{H}$ is a *semi*-direct product of H^c and R, $H^c * R$,[*)]

$$(h, r)\cdot(h', r')=(h\cdot h', h'^{-1}rh'r'), \qquad (2\cdot14)$$

where $h, h'\in H^c$ and $r, r', h'^{-1} rh'\in R$.
Here we present the corollary of theorem 2.

Corollary 4

$$N_Q=\dim[G/H]-\dim R\,.$$

One may recall Levi's theorem which tells us that any Lie algebra is a direct sum of a semisimple part and a radical.[6)] In our case condition (2·12) provides us with more detailed information on the structure of $\hat{H}$. The brief proof of this $\hat{H}$-structure theorem is given in Appendix A. In this paper, we will exhibit all possible $\hat{\mathfrak{h}}$'s for every classical simple group.

An important remark is in order: It is a result of Weyl's theorem[6)] that, for any complex simple Lie algebra, one can choose a real form, which generates a compact Lie group. In our case, however, the real structure is a priori specified by G. Hence if we pick up a complex simple subalgebra $\mathfrak{g}_s$ of $\mathfrak{g}^c$, its compact real form is, in general, not included in $\mathfrak{g}$. The following example will make this phenomenon understandable: Suppose that $G=SU(3)$,

$$\mathfrak{h}=\{\sum_{\alpha=1}^{3} a_\alpha\lambda^\alpha, a_\alpha;\ \text{real number},\ \lambda^\alpha;\ \text{Gell-Mann matrices}\}$$

and $g\in SU(3)^c$. If we define $\mathfrak{h}_s=g\mathfrak{h}g^{-1}$, $\mathfrak{h}_s$ generates a compact real form $\mathfrak{su}(2)$ but $\mathfrak{h}_s$ is, in general, not included in $\mathfrak{su}(3)$. When the compact real form of $\mathfrak{g}_s$ is not included in $\mathfrak{g}$, we call such embedding "twisted embedding", while natural embedding refers to the usual simple embedding. However, in our case, we can always choose $\hat{\mathfrak{h}}$ so that its compact real form is included in $\mathfrak{g}$ by taking appropriate $g\in G^c$ and redefining the minimal point, $\phi_0'=\rho(g)\phi_0$ in the G^c invariant manifold of the minimal points of $V_{\rm eff}$(see Statement 1). Thus it is enough to consider the case of natural embedding.

B. *Non-linear realization and invariant lagrangian*

The non-linear realization in supersymmetric theory can be made for any system by introducing necessary quasi goldstone bosons together with quasi goldstone fermions. Let the coordinates of coset space $G^c/\hat{H}$ and, at the same time, the "representative" of $G^c/\hat{H}$ be denoted by ξ. An element $g\xi$ with $\xi\in G^c/H$ and $g\in G$ can be decomposed uniquely into the variables $\xi'\in G^c/\hat{H}$ and $\hat{h}\in\hat{H}$,

$$g\xi=\xi'(\xi, g)\hat{h}(\xi, g). \qquad (2\cdot15)$$

We define the following transformation for chiral superfields ξ, with respect to $g\in G$,

$$\xi'(\xi, g)=g\xi\hat{h}^{-1}(\xi, g). \qquad (2\cdot16)^{*)}$$

*) Hereafter we denote semi-direct product by the symbol "$*$".

One can easily see that Eq. (2·16) defines correctly the operation of G on ξ. It must be remarked that the complex conjugate field $\xi^\dagger$ does not appear on the r.h.s. of Eq. (2·16), that is, the above tranformation does not mix ξ and the anti-chiral superfield $\xi^\dagger$.

Next we show how to construct a supersymmetric G-invariant lagrangian. We learn from Eq. (2·16) that a constant unitary matrix $g\epsilon G$ can be canceled by taking the bilinear form $\xi^\dagger\xi$,

$$\begin{aligned}\xi^\dagger\xi\to\xi'^\dagger\xi'&=\hat{h}^{-1\dagger}(\xi^\dagger,\ g)\xi^\dagger g^\dagger g\xi\hat{h}^{-1}(\xi,\ g)\\&=\hat{h}^{-1\dagger}(\xi^\dagger,\ g)\xi^\dagger\xi\hat{h}^{-1}(\xi,\ g).\end{aligned}\qquad(2\cdot17)^{*)}$$

On the other hand, $\hat{h}^{-1}(\xi,\ g)$ and $\hat{h}^{-1\dagger}(\xi^\dagger,\ g)$ cannot be easily canceled, since while $\hat{h}^{-1}(\xi,\ g)$ depends on the chiral superfield ξ, $\hat{h}^{-1\dagger}(\xi^\dagger,\ g)$ depends on the anti-chiral superfields $\xi^\dagger$, and ξ and $\xi^\dagger$ are mutually *independent* chiral superfields.

The following is three kinds of possible recipes:
The first one (which we name "A-type") works when there exists such an analytic representation $(\rho,\ V)$ of the group G^c that the restriction of ρ to the subgroup $\hat{H}$ of G^c contains a trivial representation. We chose a basis $\{e_a\}$ in a subspace of V in which $\rho(\hat{H})$ is a trivial representation,

$$\rho(\hat{H})e_a=e_a\,.\qquad(2\cdot18)$$

Observing that $\rho(\xi)e_a$ is transformed under group G as

$$\begin{aligned}\rho(\xi')e_a&=\rho(g)\rho(\xi)\rho(\hat{h}^{-1})e_a\\&=\rho(g)\rho(\xi)e_a\,,\end{aligned}\qquad(2\cdot19)$$

the following turns out to be candidates for a lagragnian,

$$[f(e_a{}^\dagger\rho(\xi^\dagger\xi)e_b)]_D\,,\qquad(2\cdot20)$$

where f is an arbitrary function and $[\ \,]_D$ means taking the D-component of the superfield. The following examples will be helpful for understanding

(i) $G=SU(N)$, $\hat{H}=SO(N)^c$ or $Sp(N)^c$ and ρ is a second-rank tensor representation. $\rho(\xi)e$ is given by the following matrix:

$$(\xi^TJ\xi)_{ij}=\xi_{ia}\xi_{jb}J_{ab}\,,\qquad(2\cdot21)$$

where a matrix J is an invariant metric of $SO(N)$ or $Sp(N)$, i.e.,

$$g^TJg=J\qquad\text{for } g\in SO(N)\text{ or } Sp(N).\qquad(2\cdot22)$$

(ii) $G=SU(N)$, $\hat{H}=SU(N-S)(1\le S\le N-1)$ and ρ is a fundamental representation.[4c)]
One takes the basis $(e_a)_i(a=N-S+1,\ \cdots,\ N)$ as follows:

$$(e_a)_i=\delta_{ai}\,.\qquad(2\cdot23)$$

(iii) $e=\phi_0$(see Eq. (2·4)).[4c)]

*) To be precise, this equation should be written by introducing some unitary representation ρ of the group G and the corresponding "analytic representation" of G^c.

The second is a generalization of Zumino's[2] recipe (B-type), which works in general cases. We introduce projections η_i ($i=1, 2\cdots$) which have the following properties:

$$\text{(a)} \qquad \eta_i^2=\eta_i\,,$$

$$\text{(b)} \qquad \rho(X)\eta_i=\eta_i\rho(X)\eta_i \quad X\in\hat{H}\,,$$

$$\text{(c)} \qquad \eta_i^+=\eta_i\,. \qquad \text{(for each } i\text{)} \tag{2·24}$$

Let us consider a function with η_i,

$$\ln[\det{}_{\eta_i}\rho(\xi^\dagger\xi)], \tag{2·25}$$

where $\det_{\eta_i}$ denotes a determiant defined in the subspace $\eta_i V$. Under the group G this is transformed as (suffix i is abbreviated below)

$$\begin{aligned}\ln\det{}_\eta[\rho(\xi'^\dagger\xi')]&=\ln\det{}_\eta[\eta\rho(\xi'^\dagger\xi')\eta]\\&=\ln\det{}_\eta[\eta\rho(\hat{h}^{\dagger-1})\rho(\xi^\dagger\xi)\rho(\hat{h}^{-1})\eta]\\&=\ln\det{}_\eta[\eta\rho(\hat{h}^{\dagger-1})\eta\rho(\xi^\dagger\xi)\eta\rho(\hat{h}^{-1})\eta]\\&=\ln\det{}_\eta[\rho(\xi^\dagger\xi)]+\ln\det{}_\eta[\rho(\hat{h}^{-1})]+\ln\det{}_\eta[\rho(\hat{h}^{-1\dagger})].\end{aligned} \tag{2·26}$$

Since $\ln\det_\eta[\rho(\hat{h}^{-1})]$ is a chiral superfield,

$$[\ln\det{}_\eta\rho(\xi^\dagger\xi)]_D\,, \tag{2·27}$$

is also a candidate for a G-invariant lagrangian.

The last one is C-type lagrangian. A field dependent group element $\hat{h}^{-1}$ in Eq. (2·15) can be canceled in the following term which is an extension of the projection operators in the usual non-supersymmetric theories:

$$P_i\equiv(\rho(\xi)\eta_i)[\rho(\xi^\dagger\xi)]_{\eta_i}^{-1}(\eta_i\rho(\xi^\dagger)), \qquad (i;\ \text{fixed}) \tag{2·28}$$

where $[\ ]_{\eta_i}^{-1}$ means the inverse defined in the subspace projected out by η_i. It is transformed under group G as

$$\begin{aligned}P'&=(\rho(\xi')\eta)[\rho(\xi'^\dagger\xi')]_\eta{}^{-1}(\eta\rho(\xi Z^\dagger))\\&=\rho(g)\rho(\xi)\rho(h^{-1})\eta[(\rho(\hat{h}^{\dagger-1})\rho(\xi^\dagger\xi)\rho(\hat{h}^{-1})]_\eta{}^{-1}\eta\rho(\hat{h}^{\dagger-1})\rho(\xi^\dagger)\rho(g^\dagger)\\&=\rho(g)(\rho(\xi)\eta)(\eta\rho(\hat{h}^{-1})\eta)[\rho(\hat{h}^{\dagger-1})\rho(\xi^\dagger\xi)\rho(\hat{h}^{-1})]_\eta{}^{-1}(\eta\rho(\hat{h}^{\dagger-1}))(\eta\rho(\xi^\dagger)\eta)\rho(g^\dagger)\\&=\rho(g)(\rho(\xi)\eta)[\rho(\xi^\dagger\xi)]_\eta{}^{-1}(\eta\rho(\xi^\dagger))\rho(g^\dagger)\\&=\rho(g)P\rho(g^\dagger),\end{aligned} \tag{2·29}$$

where suffix i is omitted. Note that,

$$P_i^2=P_i\,, \qquad \mathrm{Tr}\,P_i=\text{const}\,, \tag{2·30}$$

thus the candidates for non-trivial invariants are $\mathrm{Tr}\,P_iP_j(i\neq j)$, $\mathrm{Tr}(P_iP_jP_k)(i\neq j\neq k)$, etc. and the following is a candidate for a lagrangian,

$$[f(\mathrm{Tr}(P_iP_j),\ \mathrm{Tr}(P_iP_jP_k)\cdots)]_D\,, \qquad (i\neq j,\ i\neq j\neq k\cdots) \tag{2·31}$$

where f is an arbitrary function of multivariables. Some comments are in order: (i) The above A-, B-, C-type invariants in Eqs (2·20), (2·27), (2·31) are inequivalent to each other, in general. (ii) As for the B- and C-type formulae, all the independent lagrangians are exhausted by taking the fundamental representations as ρ in Eqs. (2·27), (2·28) and (2·31). For example, the invariant in Eq. (2·27) with general ρ is "equivalent" to that with the fundamental ρ_f since difference is expressed by those in Eqs. (2·20) and (2·31). (iii) The A-type and C-type formulae are invariants under G by themselves, so the general expression of a lagrangian can be written as a D term of an arbitrary function of A- and C-type invariants plus B-type term (Eq. (2·27)) suggested first by Zumino.[2)] (iv) In pure realization, only B-type terms give supersymmetric non-linear lagrangians, which clearly indicates that the arbitrariness of functions in Eqs. (2·20) and (2·31) comes from the introduction of quasi goldstone bosons[4e)] (the proof of which will be shown in a separate paper[7)]).

C. *Matter fields*

Let us consider matter chiral superfields N which are transformed according to a linear representation ρ_0 of the subgroup $\hat{H}$. We define the transformation law of N under G as follows:

$$N'=\rho_0(\hat{h}(\xi, g))N\,, \tag{2·32}$$

where $\hat{h}(\xi, g)$ is given by Eq. (2·16). It is easy to construct a G-invariant lagrangian of the matter fields N, since we can obtain the "linear base" of matter fields: Pick up some representation ρ of G^c whose restriction on $\hat{H}$ contains ρ_0 and define the operation ρ on N as the above embedding ρ_0 in ρ. Then $\rho(\xi)N$ is transformed linearly under G,

$$\begin{aligned}\rho(\xi')N'&=\rho(g\xi\hat{h}^{-1})\rho_0(\hat{h})N\\&=\rho(g)\rho(\xi)\rho_0(\hat{h}^{-1})\rho_0(\hat{h})N\\&=\rho(g)\rho(\xi)N.\end{aligned} \tag{2·33}$$

Next consider matter vector superfields V, in which $\hat{h}$ in Eq. (2·32) should be replaced by a unitary "matrix" and consequently the transformation in G mixes the chirality through chiral fields ξ and anti-chiral fields $\xi^\dagger$.*) Observing that

$$[\xi\phi(\xi, \xi^\dagger)\phi^\dagger(\xi, \xi^\dagger)\xi^\dagger]'=g[\xi\phi(\xi, \xi^\dagger)\phi^\dagger(\xi, \xi^\dagger)\xi^\dagger]g^\dagger \tag{2·34}$$

with

$$\phi(\xi, \xi^\dagger)=[\xi^\dagger\xi]^{-1/2}\,, \tag{2·35}$$

we see that $\xi\phi$ is transformed under G as

$$[\xi\phi]'=g(\xi\phi)K^{-1}(\xi, \xi^\dagger, g), \tag{2·36}$$

where K is a *unitary* "matrix" of indicated variables. Thus each of Eqs. (2·32) and (2·33) is replaced by

*) This problem was suggested by Prof. Ohnuki.

$$V'=\rho_0(K(\xi a\ \xi^\dagger, g))V\,, \tag{2·37}$$

$$\rho(\xi'\phi')V'=\rho(g)\rho(\xi\phi)V\,. \tag{2·38}$$

D. *Structure of $\hat{H}$*

In this subsection we will present all the possible candidates for $\hat{\mathfrak{h}}$ for a given classical group G and its subgroup H.

(i) $G=SU(N)$

Let $\mathfrak{h}^c$ be $\mathfrak{h}_1\oplus\cdots\oplus\mathfrak{h}_a\oplus n\{u^c(1)\}$ where $\mathfrak{h}_i=\mathfrak{su}(n_i)^c$ and $n+\sum_{i=1}^{a}n_i\leq N$. An element of Lie algebra $\mathfrak{g}^c$ $(=\mathfrak{su}(N)^c)$ is represented by a traceless complex $N\times N$ matrix

$$\mathfrak{g}^c=\left\{\begin{bmatrix} h_1 & t_{12} & \cdots & t_{1b} \\ t_{21} & h_2 & \cdots & t_{2b} \\ \vdots & \vdots & \ddots & \vdots \\ t_{b1} & t_{b2} & \cdots & h_b \end{bmatrix}\right\}, \tag{2·39}$$

in which the Lie algebra $\mathfrak{h}^c$ is embedded as

$$\begin{array}{c}\left[\begin{array}{ccccc|c} h_1 & & & & & \\ & \ddots & & & & \\ & & h_i & & & \\ & & & \ddots & & \\ & & & & h_a & \\ \hline & & & & & \\ & & & & & \\ & & & & & \end{array}\right]\begin{array}{c} 1 \\ \vdots \\ i \\ \vdots \\ a \\ a+1 \\ \vdots \\ b \end{array} \\ 1\cdots i\cdots a \quad a+1\cdots b\end{array} \qquad \left(b-a=N-\sum_{i=1}^{a}n_i\right). \tag{2·40}$$ [*]

Here each submatrix (we call it "block", hereafter) is labeled by the "block number" (i, j).[**]

Let us find which element is candidate for $\mathfrak{r}$ in Eq. (2·13). Theorem 3 shows that $\mathfrak{r}$ is an ideal and forms an invariant space under $\mathfrak{h}^c$. Hence if it contains at least one element of block t_{ij}, it includes the *whole* elements of t_{ij} (see Appendix B). Since $\mathfrak{r}$ is nilpotent, both t_{ij} and t_{ji} are not included in $\mathfrak{r}$ at the same time.

One may think, for example, that the following matrix is a candidate for $\mathfrak{r}$

---

[*] Here each h_i represents the corresponding $\mathfrak{su}(n_i)^c$ algebra. As to every $\mathfrak{u}^c(1)$ algebra in $\mathfrak{h}^c$, it is understood to be properly embedded in some diagonal elements of the $N\times N$ matrix.

[**] $U(1)$ groups do not participate in labeling the block numbers essentially because any representation of a $U(1)$ group is one dimensional. In Eq. (2·40) each $U(1)$ generator is understood to be embedded in the $N\times N$ matrix in three ways. (i) The generator of a $U(1)$ group *wholly* embedded in some diagonal $\mathfrak{su}(n_i)$ blocks, i.e., it is given by a suitable linear combination of unit sub-matrices of $\mathfrak{su}(n_i)$ blocks. (ii) The generator is expressed by a linear combination of both some diagonal $\mathfrak{su}(n_i)$ blocks and the blocks from $a+1$ to b in Eq. (2·35). (iii) The generator is *wholly* embedded in some diagonal blocks from $a+1$ to b.

$$\begin{array}{cl} \omega=\left[\begin{array}{c|ccccccc} 0 & & & & & & & \\ \hline & 0 & & & & & & \\ & & \ddots & & & & & \\ & & & 0 & & & & \\ & & & & \underset{i}{1} & & & \\ & & & & & 0 & & \\ & & & & & & \ddots & \\ & & & & & & & 0 \end{array}\right] & \begin{array}{l} 1\\ \vdots\\ a\\ a+1\\ \cdot\\ \cdot\\ b \end{array} \\ \quad 1\cdots a\ \ a+1\ \ \cdot\ \ \cdot\ \ \cdot\ \ b & \end{array} \tag{2·41}$$

In fact this matrix is commutable with $\mathfrak{h}^c$ and the algebra $\mathfrak{a}=\{c\omega;\ c$ is a complex number$\}$ has no intersection with $\mathfrak{g}$,

$$\mathfrak{a}\cap\mathfrak{g}=\phi\,. \tag{2·42}$$

However, such a matrix ω is not contained in $\mathfrak{r}$ since the representation of $\mathfrak{r}$ on $\mathfrak{g}$ is nilpotent.*)

Now our final result is

$$\mathfrak{r}=\sum_{(i,j)\in I}\oplus t_{ij}\,, \tag{2·43}$$

where I is a subset of I_0 defined by

$$I_0\equiv\{(i,\ j)\ \ ;\ \ 1\leq i\leq j\leq b,\ i \text{ or } j \text{ stand for block no.}\}, \tag{2·44}$$

specified by the condition,

$$\textit{if } (i,\ j),\ (i',\ j')\in I \quad\text{and}\quad j=i',\text{ then } (i,\ j')\in I\,. \tag{2·45}$$

This condition guarantees the algebra $\mathfrak{r}$ to be closed. Here we have adopted the convention to take t_{ij} from the upper triangular part of the matrix only ($i<j$) (see Appendix B). This is always possible if we properly arrange the order of the block numbers. Up to now we have not addressed the ordering of the factors $\mathfrak{h}_i\subset h^c$. However notice that, once we take the above convention, the freedom to choose either t_{ij} or t_{ji} is replaced by that of the ordering of the factors. Different ordering of the factors with a given pattern of I yields, in general, inequivalent realizations. The details can be better grasped if one learns the concrete forms of $\mathfrak{r}$ for each case in the following sections.

Next we consider the case in which some of the factors $\mathfrak{su}(n_i)$ in $\mathfrak{h}$ are replaced by $\mathfrak{su}(n_i)$or $\mathfrak{sp}(n_i)$. Again each t_{ij} is irreducible under h^c. Note that any element in the diagonal $\mathfrak{su}(n_i)$ block is not a candidate for the element of the nilpotent algebra even if it belongs to the space of $\mathfrak{su}(n_i)/\mathfrak{su}(n_i)$(or $\mathfrak{sp}(n_i)$). The proof can be shown with the use of theorem 3 but we will leave it to a separate paper.[7] Thus one gets the same expression of $\mathfrak{r}$ as given in Eq. (2·43).

Similarly we can discuss when some factors $\mathfrak{h}_a\subset\mathfrak{h}$ are embedded in $\mathfrak{g}^c$ with some

*) Any diagonalizable ω contained in $\hat{\mathfrak{h}}$ is acturally an element of $\mathfrak{h}^c$ according to the $\hat{H}$-structure theorem (see Appendix A).

irreducible higher representations, which will be given in a separate paper.[7)]

We readily see from corollary 4 that maximal realization corresponds to the case $I=\phi$(empty set) or $R=\{1\}$, where the number of the quasi goldstone bosons is maximal, i.e., $N_Q(=\dim G^c/H^c-\dim G/H)$ is equal to that of goldstone bosons, $N_G(=\dim G/H)$. Dim$R$ $(=\dim\hat{H}-\dim H^c)$increases, in proportion to the decrease the number of quasi goldstone bosons. We get the maximal value of $\dim R$ when $I=I_0$, yielding minimal realization. Then under what conditions does one have the chance of pure realization? Equation (2·43) teaches us that the pure realization is possible only when

(i) $\operatorname{rank}\mathfrak{g}=\operatorname{rank}\mathfrak{h}$

(ii) $I=I_0$

(iii) all the factors of H are unitary groups.

(ii) $G=SO(N)$

Let us consider the case when $H=\prod_{i=1}^{b}SO(m_i)\otimes\prod_{j=1}^{a}SU(n_j)\otimes\{U(1)\}^n$. It is convenient to adopt the following expression for the Lie algebra $G=SO(N)^c$,

$$\mathfrak{so}(N)^c=\{A:A^TJ+JA=0,\ A\in\mathfrak{gl}(N,C)\},$$

$$J=\begin{array}{c}\left[\begin{array}{c:c:c} 0 & 0 & 1 \\ \hdashline 0 & 1 & 0 \\ \hdashline 1 & 0 & 0 \end{array}\right]\end{array}\begin{array}{l} 1\ \vdots\ a \\ 1\ \vdots\ b\ \vdots\ b+c \\ 1\ \vdots\ a \end{array}\qquad \Big(c=N-2\sum_{i=1}^{a}n_i-\sum_{q=1}^{b}m_q\Big), \tag{2·46}$$[*)]

or explicitly A is written as

$$\left[\begin{array}{c:c:c} W & X & S \\ \hdashline Y & Z & -X^T \\ \hdashline T & -Y^T & -W^T \end{array}\right]\begin{array}{l} 1\ \vdots\ a \\ 1\ \vdots\ b+c \\ 1\ \vdots\ a \end{array} \tag{2·47}$$

$$Z^T=-Z,\quad S^T=-S \quad\text{and}\quad T^T=-T.^{**)}$$

With this expression, it is found that in any element of $\hat{\mathfrak{h}}$ one can put Y and T to be zero

*) The suffices stand for the block numbers as defined before (see the explanation below Eq. (2·40)), and here they are renumbered in each submatrix for later convenience. Actually, the metric J should be slightly modified under certain circumstances corresponding to the embedding of the $U(1)$ groups of H into G (see the next footnote).

**) For example, $Z=-Z^T$ means that $z=-z^T(z\in Z)$. Hereafter we adopt such an abbreviation.

by taking an appropriate basis (see Appendix B).

Now each factor of $\mathfrak{h}^c$ is embedded in Z and W in A as

$$W=\begin{array}{c}\left[\begin{array}{cccc}\mathfrak{su}(n_1) & & & \\ & \mathfrak{su}(n_2) & & 0 \\ & & \ddots & \\ & 0 & & \mathfrak{su}(n_a)\end{array}\right]\end{array}\begin{array}{l}1\\2\\ \vdots\\ a\end{array}\,, \tag{2·48}$$ [*]

$$Z=\left[\begin{array}{ccccc}\mathfrak{su}(m_1) & & & & \\ & \mathfrak{su}(m_2) & & & \\ & & \ddots & \mathfrak{su}(m_b) & \\ & & & & 0 \\ & & & & \end{array}\right]\begin{array}{l}1\\2\\ \vdots\\ b\\ b+1\\ \vdots\\ b+c\end{array}\,. \tag{2·49}$$

Let each of the submatrices W, Z, S and Y be expressed by a sum of irreducible submatrices under $\mathfrak{h}$,

$$W=\sum_{i=1}^{a}\sum_{j=1}^{a}\oplus W_{ij}\,, \tag{2·50a}$$

$$Z=\sum_{p=1}^{b+c}\sum_{q=1}^{b+c}\oplus Z_{pq}\,, \tag{2·50b}$$

$$S=\sum_{1\leq i\leq j\leq a}\oplus\tilde{S}_{ij}\,, \tag{2·50c}$$

$$X=\sum_{i=1}^{a}\sum_{p=1}^{b+c}\oplus X_{ip}\,, \tag{2·50d}$$

where the notations are understood to be quite analogous to Eq. (2·39), and

$$\tilde{S}_{ij}=\begin{cases}S_{ij}\oplus(-S_{ij}^T) & \text{for } i\neq j\\ S_{ii} & \text{for } i=j\,.\end{cases} \tag{2·51}$$

Any non-diagonal part of block Z_{ij} $(i\neq j)$ is not included in the *nilpotent* ideal $\mathfrak{r}$, since the antisymmetric property of Z (see Eq. (2·47)) automatically enlarges the group $\hat{H}$ to include $\mathfrak{so}(n_i+n_j)$ instead of $\mathfrak{so}(n_i)\oplus\mathfrak{so}(n_j)$ (see Appendix B). If one notices that the

---

[*] Precisely speaking, this matrix is undersoood to include all the $U(1)$ generators of H which are wholly embedded in the diagonal elements of the matrix. Hence in the case when some of the $U(1)$ generators are not given by linear combinations of unit sub-matrix of $\mathfrak{su}(s_i)$ blocks, one should make W larger *just* to include the whole $\mathfrak{su}(n_i)$ and $\mathrm{u}(1)$ algebra in H. As to the enlarged part of W, the block number should be assigned for every one column or row since any repressentation of a $U(1)$ group is one dimensional. According to this enlargement, the metric in (2·46) should be modified:

$$W=\begin{array}{c}\begin{array}{ccc|cccc}1 & \cdots & a & a+1 & a+2 & \cdots & a'\end{array}\\ \left[\begin{array}{ccc|cccc}\mathfrak{su}(n_1) & & & & & & \\ & \ddots & & & & & \\ & & \mathfrak{su}(n_a) & & & & \\ \hline & & & * & & & \\ & & & & * & & \\ & & & & & \ddots & \\ & & & & & & *\end{array}\right]\end{array}\begin{array}{l}1\\ \vdots\\ a\\ a+1\\ a+2\\ \vdots\\ a'\end{array}\,,\quad J=\left[\begin{array}{c|c|c} & & 1\\ \hline & 1 & \\ \hline 1 & & \end{array}\right]\begin{array}{l}1\\ \vdots\\ a'\\ 1\\ \vdots\\ b+c'\\ 1\\ \vdots\\ a'\end{array}\,.$$

commutators among S, X and Z do not produce W, the candidates to be included in $\mathfrak{r}$ are obtained in a similar manner,

$$W_I = \sum_{(i,j)\in I} \oplus W_{ij}\,, \tag{2·52}$$

where I is defined in Eqs. (2·44) and (2·45). If one adopts the block X_{ip} as a candidate for $\mathfrak{r}$, the commutator,

$$\left[\left[\begin{array}{c:c:c} W & & \\ \hdashline & & \\ \hdashline & & -W^T \end{array}\right], \left[\begin{array}{c:c:c} & X & \\ \hdashline & & -X^T \\ \hdashline & & \end{array}\right]\right] \subset \left[\begin{array}{c:c:c} & X & \\ \hdashline & & -X^T \\ \hdashline & & \end{array}\right], \tag{2·53}$$

implies that, in order for $\hat{\mathfrak{h}}$ to form a closed algebra, we must include in $\mathfrak{r}$ all the blocks of the following:

$$X_K = \sum_{(i,p)\in K} \oplus X_{ip}\,, \qquad K\subset K_0\,, \tag{2·54}$$

where K is a subset of K_0 defined by

$$K_0 = \{(i,\ p);\ i=1,\ 2\cdots a,\ p=1,\ 2\cdots,\ b+c\} \tag{2·55}$$

specified by the condition,

$$\textit{if}\,(j,\ p)\in K \quad \text{and} \quad (i,\ j)\in I, \qquad \text{then } (i,\ p)\in K\,. \tag{2·56}$$

Further the commutation relation,

$$\left[\left[\begin{array}{c:c:c} & X & \\ \hdashline & & -X^T \\ \hdashline & & \end{array}\right], \left[\begin{array}{c:c:c} & X & \\ \hdashline & & -X^T \\ \hdashline & & \end{array}\right]\right] \subset \left[\begin{array}{c:c:c} & & S \\ \hdashline & & \\ \hdashline & & \end{array}\right], \tag{2·57}$$

requires that the blocks $\tilde{S}_{ij}$ of the following are also included in $\mathfrak{r}$,

$$S_L = \sum_{(i,j)\in L} \oplus \tilde{S}_{ij}\,, \qquad L\subset L_0\,, \tag{2·58}$$

where L is such a subset of L_0 given by

$$L_0 = \{(i,\ j)\,;\ i\leq j,\ 1\leq i,\ j\leq a\} \tag{2·59}$$

that

$$\text{if} \quad (i,\ p)\in K \quad \text{and} \quad (j,\ p)\in K, \qquad \text{then } (i,\ j)\in L\,. \tag{2·60}$$

Our result is

$$\mathfrak{r} = W_I \oplus X_K \oplus S_L\,. \tag{2·61}$$

When some of the factors $\mathfrak{su}(n_k)(n_k;\text{ even})$ are replaced by $\mathfrak{sp}(n_k)$ $(k=k_1,\ k_2,\ \cdots k_d)$, the discussion can be made quite in a parallel way, only by noticing the fact: Each of the diagonal blocks $S_{k_i k_i}$ is now reducible, which is decomposed into singlet parts $S^{(s)}_{k_i k_i}$ proportional to the Sp-metric and the rest $S^{(n,s)}_{k_i k_i}$, so Eq. (2·51) is replaced by

$$\tilde{S}_{ij}=\begin{cases} S_{ij}\oplus(-S^T_{ij}) & \text{for } i\neq j \\ S_{ii} & \text{for } i=j\neq k_i \\ S^{(s)}_{k_i k_i}\oplus S^{(n,s)}_{k_i k_i} & \text{for } i=i=k_i\,. \end{cases} \tag{2·62}$$

Accordingly the S_L part of (2·61) should be replaced by

$$S_{L,M}=\sum_{(i,j)\in L}\oplus S_{ij}+\sum^{(s)}_{k_1\in M}\oplus S^{(s)}_{k_i k_i}+\sum^{(n,s)}_{k_i\in M}\oplus S^{(n,s)}_{k_i k_i}\,, \tag{2·63}$$

where each of $M^{(s)}$ and $M^{(ns)}$ is a subset of M_L defined by

$$M_L=\{k_i;\ (k_i,\ K_i)\in L_0 \ \text{ and } (k_i,\ K_i)\in L,\ k_i=k_1,\cdots,\ k_d\}. \tag{2·64}$$

By similar discussion, it turns out that the pure realization is possible when rank $\mathfrak{g}$=rank $\mathfrak{h}$ and $\mathfrak{h}$ includes at most one $\mathfrak{su}(m)$ factor with the condition

$$G/H=SO(N)/(SO(m)\otimes\prod_{i=1}^{a}SU(n_i)\otimes\{U(1)\}^n) \tag{2·65}$$

with

$$N-m=2\sum_{i=1}^{a}(n_i-1)+2n\,,$$

$$n\geq a\,. \tag{2·66}$$

(iii) $G=Sp(2N)$

We can give discussion almost similar to the previous subsection. Let H be $\prod_{p=1}^{b} Sp(2m_p)\otimes\prod_{i=1}^{a}SU(n_i)$(or $SO(n_j)$)$\otimes\{U(1)\}^n$.*) The Lie algebra of $G^c=Sp(2N)^c$ is expressed as

$$\mathfrak{sp}(2N)^c=\{A;\ A^T\Omega+\Omega A=0,\ A\in\mathfrak{gl}(2N,\ C)\} \tag{2·67}$$

with

$$\Omega=\begin{array}{c}\left[\begin{array}{c:c:c} 0 & 0 & 1 \\ \hdashline 0 & \Omega_0 & 0 \\ \hdashline -1 & 0 & 0 \end{array}\right]\end{array}\begin{array}{l} 1\\ \vdots\\ a\\ 1\\ \vdots\\ 2b+c\\ 1\\ \vdots\\ a\end{array}\,,\qquad \Omega_0=\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}, \tag{2·68}$$
**)

$$c=2N-\sum_{i=1}^{A}2n_i-\sum_{p=1}^{b}2m_p$$

or explicitly A is written as

*) It is quite similar to treat the $U(1)$ groups in H as before.

**) Of course the expression $\tilde{S}_{ij}$ defined in (2·51) should be replaced by

$$\tilde{S}_{ij}=\begin{cases} S_{ij}\oplus(S_{ij})^T & \text{for } i\neq j\,, \\ S_{ij} & \text{for } i=j\,. \end{cases}$$

$$A=\left[\begin{array}{c:c:c} W & X & S \\ \hdashline Y & Z & \Omega_0 X^T \\ \hdashline T & -Y^T\Omega_0 & -W^T \end{array}\right], \qquad \Omega_0 Z+Z^T\Omega_0=0,\quad S^T=S \text{ and } T^T=T. \quad (2\cdot69)$$

Embedding $\mathfrak{h}=\sum_{p=1}^{b}\mathfrak{sp}(2m_p)\oplus\sum_{i=1}^{a}\mathfrak{h}(n_i)\oplus n\{\mathfrak{u}(1)\}$(with $\mathfrak{h}(n_i)=\mathfrak{su}(n_i)$ or $\mathfrak{so}(n_i)$) in $\mathfrak{g}^c$ as

$$W=\begin{array}{c}\left[\begin{array}{cccc} \mathfrak{su}(n_1) & & & \\ & \mathfrak{su}(n_2) & & \\ & & \ddots & \\ & & & \mathfrak{su}(n_a) \end{array}\right]\begin{array}{c}1\\2\\ \vdots \\ a\end{array}, \\ \begin{array}{cccc}1 & 2 & \cdots & a\end{array}\end{array} \qquad (2\cdot70)$$

and

$$Z=\begin{array}{c}\left[\begin{array}{cccc:c} \mathfrak{sp}(2m_1) & & 0 & & \\ & \mathfrak{sp}(2m_2) & & & \\ 0 & & \ddots & & \\ & & & \mathfrak{sp}(2m_b) & \\ \hdashline & & & & 0 \end{array}\right]\begin{array}{c}1\\2\\ \vdots \\ b\\ b+1\\ \vdots \\ 2+c\end{array}, \\ \begin{array}{ccccc}1 & 2 & \cdots & b & b+1\cdots b+c\end{array}\end{array} \qquad (2\cdot71)$$

we get our final result for $\mathfrak{r}$,

$$\mathfrak{r}=W_I\oplus X_K\oplus S_{L,M}, \qquad (2\cdot72)^{*)}$$

where W_I, V_K and $S_{L,M}$ are defined quite analogously to Eq. (2·52), (2·54) and (2·63). It is evident that pure realization is possible when rank $\mathfrak{g}$=rank $\mathfrak{h}$ and at most one $\mathfrak{sp}$ factor is included in $\mathfrak{h}$ with the condition:

$$G/H=Sp(2N)/(Sp(2m)\otimes\prod_{i=1}^{a}SU(n_i)\otimes\{U(1)\}^n),$$

$$N-m=\sum_{i=1}^{a}(n_i-1)+n,$$

$$a\leq n. \qquad (2\cdot73)$$

Here we summarize the results in the form of a theorem.

Theorem 5 Let $\mathfrak{g}$ be a simple classical algebra, the nilpotent ideal $\mathfrak{r}$ is given by

$$\mathfrak{r}=W_I \qquad \text{for } \mathfrak{g}=\mathfrak{su}(N), \qquad (2\cdot74)$$

and

*) Of course one should notice that in this case the diagonal block $S_{k_ik_i}$ is decomposed into singlet part proportional to *SO metric* and the rest.

$$\mathfrak{r} = W_I \oplus X_K \oplus S'_{L,M} \qquad \text{for } \mathfrak{g} = \mathfrak{su}(N) \text{ or } \mathfrak{sp}(2N). \tag{2·75}$$

Here I, K and (L, M) are subsets of I_0, K_0 and $(L_0 M_L)$ defined in Eqs. (2·44), (2·55), (2·59) and (2·64) under the constraints (2·45), (2·56) and (2·60). There it is assumed that the embedding of each factor of $\mathfrak{h}$ in $\mathfrak{g}$ is "natural". The prime in (2·75) means that S_L or $S_{L,M}$ is properly adopted according to the corresponding H.

E. *Extension to semisimple Lie groups*

This subsection deals with the case where $G = G_1 \otimes G_2$ and the H is embedded in G in a non-trivial way. Extension to general cases of G given by the direct product of multifactors $G_1 \otimes G_2 \otimes G_3 \otimes \cdots$, is straightforward. Consider the group G.

$$G = \{g_1 \otimes g_2;\ g_1 \in G_1,\ g_2 \in G_2\}, \tag{2·76}$$

in which the subgroup H is embedded in such a way that $H = H_V \otimes H_L \otimes H_R \subset G$, where $H_V = \{g \otimes g;\ g \in H_V\}$, $H_L = \{g \otimes 1;\ g \in H_1\}$ and $H_R = \{1 \otimes g;\ g \in H_2\}$. The complexification of G is a direct product of the complexifications of G_1 and G_2,

$$G^c = G_1{}^c \otimes G_2{}^c\,. \tag{2·77}$$

Let R_1 and R_2 be the nilpotent groups corresponding, respectively, to the coset spaces $G_1/H_V \otimes H_1$ and $G_2/H_V \otimes H_2$ as before. Then $\hat{H}$ is given by

$$\hat{H} = H^c * (R_1 \otimes R_2), \qquad H^c = H_V{}^c \otimes H_L{}^c \otimes H_R{}^c\,, \tag{2·78}$$

which turns out to form a group, since each of the corresponding Lie algebra $\mathfrak{r}_1$ and $\mathfrak{r}_2$ is the ideal of $\hat{\mathfrak{h}}$.

Let $\xi_1 \otimes \xi_2$ be represetatives of the coset space $G^c/\hat{H}$, then each of ξ_1 and ξ_2 is transformed under $G(\ni g_1 \otimes g_2)$ as

$$\xi_1{}' = g_i \xi_i \hat{h}_i{}^{-1}, \qquad (\text{for } i = 1,\ 2,\ \text{fixed}) \tag{2·79}$$

where

$$\hat{h}_i \in H_V{}^c \otimes H_i{}^c * R_i\,. \tag{2·80}$$

The G-invariant non-linear lagrangians are obtained in the same way as before according to the three kinds of recipes. For example, the B-type invariants are given by

$$[\ln \det{}_{\eta_1}[\xi_1{}^\dagger \xi_1]]_D\,, \qquad [\ln \det{}_{\eta_2}[\xi_2{}^\dagger \xi_2]]_D\,,$$

where the projections η_i satisfy the conditions

$$(hr_i)\eta_i = \eta_i (hr_i) \eta_i \qquad \text{with } h \in H^c\,. \ (\text{for fixed } i) \tag{2·81}$$

Furthermore, if there exists a matrix Γ which satisfies

$$\rho_1(hr_1)^{-1} \Gamma \rho_2(hr_2) = \Gamma\,, \tag{2·82}$$

then $\rho_1(\xi_1)\Gamma\rho_2(\xi_2)^{-1}$ is one of the correspondents to $\rho(\hat{H})e$ in Eq.(2·18), since it is transformed under the group G as

$$[\rho_1(\xi_1)\Gamma\rho_2(\xi_2)^{-1}]' = \rho_1(g_1)[\rho_1(\xi_1)\Gamma\rho_2(\xi_2)^{-1}]\rho_2(g_2)^{-1}\,. \tag{2·83}$$

Here we present an example which is one of the most familiar types of $(G_1 \otimes G_2)/H$.

Example: $G = SU(N) \otimes SU(N)$ and $H = \{g \otimes g,\ g \in SU(N)\}$.

Since $G_1{}^c = G_2{}^c = H^c$, both of R_i are $\{1\}$. Hence only maximal realization is possible in this case. We can choose $\xi \otimes \xi^{-1}$ as a representative of $G^c/\hat{H}$. The transformation law of the goldstone superfields is given by

$$(\xi' \otimes \xi'^{-1}) = (g_L \otimes g_R)(\xi \otimes \xi^{-1})(h^{-1}(g_L \otimes g_{R'} \xi) \otimes h^{-1}(g_L \otimes g_{R'} \xi)), \tag{2·84}$$

where

$$g_L \otimes g_R \in G. \tag{2·85}$$

From the above equation we have

$$\xi'^2 = g_L \xi^2 g_R{}^{-1}, \tag{2·86a}$$

$$h(g_1 \otimes g_2,\ \xi) = (g_L \xi^2 g_R{}^{-1})^{-1/2} g_L \xi = (g_L \xi^2 g_R{}^{-1})^{1/2} g_R \xi^{-1}. \tag{2·86b}$$

Notice that ξ^2 is transformed linearly and just corresponds to Cronin's M matric[8] in a usual non-supersymmetric theory.

As to a matter chiral superfield N which is in a ρ-represention under H, it is transformed under the group G as

$$N' = (\rho(h) \otimes \rho(h)) \cdot N \quad \text{and} \quad (\rho(h) \otimes \rho(h))\eta = \eta(\rho(h) \otimes \rho(h))\eta\,, \tag{2·87}$$

where h is given by Eq. (2·81b). Thus,

$$\begin{aligned}\{(\rho(\xi) \otimes \rho(\xi^{-1})) \cdot N\}' &= \{\rho(\xi') \otimes \rho(\xi'^{-1})\} \cdot N' \\ &= \{\rho(g_L) \otimes \rho(g_R)\} \cdot (\rho(\xi) \otimes \rho(\xi^{-1})) \cdot N\,.\end{aligned} \tag{2·88}$$

§ 3. Pure realization: $G = U(N)$

We have clarified the general properties of non-linear realization in supersymmetric theories in the previous section. In this section we show how to construct the lagrangian explicitly, which will be helpful for practical use and also for further understanding the general discussion in §2. Here in this paper, we exhibit all kinds of pure realizations leaving the study on the other ones (maximal, intermediate and minimal) in a forthcoming paper.[7] We start with the case of $G/H = U(N)/(U(n_1) \otimes U(n_2) \otimes \cdots \otimes U(n_a))(\sum_{i=1}^{a} n_i = N)$.

Before going into general case (i.e. $H = U(n_1) \otimes \cdots \otimes U(n_a)$) we consider the $a=2$ case $(U(m+n)/(U(m) \otimes U(n))$,[4b] which was already investigated by Zumino[2] some years ago. He took notice of the properties of the Grassmann manifold $G_{m,n} \simeq U(m+n)/(U(m) \otimes U(n))$ and wrote down the non-linear lagrangian in an intuitive way. Here we try to construct the same lagrangian in a more systematic way according to the recipe explained in §2.

In this case $\mathfrak{g}^c$ and $\mathfrak{h}^c$ are parametrized as follows:

$$\mathfrak{g}^c=\left\{\begin{array}{cc} m & n \\ \left[\begin{array}{cc} a & b \\ c & d \end{array}\right] & \begin{array}{l} m \\ n \end{array} \end{array},\ \text{all elements are complex numbers}\right\}=\mathfrak{u}(m+n)^c\,, \tag{3·1}$$

$$\mathfrak{h}^c=\left\{\begin{array}{cc} m & n \\ \left[\begin{array}{cc} a & 0 \\ 0 & d \end{array}\right] & \begin{array}{l} m \\ n \end{array} \end{array};\ a\in\mathfrak{u}(m)^c,\ d\in\mathfrak{u}(n)^c\right\}=\mathfrak{u}(m)^c\oplus\mathfrak{u}(n)^c\,. \tag{3·2}$$

Let us give candidates for the maximal groups of R. Since the matrices b and c in Eq. (3·1)*) are respectively in the irreducible representations $(\boldsymbol{m}, \boldsymbol{n}^*)$ and $(\boldsymbol{m}^*, \boldsymbol{n})$ of H, each component of them cannot be independently combined with $\mathfrak{h}^c$ to make an algebra $\hat{\mathfrak{h}}$. Of course both of b and c cannot be combined with $\hat{\mathfrak{h}}^c$. Thus the possible candidates for $\hat{\mathfrak{h}}(\supset\mathfrak{r}^c)$ are given by

$$\hat{\mathfrak{h}}_1=\begin{array}{c} \begin{array}{cc} m & n \end{array} \\ \left[\begin{array}{cc} a & b \\ 0 & d \end{array}\right] \end{array} \quad\text{or}\quad \hat{\mathfrak{h}}_2=\begin{array}{cc} m & n \\ \left[\begin{array}{cc} a & 0 \\ c & d \end{array}\right] & \begin{array}{l} m \\ n \end{array} \end{array}. \tag{3·3}$$

If we interchange m and n, $\hat{h}_2$ can be represented by upper triangular matrices,

$$\hat{h}_2=\begin{array}{cc} n & m \\ \left[\begin{array}{cc} a & b \\ 0 & d \end{array}\right] & \begin{array}{l} n \\ m \end{array} \end{array}. \tag{3·4}$$

Let us exhibit non-linear realization for the case $\hat{\mathfrak{h}}=\hat{\mathfrak{h}}_1$, the results with $\hat{\mathfrak{h}}_2$ being given by interchanging m and n, the physical meaning of which will be discussed later. According to corollary 4, this case corresponds to a pure realization ($N_Q=0$). The projection η is given by

$$\eta=\begin{array}{cc} m & n \\ \left[\begin{array}{cc} 1 & 0 \\ 0 & 0 \end{array}\right] & \begin{array}{l} m \\ n \end{array} \end{array}. \tag{3·5}$$

The goldstone superfields**) are identified with the representative of $G^c/\hat{H}$,

---

*) Here the matrices b and c simultaneously mean such elements of $\mathfrak{g}^c$ as is represented by

$$\begin{array}{cc} m & n \\ \left[\begin{array}{cc} 0 & b \\ 0 & 0 \end{array}\right] & \begin{array}{l} m \\ n \end{array} \end{array} \text{ and } \begin{array}{cc} m & n \\ \left[\begin{array}{cc} 0 & 0 \\ c & 0 \end{array}\right] & \begin{array}{l} m \\ n \end{array} \end{array},\ \text{respectively.}$$

**) Hereafter we take the convention that all chiral superfields are left-handed unless we make special mention of it.

$$\xi = \exp i \begin{array}{cc} m & n \\ \left[\begin{array}{cc} 0 & 0 \\ c & 0 \end{array}\right] & \begin{array}{c} m \\ n \end{array} \end{array} = \begin{array}{cc} m & n \\ \left[\begin{array}{cc} 1 & 0 \\ \pi & 1 \end{array}\right] & \begin{array}{c} m \\ n \end{array} \end{array} \in G^c/\widehat{H}\,. \tag{3·6}$$

Here we choose π as goldstone superfields,[*] which leads us to the same parametrization as Zumino's. Under the $U(m+n)$ group ξ is transformed as

$$g\xi = \xi'(g, \xi)\widehat{h}(g, \xi), \qquad (g \in U(m+n)) \tag{3·7}$$

where $\xi'(\in G^c/\widehat{H})$ and $\widehat{h}(\in \widehat{H})$ are functions of indicated variables. In the matrix form Eq. (3·7) is represented by

$$\begin{array}{c} \begin{array}{cc} m & n \end{array} \\ \left[\begin{array}{cc} A & B \\ C & D \end{array}\right] \end{array} \cdot \begin{array}{cc} \begin{array}{cc} m & n \end{array} & \\ \left[\begin{array}{cc} 1 & 0 \\ \pi & 1 \end{array}\right] & \begin{array}{c} m \\ n \end{array} \end{array} = \begin{array}{c} \begin{array}{cc} m & n \end{array} \\ \left[\begin{array}{cc} 1 & 0 \\ \pi' & 1 \end{array}\right] \end{array} \cdot \begin{array}{c} \begin{array}{cc} m & n \end{array} \\ \left[\begin{array}{cc} A+B\pi & B \\ 0 & D' \end{array}\right] \end{array}, \tag{3·8}$$

where

$$\pi' = (C+D\pi)/(A+B\pi),$$
$$D' = D - \pi' B\,. \tag{3·9}$$

Note that the goldstone superfields are in the $(\boldsymbol{m}^*, \boldsymbol{n})$ representation of the isotropy group $U(m)\otimes U(n)$, and π^* does not appear on the r.h.s. of Eq. (3·9).

The supersymmetric non-linear lagrangian is obtained in terms of the following vector superfield:

$$V(\xi, \xi^\dagger) = \ln\det_\eta \xi^\dagger \xi$$
$$= \ln\det\left\{ \begin{array}{c} \begin{array}{cc} m & n \end{array} \\ m\,[1 \quad 0] \end{array} \begin{array}{c} \begin{array}{cc} m & n \end{array} \\ \left[\begin{array}{cc} 1 & \pi \\ 0 & 1 \end{array}\right] \end{array} \begin{array}{c} \begin{array}{cc} m & n \end{array} \\ \left[\begin{array}{cc} 1 & 0 \\ \pi & 1 \end{array}\right] \end{array} \begin{array}{cc} m & \\ \left[\begin{array}{c} 1 \\ 0 \end{array}\right] & \begin{array}{c} m \\ n \end{array} \end{array} \right\} \tag{3·10}$$
$$= \ln\det \xi_\eta{}^\dagger \xi_\eta\,,$$

where

$$\xi_\eta = \begin{array}{cc} m & \\ \left[\begin{array}{c} 1 \\ \pi \end{array}\right] & \begin{array}{c} m \\ n \end{array} \end{array}. \tag{3·11}$$

The superfield V is transformed under the $U(m+n)$ group as

$$V(\xi', \xi'^\dagger) = V(\xi, \xi^\dagger) + \ln\det\,[A+B\pi]^{-1} + \ln\det[A^\dagger + \pi^\dagger B^\dagger]^{-1}\,. \tag{3·12}$$

Then the lagrangian is given by

---

[*] Of course we may choose c as goldstone superfields.

$$\mathcal{L}=(\text{const})[V(\xi,\ \xi^\dagger)]_D\,, \tag{3·13}$$

which exactly coincides with the one given by Zumino. One comment is in order: If we adopt $\hat{\mathfrak{h}}_2$ in place of $\hat{\mathfrak{h}}_1$, projection (3·5) and the goldstone superfields (3·6) are replaced by

$$\eta^{(2)}=\begin{array}{cc} \begin{array}{cc} n & m \end{array} & \\ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} & \begin{array}{c} n \\ m \end{array} \end{array},\qquad \xi^{(2)}=\begin{array}{cc} \begin{array}{cc} n & m \end{array} & \\ \begin{bmatrix} 1 & 0 \\ \pi^{(2)} & 1 \end{bmatrix} & \begin{array}{c} n \\ m \end{array} \end{array}. \tag{3·14}$$

Thus $\pi^{(2)}$ is in the $(\boldsymbol{m},\ \boldsymbol{n}^*)$ representation of the isotropy group $U(m)\otimes U(n)$. Though $\pi^{(2)*}$ is in the same representation as π in (3·6), it has an opposite chirality (chiral superfield of the opposite chirality) to π. Hence in the presence of matter fields, the system with $\pi^{(2)}$ is, in general, distinguished from that with π.

Here we show that the bosonic part of lagrangian (3·13) is in the same form as constructed by the G-covariant projection operator P in a *usual non-supersymmetric way.* [For a moment ξ represents (complex) scalar components of the superfield]. The projection operator,

$$P\equiv\xi_\eta\frac{1}{\xi_\eta{}^\dagger\xi_\eta}\xi_\eta{}^\dagger\,, \tag{3·15}$$

is transformed under the $U(m+n)$ group as

$$P'=\xi_\eta{}'\frac{1}{\xi_\eta{}^{\dagger\prime}\xi_\eta{}'}\xi_\eta{}'^\dagger=gPg^\dagger\,,\qquad g\in U(m+n). \tag{3·16}$$

The lagrangian is given by

$$\mathcal{L}=\mathrm{Tr}\ \partial_\mu P\partial_\mu P=2\mathrm{Tr}(1-P)\partial_\mu\xi_\eta\frac{1}{\xi_\eta{}^\dagger\xi_\eta}\partial_\mu\xi_\eta{}^\dagger\,, \tag{3·17}$$

where use has been made of the equation

$$\begin{aligned}\delta P&=\delta\xi_\eta\frac{1}{\xi_\eta{}^\dagger\xi_\eta}\xi_\eta{}^\dagger+\xi_\eta\frac{1}{\xi_\eta{}^\dagger\xi_\eta}\{-\delta(\xi_\eta{}^\dagger\xi_\eta)\}\frac{1}{\xi_\eta{}^\dagger\xi_\eta}\xi_\eta{}^\dagger+\xi_\eta\frac{1}{\xi_\eta{}^\dagger\xi_\eta}\delta\xi_\eta{}^\dagger\\&=(1-P)\delta\xi_\eta\frac{1}{\xi_\eta{}^\dagger\xi_\eta}\xi_\eta{}^\dagger+\xi_\eta\frac{1}{\xi_\eta\xi_\eta}\delta\xi_\eta{}^\dagger(1-P).\end{aligned} \tag{3·18}$$

On the other hand, the bosonic part of (3·13) is given by

$$\begin{aligned}\mathcal{L}_B&\propto\left[\partial_\mu\xi_\eta{}^\dagger\frac{\delta}{\delta\xi_\eta{}^\dagger}\right]\left[\partial_\mu\xi_\eta\frac{\delta}{\delta\xi_\eta}\right]\ln\det(\xi_\eta{}^\dagger\xi_\eta)\\&=\left[\partial_\mu\xi_\eta{}^\dagger\frac{\delta}{\delta\xi_\eta{}^\dagger}\right]\mathrm{Tr}\frac{1}{\xi_\eta{}^\dagger\xi_\eta}\xi_\eta{}^\dagger\partial_\mu\xi_\eta\\&=\mathrm{Tr}\frac{1}{\xi_\eta{}^\dagger\xi_\eta}(-\partial_\mu\xi_\eta{}^\dagger\xi_\eta)\frac{1}{\xi_\eta{}^\dagger\xi_\eta}\xi_\eta{}^\dagger\partial_\mu\xi_\eta+\mathrm{Tr}\frac{1}{\xi_\eta{}^\dagger\xi_\eta}\partial_\mu\xi_\eta{}^\dagger\partial_\mu\xi_\eta\\&=\mathrm{Tr}(1-P)\partial_\mu\xi_\eta\frac{1}{\xi_\eta{}^\dagger\xi_\eta}\partial_\mu\xi_\eta{}^\dagger\,,\end{aligned} \tag{3·19}$$

which coincides with that in (3·17).

Next we proceed to the $a=3$ case $U(l+m+n)/U(l)\otimes U(m)\otimes U(n)$.[4b)] $\mathfrak{g}^c$ and $\mathfrak{h}^c$

are parametrized as follows:

$$\mathfrak{g}^c=\left\{\begin{array}{cc} \begin{matrix} l & m & n \end{matrix} & \\ \begin{bmatrix} h_l & t_{lm} & t_{lm} \\ t_{ml} & h_m & t_{mn} \\ t_{nl} & t_{nm} & h_n \end{bmatrix} & \begin{matrix} l \\ m; \\ n \end{matrix} \end{array} \text{ all elements are complex numbers}\right\}$$

$$=\mathfrak{u}(l+m+n)^c\,, \tag{3·20}$$

$$\mathfrak{h}^c=\left\{\begin{array}{cc} \begin{matrix} l & m & n \end{matrix} & \\ \begin{bmatrix} h_l & 0 & 0 \\ 0 & h_m & 0 \\ 0 & 0 & h_n \end{bmatrix} & \begin{matrix} l \\ m; \\ n \end{matrix} \end{array} \ h_i \in \mathfrak{u}(i)^c \quad i=l,\ m,\ n\right\}. \tag{3·21}$$

According to the general discussion in the previous section, the maximal one of candidates for $\mathfrak{r}$ is given by

$$\mathfrak{r}=\left\{\begin{array}{cc} \begin{matrix} l & m & n \end{matrix} & \\ \begin{bmatrix} 0 & t_{lm} & t_{lm} \\ 0 & 0 & t_{mn} \\ 0 & 0 & 0 \end{bmatrix} & \begin{matrix} l \\ m \\ n \end{matrix} \end{array}\right\}, \tag{3·22}$$

and thus

$$\hat{\mathfrak{h}}^{\max}=\left\{\begin{array}{cc} \begin{matrix} l & m & n \end{matrix} & \\ \begin{bmatrix} h_l & t_{lm} & t_{ln} \\ 0 & h_m & t_{mn} \\ 0 & 0 & h_n \end{bmatrix} & \begin{matrix} l \\ m \\ n \end{matrix} \end{array}\right\}. \tag{3·23}$$

With the above $\hat{\mathfrak{h}}^{\max}$ we can carry out the pure realization ($N_Q=0$). Here we will make one comment: Besides $\hat{\mathfrak{h}}^{\max}$ in (3·23) we have precisely the following five kinds of candidates for $\hat{\mathfrak{h}}$ providing pure realization [all of dim$[\hat{\mathfrak{h}}]$ below are equal to dim $[\hat{\mathfrak{h}}^{\max}]$],

$$\hat{\mathfrak{h}}^{(nml)}=\left\{\begin{array}{cc} \begin{matrix} l & m & n \end{matrix} & \\ \begin{bmatrix} h_l & 0 & 0 \\ t_{ml} & h_m & t_{mn} \\ t_{nl} & 0 & h_n \end{bmatrix} & \begin{matrix} l \\ m \\ n \end{matrix} \end{array}\right\}, \quad \hat{\mathfrak{h}}^{(nlm)}=\left\{\begin{array}{cc} \begin{matrix} l & m & n \end{matrix} & \\ \begin{bmatrix} h_l & t_{lm} & 0 \\ 0 & h_m & 0 \\ t_{nl} & t_{nm} & h_n \end{bmatrix} & \begin{matrix} l \\ m \\ n \end{matrix} \end{array}\right\},$$

$$\hat{\mathfrak{h}}^{(lnm)}=\left\{\begin{array}{cc} \begin{matrix} l & m & n \end{matrix} & \\ \begin{bmatrix} h_l & t_{lm} & t_{ln} \\ 0 & h_m & 0 \\ 0 & t_{nm} & h_n \end{bmatrix} & \begin{matrix} l \\ m \\ n \end{matrix} \end{array}\right\},$$

$$\hat{\mathfrak{h}}^{(mnl)}=\left\{\begin{array}{c}\begin{matrix} l & m & n\end{matrix}\\ \begin{bmatrix} h_l & 0 & 0 \\ t_{ml} & h_m & 0 \\ t_{nl} & t_{nm} & h_n \end{bmatrix}\end{array}\begin{matrix} \\ l \\ m \\ n\end{matrix}\right\},\quad \hat{\mathfrak{h}}^{(mln)}=\left\{\begin{array}{c}\begin{matrix} l & m & n\end{matrix}\\ \begin{bmatrix} h_l & 0 & t_{ln} \\ t_{ml} & h_m & t_{mn} \\ 0 & 0 & h_n \end{bmatrix}\end{array}\begin{matrix} \\ l \\ m \\ n\end{matrix}\right\}. \tag{3·24}$$

Every set of goldstone superfields $\xi^{(ijk)}$ corresponding to the above $\hat{\mathfrak{h}}^{(ijk)}$ $(i, j, k, =l, m, n$ and $(lmn)\equiv\max)$ is in the representation of the isotropy group $U(l)\otimes U(m)\otimes U(n)$ different from eath other. Hence each non-linear lagrangian is inequivalent.*) If one properly rearranges the matrices in terms of l, m and n for each $\hat{\mathfrak{h}}$ in (3·24), every $\hat{\mathfrak{h}}$ can be represented by an upper triangular matrix as seen in (3·23),

$$\hat{\mathfrak{h}}^{(ijk)}=\left\{\begin{array}{c}\begin{matrix} i & j & k\end{matrix}\\ \begin{bmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{bmatrix}\end{array}\begin{matrix} \\ i \\ j \\ k\end{matrix}\quad (i, j, k=l, m, n)\right\}. \tag{3·25}$$

Therefore it is sufficient to consider the case of $\hat{\mathfrak{h}}^{\max}$ in (3·23). We have two kinds of projections,

$$\eta_1=\begin{array}{c}\begin{matrix} l & m & n\end{matrix}\\ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\end{array}\begin{matrix} \\ l \\ m \\ n\end{matrix},\quad \eta_2=\begin{array}{c}\begin{matrix} l & m & n\end{matrix}\\ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\end{array}\begin{matrix} \\ l \\ m \\ n\end{matrix}. \tag{3·26}$$

The goldstone superfields are represented by

$$\xi=\exp i\begin{array}{c}\begin{matrix} l & m & n\end{matrix}\\ \begin{bmatrix} 0 & 0 & 0 \\ t_{ml} & 0 & 0 \\ t_{nl} & t_{nm} & 0 \end{bmatrix}\end{array}\begin{matrix} \\ l \\ m \\ n\end{matrix}=\begin{array}{c}\begin{matrix} l & m & n\end{matrix}\\ \begin{bmatrix} 1 & 0 & 0 \\ \pi_{ml} & 1 & 0 \\ \pi_{nl} & \pi_{nm} & 1 \end{bmatrix}\end{array}\begin{matrix} \\ l \\ m \\ n\end{matrix}\quad (\in G^c/\hat{H}^{\max}) \tag{3·27}$$

They are transformed under the $U(l+m+n)$ group as

$$g\xi=\xi'(g, \xi)\hat{h}(g, \xi),$$

where

$$g\in U(l+m+n),\qquad \hat{h}(g, \xi)\in\hat{H}^{\max}=\{\exp(ih^{\max})\}, \tag{3·28}$$

which is represented in a matrix form as

$$\begin{array}{c}\begin{matrix} l & m & n\end{matrix}\\ \begin{bmatrix} H_l & T_{lm} & T_{ln} \\ T_{ml} & H_m & T_{mn} \\ T_{nl} & T_{nm} & H_n \end{bmatrix}\end{array}\begin{array}{c}\begin{matrix} 0 & m & n\end{matrix}\\ \begin{bmatrix} 1 & 0 & 0 \\ \pi_{ml} & 1 & 0 \\ \pi_{nl} & \pi_{nm} & 1 \end{bmatrix}\end{array}\begin{matrix} \\ l \\ m \\ n\end{matrix}=\begin{array}{c}\begin{matrix} l & m & n\end{matrix}\\ \begin{bmatrix} 1 & 0 & 0 \\ \pi'_{ml} & 1 & 0 \\ \pi'_{nl} & \pi'_{nm} & 1 \end{bmatrix}\end{array}\begin{array}{c}\begin{matrix} l & m & n\end{matrix}\\ \begin{bmatrix} \hat{H}_l & \hat{T}_{lm} & \hat{T}_{ln} \\ 0 & \hat{H}_m & \hat{T}_{mn} \\ 0 & 0 & \hat{H}_n \end{bmatrix}\end{array}\begin{matrix} \\ l \\ m \\ n\end{matrix}, \tag{3·29}$$

*) Though $\xi^{(ijk)}$ and $\xi^{(kji)*}$ are in the same representation of the $U(l)\otimes U(m)\otimes U(n)$, they are of opposite chirality to each other. Hence in the presence of matter fields, the system with $\xi^{(ijk)}$ is, in general, distinguished from that with $\xi^{(kji)}$.

where calculation of each component of the r.h.s. is straightforward and is left to the readers.

The supersymmetric non-linear lagrangian is given by

$$\mathcal{L}=c_1[\ln\det{}_{\eta_1}(\xi^\dagger\xi)]_D+c_2[\ln\det{}_{\eta_2}(\xi^\dagger\xi)]_D\,, \tag{3·30}$$

where c_i's are constant parameters whose ratio is arbitrary. The superfields $\ln\det{}_{\eta_i}(\xi^\dagger\xi)$ can be rewritten as

$$\ln\det{}_{\eta_i}(\xi^\dagger\xi)=\ln\det(\xi^{i}_{(i)}\xi_{(i)}),\qquad (i=1,\,2)$$

where

$$\xi_{(1)}=\begin{array}{c} l \\ \begin{bmatrix} 1 \\ \pi_{ml} \\ \pi_{nl} \end{bmatrix} \end{array}\begin{array}{l} \\ l \\ m, \\ n \end{array}\qquad \xi_{(2)}=\begin{array}{c} \begin{array}{cc} l & m \end{array} \\ \begin{bmatrix} 1 & 0 \\ \pi_{ml} & 1 \\ \pi_{nl} & \pi_{nm} \end{bmatrix} \end{array}\begin{array}{l} \\ l \\ m\,. \\ n \end{array} \tag{3·31}$$

Note here that the G-covariant projection operators are given by

$$P_1=\xi_{(1)}\frac{1}{\xi^\dagger_{(1)}\xi_{(1)}}\xi^\dagger_{(1)},\qquad P_2=\xi_{(2)}\frac{1}{\xi^\dagger_{(2)}\xi_{(2)}}\xi^\dagger_{(2)}\,. \tag{3·32}$$

They satisfy the equation

$$\mathrm{Tr}\,P_1P_2=\mathrm{Tr}\,P_1=(\mathrm{const}), \tag{3·33}$$

which implies that the C-type recipe gives only a trivial invariant.

It is now easy to proceed to the general case $(U(N)/(U(n_1)\otimes\cdots\otimes U(n_1)))$. The algebras $\mathfrak{g}^c$ and $\hat{\mathfrak{h}}$ are parametrized as

$$\mathfrak{g}^c=\left\{\begin{array}{c} \begin{array}{ccc} n_1 & n_2 & \cdots n_a \end{array} \\ \begin{bmatrix} h_1 & t_{12} & \cdots t_{1a} \\ t_{21} & h_2 & \cdots t_{2a} \\ \vdots & \vdots & \ddots \vdots \\ t_{a1} & t_{a2} & \cdots h_a \end{bmatrix} \end{array}\begin{array}{l} \\ n_1 \\ n_2 \\ \vdots \\ n_a \end{array}\right\}=\mathfrak{u}(N)^c,\qquad \Big(N=\sum_{i=1}^{a}n_i\Big) \tag{3·34}$$

$$\hat{\mathfrak{h}}=\left\{\begin{array}{c} \begin{array}{ccc} n_1 & n_2 & \cdots n_1 \end{array} \\ \begin{bmatrix} h_1 & t_{12} & \cdots t_{1a} \\ & h_2 & \cdots t_{2a} \\ 0 & & \ddots \vdots \\ & & h_a \end{bmatrix} \end{array}\begin{array}{l} \\ n_1 \\ n_2 \\ \vdots \\ n_a \end{array}\right\}. \tag{3·35}$$

We have a! kinds of different candidates for $\hat{\mathfrak{h}}$ which lead, in general, to inequivalent non-linear lagrangians. Again it is enough to study only the above case. There are $a-1$ kinds of projections,

$$\eta_1=\begin{array}{c} n_1 \\ n_1\left[\begin{array}{c:c} 1 & \\ \hdashline & 0 \end{array}\right] \end{array},\quad \eta_2=\begin{array}{c} n_1+n_2 \\ n_1+n_2\left[\begin{array}{c:c} 1 & \\ \hdashline & 0 \end{array}\right] \end{array},\cdots,\quad \eta_{a-1}=\begin{array}{c} n_a \\ \left[\begin{array}{c:c} 1 & 0 \\ \hdashline 0 & 0 \end{array}\right] \end{array} n_a\,. \tag{3·36}$$

The goldstone superfields are represented by

$$\xi=\exp i\begin{array}{c} \begin{array}{ccccc} n_1 & n_2 & \cdots & & n_a \end{array} \\ \left[\begin{array}{ccccc} 0 & & & & \\ t_{21} & 0 & & & \\ t_{31} & t_{32} & 0 & & \\ \vdots & \vdots & & \ddots & \\ t_{a1} & t_{a2} & \cdots & t_{aa-1} & 0 \end{array}\right] \end{array}\begin{array}{c} n_1 \\ n_2 \\ n_3 \\ \vdots \\ n_a \end{array} = \begin{array}{c} \begin{array}{ccccc} n_1 & n_2 & \cdots & n_{a-1} & n_a \end{array} \\ \left[\begin{array}{ccccc} 1 & & & & \\ \pi_{21} & 1 & 0 & & \\ \pi_{31} & \pi_{32} & \ddots & & \\ \vdots & \vdots & & 1 & \\ \pi_{a1} & \pi_{a2} & \cdots & \pi_{aa-1} & 1 \end{array}\right] \end{array}\begin{array}{c} n_1 \\ n_2 \\ n_3 \\ \vdots \\ n_{a-1} \\ n_a \end{array} \in G^c/\widehat{H}\,, \tag{3·37}$$

which are transformed under the $U(N)$ group as

$$g=\xi'(g,\ \xi)\widehat{h}(g,\ \xi), \tag{3·38}$$

$$g\in U(N), \qquad \xi'\in G^c/\widehat{H}, \qquad \widehat{h}\in\widehat{H}\,. \tag{3·39}$$

The supersymmetric non-linear lagrangian is given by

$$\mathcal{L}=c_1[\ln \det{}_{\eta_1}(\xi^\dagger\xi)]_D+c_2[\ln \det{}_{\eta_2}(\xi^\dagger\xi)]_D+\cdots+c_{a-1}[\ln \det{}_{\eta_{a-1}}(\xi^\dagger\xi)]_D\,, \tag{3·40}$$

where c_i's are constant parameters whose ratios are arbitrary.

§4. Pure realization: $G=O(N)$ or $Sp(2N)$

In this section we consider the cases

$$G/H=O(N)/(O(m)\otimes U(n_1)\otimes U(n_2)\otimes\cdots\otimes U(n_a)), \qquad (m+2\sum_{i=1}^{a}n_i=N) \tag{4·1a}$$

$$G/H=Sp(2N)/(Sp(2m)\otimes U(n_1)\otimes U(n_2)\otimes\cdots\otimes U(n_a)). \qquad (m+\sum_{i=1}^{a}n_i=N) \tag{4·1b}$$

Besides the $U(N)/(U(n_1)\otimes\cdots\otimes U(n_a))$ in §3, the above two are the only manifolds which allow pure realization for the case of G and H being classical groups as is easily seen in §2. We parametrize the $O(N)$ ($Sp(2N)$) group in terms of $U(N)$ ($U(2N)$) with the constraints,

$$O(N)=\{g;\ g\in U(N)\ \text{ s.t. }\ g^TJg=J \text{ with } J=\begin{array}{c} \begin{array}{ccc} n & m & n \end{array} \\ \left[\begin{array}{ccc} & & 1 \\ & 1 & \\ 1 & & \end{array}\right] \end{array}\begin{array}{l} n \\ m(\sum_{i=1}^{a}n_a=n)\}, \\ n \end{array} \tag{4·2a}$$

$$Sp(2N)=\{g;\ g\in U(2N)\ \text{ s.t. }\ g^T\Omega g=\Omega \text{ with } \Omega=\begin{array}{c} \begin{array}{cccc} n & m & m & n \end{array} \\ \left[\begin{array}{cccc} & & & 1 \\ & & 1 & \\ & -1 & & \\ -1 & & & \end{array}\right] \end{array}\begin{array}{l} n \\ m \\ m \\ n \end{array}(\sum_{i=1}^{a}n_a=n)\}. \tag{4·2b}$$

Hence each of $\mathfrak{g}^c$ is parametrized as

$$\mathfrak{o}(N)^c = \left\{ \begin{array}{cccc} n & m & n & \\ \left[\begin{matrix} W \\ Y \\ T \end{matrix}\right. & \begin{matrix} X \\ Z \\ -Y^T \end{matrix} & \left.\begin{matrix} S \\ -X^T \\ -W^T \end{matrix}\right] & \begin{matrix} n \\ m \\ n \end{matrix} \end{array} ;\ W\in\mathfrak{u}(n)^c,\ \ Z\in\mathfrak{o}(m)^c,\ \ S^T=-S,\ \ T^T=-T \right\},$$

(4·3a)

$$\mathfrak{sp}(2N)^c = \left\{ \begin{array}{cccc} n & 2m & n & \\ \left[\begin{matrix} W \\ Y \\ T \end{matrix}\right. & \begin{matrix} X \\ Z \\ V \end{matrix} & \left.\begin{matrix} S \\ U \\ -W^T \end{matrix}\right] & \begin{matrix} n \\ 2m \\ n \end{matrix} \end{array} : \begin{array}{l} W\in\mathfrak{u}(n)^c,\ \ S^T=S,\ \ T^T=T, \\ Z\in\mathfrak{sp}(2m)^c,\ \ U=\Omega_0 X^T,\ \ V=-Y^T\Omega_0 \end{array}, \right.$$

$$\left. \Omega_0 = \begin{array}{ccc} m & m & \\ \left[\begin{matrix} 0 \\ -1 \end{matrix}\right. & \left.\begin{matrix} 1 \\ 0 \end{matrix}\right] & \begin{matrix} m \\ m \end{matrix} \end{array} \right\}, \tag{4·3b}$$

where all matrix elements are complex numbers. Note that every element of the $\mathfrak{o}(N)^c$ or $\mathfrak{sp}(2N)^c$ satisfies the following conditions:

$$A^T J + JA = 0\,, \qquad A\in\mathfrak{o}(N)^c\,. \tag{4·4a}$$

$$A^T \Omega + \Omega A = 0\,, \qquad A\in\mathfrak{sp}(2N)^c\,. \tag{4·4b}$$

Each of $\mathfrak{h}^c$ is given by

$$\mathfrak{h}^c_{(o)} = \left\{ \begin{array}{cccc} n & m & n & \\ \left[\begin{matrix} H \\ 0 \\ 0 \end{matrix}\right. & \begin{matrix} 0 \\ Z \\ 0 \end{matrix} & \left.\begin{matrix} 0 \\ 0 \\ -H^T \end{matrix}\right] & \begin{matrix} n \\ m \\ n \end{matrix} \end{array} ;\ H = \begin{array}{cc} n_1 \quad n_2 \cdots n_a & \\ \begin{bmatrix} H_1 & & 0 \\ & H_2 & \\ & \ddots & \\ 0 & & H_a \end{bmatrix} & \begin{matrix} n_1 \\ n_2 \\ \vdots \\ n_a \end{matrix} \end{array},\ \begin{array}{l} H_i\in\mathfrak{u}(n_i)^c \\ Z\in\mathfrak{o}(m)^c \end{array} \right\}, \tag{4·5a}$$

$$\mathfrak{h}^c_{(sp)} = \left\{ \begin{array}{cccc} n & 2m & n & \\ \left[\begin{matrix} H \\ 0 \\ 0 \end{matrix}\right. & \begin{matrix} 0 \\ Z \\ 0 \end{matrix} & \left.\begin{matrix} 0 \\ 0 \\ -H^T \end{matrix}\right] & \begin{matrix} n \\ 2m \\ n \end{matrix} \end{array} ;\ H = \begin{array}{cc} n_1 \quad n_2 \cdots n_a & \\ \begin{bmatrix} H_1 & & \\ & H_2 & \\ & \ddots & \\ 0 & & H_a \end{bmatrix} & \begin{matrix} n_1 \\ n_2 \\ \vdots \\ n_a \end{matrix} \end{array},\ \begin{array}{l} H_i\in\mathfrak{u}(n_i)^c \\ Z\in\mathfrak{sp}(2m)^c \end{array} \right\}. \tag{4·5b}$$

According to the general discussion in §2, the pure realization is provided by the following $\hat{\mathfrak{h}}$:

$$\hat{\mathfrak{h}}_{(O)}=\left\{\begin{bmatrix} H_P & X & S^T \\ 0 & Z & -X^T \\ 0 & 0 & -H_p \end{bmatrix}\right\}, \quad \hat{\mathfrak{h}}_{(Sp)}=\left\{\begin{bmatrix} H_p & X & S \\ 0 & Z & \Omega_0 X^T \\ 0 & 0 & -H^T \end{bmatrix}\right\}, \tag{4·6}$$

where

$$H_p=\begin{array}{c} \begin{array}{cccc} n_1 & n_2 & n_3 & \cdots n_a \end{array} \\ \begin{bmatrix} H_1 & T_{12} & T_{13}\cdots & T_{1a} \\ & H_2 & T_{23}\cdots & T_{2a} \\ & & & \vdots \\ & & \ddots & T_{a-1a} \\ & 0 & & H_a \end{bmatrix} \end{array} \begin{array}{c} n_1 \\ n_2 \\ \vdots \\ n_{a-1} \\ n_a \end{array} . \tag{4·7}$$

There are a! kinds of different candidates for $\hat{\mathfrak{h}}$ which are obtained by taking the permutations between n_i's $(i=1, \cdots, a)$ in the same pattern of matrices as that in (4·7). The projections corresponding to $\hat{\mathfrak{h}}$'s in Eq. (4·6) are given by*)

$$\eta_i^{(O)}=\begin{array}{c} \begin{array}{cc} n & n+m \end{array} \\ \begin{bmatrix} 1_i & 0 \\ 0 & 0 \end{bmatrix} \end{array} \begin{array}{c} n \\ m+n \end{array}, \quad \eta_{a+1}^{(O)}=\begin{array}{c} \begin{array}{ccc} n & m & n \end{array} \\ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{array} \begin{array}{c} n \\ m \\ n \end{array}, \tag{4·8a}$$

$$\eta_i^{(Sp)}=\begin{array}{c} \begin{array}{cc} n & n+2 \end{array} \\ \begin{bmatrix} 1_i & 0 \\ 0 & 0 \end{bmatrix} \end{array} \begin{array}{c} n \\ n+2m \end{array}, \quad \eta_{a+1}^{(Sp)}=\begin{array}{c} \begin{array}{ccc} n & 2m & m \end{array} \\ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{array} \begin{array}{c} n \\ 2m \\ n \end{array} \tag{4·8b}$$

with

$$1_i=\begin{array}{c} n_1\cdots n_i \\ \left[\begin{array}{c:c} 1 & 0 \\ \hdashline 0 & 0 \end{array}\right] \end{array} \begin{array}{c} n_1 \\ \vdots \\ n_i \end{array} \quad (i=1, \cdots, a). \tag{4·9}$$

The goldstone superfields are represented by

$$\xi_{(O)}=\exp i \begin{array}{c} \begin{array}{ccc} n & m & n \end{array} \\ \begin{bmatrix} W_p & 0 & 0 \\ Y & 0 & 0 \\ T & -Y^T & -W_p^T \end{bmatrix} \end{array} \begin{array}{c} n \\ m \\ n \end{array} = \begin{bmatrix} w_p & 0 & 0 \\ y & 1 & 0 \\ t & -(yw_p^{-1})^T & (w_p^T)^{-1} \end{bmatrix} \in G^c/\hat{H}_{(O)}, \tag{4·10a}$$

$$\xi_{(Sp)}=\exp i \begin{array}{c} \begin{array}{ccc} n & 2m & n \end{array} \\ \begin{bmatrix} W_p & 0 & 0 \\ Y & 0 & 0 \\ T & -Y^T\Omega_0 & -W_p^T \end{bmatrix} \end{array} \begin{array}{c} n \\ 2m \\ n \end{array} = \begin{bmatrix} W_p & 0 & 0 \\ y & 1 & 0 \\ t & (\Omega_0 y w_p^{-1})^T & (w_p^T)^{-1} \end{bmatrix} \in G^c/\hat{H}_{(Sp)}, \tag{4·10b}$$

*) If $G=O(N)$, $Sp(2)$ and $Sp(4)$, $a_1=1$ and $n_1=1$, then G/H are irreducible Kähler manifolds and two invariants constructed by using η_1 and η_2 are equivalent to each other.

$$W_p=\begin{array}{c}\begin{array}{cccc} n_1 & n_2\cdots n_{a-1} & & n_a\end{array}\\ \left[\begin{array}{cccc} 0 & & & \\ t_{21} & 0 & 0 & \\ t_{31} & t_{32} & & \\ \vdots & \vdots & \ddots & \\ t_{a1} & t_{a2}\cdots & t_{aa-1} & 0 \end{array}\right]\end{array}\begin{array}{c} n_1\\ n_2\\ n_3\\ \vdots\\ n_a\end{array},\quad w_p=\begin{array}{c}\begin{array}{cccc} n_1 & n_2\cdots n_{a-1} & & n_a\end{array}\\ \left[\begin{array}{cccc} 1 & & & \\ \pi_{21} & 1 & 0 & \\ \pi_{31} & \pi_{32} & & \\ \vdots & \vdots & \ddots & \\ \pi_{a1} & \pi_{a2}\cdots & \pi_{aa-1} & 1 \end{array}\right]\end{array}\begin{array}{c} n_1\\ n_2\\ \\ \vdots\\ n_a\end{array} \tag{4·11}$$

with the constraints,

$$w_p{}^T t+t^T w_p+y^T y=0 \qquad \text{for } G=O(N), \tag{4·12a}$$

$$-w_p{}^T t+t^T w_p+y^T \Omega_0 y=0 \qquad \text{for } G=Sp(2N). \tag{4·12b}$$

Hence if we parametrize the matrix elements t and y in Eq. (4·10) as

$$t=\begin{array}{c}\begin{array}{ccccc} n_1 & n_2 & \cdots & n_a\end{array}\\ \left[\begin{array}{cccc} \alpha_1 & \gamma_{12} & \cdots & \gamma_{1a}\\ \beta_{21} & \alpha_2\,\gamma_{23} & \cdots & \gamma_{2a}\\ \beta_{31} & \beta_{32} & \cdots & \vdots\\ \vdots & \vdots & \vdots & \gamma_{a-1a}\\ \beta_{a1} & \beta_{a2} & \beta_{aa-1} & \alpha_a \end{array}\right]\end{array}\begin{array}{c} n_1\\ n_2\\ \vdots\\ n_{a-1}\\ n_a\end{array},\qquad y=\begin{array}{c}\begin{array}{cc} n_1 & n_2\cdots n_a\end{array}\\ \left[\begin{array}{cc} & \\ \rho_1 & \rho_2\cdots\rho_a\\ & \end{array}\right]\end{array}$$

$$m(G=O(N) \text{ or } 2m(G=Sp(2N)), \tag{4·13}$$

we see that each of π_{ij}, β_{ij}, ρ_l and antisymmetric [symmetric] part of α_l have $2n_in_j$, $2n_in_j$, $2n_lm[4n_lm]$ and $n_l(n_l-1)$ $[n_l(n_l+1)]$ independent goldstone modes while γ_{ji} and symmetric [antisymmetric] part of α_l are only dependent modes for the case $G=O(N)[Sp(2N)](i=2,\cdots,a;\ j=1,\cdots,a-1;\ l=1,\cdots,a)$. Note that Eqs. (4·12) are the same as the conditions,

$$\xi^T_{(O)}J\xi_{(O)}=J\,, \tag{4·14a}$$

$$\xi^T_{(Sp)}\Omega\xi_{(Sp)}=\Omega\,. \tag{4·14b}$$

The transformation law of goldstone superfields is given by

$$g\xi=\xi'(g,\xi)\hat{h}(g,\xi), \qquad g\in G\,, \qquad \hat{h}\in\hat{H}\,. \tag{4·15}$$

Of course the transformed fields satisfy the corresponding conditions (4·14),

$$\begin{aligned}&\xi'^T\Gamma\xi'=(\hat{h}^{-1})^T\xi^T g^T\Gamma g\xi\hat{h}^{-1}=(\hat{h}^{-1})^T\xi^T\Gamma\xi\hat{h}^{-1}\,,\\ &(\hat{h}^{-1})^T\Gamma\hat{h}^{-1}=\Gamma\,,\end{aligned} \tag{4·16}$$

where

$$\Gamma=\begin{cases} J & \text{for } G=O(N),\\ \Omega & \text{for } G=Sp(2N).\end{cases} \tag{4·17}$$

The supersymmetric non-linear lagrangian is given by

$$\mathcal{L}_{(A)}=\sum_{i=1}^{a+1}c_i[\ln\det{}_{\eta_i^{(A)}}(\xi^\dagger_{(A)}\xi_{(A)})]_D, \quad (A=O \text{ or } Sp) \tag{4·18}$$

where c_i's are constant parameters whose relative values are arbitrary.

§ 5. Matter fields

We have studied the non-linear lagrangians for the goldstone superfields in the previous sections. In this section we show how to construct invariant lagrangians for matter fields. The mass terms of matter chiral superfields are obtained throuhg F terms, whereas kinetic terms come from D terms. We have already shown in §2 that a matter field N in the representation ρ_0 of $\hat{H}$ transforms under G as

$$N'=\rho_0(\hat{h}(\xi,\ g))N\ , \qquad g\in G\ , \tag{5·1}$$

where the transformation property of $h(\xi,\ g)$,

$$h(\xi,\ g_1 g_2)=h(g_2\xi h^{-1}(\xi,\ g_2),\ g_1)\cdot h(\xi,\ g_2), \tag{5·2}$$

indicates that above N is in a representation of G, i.e.,

$$N\overset{g_1}{\to}N'=\rho_0(\hat{h}(\xi,\ g_1)N\overset{g_2}{\to}\rho_0(\hat{h}(g_2\xi h^{-1}(\xi,\ g_2),\ g_1)\rho_0(\hat{h}(\xi,\ g_2))N$$
$$=\rho_0(\hat{h}(\xi,\ g_1 g_2))N. \tag{5·3}$$

In §2 we have shown that a linear basis $\rho(\xi)\ N$ in G is constructed by picking up some representation ρ of G whose restriction to $\hat{H}$ contains ρ_0.

Now let us discuss the case of pure realization of $G=U(N)$ $(N=\sum_{i=1}^{a}n_i)$ and $H=U(n_1)\otimes\cdots\otimes U(n_a)$. In this case elements $\hat{h}\in\hat{\mathfrak{h}}$ are expressed as

$$\hat{h}=\begin{bmatrix} \mathrm{u}(n_1) & & & & \\ & \ddots & & & * \\ & & \mathrm{u}(n_i)\ddots & * & \\ & & 0 & \ddots\mathrm{u}(n_j) & \\ 0 & & & & \ddots \\ & & & & \mathrm{u}(n_a) \end{bmatrix}. \tag{5·4}$$

Now define submatrices $\rho_{ij}(\hat{h})(i\leq j)$ as those indicated by dotted line in Eq. (5·4),

$$\rho_{ij}(\hat{h})=\begin{bmatrix} \mathrm{u}(n_i) & & * \\ & \ddots & \\ 0 & & \mathrm{u}(n_j) \end{bmatrix}. \tag{5·5}$$

From (5·4) and (5·5) it turns out that $\rho_{ij}(\hat{h}(\xi,\ g))$ satisfies the same algebraic relation as shown in Eq. (5·2), which implies $\rho_{ij}(\hat{h})$ is a candidate for ρ_0. Also its contragradient representation $(\rho_{ij}^{-1}(\hat{h}))^T$ obeys the same relation (5·2). Mass terms can be, in general, obtained as follows: If there are a pair of matter fields $N(i,\ j)$ and $\tilde{N}(k,\ l)$, which are in ρ_{ij} and $(\rho_{kl}^{-1})^T$ representations, respectively, with $i,\ j$ and $k,\ l$ being ordered as

$$i\leq k\leq j\leq l\ . \tag{5·6}$$

We define operators η_{ij} which project out the blocks $(i\cdots j)$ from $(i\cdots j\cdots l)$, and express the matter fields $N(i,\ j)$ and $\tilde{N}(k,\ l)$ as

$$N(i,\ j)=\eta_{ij}\left[\begin{array}{c} \\ N(i,\ j) \\ \\ 0 \\ \vdots \\ 0\end{array}\right]\begin{array}{l} i \\ \\ j \\ j+1 \\ \\ l\end{array}, \tag{5·7a}$$

$$N(k,\ l)=\eta_{kl}\left[\begin{array}{c} 0 \\ \vdots \\ 0 \\ \\ N(k,\ l) \\ \\ \end{array}\right]\begin{array}{l} i \\ \\ k-1 \\ k \\ \\ l\end{array}. \tag{5·7b}$$

They are transformed under G as

$$N'(i,\ j)=\rho_{ij}(\hat{h})N(i,\ j)=\rho_{il}(\hat{h})\eta_{ij}\begin{bmatrix} N(i,\ j) \\ 0 \\ \vdots \\ 0\end{bmatrix},$$

$$\tilde{N}'(k,\ l)=(\rho_{kl}^{-1}(\hat{h}))^T\tilde{N}(k,\ l)=(\rho_{il}^{-1}(\hat{h}))^T\eta_{kl}\begin{bmatrix} 0 \\ \vdots \\ 0 \\ \tilde{N}(k,\ l)\end{bmatrix}. \tag{5·8}$$

If one notices that

$$\rho_{ij}\eta_{ij}=\eta_{ij}\rho_{il}\eta_{ij} \tag{5·9}$$

then

$$\tilde{N}(k,\ l)^T\cdot N(i,\ j)\equiv(0\cdots0\tilde{N}(k,\ l)^T)\eta_{kl}\eta_{ij}\begin{bmatrix} N(i,\ j) \\ 0 \\ \vdots \\ 0\end{bmatrix} \tag{5·10}$$

turns out to be invariant under G,

$$\tilde{N}'(k,\ l)^T\cdot N'(i,\ j)=[0\cdots0\tilde{N}(k,\ l)^T]\eta_{kl}\rho_{il}^{-1}(\hat{h})\eta_{kl}\eta_{ij}\rho_{il}(\hat{h})\eta_{ij}\begin{bmatrix} N(i,\ j) \\ 0 \\ \vdots \\ 0\end{bmatrix}$$

$$=(0\cdots0\tilde{N}^T(k,\ l))\eta_{kl}\rho_{il}^{-1}(\hat{h})\cdot\rho_{il}(\hat{h})\eta_{ij}\begin{bmatrix} N(i,\ j) \\ 0 \\ \vdots \\ 0\end{bmatrix}$$

$$=(0\cdots 0 N^T(k,\ l))\eta_{kl}\eta_{ij}\begin{bmatrix} N(i,\ j) \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \tilde{N}^T(k,\ l)\cdot N(i,\ j). \qquad (5\cdot 11)$$

On the other hand, if one tries to get kinetic terms, one has to construct linear basis of G from matter chiral fields $N(i,\ j)$ or $N(k,\ l)$. The following statement is evident:

*Statment 6**) Let the fundamental representation of G be denoted by ρ_f. Then the restriction to $\hat{H}$ of the following types of antisymmetric representations (ASR) of ρ_f (or $(\rho_f{}^{-1})^T$) include ρ_{ij} (or $(\rho_{\bar{i}\bar{j}}^{-1})^T$),**)

$$(n_1+n_2+\cdots+n_{i-1}+1)\text{-th ASR of } \rho_f|_{\hat{H}}\supset\rho_{ij}\,,$$

$$(n_1+n_2+\cdots+n_i-1)\text{-th ASR of } \rho_f|_{\hat{H}}\supset(\rho_{\bar{i}\bar{j}}^{-1})^T\,,$$

$$(n_l+n_{l+1}+\cdots+n_a-1)\text{-th ASR of } (\rho_f{}^{-1})^T|_{\hat{H}}\supset\rho_{kl}\,,$$

$$(n_{l+1}+n_{l+2}+\cdots+n_a+1)\text{-th ASR of } (\rho_f{}^{-1})^T|_{\hat{H}}\supset(\rho_{\bar{k}\bar{l}}^{-1})^T\,. \qquad (5\cdot 12)$$

The statement provides linear basis under G (see Eq. (2·32)) for any matter field $N(i,\ j)$ or $\tilde{N}(k,\ l)$. Extension to more general cases including matter fields in higher representations is now easy.

Now an explicit example would be helpful to the readers. Take the pure realization of the case of $G/H=U(l+m+n)/U(l)\otimes U(m)\otimes U(n)$. One of the expressions of $\hat{\mathfrak{h}}$ is given by Eq. (3·23) and the corresponding goldstone field is expressed as Eq. (3·27). Thus various types of ρ_{ij}'s are

$$\rho_{11}=[h_l], \qquad \rho_{22}=[h_m], \qquad \rho_{33}=[h_n],$$

$$\rho_{12}=\begin{bmatrix} h_l & t_{ln} \\ 0 & h_n \end{bmatrix}, \qquad \rho_{23}=\begin{bmatrix} h_m & t_{mn} \\ 0 & h_n \end{bmatrix},$$

$$\rho_{13}=\begin{bmatrix} h_l & t_{lm} & t_{ln} \\ 0 & h_m & t_{mn} \\ 0 & 0 & h_n \end{bmatrix}. \qquad (5\cdot 13)$$

If we denote the corresponding matter fields as $N(i,\ j)$ and $\tilde{N}(k,\ l)$ (for $(\rho_{\bar{k}\bar{l}}^{-1})^T$), then we get various types of mass terms as follows:

$$\tilde{N}(1,\ 1)^T\cdot N(1,\ 1),\quad \tilde{N}(1,\ 2)^T\cdot N(1,\ 1),\quad \tilde{N}(1,\ 3)^T\cdot N(1,\ 1), \qquad (5\cdot 14\text{a})$$

$$\tilde{N}(2,\ 2)^T\cdot N(1,\ 2),\quad \tilde{N}(2,\ 3)^T\cdot N(1,\ 2),\quad \tilde{N}(2,\ 2)^T\cdot N(2,\ 2),\quad \tilde{N}(2,\ 3)^T\cdot N(2,\ 2), \qquad (5\cdot 14\text{b})$$

$$\tilde{N}(3,\ 3)^T\cdot N(2,\ 3),\quad \tilde{N}(3,\ 3)^T\cdot N(3,\ 3), \qquad (5\cdot 14\text{c})$$

$$\tilde{N}(1,\ 2)^T\cdot N(1,\ 2),\quad \tilde{N}(1,\ 3)^T\cdot N(1,\ 2), \qquad (5\cdot 14\text{d})$$

*) Our previous argument[9] does not give a general method to obtain mass terms of matter fields. One of the examples which was proposed by Yanagida has led us to the general discussion presented here.

**) In general the transformation property under $U(1)$ of $\hat{H}$ of the representations ρ_{ij} or $(\rho_{\bar{i}\bar{j}}^{-1})^T$ in Eq. (5·12) is different from those of Eq. (5·5). $U(1)$ charges of matter fields introduced here are supposed to be assigned so as to be consistent with Eq. (5·12).

$$\tilde{N}(2,3)^T\cdot N(2,3),\quad \tilde{N}(3,3)^T\cdot N(2,3), \tag{5·14e}$$

$$\tilde{N}(1,3)^T\cdot N(1,3), \tag{5·14f}$$

each of which has the mass term of the type $N_{l*}^T N_l$ (5·14a), $N_{m*}^T N_m$ (5·14b), $N_{n*}^T N_n$ (5·14c), $N_{l*}^T N_l + N_{m*}^T N_m$ (5·14d), $N_{m*}^T N_m + N_{n*}^T N_n$ (5·14e), or $N_{l*}^T N_l + N_{m*}^T N_m + N_{n*}^T N_n$ (5·14f).

Now we consider a chiral superfield linearly transforming under G,

$$\xi N_f \equiv \begin{bmatrix} 1 & 0 & 0 \\ \pi_{ml} & 1 & 0 \\ \pi_{nl} & \pi_{nm} & 1 \end{bmatrix} \begin{bmatrix} n_l \\ N_m \\ N_n \end{bmatrix} \tag{5·15}$$

with N_f which transforms under G as

$$N_f' = \hat{h}(\pi, g) N_f\,. \tag{5·16}$$

Then linearly transforming fields in G,

$$\phi(1,1)=\xi N(1,1)=\begin{bmatrix} N_l^{(1)} \\ \pi_{ml}N_l^{(1)} \\ \pi_{nl}N_l^{(1)} \end{bmatrix},\quad \phi(1,2)=\xi N(1,2)=\begin{bmatrix} N_l^{(2)} \\ N_m^{(2)}+\pi_{ml}N_l^{(2)} \\ N_n^{(2)}+\pi_{nm}N_m^{(2)}+\pi_{nl}N_l^{(2)} \end{bmatrix},$$

$$(3,3)=\xi N(3,3)=\begin{bmatrix} n_l^{(3)} \\ \pi_{ml}N_l^{(3)}+N_m^{(3)} \\ \pi_{nl}N_l^{(3)}+\pi_{nm}N_m^{(3)}+N^{(3)} \end{bmatrix} \tag{5·17}$$ *)

are introduced in a simple way ($\eta_{13}=1$). Similarly we have

$$(\xi^{-1})^T\tilde{N}_f \equiv \begin{bmatrix} 1 & -\pi_{ml}^T & -\pi_{nl}^T-\pi_{nm}^T\pi_{ml}^T \\ 0 & 1 & -\pi_{nm}^T \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \tilde{N}_{l^*} \\ \tilde{N}_{m^*} \\ \tilde{N}_{n^*} \end{bmatrix}, \tag{5·18}$$

and linearly transforming chiral fields,

$$\tilde{\phi}(1,3)=(\xi^{-1})^T\begin{bmatrix} \tilde{N}_{l^*}^{(1)} \\ \tilde{N}_{m^*}^{(1)} \\ \tilde{N}_{n^*}^{(1)} \end{bmatrix} = \begin{bmatrix} \tilde{N}_{l^*}^{(1)}-\pi_{ml}^T N_{m^*}^{(1)}-(\pi_{nl}^T+\pi_{nm}^T\pi_{ml}^T)\tilde{N}_{n^*}^{(1)} \\ \tilde{N}_{m^*}^{(1)}-\pi_{nm}^T\tilde{N}_{m^*}^{(1)} \\ \tilde{N}_{n^*}^{(1)} \end{bmatrix},$$

$$\tilde{\phi}(2,3)=(\xi^{-1})^T\tilde{N}(2,3)=\begin{bmatrix} -\pi_{ml}^T\tilde{N}_{m^*}^{(2)}-(\pi_{nl}^T+\pi_{nm}^T\pi_{ml}^T)\tilde{N}_{n^*}^{(2)} \\ \tilde{N}_{m^*}-\pi_{nm}^T\tilde{N}_{n^*}^{(2)} \\ \tilde{N}_{n^*}^{(2)} \end{bmatrix},$$

$$\tilde{\phi}(3,3)=(\xi^{-1})^T\tilde{N}(3,3)=\begin{bmatrix} -(\pi_{nl}^T+\pi_{nm}^T\pi_{ml}^T)\tilde{N}_{n^*}^{(2)} \\ -\pi_{nm}^T\tilde{N}_{n^*}^{(2)} \\ \tilde{N}_{n^*}^{(2)} \end{bmatrix}. \tag{5·19}$$

As for the linear basis relevent to the matter fields such as $N(2,2)$, $N(2,3)$, $N(3,3)$, $\tilde{N}(1,2)$, $\tilde{N}(2,2)$ or $\tilde{N}(1,1)$, we have to choose an appropriate higher representation

*) The superfices ($\cdot$) are added to matter fields N only to distinguish independent fields.

according to statement 6.

Appendix A

—— *Outline of the Proof of $\hat{H}$-Structure Theorem* ——

In the following, all the standard properties of Lie algebra are assumed to be known to the readers.

Discussion 0 First $\mathfrak{g}$ is assumed to be a compact semi-simple algebra. As to $U(1)$ factors we will deal with them in the last part of this appendix.

Discussion 1 Levi's theorem tells how to decompose $\hat{\mathfrak{h}}$

$$\hat{\mathfrak{h}}=\mathfrak{h}_0\oplus\mathfrak{r}\,, \tag{A·1}$$

where $\mathfrak{h}_0$ is a semi-simple algebra and $\mathfrak{r}$ is a radical. Since $\mathfrak{r}$ is solvable, Engel's theorem enables us to decompose $\mathfrak{r}$ further as

$$\mathfrak{r}=\mathfrak{r}^{(0)}\oplus\mathfrak{r}^{(1)}\,, \qquad \mathfrak{r}^{(1)}=[\mathfrak{r},\ \mathfrak{r}], \tag{A·2}$$

where $\mathfrak{r}^{(1)}$ is nilpotent.

Definition 2 With any X, $Y\in\mathfrak{g}$,

$$ad(X)Y\equiv[X,\ Y]. \tag{A·3}$$

Statement 3 Suppose that there exists such n as leads to $[ad(X)]^n=0$ and that an irreducible representation $(\rho,\ V)$ of $\mathfrak{g}$ is traceless, then all the eigenvalues of $\rho(X)$ vanish.
proof) Let V_λ be a subspace of V and the eigenvalue of $\rho(X)$ be λ in V_λ,

$$(\rho(X)-\lambda)^m V_\lambda=0\,, \qquad V_\lambda\subset V\,, \tag{A·4}$$

then

$$(\rho(X)-\lambda)^{n+m}\rho(Y)V_\lambda=\sum_i {}_{n+m}C_i\rho(ad(X)^iY)(\rho(X)-\lambda)^{n+m-i}V_\lambda=0\,. \tag{A·5}$$

Thus

$$\rho(Y)V_\lambda\subset V_\lambda\,. \tag{A·6}$$

Since ρ is an irreducible traceless representation,

$$V_\lambda=V \quad \text{and} \quad \lambda=0\,. \tag{A·7}$$

Discussion 4 With any $X\in\mathfrak{r}^{(0)}$ we can decompose $\mathfrak{g}$ as follows:

$$\mathfrak{g}=\sum_\alpha\mathfrak{g}_\alpha\,, \qquad (ad(X)-\alpha)^{n_\alpha}g_\alpha=0\,. \tag{A·8}$$

If one defines D in such a way that

$$DX_\alpha=\alpha X_\alpha\ (X_\alpha\in g_\alpha), \tag{A·9}$$

then it is well known that D is an inner derivation and there exists $X^0\in\mathfrak{g}$ with which D is expressed as

$$D=ad(X^0). \tag{A·10}$$

Hence we can decompose X as

$$X = X^0 + X^+ \quad \text{and} \quad [X^0, X^+] = 0\,. \tag{A·11}$$

Statement 5 If there exists $X \in \mathfrak{g}$ with which $\rho(X)\phi_0 = 0$, then

$$\rho(X^0)\phi_0 = 0 \quad \text{and} \quad \rho(X^+)\phi_0 = 0\,. \tag{A·12}$$

Proof) It is enough to consider for each irreducible component of ρ. Any finite dimensional representation of a semi-simple algebra known to be traceless and

$$[\rho(X^0), \rho(X^+)] = 0\,. \tag{A·13}$$

Let $\rho(X^+)$ be represented by Jordan's standard form, then this statement is self-evident by noticing Statement 3.

Discussion 6 One can choose a Cartan subalgebra $\mathfrak{h}_{\rm sub}$ so that any element X^0 is given by a linear combination of $H_i \in \mathfrak{h}_{\rm sub}$.

$$X^0 = \sum_i c_i H_i \text{ and eigenvalues of } \rho(H_i) \text{ are real.} \tag{A·14}$$

By taking $\rho(H_i)$-diagonal basis and recalling (A·12), we are led to the result:

$$\rho(\bar{X}^0)\phi_0 = 0\,, \qquad \text{where} \quad \bar{X}^0 = c_i^* H_i\,, \tag{A·15}$$

so $\bar{X}^0$ should be also included in $\hat{\mathfrak{h}}$,

$$\bar{X}^0 \in \hat{\mathfrak{h}}\,. \tag{A·16}$$

Discussion 7 Let us consider ω of the following form,

$$\omega = \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{2} \end{bmatrix}. \tag{A·17}$$

Then the closure of $\{e^{i\theta\omega};\ \theta = \text{real}\}$ is equal to $U(1) \otimes U(1)$ (Kroncker's discussion). Hence if we write

$$X^0 + \bar{X}^0 = \sum_i a_i \Lambda_i\,, \tag{A·18}$$

then each Λ_i corresponds to the generator of *a* $U(1)$ group and

$$\Lambda_i \in \mathfrak{r}\,. \tag{A·19}$$

Statement 8 All Λ_i commute with both the semi-simple part of $\hat{\mathfrak{h}}$ and $\mathfrak{r}^{(0)}$,

$$[\mathfrak{h}_0, \Lambda_i] = [\mathfrak{r}^{(0)}, \Lambda_i] = 0\,. \tag{A·20}$$

Proof)

(i) All the representations of Λ_i are of one dimension. Since

$$[\Lambda_i, \mathfrak{r}] \subset \mathfrak{r} \quad \text{and} \quad [\Lambda_i, \mathfrak{r}^{(1)}] \subset \mathfrak{r}^{(1)}, \tag{A·21}$$

we can choose such $\mathfrak{r}^{(0)}$ that satisfies

$$[\Lambda_i, \mathfrak{r}^{(0)}] \subset \mathfrak{r}^{(0)}\,. \tag{A·22}$$

If $[\Lambda_i, \mathfrak{r}^{(0)}] \neq 0$, there exists some $X \in \mathfrak{r}^{(0)}$ which satisfies

$$[\Lambda_i, X] = \lambda_i X\,. \tag{A·23}$$

Since

$$[\Lambda_i, X] \subset \mathfrak{r}^{(1)} \quad \text{and} \quad X \in \mathfrak{r}^{(0)}, \tag{A·24}$$

then

$$\lambda_i = 0 \quad \text{and} \quad [\Lambda_i, \mathfrak{r}^{(0)}] = 0\,.$$

(ii) If there exists such an element E that

$$[\Lambda_i, E] = \lambda_i E \quad \text{with} \quad \lambda_i \neq 0 \tag{A·25}$$

and

$$E = H + \omega \qquad (H \in \mathfrak{h}_0,\ \omega \in \mathfrak{r}), \tag{A·26}$$

then $\Lambda \in \mathfrak{r}$ and $[\mathfrak{h}_0, \mathfrak{r}] \subset \mathfrak{r}$ imply

$$[\Lambda_i, E] \in \mathfrak{r}\,. \tag{A·27}$$

Thus $H = 0$ and

$$[\mathfrak{h}_0, \Lambda_i] = 0\,. \tag{A·28}$$

Discussion 9 By repeating the above procedure, and noticing the nilpotency of $ad(X)$ where $X \in [\mathfrak{r}, \mathfrak{r}]$ (Engel's theorem), $\mathfrak{r}^{(0)}$ is decomposed into $U(1)$ part and $\mathfrak{r}^+$ (see Eq. (A·11)), where $\mathfrak{r}^+$, together with $\mathfrak{r}^{(1)}$, is nilpotent. Thus we obtain the following statement.

Statement 10 $\hat{\mathfrak{h}}$ can be decomposed as

$$\hat{\mathfrak{h}} = \mathfrak{h}_1 \oplus \mathfrak{r}_{NP}\,, \tag{A·29}$$

where

$$\mathfrak{h}_1 = \mathfrak{h}_0 \oplus \sum \mathfrak{u}_i{}^c(1), \tag{A·30}$$

and all the eigenvalues of $ad(X)$ $(X \in \mathfrak{r}_{NP})$ vanish.

Discussion 11 Since any two Cartan subalgebras, $\mathfrak{h}_{\mathrm{sub}}$ and $\mathfrak{h}'_{\mathrm{sub}}$, are connected by automorphism g,

$$\mathfrak{h}'_{\mathrm{sub}} = g\,\mathfrak{h}_{\mathrm{sub}}\,g^{-1} \qquad (g \in G^c), \tag{A·31}$$

there exists $g \in \mathfrak{g}^c$ that

$$g\,\mathfrak{h}_{1R}\,g^{-1} \subset \mathfrak{g}\,, \tag{A·32}$$

where $\mathfrak{h}_{1R}$ is a compact real form of $\mathfrak{h}_1$.

Discussion 12 Up to now $\mathfrak{g}$ has been assumed to be a semi-simple algebra. We can make the same discussion when $\mathfrak{g}$ includes $U(1)$ factors; if any $U(1)$ factor is to be included in $\hat{\mathfrak{h}}$, it cannot belong to nilpotent part but to $\mathfrak{h}_1$ and the discussion is not altered. Thus we get the same results for $\hat{\mathfrak{h}}$ structure.

Appendix B

—— *General Expressions of Matrices of* $\hat{\mathfrak{h}}$ ——

Definition 1 Consider spaces, V, W, W' and W'' and an algebra $\hat{\mathfrak{h}}$ which are subject to the following conditions:

$$W \subset V, \qquad \hat{\mathfrak{h}} W \subset W$$

and, if $W = W' + W''$, $W' \cap W'' = \phi$ with $\mathfrak{h} W'^{('')} \subset W'^{('')}$, then $w'^{('')} = 0$ or W. Then we call the $\hat{\mathfrak{h}}$-invariant subspace W"irreducible in terms of direct sum" (IDS).

B-I. *The case* $\mathfrak{g} = \mathfrak{su}(N)$

Discussion 2 Let $\hat{\mathfrak{h}}$ be a subalgebra of $\mathfrak{g}$ and $V(=c^N)$ be the fundamental representation space.
Decompose V into *IDS* $\hat{\mathfrak{h}}$-invariant subspaces V_α,

$$V = \sum_{\alpha=1}^{t} V_\alpha . \tag{B·1}$$

Discussion 3 Take the non-zero $\hat{\mathfrak{h}}$-invariant subspace $W_{\alpha 1}$ the dimension of which is the minimal among those subspaces of V_α. Then a space $V_\alpha / W_{\alpha 1}$ is $\hat{\mathfrak{h}}$-invariant.

Discussion 4 From the $\hat{\mathfrak{h}}$-invariant space $V_\alpha / W_{\alpha 1}$ we have another invariant subspace $V_\alpha / W_{\alpha 2}$ according to the same procedure as of Discussion 3. Repeating the above procedure, we finally obtain the following sequence:

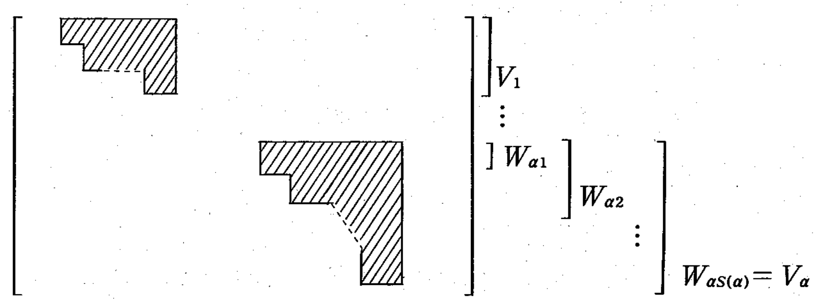
$$W_{\alpha 1} \subset W_{\alpha 2} \subset W_{\alpha 3} \cdots \subset W_{\alpha s(\alpha)} = V_\alpha , \tag{B·2}$$

where each of $W_{\alpha a}$ is a $\hat{\mathfrak{h}}$-invariant subspace

$$\hat{\mathfrak{h}} W_{\alpha a} \subset W_{\alpha a} \ (a = 1, \cdots, s(\alpha)). \tag{B·3}$$

Discussion 5 By introducing successively appropriate bases of $W_{\alpha a}$ in conformity with those of $W_{\alpha a-1}$, $\hat{\mathfrak{h}}$ is represented by

$\hat{\mathfrak{h}}$;



. (B·4)

B-II. The case $\mathfrak{g} = \mathfrak{su}(N)$ or $\mathfrak{sp}(N)$

Deffnition 6 Let $\hat{\mathfrak{h}}$ be a subalgebra of $\mathfrak{g}$ and $V(=c^N)$ be the corresponding fundamental representation space. Taking a $\hat{\mathfrak{h}}$-invariant subspace W of V, we define $N(W)$ as

$$N(W) = \{v;\ (v, \Gamma^\forall e) = 0,\ e \in W,\ v \in V\}, \tag{B·5}$$

where Γ is the corresponding metric (see Eq. (4·2)),

$$\Gamma = \begin{cases} J & \text{for } G = O(N), \\ \Omega & \text{for } G = Sp(2N). \end{cases} \tag{B·6}$$

Statement 7

$$\hat{\mathfrak{h}} N(W) \subset N(W) \text{ and } \dim N(W) = \dim V - \dim W .$$

Proof) The latter equation is self-evident by definition. The proof of the former equation is as follows: Let us take $\hat{h}:\in\hat{\mathfrak{h}}$, $n\in N(W)$ and $e\in W$. Since $\hat{h}e\in W$ and $\hat{h}^T\Gamma+\Gamma\hat{h}=0$, then we have

$$(\hat{h}n,\ \Gamma e)=(n,\ \hat{h}^T\Gamma e)=-(n,\ \Gamma\hat{h}e)=0\,, \tag{B·7}$$

which indicates that $\hat{\mathfrak{h}}N(W)\subset N(W)$ (see Eq. (B·5)).

Discussion 8 Let $\tilde{V}$ be IDS $\hat{\mathfrak{h}}$-invariant subspace of V and suppose that the $\hat{h}$-invariant non-trivial $(\neq\phi)$ subspace $\tilde{V}'$ of $\tilde{V}$ is of the minimal dimension among those $\hat{h}$-invariant subspaces of $\tilde{V}$. Then

$$\hat{\mathfrak{h}}(N(\tilde{V}')\cap\tilde{V}')\subset(N(\tilde{V}')\cap\tilde{V}') \tag{B·8}$$

implies

$$N(\tilde{V}')\cap\tilde{V}'=\phi \quad\text{or}\quad N(\tilde{V}')\supset\tilde{V}\,. \tag{B·9}$$

Which of the above two cases is realized? In the former case, we have, from the dimensional analysis,

$$\tilde{V}=\tilde{V}'+N(\tilde{V}) \tag{B·10}$$

with each of $\tilde{V}'$ and $N(\tilde{V}')$ being $\hat{\mathfrak{h}}$-invariant. Thus (B·10) contradicts the assumption that $\hat{V}$ is IDS $\hat{\mathfrak{h}}$-invariant subspace. Hence we are led to the latter case, $N(V')\supset\tilde{V}'$, which implies that $\tilde{V}'$ is a zero-norm space.

Definition 9 Let W be the maximal $\hat{\mathfrak{h}}$-invariant zero-norm subspace of V,

$$\hat{\mathfrak{h}}W\subset W \quad\text{and}\quad (e,\ \Gamma e')=0 \qquad \text{for } e,\ e'\in W\subset V\,. \tag{B·11}$$

Define a subspace E of V as

$$V=E+N(W) \quad\text{and}\quad E\cap N(W)=0\,. \tag{B·12}$$

Discussion 10 From Statement 7,

$$\dim E=\dim W\,. \tag{B·13}$$

One can introduce the basis $\{e_i\}$ of W and $\{f_i{}^0\}$ of E in such a way that

$$(e_i,\ \Gamma f_j{}^0)=\delta_{ij}\,, \qquad 1\le i,\quad j\le\dim E \tag{B·14}$$

with the use of the properties of $N(W)$ (Definition 6). Here we introduce the bases of the rest of the space $N(W)$,

$$\{n_\alpha{}^0,\ \alpha=1,\ 2,\cdots,\ \dim N(W)-\dim W\}, \tag{B·15}$$

which, combined with the basis $(e_i\}$ of W, form the whole bases of $N(w)$. The metric tensor constructed by the elements defined by the inner products of those bases, is given as

$$\begin{bmatrix} (e\Gamma e) & (e\Gamma n^0) & (e\Gamma f^0) \\ (n^0\Gamma e) & (n^0\Gamma n^0) & (n^0\Gamma f^0) \\ (f^0\Gamma e) & (f^0\Gamma n^0) & (f^0\Gamma f^0) \end{bmatrix} = \begin{bmatrix} 0 & 0 & \delta_{ij} \\ 0 & \Gamma'_{\alpha\beta} & A_{\alpha j} \\ \varepsilon\delta_{ij} & \varepsilon A_{i\beta} & B_{ij} \end{bmatrix}, \tag{B·16}$$

where $1\le i$, $j\le\dim W$, $1\le\alpha$, $\beta\le\dim[N(W)/W]$ and $\varepsilon=1(-1)$ for $\mathfrak{so}(N)(\mathfrak{sp}(N))$.

Now let us redefine other bases $\{n_\alpha\}$ and $\{f_i\}$ so that the submatrices A and B may vanish. This can be done, for instance, by choosing them as

$$\begin{cases} f_i = f_i^{\,0} - \dfrac{1}{2} B_{ij} e_j \\ n_\alpha = n_\alpha^{\,0} - A_{\alpha j} e_j \,. \end{cases} \tag{B·17}$$

Observing that

$$\hat{h}^T \Gamma + \Gamma \hat{h} = 0 \qquad \text{for } \hat{h} \in \hat{\mathfrak{h}}\,, \tag{B·18}$$

we finally get to Statement 12.

Statement 12 If one takes an appropriate set of the bases $\{e\}$, $\{n\}$ and $\{f\}$, one can express a general expression in the matrix form of an algebra $\hat{\mathfrak{h}}$ with the relevant metric Γ as

$$\hat{\mathfrak{h}}: \quad \begin{array}{cc} \begin{bmatrix} W & X & S \\ 0 & Z & -\Gamma_0 X^T \\ 0 & 0 & -W^T \end{bmatrix} & \begin{array}{c} e \\ n \\ f \end{array} \end{array} \tag{B·19}$$

and

$$\Gamma: \quad \begin{array}{cc} \begin{array}{ccc} e & n & f \end{array} & \\ \begin{bmatrix} & & 1 \\ & \Gamma_0 & \\ \varepsilon 1 & & \end{bmatrix} & \begin{array}{c} e \\ n \\ f \end{array} \end{array} \tag{P·20}$$

$$S^T + \varepsilon S = 0, \quad Z^T \Gamma_0 + \Gamma_0 Z = 0\,, \quad \Gamma_0^{\,T} = \varepsilon \Gamma_0 \quad \text{and} \quad \Gamma_0^{\,2} = \varepsilon\,, \tag{B·21}$$

where

$$\varepsilon = \begin{cases} 1 & \text{for } \mathfrak{g} = \mathfrak{so}(N), \\ -1 & \text{for } \mathfrak{g} = \mathfrak{sp}(N). \end{cases} \tag{B·22}$$

Statement 13 The submatrix Z which appears in Eq. (B·19) in Statement 12 is completely reducible.

Proof)

If Z is not completely reducible, there exist some zero-norm vectors in the manifold $N(W)/W$ (from Discussion 8). This contradicts the maximality of W.

References

1) S. Coleman, J. Wess and B. Zumino, Phys. Rev. **177** (1969), 2239.
2) B. Zumino, Phys. Lett. **87B** (1979), 203.
3) P. Fayet and S. Ferrara, Phys. Rep. **32** (1977), 249.
4a) B. A. Ovrut and J. Wess, Phys. Rev. **D 25** (1982), 409.
b) C. Lee and H. S. Sharatchandra, Munich Preprint MPI-PAE/PTh 54/83.
c) W. Lerche, ibid 59/83, Nucl. Phys. **B** (to be published).
d) T. Kugo, I. Ojima and T. Yanagida, Phys. Lett. **135B** (1984), 402.
e) W. Buchmüller, R. D. Peccei and T. Yanagida, Nucl. Phys. **B227** (1983), 503.
5) M. Bando, T. Kuramoto, T. Maskawa and S. Uehara, Phys. Lett. **138B** (1984), 94.
6) C. Hochshild, *The Structure of Lie Groups* (Holden-Day, Inc. 1965).
J. E. Humphreys, *Introduction to Lie Algebra and Representation Theory* (Springer-Verlag, GTM series).
7) M. Bando, T. Kuramoto, T. Maskawa and S. Uehara, Preprint RIFP-560.
8) J. A. Cronin, Phys. Rev. **161** (1967), 1483.
9) M. Bando. T. Kuramoto, T. Maskawa and S. Uehara, Preprint RIFP-548.