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H.L. BERK, B.N. BREIZMAN, J. FITZPATRICK,[a] M.S. PEKKER, H.V. WONG, and K.L. WONG[b]
Institute for Fusion Studies
The University of Texas at Austin
Austin, Texas 78712 USA

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[a] Permanent Address: Univ. of California, Berkeley, CA 94720
[b] Princeton Univ., Princeton, NJ 08543

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H.L. Berk, B.N. Breizman, J. Fitzpatrick,[a] M.S. Pekker, H.V. Wong, and K.L. Wong[b]

Institute for Fusion Studies, The University of Texas at Austin
Austin, Texas 78712 USA

Abstract

A method is presented for predicting the saturation levels and particle transport in weakly unstable systems where there are a discrete number of modes. Conditions are established for either steady sate or pulsating responses when several modes are excited for cases where there is and there is not resonance overlap. The conditions for achieving different levels of saturation are discussed. Depending on details, the saturation level can be quite low, where only a small fraction of the available free energy is released to waves, or the saturation level can be quite high, with almost a complete conversion of free energy to wave energy coupled with rapid transport.

[a]Permanent Address: University of California, Berkeley, Ca 94720
[b]Princeton University, Princeton, NJ 08543

I. Introduction

The standard process for sustained ignition in a controlled fusion system[1] is the production and containment of charged fusion products (e.g. 3.5 MeV alpha particles in the D-T reaction; 15 MeV proton products in the D-He^3 reaction) with the transfer of their energy to the background plasma through classical processes, such as electron drag. In this manner ignition is sustained and a steady-state distribution function of the energetic particle component is formed. A primary concern in this process is whether the free energy of instability associated with such an energetic particle distribution, causes the spontaneous build-up of waves which can lead to the deposition of the energy of the fusion charged products to the wall rather than to the plasma.[2] Clearly this process is of fundamental concern in the fusion program and it is important that the nonlinear processes that can lead to anomalous diffusion be understood. The purpose of this presentation is to discuss a basic physics approach that is being developed to systematically analyze this problem both qualitatively and quantitatively.

In dealing in nonlinear theories, it is frequently assumed that the nonlinear processes are so rapid, that the longer term evolution due to classical transport processes can be ignored. Further, a typical turbulence scenario assumes that there is continuous spectrum of waves,[3,4] so that nonlinear theories for weak turbulence, such as quasi-linear theory, can be used. In the problem we are dealing with, the nonlinear description of Toroidal Alfvén Eigenmodes in a tokamak,[5–8] we expect a discrete spectrum of weakly unstable waves (where the growth γ is much less than the real frequency, ω) and the competition between classical and nonlinear processes to persist in the long term evolution of the distribution function and even in the establishment of the saturation level of the waves. This competition needs to be included in the nonlinear analysis.[9] Another mechanism that profoundly affects the nonlinear evolution of the system and needs to be included in the analysis is damping from the background plasma. All these mechanisms determine whether the excitation spectrum appears at a steady level[9] or as pulsations,[10] and whether the spectrum is a continuum that can give rise to global diffusion[11,12] or a discrete set of modes[12] which does not have a large effect on transport. Specific criteria for these responses have been obtained and is discussed below.

A primary physics process in weak turbulence is the particle-wave resonance interaction. The paradigm problem for this process is the interaction of a freely flowing particle with an electrostatic wave. If at the resonance position (for the electrostatic problem the resonance condition is $\Omega \equiv \omega - \mathbf{k}\cdot\mathbf{v} = 0$ with $\mathbf{k}$ the wavenumber, ω, the wave frequency and $\mathbf{v}$ the particle velocity), $\partial f/\partial E < 0$ (f is the particle distribution function, and E the particle energy) there is Landau damping,[13] while if $\partial f/\partial E > 0$, there can be wave growth.[3,4] This resonant particle-wave interaction mechansim persists in every wave problem with a kinetic distribution function in long mean free path systems. In the problem we are most interested in, low frequency waves (ω much less than the cyclotron frequency) in a torus, the resonance condition for the particle-wave interaction is[14]

$$\Omega_\ell(E,\mu,P_\phi) = \omega - n\varpi_\phi(E,\mu,P_\phi) - \ell\varpi_\theta(E,\mu,P_\phi) = 0$$

where ϕ and θ refer to toroidal and poloidal angles of the torus, n is the toroidal quantum number, and ℓ is a set of integers, and E, μ, P_ϕ are particle's energy, magnetic moment and angular momentum. The particle-wave resonance produces a drive for possible instability if

$$\frac{\partial F}{\partial E} - \frac{n}{\omega}\frac{\partial F}{\partial P_\phi} = \left.\frac{\partial F}{\partial E}\right|_{P_\phi - nE/\omega} > 0.$$

($\big|_{P_\phi - nE/\omega}$ denotes at constant $P_\phi - \frac{n}{\omega}E$). This condition, which can always be satisfied for any distribution that has a dependence on P_ϕ for arbitrary n and ω, is what gives rise to the "universal instability" drive.[15,16] For $\omega < nE/P_\phi$, the particles primarily move spatially in response to the wave, and if such motion becomes stochastic, spatial diffusion arises (this is the basic cause of anomalous plasma transport). An important point we emphasize, is that the mathematical structure of the particle-wave interaction for the bump-on-tail problem and for particle-wave interaction with low frequency toroidal waves, have a great deal in common. We will see that many aspects of the toroidal problem can be understood by analyzing the simpler 1-dimensional electrostatic bump-on-tail problem. The main extension that is needed, the treatment of many resonance interactions simultaneously, is relatively straightforward.

An important issue for the response of a system is whether the instability spectrum produces mode overlap.[17] Without mode overlap, there is no global diffusion of particles. The saturation level of modes is then quite low, and the "free energy"[18] that produces $\partial F/\partial E > 0$, is hardly tapped. However, if overlap suddenly arises, a release of free energy to wave energy arises, which causes rapid diffusion of particles, in a relatively short time interval.[11,12]

The structure of the paper is as follows: In Sec. II we discuss the nonlinear theory with sources and sinks for the bump-on-tail problem when there is no mode overlap. In this section we present a discussion of the effect of frequency sweeping,[19,20,21] a process, first applied to extract energy from a free electron laser,[19] that enhances the saturation level of single modes.[9,22] It is a process that can be controlled externally and applied to channelling of the fusion product energy[23] and even to the direct conversion of energy from charged fusion products.[22]

In Sec. III we discuss how the effect of mode-overlap can be described. We summarize our effort to build a quasi-linear transport code that models the correct behavior in both the mode overlap and non-overlap regimes. In Sec. IV we show how the formalism for the bump-on-tail problems applies to the toroidal problem, both for single-modes and for the model quasi-linear equations. In Sec. V we discuss the relevance of our theory to experiment.

II. Saturation of a Single Mode

The first topic to be discussed is that of the saturation of a single mode in a weakly unstable system, when there is no mode overlap. In this section we consider the one-dimensional bump-on-tail instability, and in Sec. IV we show how the analysis generalizes to more complicated systems.

For the nonlinear properties, the most important aspect is the nonlinear properties of the particles that are nearly resonant with the wave. For these particles, $\partial F/\partial E > 0$, where E is the kinetic energy of the particles. Thus the distribution has free energy that can spontaneously convert to wave energy. The linear growth, γ_L, in the absence of dissipation, is proportional to $\partial F/\partial E$. The effects of dissipation will first be neglected and then later incorporated into this discussion.

As a wave grows, the distribution will mix in the trapping region, $|v - \omega/k| \approx \omega_b/k$ with $\omega_b = (\frac{eEk}{m})^{1/2}$ the trapping frequency. Roughly, this mixing causes the trapped particle distribution to flatten inside the separatrix, and leave the distribution nearly unaffected outside the separatrix. The flattening depletes the drive within the separatrix, but hardly affects the particles outside the separatrix. The frequency of the wave is determined from the background plasma characteristics and not from the properties of the resonant particles. Thus, at saturation, the resonant phase velocity remains fixed in a region where the distribution is flat, so that the wave can no longer grow.

Note that to the extent the distribution is truly flat inside the separatrix, with a value, $F = F(\frac{1}{2}m\omega^2/k^2)$, and the distribution responds adiabatically outside the separatrix, the saturation level can be computed from energy conservation arguments. One finds, by equating the sum of the kinetic and potential energy in the final state, to the kinetic energy of the initial state, that $\omega_B = 2.88\gamma_L$. This scaling has now been established in several numerical simulation experiments.[24–30] In the bump-on-tail instability the result is $\omega_B = 3.2\gamma_L$.[26,30]

These results indicate that the relaxation ansatz gives the correct scaling for the saturation. The relaxation of the distribution is only somewhat more complicated than the assumed ansatz as a small portion of the passing region mixes as well. By empirically adjusting the width of the flattened region, as will be described in Sec. III, where the quasi-linear model is discussed, the correct saturation level can be obtained.

We call the saturation arising from wave trapping, the natural saturation mechanism. This terminology is used because this mechanism is the dominant nonlinearity for sufficiently weak $\gamma_L \ll \omega$, and isolated instabilities, where $\gamma_L \ll \Delta\omega$, with $\Delta\omega$ the frequency difference between different resonance frequencies. This follows because the effect of trapping is proportional to $|E|^{1/2}$, rather than in powers of $|E|^2$ which arises in other nonlinear mode coupling problems.[31,32,33]

With dissipation present, a wave damps at a rate γ_d, in the absence of an instability source. For instability to arise, one needs $\gamma_L > \gamma_d$ with the growth rate given by $\gamma = \gamma_L - \gamma_d$. We find that the natural saturation mechanism due to trapping, $\frac{\omega_b}{\gamma} \approx$ const, still applies, as seen from the result of particle simulation experiments[29] shown in Fig. 1 where the peak value of ω_b versus $1 - \gamma_d/\gamma_L$ is plotted. In these cases a mode appears as a pulse rather than reaching a steady level. The peak saturation level occurs when $\omega_b \sim 3\gamma$, whereupon the instability is quenched as the result of the formation of a plateau at the resonance velocity. Then, the dissipation mechanism, which is assumed to be a nonresonant process that is not likely to be affected by nonlinearity, will cause the wave to damp at the dissipation rate γ_d.

In practice instability can be established by having a weak source that produces the energetic particles which then relax through collisional processes to the unstable distribution we are considering. When instability arises and a plateau in the resonant region is formed, the source will cause the original distribution function to reconstitute at a rate $\nu_{\text{eff}} = \nu_d(\frac{\overline{\Omega}}{\omega_b}) + \nu_s(\frac{\overline{\Omega}^2}{\omega_b^2}) + \nu_a$, where ν_d is the drag rate that causes the resonant particles to reduce their speed by a factor of $1/e$, and ν_s is the global velocity diffusion rate (the diffusion is due to a superposition of collisional and heating processes that cause both pitch angle scattering and energy diffusion) and ν_a is a particle annihilation rate, and $\overline{\Omega}$ is the resonance spread of a fully relaxed energetic particle distribution function (for the bump-on-tail problem for plasma waves, $\overline{\Omega} = \overline{\omega - kv} \approx \omega$; in other problems $\overline{\Omega}$ can be substantially different from ω). Thus, the flux of new particles into the resonance region, together with the relaxation processes, causes the instability to either reoccur, or even persist indefinitely. If $\nu_{\text{eff}} \ll \gamma_d \ll \gamma_L$, instability is first quenched by plateau formation, but after a short time the transport processes will cause particles to relax to the unstable state. A weak precursor instability can occur when the distribution function in the resonance region is only a fraction of the prevailing slope surrounding the resonance. However, the relaxation of this precursor mode only affects a small fraction of the original resonance region. Hence, the plateau formed from the relaxation of the precursor mode, does not flatten most of the interval of the newly forming slope and the build-up of the overall slope, over most of the phase space region, continues in time. It is only when enough time has elapsed that the slope in the resonance region is a reasonable fraction ($\sim \frac{1}{2}$) of the prevailing slope surrounding the resonance region, does enough free energy accumulate locally to allow the overall interval to flatten again and produce a major pulse of instability. Thus with a weak relaxation process present, i.e. $\nu_{\text{eff}} \ll \gamma_d \ll \gamma_L$, pulsations with precursors arise, with a significant spread of repeated pulse heights. In computer simulation, it is found that the average ω_b height of significant size pulses is given by $\varpi_b \sim 1.4\gamma_L$. The repetitiveness of the pulses and the flattening of the distribution function is shown in Fig. 2.

When $\nu_{\text{eff}} > \gamma_d$, the source of new particles that enter the resonance region from collisional processes, prevent a plateau from completely flattening. This result leads to a steady level in the mode amplitude being established and this level exceeds that natural saturation level. Thus, the sources that are supplying new particles, prevent complete plateau formation within the resonance interval and the source pumps the wave to a level above the natural saturation level. An analytic calculation of the saturation level has been made when $\nu_{\text{eff}} \ll \omega_b$,[9] and the results are easiest to understand when diffusive processes dominate drag processes. Then the nonlinear growth rate is found to be given by

$$\gamma_{\text{NL}} \sim \gamma_L \frac{\nu_{\text{eff}}}{\omega_b}. \tag{1}$$

Hence, as an unstable mode grows, the energy rate reduces, until the damping rate, γ_d, is as large as the drive. When the drive and damping are equal, a steady level for the trapping frequency is then established, which is

$$\omega_b \sim \left(\frac{\gamma_L}{\gamma_d}\,\overline{\Omega}^2\,\nu_s\right)^{1/3}. \tag{2}$$

This level exceeds the natural level when $\nu_s > \gamma_d$. If $\nu_s < \gamma_d$, it turns out that the steady-state level predicted above is unstable and the system evolves in the pulsation mode described previously. One can also show that when $\nu_{\text{eff}} < \gamma_d$, the natural level of saturation cannot produce a steady oscillation as then more energy would be absorbed by dissipation than can be injected into the system by the external source. Thus, when $\nu_{\text{eff}} < \gamma_d$, we return to the previously described pulsation scenario.

The steady-state response of the system, when $\nu_{\text{eff}} > \gamma_d$, has been confirmed in computer simulations. In this case $\nu_{\text{eff}} = \nu_a$, the particle annihilation rate. The results are shown in Fig. 3.

The situation is more complicated when effects of drag dominate, which arise if $\nu_d > \nu_s\overline{\Omega}/\gamma_L$. It turns out this case is similar to what can happen when the real frequency changes, and we defer this discussion to later in this section.

Special analysis, which has recently been completed, is needed for the case near marginal stability, where $\gamma \equiv \gamma_L - \gamma_d \ll \gamma_L$. Here when a steady source of instability is present, and $1 - \gamma_d/\gamma_L$ small, one would expect from the bifurcation theory of dissipative systems, that a steady solution should exist whose magnitude is proportional to $(1-\gamma_d/\gamma_L)^p$ with $p > 0$. The saturation level in this case can be calculated from a perturbation theory that expands about the unperturbed solution with transport processes present. Such a perturbation procedure is applicable if a particle's lifetime in resonance is short compared to a trapping time. One then finds that the system's response can be described by an equation with a temporally nonlocal integral equation containing a cubic nonlinearity. In Ref. 30 an analysis of this equation is performed.

When $\frac{\nu_{\rm eff}}{\gamma} > 4.38$, the integral equation admits a steady level of oscillation that is below the natural level. The trapping frequency is found to be given by,

$$\omega_B = 8^{1/4}\,\nu_{\rm eff}\left(1 - \frac{\gamma_L}{\gamma_d}\right)^{1/4}. \tag{3}$$

However, this level is found to be unstable if $\nu_{\rm eff}/\gamma < 4.4$. For $4.4 < \frac{\nu_{\rm eff}}{\gamma} < 2.4$, pulsation levels at about the steady state level is observed with more complicated structure associated with smaller values of $\nu_{\rm eff}/\gamma$. However, when $\frac{\nu_{\rm eff}}{\gamma} \leq 2.4$, a truly major change in the response arises. The reduced dynamics, described by the nonlinear integral equation is found to blow up in a finite time for perturbations that begin at a low amplitude (even for $\frac{\nu_{\rm eff}}{\gamma} > 2.4$, it found that the perturbation blows up in a finite time if the initial perturbation is high enough). Examples of solutions to the integral equation are shown in Fig. 4(a)-(e).

In reality when there is rapid blow-up in the solution to the nonlinear integral equation, it means that the true solution of the system will increase rapidly until saturation due to particle trapping occurs. Then the oscillation level will damp due to depletion of the drive, and over the longer term, pulsations again arise with the maximum saturation level determined by $\omega_b \simeq 3(\gamma_L - \gamma_d)$.

The discussion up to now assumed that the frequency does not change during resonance. Hence the depletion of the drive is only in the resonance region. As a result only a very small fraction of the available free energy is converted to wave energy. When $\nu_{\rm eff} \ll \gamma_d \ll \gamma_L$, it can be shown that the wave energy, W_E, released per pulse in terms of the free energy,

W_F (typically the free energy, W_F, is comparable to the total kinetic energy of the energetic particle distribution), is given by

$$W_E \sim \left(\frac{\gamma_L}{\Omega}\right)^3 W_F \tag{4}$$

When $\nu_{\text{eff}} > \gamma_d$, we have noted that $\omega_b \sim \gamma_L \nu_{\text{eff}}/\gamma_d$, which allows an increased wave energy at saturation. As $\nu_{\text{eff}} = \nu_s \Omega^2/\omega_b^2$, the level of wave energy is given by

$$W_E \sim \frac{\gamma_L}{\Omega}\frac{\nu_s}{\gamma_d} W_F, \tag{5}$$

which is still a fairly low level.

Another way to enhance the wave energy conversion of a single mode is to have the frequency change.[22] Our simulation results show that in systems with weak instability, where the background plasma parameters are constant in time, the frequency does not change during the saturation of the mode. However, mechanisms that cause the frequency to shift can well exist. For example, the parameters that define the plasma can be changing in time because the system has not reached a steady-state or because other instabilities cause transients in the background plasma parameter. The frequency shift allows the position of the particles in resonance to change and this leads to the possibility that additional free energy can be tapped.

The rate of energy conversion of free energy is determined by two processes as the frequency is swept; a non-adiabatic process that gives energy conversion at the linear rate or less and an adiabatic process, that can give energy conversion rates that are even faster than linear energy conversion rate. First we will discuss the nonadiabatic process.

In the nonadiabatic process, mechanisms exist that allows particles to enter the resonant region, and then leave it. If the particles remain in resonance in a time less than the bounce time, the fields do not have a chance to deplete the drive. As a result, the energy transfer rate from the free energy reservoir to the wave remains $2\gamma_L$. If the particles remain in resonance for a time, T, is longer than the trapping time, the energy transfer rate is reduced to $\gamma_L/\omega_b T$. The input of particles can be due to collisions, in which case $T^{-1} \simeq \nu_{\rm eff} \sim \nu_s \overline{\Omega}^2/\omega_b^2$. It can also be due to the sweeping of the frequency, and then new particles enter the resonance region at the rate $\nu\omega/\omega_b$ where $\nu = \frac{1}{\omega}\frac{d\omega}{dt}$. It is interesting to note that particle drag has the same effect as frequency sweeping as the drag allows new particles enter the resonance region. Thus, with drag processes we can use the same relations as we can use with frequency sweeping, where we alter the definition of ν to $\nu = v^{-1}dv/dt$. The integration of the wave energy transfer by this mechanism allows for a level of $\omega_b \sim (\gamma_L\overline{\Omega})^{1/2}$, or equivalently $W_E \sim (\frac{\gamma_L}{\overline{\Omega}})^{3/2} W_F$. This level is appreciably greater than the natural saturation mechanism.

In addition to this nonadiabatic process, it can be shown that in the absence of diffusion, that particles trapped in the wave will adiabatically follow the resonance as either the frequency is swept or as the particle slows down due to drag, as long as $\nu_s(\frac{\overline{\Omega}}{\omega_b^2})^2 < \nu$. The result is that phase space discontinuities will build up, which if they get strong enough, will dominate the energy transfer from the free energy reservoir to the wave. If collisions or damping are not important, adiabatic frequency sweeping (or drag) produce $\omega_B \sim (\overline{\Omega}^2\gamma_L)^{1/3}$ or equivalently, a wave energy level is given by

$$W_E \sim \frac{\gamma_L}{\overline{\Omega}} W_F. \tag{6}$$

This level exceeds the level that can be reached by the nonadiabatic frequency mechanism. This sweeping mechanism has been simulated numerically by using particle drag. In Fig. 5 we observe that an enhanced depression on the distribution function forms. In Fig. 6 we see the energy transferred to the waves follows the predictions of analytic theory.

In order to achieve the relatively high level of wave energy conversion, it is necessary to form a large enough trapping well so that collisions do not cause the loss of trapped particles. If diffusion processes limit the size of the phase space discontinuity that can be achieved, the energy transfer rate due to adiabatic sweeping is reduced.[22] For $\nu_s(\frac{\overline{\Omega}}{\omega_b})^2 > \nu$, the level of ω_b

that can be achieved is $\omega_b \simeq \nu\gamma_L/\nu_s$ and the wave energy level that can be reached is given by

$$W_E = \left(\frac{\gamma_L \nu}{\bar{\Omega}\nu_s}\right)^3 W_F. \tag{7}$$

This level still exceeds the nonadiabatic level if $\nu/\nu_s > (\bar{\Omega}/\gamma_L)^{1/2}$.

The mechanism for adiabatic energy transfer may have an important application to energy channelling[23] ideas that have recently been proposed. Because the energy transfer rate can be made faster than linear theory would predict, it is possible to tap the free energy of a linearly stable system, i.e. where $\gamma_d > \gamma_L$. Such extraction is not possible through the usual linear mechanisms that have been previously proposed. In addition, the problem of particle scattering due to collisions, can be mitigated by proper time shaping of the frequency sweep as discussed in Ref. 30. Another area where frequency sweeping coupled with energy enhancement may be important is in the explanation of "chirping" phenomena observed in fishbone[31] experiments and in some Alfvén instability experiments.[32] The mechanisms for the change of frequency still needs to be clarified, but given a mechanism that changes the frequency, we now see that the wave energy release is enhanced.

III. Quasilinear Theory

The previous section discussed the nonlinear properties of a single mode. If modes do not overlap, i.e. $\omega_b < \Delta\Omega \simeq \frac{\bar{\Omega}}{N}$ with N the number of resonant modes, then these considerations also apply with several resonant regions being simultaneously active, as these regions hardly interact with each other. However, when there is mode overlap, $\omega_b \gtrsim \Delta\Omega$, then the problem has to be analyzed differently. In this limit one may expect quasi-linear theory to apply. However even here, care is required in the analysis. In order for conventional quasi-linear theory to be applicable, it is necessary for the excitation spectrum to be of large enough intensity so that mode overlap is satisfied at all times. If it is not satisfied, the usual quasi-linear equations are not always valid. In this section we will show how a model set of quasi-linear equations can be constructed, that reproduce the correct dynamics both when there is and when there is no mode overlap.

For the bump-on-tail problem, a steady-state analytic solution has been obtained in the presence of a background source and classical transport mechanisms. The solution shows that the combination of anomalous quasi-linear transport and classical transport allows the distribution to find a solution where the modes are nearly marginally stable. The scaling found for the anomalous loss rate, $\nu_{\rm anm}$, of the energetic particles due to this combined diffusion processes, is

$$\nu_{\rm anm} \sim \frac{\gamma_L}{\gamma_d}\nu_0 \tag{8}$$

where ν_0 is the overall loss rate due to transport from classical processes. This is a significant enhancement of loss when $\frac{\gamma_d}{\gamma_L} \ll 1$. However, in order for the turbulence level to be large enough to justify the use of steady-state quasi-linear theory in the presence of N roughly equally spaced modes, one requires

$$N \gtrsim \left(\frac{\gamma_L}{\gamma_d}\frac{\bar{\Omega}}{\nu_0}\right)^{1/3}. \tag{9}$$

This criteria can well be violated in a system with a discrete number of modes.

If Eq. (9) is violated, it means that when there is overlap, pulsations occur, with the conventional quasi-linear theory being able to describe the interval where there are large oscillations, while a new modified theory is needed to describe the system's evolution when there is no overlap.

Let us consider the evolution of a system that has just formed a flattened plateau distribution function. Initially there is only a small drive and the background damping keeps the wave stable as the sources and classical transport processes allow the distribution function to build-up the free energy associated with a destabilizing slope. At some point, γ_L, the linear drive becomes greater than the damping rate, γ_d, so that $\gamma = \gamma_L - \gamma_d > 0$, and the system goes into low level steady or benign pulsations. The distribution function can flatten locally in the resonance region, but the overall slope of the distribution function continues to grow until $\gamma = \gamma_L - \gamma_d > \omega/N$, with N the number of unstable modes, which for now are assumed to be roughly equally spaced. At this stage a violent event happens, that can be characterized as a phase space explosion. Because, mode overlap effects the distribution function over the entire phase space region where resonances arise, nearly all of the free wave energy of the distribution function can be converted to wave energy on a time scale of $\sim \gamma_L^{-1}$.

Recall that at the verge of overlap, the excited waves of a single mode when $\gamma_L > \nu_{\rm eff}$, has a wave energy arising from the natural saturation mechanism $W_E \sim (\gamma_L/\Omega)^3 W_F \sim W_F/N^3$. Even if all N of the modes were simultaneously excited, the total wave energy conversion would be $W_E \sim N(\gamma_L/\Omega)^3 W_F \sim W_F/N^2$. This is still a relatively small wave energy release compared to the free energy available in the distribution function. However, because of the mode overlap, the plateau can extend over the entire distribution function if the resonance positions of unstable modes extend throughout the phase space of the unstable distribution function. Then particles with higher energy transfer to lower energy over the entire phase space, with excess energy being converted to wave energy. This created wave energy is then at a level that is comparable to the free energy available in the kinetic distribution function.

Let us examine the ignition of this explosion in more detail. Suppose we have two modes that are at the verge of overlap. The plateaus that are formed are shown in Fig. 7(a), and associated with these two plateaus is a wave energy release. As a result of the overlap, the plateau that forms is at least twice as wide, as shown if Fig. 7(b), and hence the wave energy release is about four times the wave energy of two adjacent modes that do not quite overlap. This increase mode energy release, is strong enough to excite nearest neighbor modes, and the mode overlap region quickly spreads out, until the entire distribution function is in resonance with self-excited waves and the wave energy release only saturates when $W_E \approx W_F$. Note that once the explosion starts, the wave energy release quickly reaches a level that the diffusion is properly described by the standard quasi-linear theory.

We now discuss how quasi-linear equations can be altered to describe regimes when there is no mode overlap. Results of such modelling has recently been reported.[12] In the conventional quasi-linear theory the diffusion coefficient consists of a superposition of partial diffusion coefficients whose correlation time is proportional to a delta function, i.e. $D \propto \delta(\Omega)$, where in the bump-on-tail problem $\Omega = \omega - kv$. In the modified quasi-linear theory, this partial diffusion coefficient is broadened to reflect that because the wave tends to saturate in a growth time, the wave causes local stochasticity in the particle motion that mixes the distribution function within a separatrix width of resonance. Further resonance broadening of a particle is caused by a growing wave, with a growth rate $\gamma = \gamma_L - \gamma_d$, and with a classical diffusion process that causes particles in resonance to be lost in a time ν_{eff}^{-1}.

Hence the broadening occurs in a region $\Delta\Omega$, given by

$$\Delta\Omega = 4\eta\omega_b + \lambda(\gamma + \nu_{\text{eff}}). \tag{10}$$

Here θ and λ are numerical factors that are chosen to agree with analytic and numerical calculations. Thus the partial diffusion coefficient from a given resonance, is altered from the conventional quasi-linear formula as follows,

$$D_k = \frac{\pi}{2}\frac{\omega_{bk}^4}{k^2}\delta(\Omega_k) \rightarrow D_j = \frac{\pi}{2}\frac{\omega_{bj}^4 g(\frac{\omega_j - kv}{\Delta\Omega_j})}{k_j^2\Delta\Omega_j} \tag{11}$$

where $g(x)$ is a localized function, with $\int_{-\infty}^{\infty} dx g(x) = 1$. (In practice $g(x) = \theta(\frac{1}{4} - x^2)$ with θ a step function, or $g(x) = \frac{3}{2} - 6x^2$ in the interval, $-\frac{1}{2} < x < 1/2$ are used). The choices $\eta = 0.8$ and $\lambda = 5.3$, produce the best fit of single mode results arising from the more basic numerical and analytic simulation calculations.

The evolution of the wave energy (normalized to ω_{bj}^4) of each mode, is given by the relation,

$$\frac{\partial\omega_{bj}^4}{\partial t} = 2\gamma_j\omega_{bj}^4 - 2\gamma_d\omega_{bj}^4 \tag{12}$$

where

$$\gamma_j = \frac{2e^2\pi^2\omega_j}{m|k|} \int dvg\left(\frac{\omega_j - kv}{\Delta\Omega_j}\right)\frac{\partial f}{\partial v} \Big/ \Delta\Omega_j$$

where $\int$ denotes the integral over the region of resonance.

In the absence of damping from the background plasma and external source, this equation and the quasi-linear equation for the broadened distribution function, lead to the conservation of the total momentum, P, which is the sum of kinetic and wave momentum;

$$P = m\int_{-\infty}^{\infty} dvvf + \sum_j k/\omega\, W_{Ej}$$

An interesting phenomena that has been investigated with this quasi-linear model is the possibility of mode overlap being achieved through a "domino" effect. In the previous discussion of mode overlap, when $\gamma_L - \gamma_d \sim \gamma_L$, we assumed that the mode overlap arises when modes were roughly equally spaced. But we also noted that once overlap starts, an excess of free energy is released when the wave energy is fully developed where the average bounce frequency of a single mode, ω_{bj}, of a single mode is $\sim \gamma_L N^{1/2}$. This result indicates that if $N \gg 1$, it is possible to achieve mode overlap even when the mean separation between resonant velocities is of order $\gamma_L N^{1/2}$. To get a domino effect, we need at least a pair of modes that have a resonant frequency separation $\Omega_{j+1} - \Omega_j \approx \gamma_L$ between them. However, subsequent spacings can be wider, since the wave energy level increases as a plateau forms. Now, during the formation of the plateau, there needs to be a rapid change of f at the interface where the plateau has formed and the region that is unaffected by the plateau as shown schematically in Fig. 8. At this interface there is a rapid change in f of order $\Delta v_{p\ell}\frac{\partial F}{\partial v}$, where $\Delta v_{p\ell}$ is the velocity width of the plateau. Such a jump in f gives rise to a linear growth rate γ_L' for the altered distribution function given by

$$\gamma_L' \simeq (\gamma_L k \Delta v_{p\ell})^{1/2}. \qquad (13)$$

Note that γ_L' is much larger than the growth rate for the smooth distribution function. The natural saturation level for this mode is then, $\omega_b \sim \gamma_L' \sim (\gamma_L k \Delta v_{p\ell})^{1/2}$. Thus the spacing for the next resonant mode frequency has to be $\Delta v_r \lesssim (\gamma_L k \Delta v_{p\ell})^{1/2}$ in order for overlap to continue. If the resonant frequency separations satisfy this relation as $\Delta v_{p\ell}$ increases during the development of the turbulent spectrum, one finds that all the free energy can be tapped if there are as few as $\alpha(\frac{\bar{\Omega}}{\gamma_L})^{1/2}$ modes with α an order unity numerical factor.

An example of such a domino effect has been demonstrated numerically for the model quasi-linear equations we have described. The results are shown in Fig. 9. In this figure we observe that only the two innermost modes can give overlap by the natural saturation level. However, the excess wave energy that is produced allows for all the modes to overlap, and the global diffusion causes transport of the particles to an absorbing wall. The result is the near complete loss of particles because overlap is achieved.

Another type of domino effect is possible when there are many modes, but most of them are stabilized by linear damping. We have observed that if overlap occurs, that there is enhancement of the linear growth rate arising from the rapid variation of the distribution function at the interface of the plateau and undisturbed distribution function. This enhanced instability drive, $\gamma' \simeq (\gamma_L k \Delta v_{p\ell})^{1/2}$, can cause a distribution function, that is normally stable when $\gamma_d > \gamma_L$, to be unstable. Hence the phase space explosion can continue into the otherwise unstable region. This mechanism is a promising candidate in several experiments to explain rapid particle losses observed with wave energy bursts.

Previously we pointed out that wave trapping effects dominate mode coupling effects if the amplitude of the perturbation is small. However, we have shown that in quasi-theory, especially during a rapid explosion where all the free energy is tapped, relatively high perturbed fields are achieved. It is in this case that the quantitative evolution of the relaxation rate can be significantly altered by other nonlinear mode coupling and Compton scattering.[33–35] A complete theory of the nonlinear evolution of the system needs to include such effects.

Nonetheless, the gross feature, that the free energy of the distribution function rapidly converts to wave energy, should still persist even in a more detailed theory.

IV. Generalization of Method

The previous discussion for the bump-on-tail problem, can be generalized to almost any problem in plasma physics, including the interesting problem of alpha particles interacting with Alfvén waves. The most important generalization is to obtain a nonlinear description of the simultaneous resonances of particles at different regions of phase space. We describe here such a treatment. For simplicity we limit ourselves to low frequency waves, i.e. waves with frequency less than the cyclotron frequency of the energetic particle, so that the magnetic moment μ, can be treated as constant.

For a given toroidal mode, where the perturbation of all quantities to leading order is proportional to Re $\exp(-i\omega t + in\phi)$ (ϕ is the toroidal angle, and n the toroidal mode number), it follows from basic principles that $H' = H - \frac{\omega}{n} P_\phi$ is also conserved in the perturbation, where we now use H, the Hamiltonian, to represent the energy. Hence, in response to a single mode, only P_ϕ (at constant μ and H') can vary. The most important response is from those particles near resonance, defined as,

$$\Omega(P_\phi, H, \mu) \equiv \omega - n\varpi_\phi(P_\phi, H, \mu) - \ell\varpi_\theta(P_\phi, H, \mu) = 0 \tag{14}$$

with ℓ an integer and ϖ_ϕ and ϖ_θ the mean toroidal and poloidal (represented by the poloidal angle θ) drift frequencies. Let $P_{\phi,\ell}(H, \mu)$, satisfy Eq. (14). The locus of such solutions is shown schematically by the solid curve in Fig. 10.

The instantaneous power transfer from a particle to a wave, can be written in terms of the constants of motion and "action" angles, ϕ_a and θ_a (these angles are modulated toroidal and poloidal angles respectively chosen so that $\dot{\theta}_a \equiv \bar{\omega}_\theta, \dot{\phi}_a \equiv \bar{\omega}_\phi$ do not change during a particle's unperturbed motion). The form of the power transfer is,

$$e\mathbf{v}\cdot\widehat{\mathbf{E}}(\mathbf{r})\exp(-i\omega t+in\phi)=\sum_{\ell}\mathcal{P}_{n,\ell}(H,\mu,P_\phi)\exp\left[-i\omega t+in\phi_a+i\ell\theta_a\right]. \tag{15}$$

The nonlinear response of a near resonant particle to a toroidal wave can be written in terms of the amplitude $\mathcal{P}_{n,\ell}$. One can show that a particle with a P_ϕ value near $P_{\phi,\ell}(H,\mu)$ (at fixed H' and μ) satisfies the equation,[36]

$$P_\phi=P_{\phi\ell}+\left[\Delta P_{\phi 0}^2+2\omega_B^2(P_{\phi,\ell})\,\cos\chi_l\Big/\left|\frac{\partial\Omega(P_{\phi,\ell})}{\partial P_\phi}\right|^2\right]^{1/2}$$

where $\Delta P_{\phi 0}^2$ is a constant $(\Delta P_{\phi 0}\geq -2\omega_B^2(P_{\phi,\ell})/\left|\frac{\partial\Omega(P_{\phi,\ell})}{\partial P_\phi}\right|)$, $\omega_B^2(P_{\phi,\ell})=\left|\frac{n\mathcal{P}_{n,\ell}(P_{\phi,\ell})}{\omega}\frac{\partial\Omega(P_{\phi,\ell})}{\partial P_\phi}\right|$, and $\chi_\ell=\omega t-n\phi_a+\ell\theta_a+\chi_{\ell 0}$ ($\chi_{\ell 0}$ is a phase constant). The phase function χ_ℓ satisfies the pendulum equation,

$$\frac{d^2\chi_\ell}{dt^2}+\omega_B^2(P_{\phi,\ell})\sin\chi_\ell=0. \tag{16}$$

Note that $\omega_B(P_{\phi,\ell})$ is the bounce frequency of a deeply trapped particle in the pendulum and from Eq. (15), it follows that it is proportional to the square root of the perturbing amplitude. This structure for the response is identical to the nonlinear response of particles trapped in a single electrostatic wave. This form has been previously derived under less general assumptions.[37,29] It is also a direct application of well-established nonlinear resonance theory.[17]

As in the bump-on-tail problem, one can find the scaling for the saturation level by assuming that some fraction of the separatrix width is flattened all along the resonance curve. The linear theory for the mode amplitude determines how the amplitude $|\mathcal{P}_{n,\ell}|$ varies along the resonance line. The separatrix width is given by $\Delta P_{\phi 0}=2|\omega_B(P_{\phi,\ell})\Big/\frac{\partial\Omega}{\partial P_\phi}|$. As linear theory determines $|\mathcal{P}_{n,\ell}|$, the natural width is then known along the resonance curve. The broadening can then be expressed as we found in Eq. (10), with $\Delta\Omega=\Delta P_\phi\,\partial\Omega/\partial P_\phi$. The broadened region of resonance is indicated schematically within the dashed curve of Fig. 10.

As in the previous section, we can take the difference between the initial state and final state, where the distribution function is flattened over some fraction of the separatrix, and find that the natural saturation mechanism due to trapping scales as, $\gamma_L \propto \overline{\omega}_b$ where the bar denotes a suitably averaged bounce frequency over the regions of resonance.

It is worth noting that adiabaticity of trapped particles also applies, if the frequency (or some other external parameter) varies slowly. The adiabatic invariant, J is

$$J = \oint \frac{dX}{2\pi} \left\{ P_{\phi\ell} + \left[\Delta P_{\phi 0}^2 - \left| n\mathcal{P}_{n,\ell}(P_{\phi,\ell}) \middle/ \omega \right| \cos X \right]^{1/2} \right\}. \tag{17}$$

Hence, as in the bump-on-tail case, in toroidal systems a similar enhancement of the wave energy due to frequency sweeping can be achieved. This property should have important consequences to either energy channelling or direct conversion of energy of charged fusion products (especially of the 15 Mev proton products that result in a D-He^3 reaction), and needs to be pursued further.

The problem of how to bridge quasi-linear theory when there is mode overlap and the description of single non-overlapping modes, is solved in a manner similar to the one dimensional case. The distribution function $f(P_\phi, H, \mu)$, is taken to satisfy the following equation,

$$\begin{aligned} \frac{\partial f}{\partial t} &= \frac{\pi}{2} \sum_{n,\ell} \left(\frac{\omega}{n} \frac{\partial}{\partial H} + \frac{\partial}{\partial P_\phi} \right) D_{n,\ell} \left(\frac{\omega}{n} \frac{\partial}{\partial H} + \frac{\partial}{\partial P_\phi} \right) f + C(f) \\ &= \frac{\pi}{2} \sum_{n,\ell} \left. \frac{\partial}{\partial P_\phi} \right|_{H'} D_{n,\ell} \left. \frac{\partial}{\partial P_\phi} \right|_{H'} f - Cf \end{aligned} \tag{18}$$

where $C(f)$ represents the effects of classical relaxation (this term will be discussed further below). The partial diffusion coefficient, $D_{n,\ell}$, is given in the line broadened quasi-linear theory by,

$$D = \frac{|\omega_{B,\ell}^4(P_{\phi,\ell})|^2}{\Delta P_\phi \left| \frac{\partial \Omega(P_{\phi,\ell})}{\partial P_\phi} \right|^3} \, g \left(\frac{P_\phi - P_{\phi,\ell}}{\Delta P_\phi} \right)$$

with $g(x)$ the previously defined window function (see after Eq. (11)), and the width, $\Delta P_\phi \frac{\partial \Omega}{\partial P_\phi} = 4\eta\omega_b + \lambda(\gamma + \nu_{\text{eff}})$, determined by the same arguments as previously given.

For the classical collision operator, $C(f)$, we use a Krook-like operator,

$$C(f) = -\nu(f - f_0) - \nu_{\rm eff}\,\Delta P_\phi \left(f \Big/ \int d\,P_\phi f - f_0 \Big/ \int d\,P_\phi f_0 \right)$$

where f_0 is the distribution function expected if only classical relaxation processes were present, ν_0 is the rate of formation of the predicted distribution function from classical processes only, and the symbol $\int dP_\phi$ represents an integral only over the phase space window of each mode and is present to preserve particles during relaxation due to the combined classical and wave dependent processes.

One may also note that in treating this quasi-linear system for the alpha particle-Alfvén wave interaction, it is frequently justified to neglect $\frac{\omega}{n}\frac{\partial}{\partial H}$ compared to $\frac{\partial}{\partial P_\phi}$ in Eq. (18), because the instability drive only arises if ω is less than the diamagnetic drift. In this case the diffusion equation that needs to be solved is a set of one dimensional equations in P_ϕ, with μ and H as parameters. Such a code is now being implemented.

In addition to the quasi-linear diffusion equations, one needs to solve for the time evolution of the mode amplitudes. The growth rates that appear in these equations can be expressed in terms of the identical line broadenings that appear in the quasi-linear equations. The complete system of equations then has the property that in absence of external damping and external transport, that the total particle and wave toroidal canonical momentum is conserved.

V. Discussion

The overview of the nonlinear theory indicates many features that are consistent with experimental data, although detailed quantitative comparison have not as yet been made. In many of the observations of TAE modes one observes intervals of rapid oscillations that strongly correlate with particle loss.[38,39]

Evidence of this was first observed in the experiment of neutral-beam-driven TAE modes.[38]

Figure 11 shows the correlation between the TAE mode amplitude from the Mirnov coil data (bottom trace) and the neutron emission rate from the plasma (top trace). Each burst of TAE activity is accompanied by a drop in the neutron emission which indicates the loss of fast deuterium ions. A 10-channel BES (beam emission spectroscopy) array was used to measure the mode number.[40] This technique works only when there is a single dominant TAE mode in the plasma. Modes with $n = 2, 3$ and 4 have been observed under various conditions,[38,40] The largest loss rates correlate to the cases where the dominant modes have comparable amplitudes. For example, we note in Fig. 12(a), that the spectral decomposition of the Mirnov coil signal at time $t \sim 3.6965$ sec is dominated by the signal at 72 kHz, with the other modes at lower frequencies not as strongly excited. Here the decrease of the neutron signal is $\sim 5\%$. On the other hand, Fig. 12(b) shows the TAE burst at $t \sim 3.7014$ sec. Its frequency spectrum (Fig. 13(b)) reveals two modes with comparable amplitudes at 60 kHz and 72 kHz. The corresponding drop in neutron emission is almost three times larger than the event at 3.6965 sec. This observation is consistent with the hypothesis that significant fast particle loss can be due to mode overlap which causes global quasi-linear diffusion. However, it should be pointed out that the event at 3.014 sec is accompanied by other MHD activities which can also contribute to the enhanced fast ion loss. The repetitive TAE bursts apparently regulate the fast ion pressure gradient and keep it near the instability threshold.[41]

TAE modes have also been observed during hydrogen minority (ICRF (ion cyclotron radio frequency) heating where energetic protons are produced with energy $\sim 0.5 - 1$ MeV.[42] In this experiments the TAE mode amplitudes are very steady and only evolve slowly, on a time scale in which the plasma equilibrium changes. A fast particle probe placed near the plasma edge detects a continuous fast particle flux which correlates with the TAE mode amplitude.[43] Faster variation of the mode amplitude and the corresponding fast ion loss are found to correlate with sawtooth activities. In these ICRF heating experiments, the TAE mode amplitude can be roughly estimated from reflectometer measurements of the density oscillations associated with the mode. $\delta B_r/B_0$ is typically $\lesssim 10^{-4}$. The bounce frequency of the resonant particles trapped by the TAE mode is several times larger than the initial linear growth rate, consistent with estimates based on single mode saturation ($\omega_b \sim \nu_{\text{eff}}\, \gamma_L/\gamma_d$) due to wave-particle trapping. However, the analysis is complicated by usually having several TAE modes coexisting with their frequency separations comparable to the bounce frequency. The experiment observes particle loss, and there is not yet an explanation of why single mode saturation would lead to enhanced particle loss. If losses are due to a steady quasi-linear regime applying in experiments, more detailed calculations in the theory is needed to establish the saturation level that arises in experiments.

Clearly, much is to be done to corroborate whether the theory presented here can lead to quantitative description of experimental data. It is however clear that the nonlinear and transport theories presented here produce many different scenarios. The theories are based on first principle concepts of the particle-wave interaction, and they appear the basis of a promising method for establishing a way by which most experimental data can be explained and how predictions for ignition experiments can be established. The theory can now also be extended to investigate problems in energy channeling and direct conversion of energy in fusion systems.

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FIGURE CAPTIONS

FIG. 1. Plot of simulation results for $\max \omega_b$ vs. $(1 - \gamma_d/\gamma_L)$.

FIG. 2. Single mode pulsation. (a) Response of wave energy vs. time. (b) Particle distribution just at onset of instability at $\gamma_L t = 152$. Slope in resonance region is large enough to produce a major pulse. (c) Particle distribution at end of pulsation. The distribution has flattened in the resonance region, and the source has not yet rebuilt the slope.

FIG. 3. Numerical simulation of steady state oscillations. (a) Wave energy vs. time. (b) Steady-state particle distribution. Slope in resonance region is a balance between flattening due to finite amplitude wave and building-up of the slope from the source of new particles. (c) Comparison of saturation levels between theory (curve) and particle simulation (dots).

FIG. 4. Solution of nonlinear integral equation when $0 < \gamma_L - \gamma_d \ll \gamma_L$. In (a), $\hat{\nu} \equiv \nu_{\text{eff}}/(\gamma_L - \gamma_d) = 5$, and mode amplitude goes to a steady level, $\omega_b = 8^{1/4}\,\nu_{\text{eff}}(1 - \gamma_d/\gamma_L)^{1/4}$. (b) $\hat{\nu} = 4.3$ where steady state is unstable and oscillates regularly about one of the steady state levels. (c) $\hat{\nu} = 3$ and oscillations have regular amplitudes that are comparable to the steady state level. (d) $\hat{\nu} = 2.5$ where a rather chaotic response is observed. (e) $\hat{\nu} = 2.4$ where the chaotic response gives rise to blow-up in a finite time. (f) $\hat{\nu} = 2.4$ with only the late time shown, when solution approaches a self-similar solution.

FIG. 5. Simulation result from wave energy exchange from resonance sweeping. (a) $\gamma t = 6$, local plateau from mode saturation established. (b) $\gamma t = 18$, distribution function after nonadiabatic amplification of mode has increased the depth of phase-space

trapping. (c) $\gamma t = 36$, adiabatic mechanism has further increased amplitude of mode and depth of phase-space trapping.

FIG. 6. Comparison with analytically predicted rate of evolution of wave energy for simulation in Fig. 5.

FIG. 7. Schematic diagram of plateau formation with two adjacent mode. (a) no overlap. (b) overlap.

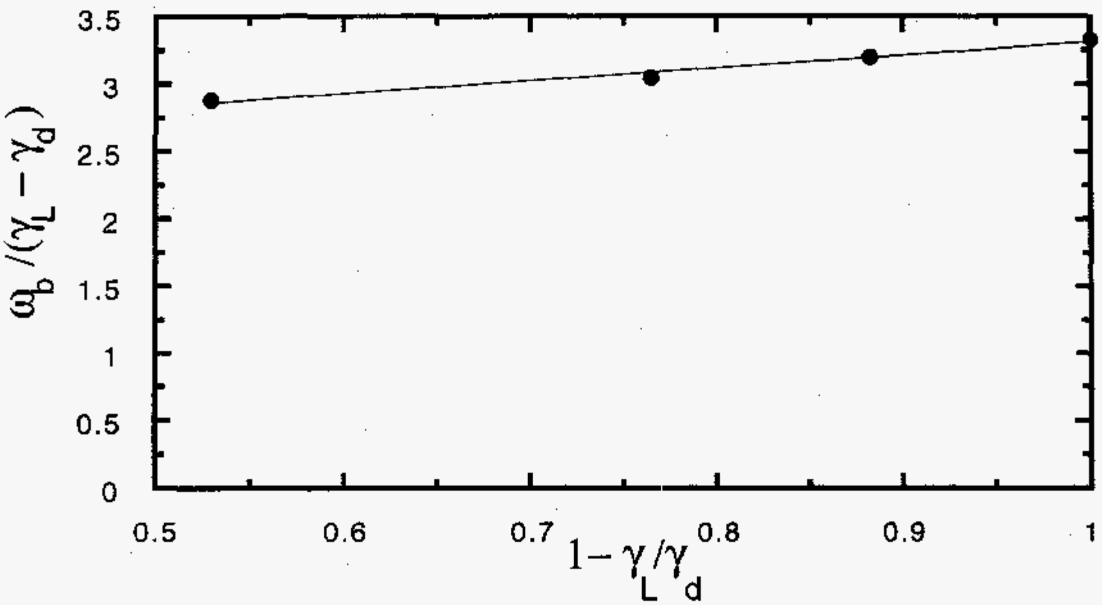
FIG. 8. Build-up of rapid change of distribution function at interface between active wave region and unperturbed region. This interface causes an enhanced growth rate and enhanced saturation level of interface mode.

FIG. 9. (a) Increased wave energy release allows widely spaced modes to overlap which causes a collapse of the distribution function shown in (b) due to the diffusion rate and loss to the boundaries.

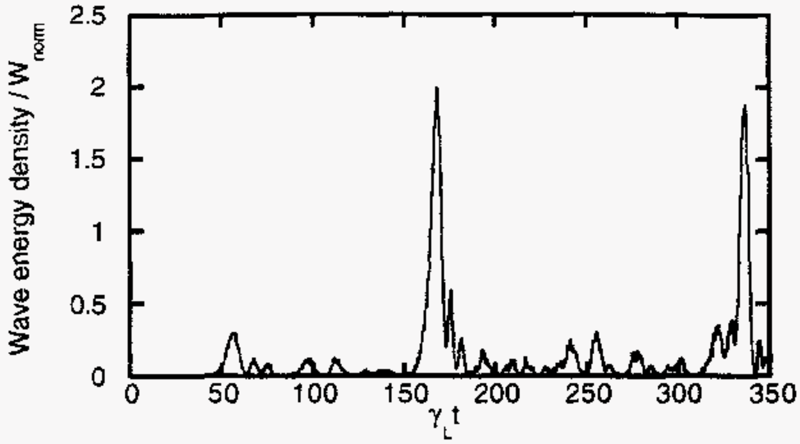
FIG. 10. Schematic diagram of the resonance broadening in phase space, is shown in (a). Equation (14) determines the center of the resonance, while the shape of the linear eigenfunction and the bounce frequency of particle in the wave determines the width of the resonance.

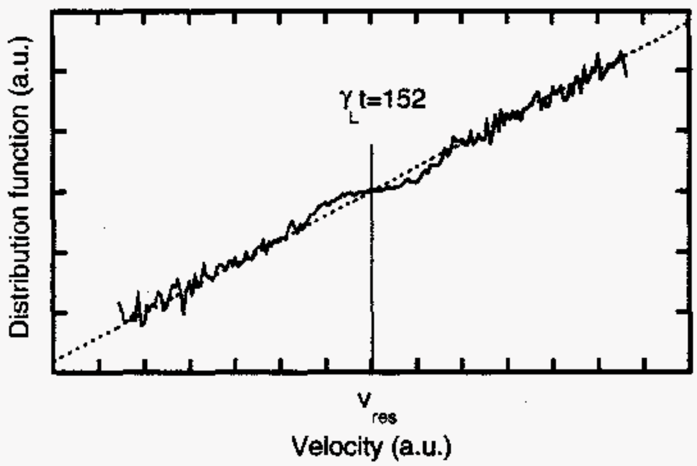
FIG. 11. Correlation of particle loss with TAE burst. The top trace shows modulations in the neutron count (proportional to the density of energetic deuterium ions), while the bottom trace shows magnetic field oscillations picked up by Mirnov coils. Data originally shown in Ref. 38.

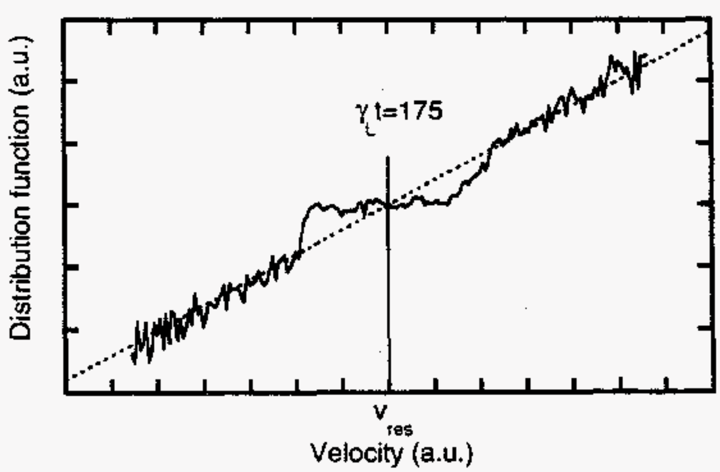
FIG. 12. Spectral analysis of Mirnov coil signals at times (a) $t \sim 3.6965\,\mathrm{s}$ and (b) $t \sim 3.7014\,\mathrm{s}$.

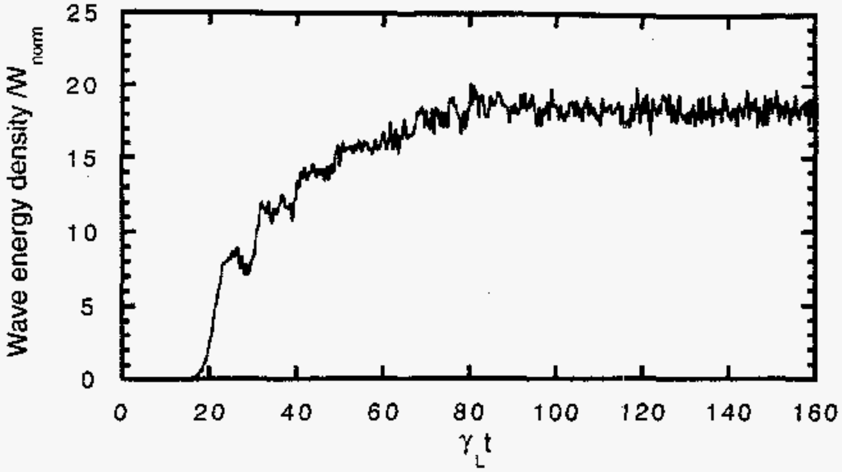


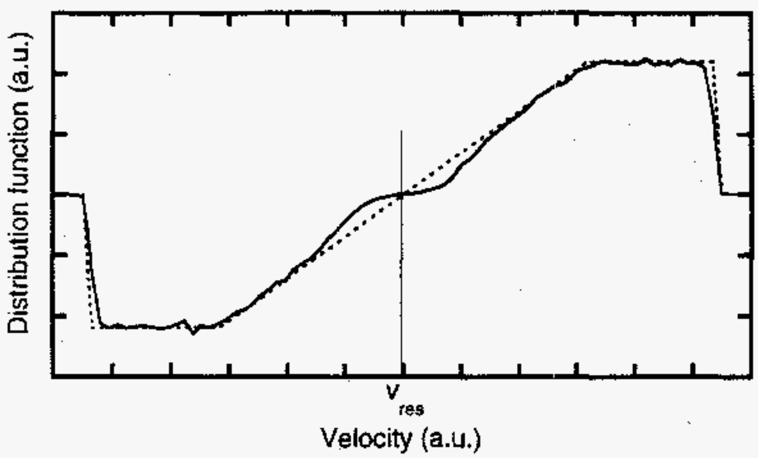
3.5
3
2.5
2
1.5
1
0.5
0
0.5
0.6
0.7
0.8
0.9
1
$\omega_b/(\gamma_L-\gamma_d)$
$1-\gamma_L/\gamma_d$

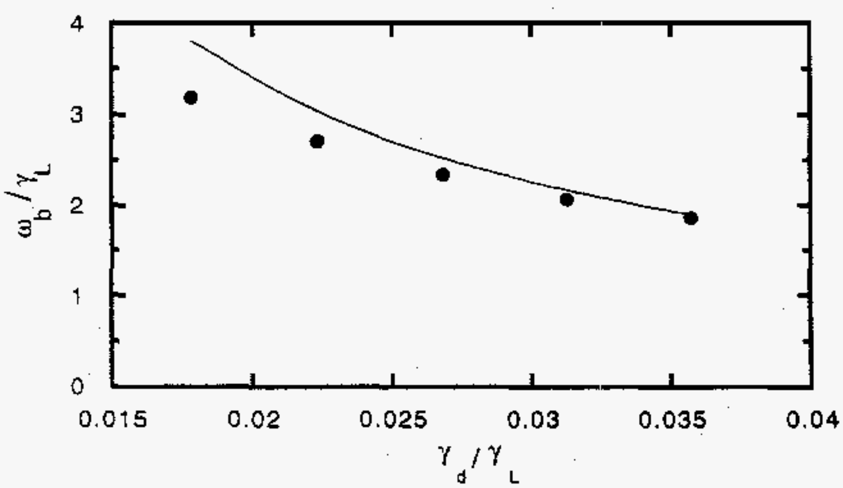

Wave energy density / W_{norm}
2.5
2
1.5
1
0.5
0
0
50
100
150
200
250
300
350
$\gamma_L t$

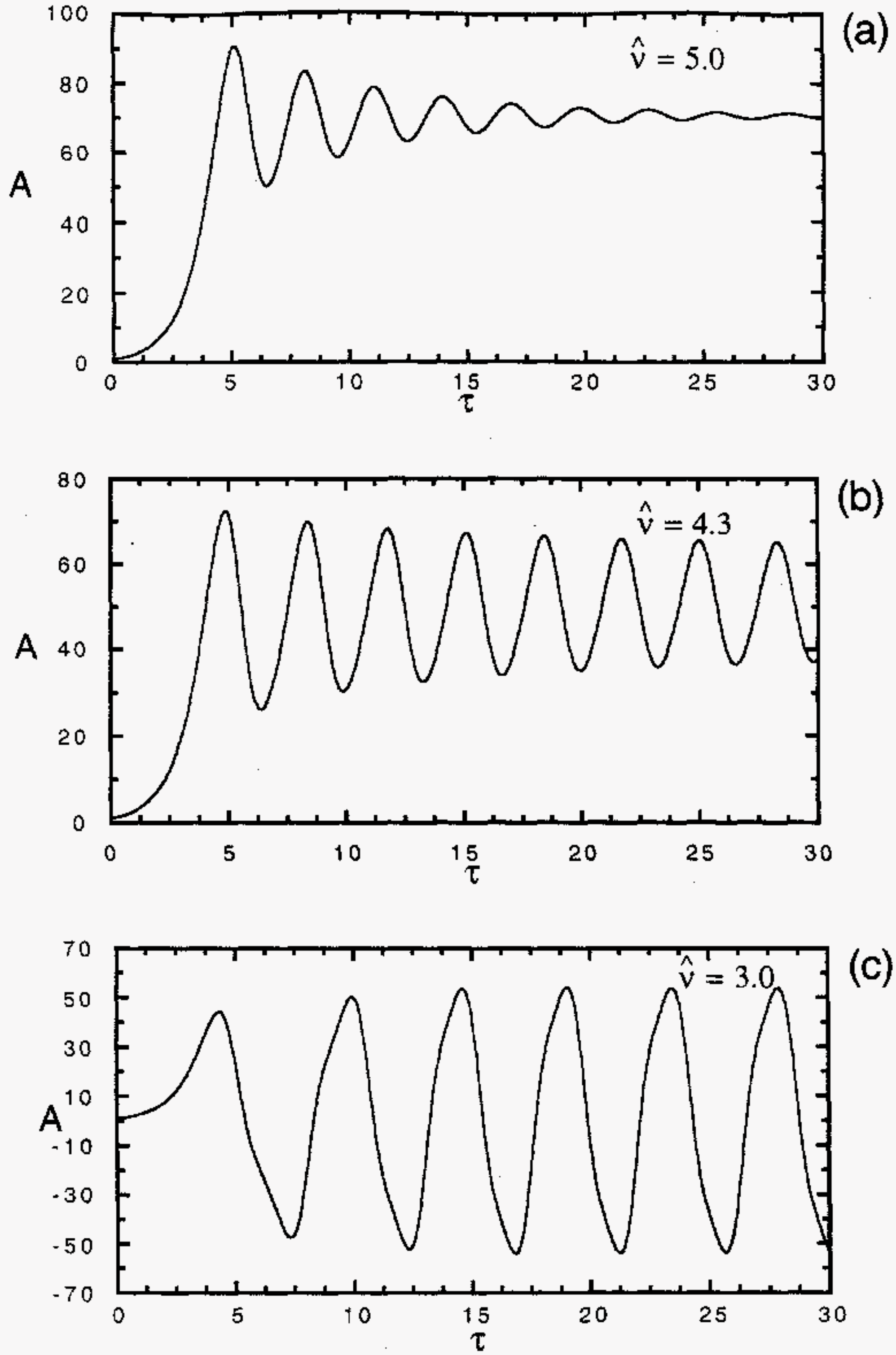

Distribution function (a.u.)
$\gamma_L t$=152
v_{res}
Velocity (a.u.)

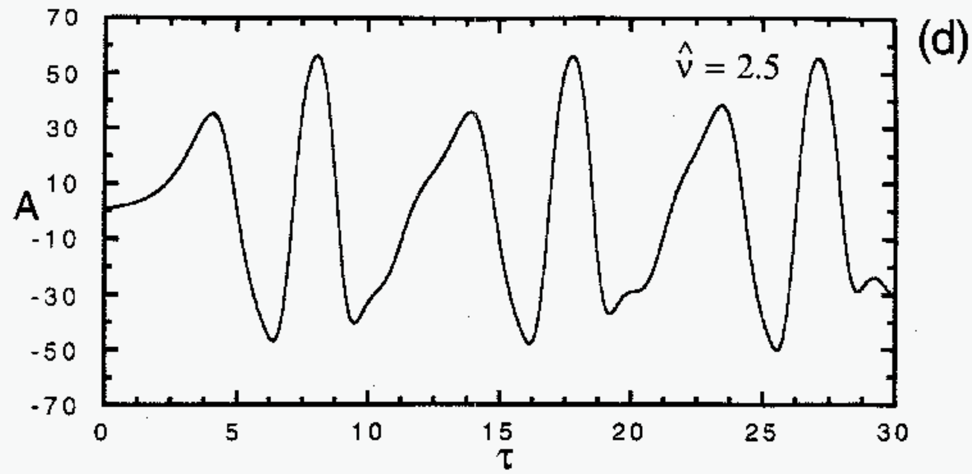

Distribution function (a.u.)
$\gamma_L t$=175
v_{res}
Velocity (a.u.)

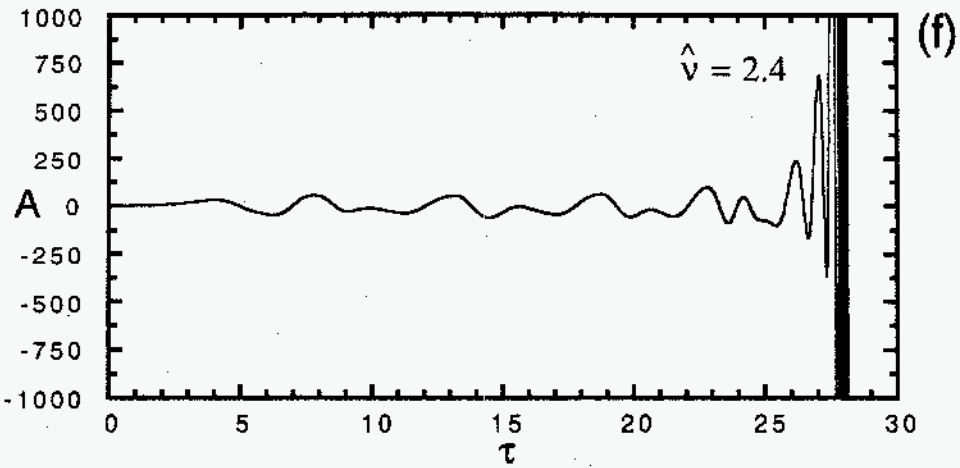

Wave energy density / W_{norm}
$\gamma_L t$
0
5
10
15
20
25
0
20
40
60
80
100
120
140
160

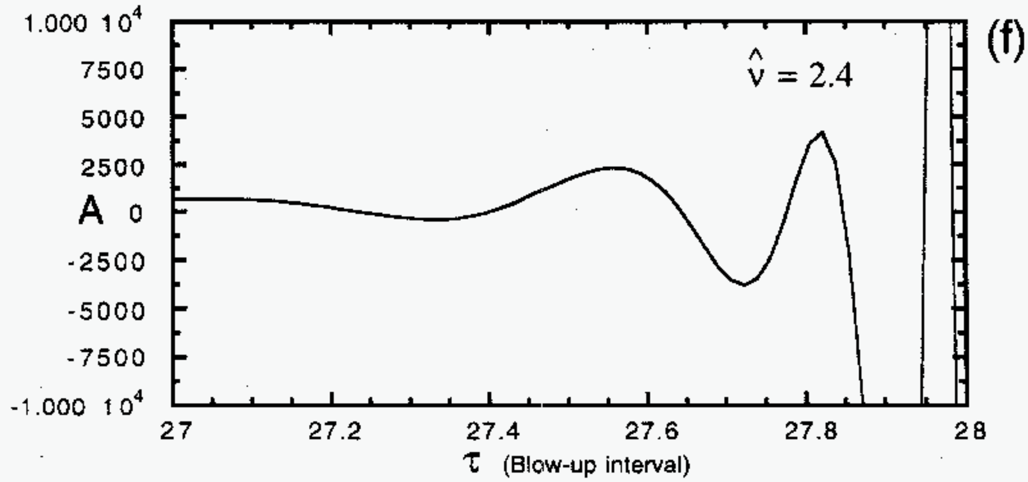

Distribution function (a.u.)
v_{res}
Velocity (a.u.)

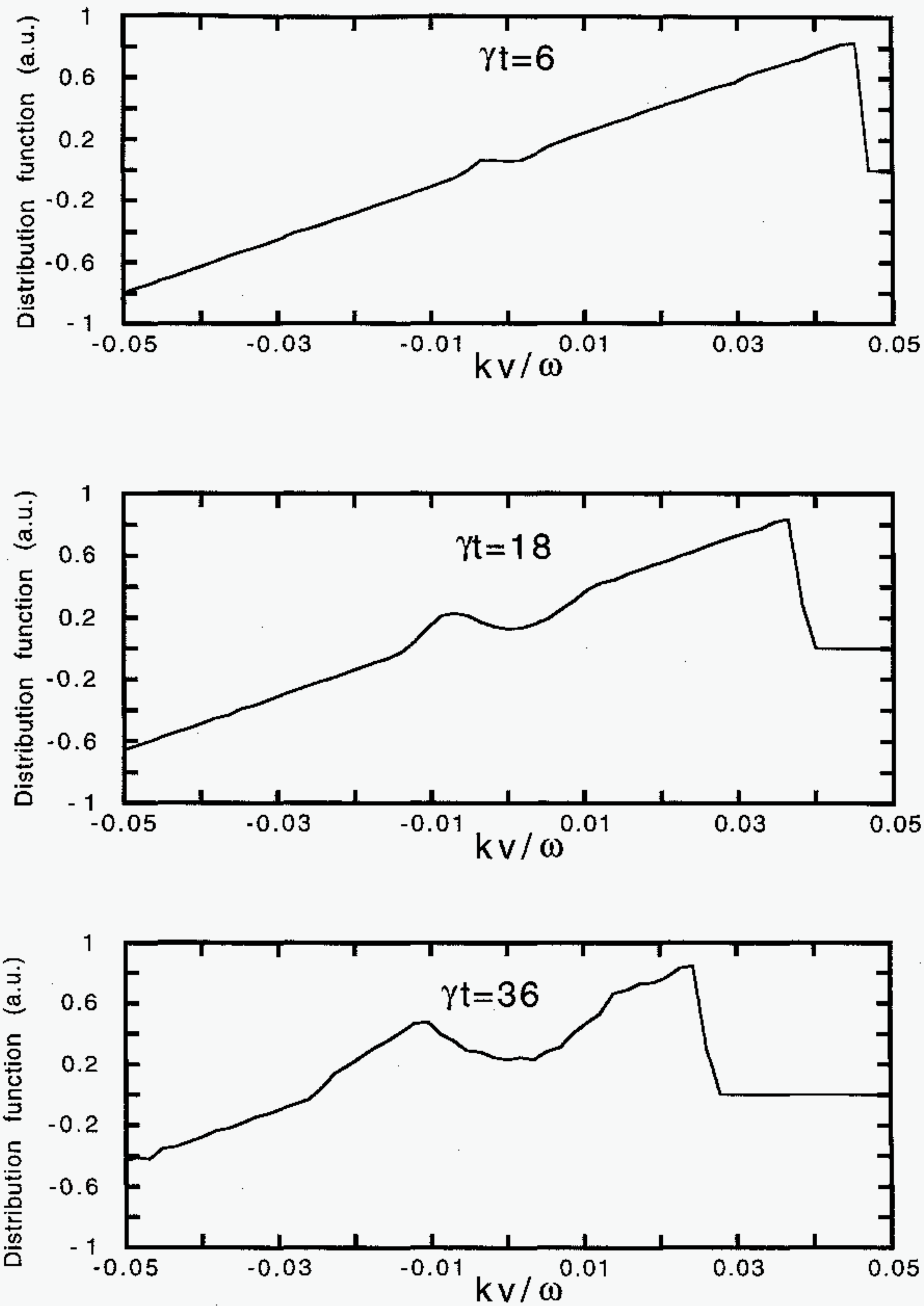

ω_b / γ_L
γ_d / γ_L
0
1
2
3
4
0.015
0.02
0.025
0.03
0.035
0.04


(a)
$\hat{\nu} = 5.0$
A
100
80
60
40
20
0
0 5 10 15 20 25 30
τ
(b)
$\hat{\nu} = 4.3$
A
80
60
40
20
0
0 5 10 15 20 25 30
τ
(c)
$\hat{\nu} = 3.0$
A
70
50
30
10
-10
-30
-50
-70
0 5 10 15 20 25 30
τ

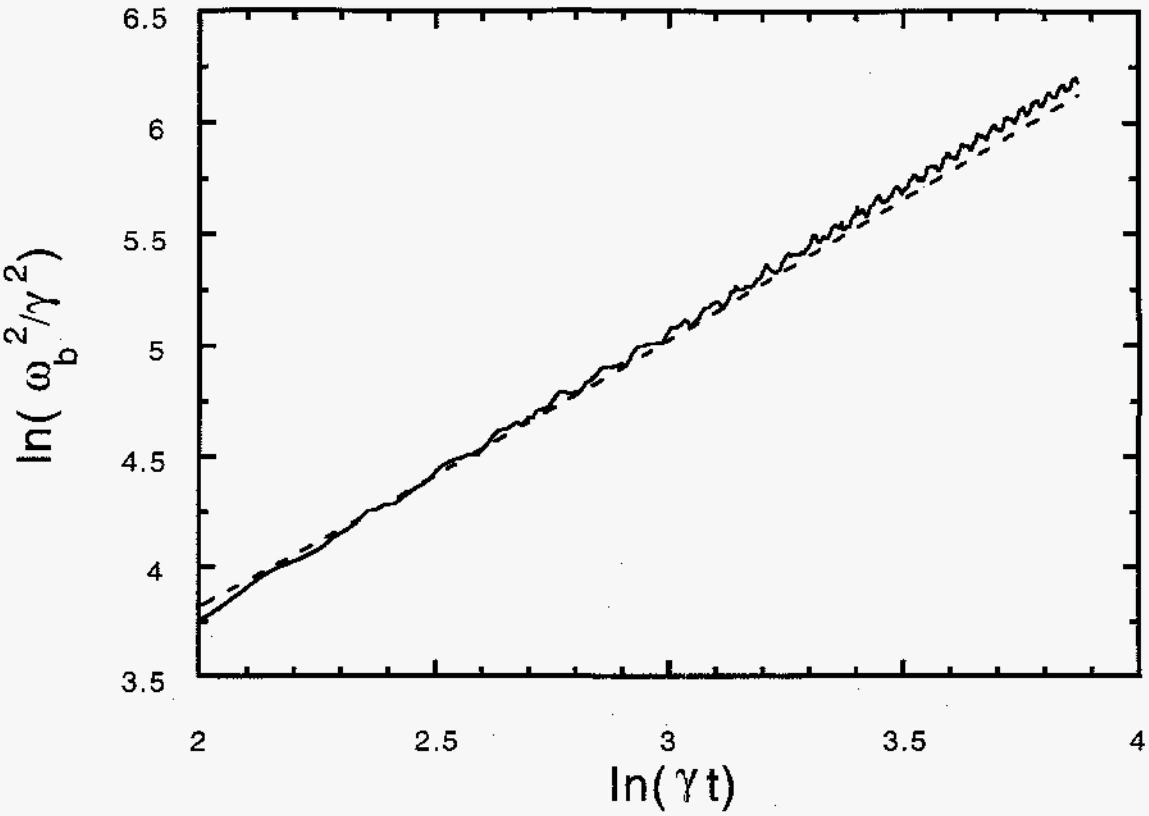

(d)
$\hat{\nu} = 2.5$
A
τ
70
50
30
10
-10
-30
-50
-70
0
5
10
15
20
25
30


(f)
$\hat{\nu} = 2.4$
A
τ
1000
750
500
250
0
-250
-500
-750
-1000
0
5
10
15
20
25
30

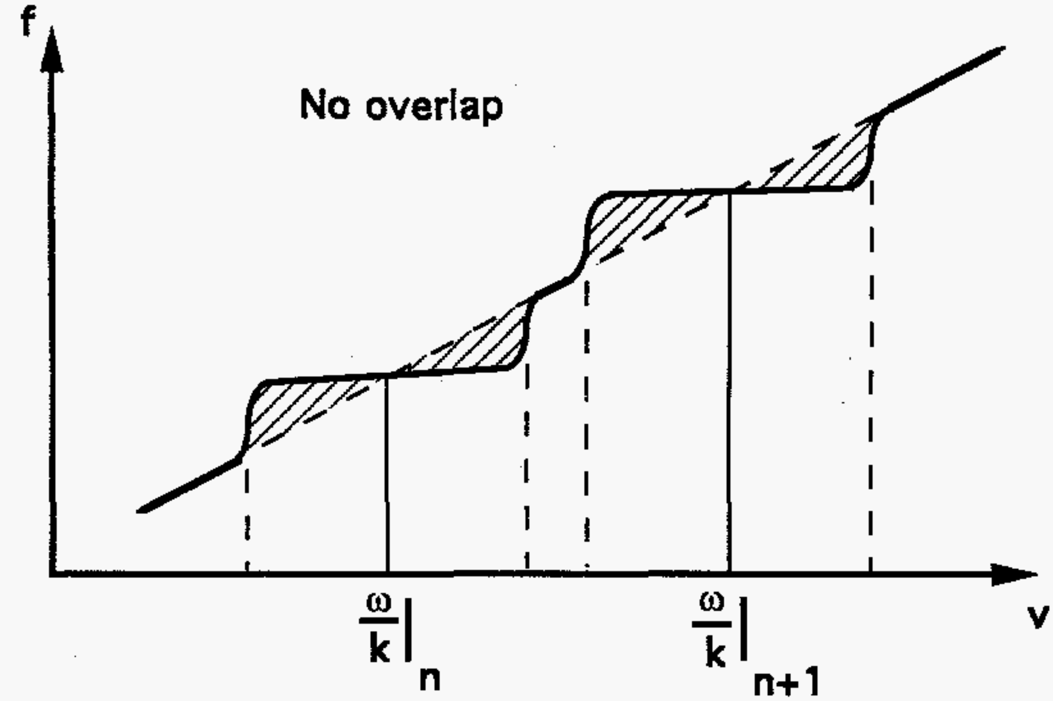

(f)
$\hat{\nu} = 2.4$
A
1.000 10^4
7500
5000
2500
0
-2500
-5000
-7500
-1.000 10^4
27
27.2
27.4
27.6
27.8
28
τ (Blow-up interval)

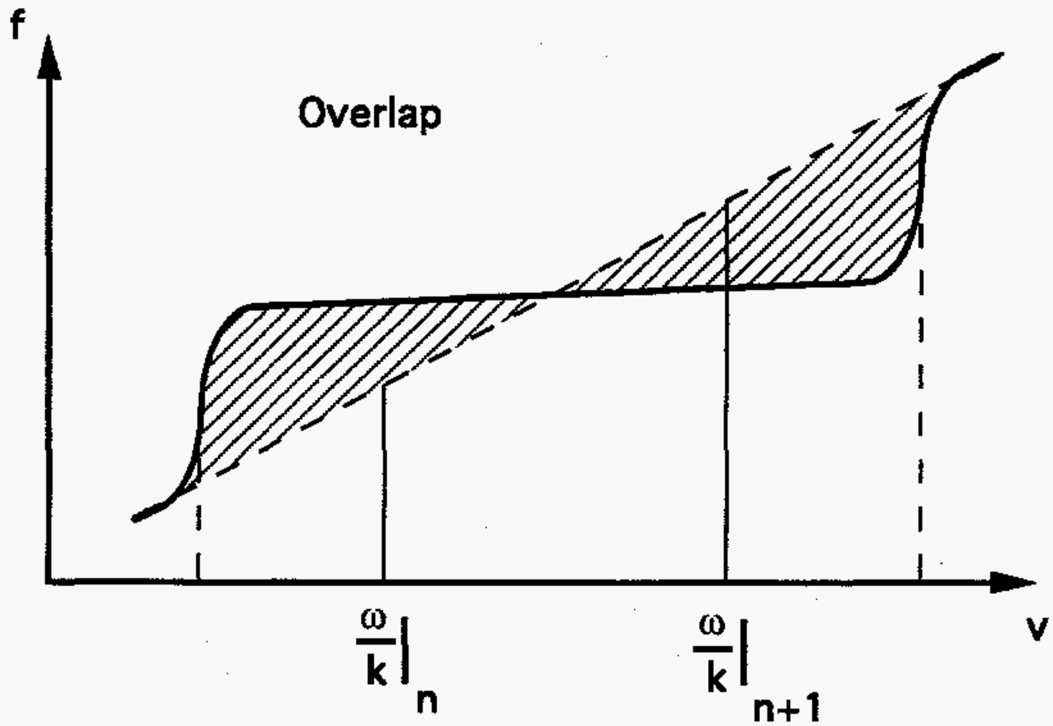


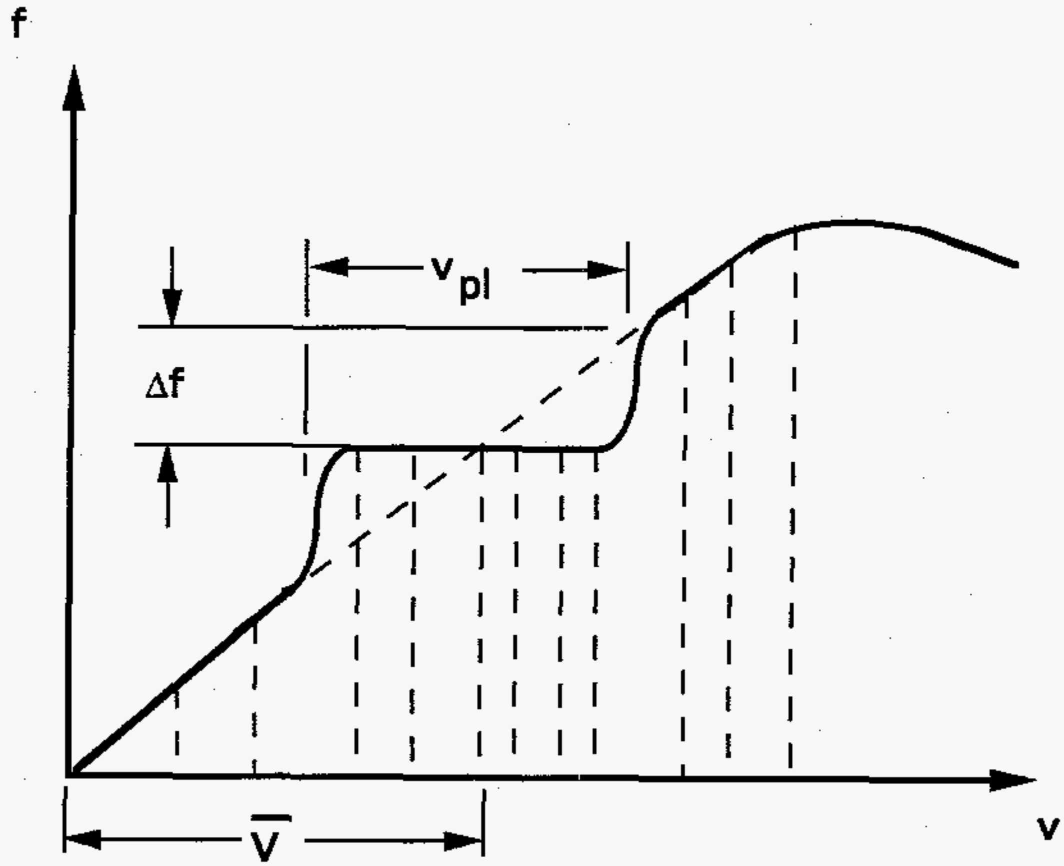
$\gamma/\omega = 3\ 10^{-4}$ $\nu/\omega = 2\ 10^{-7}$



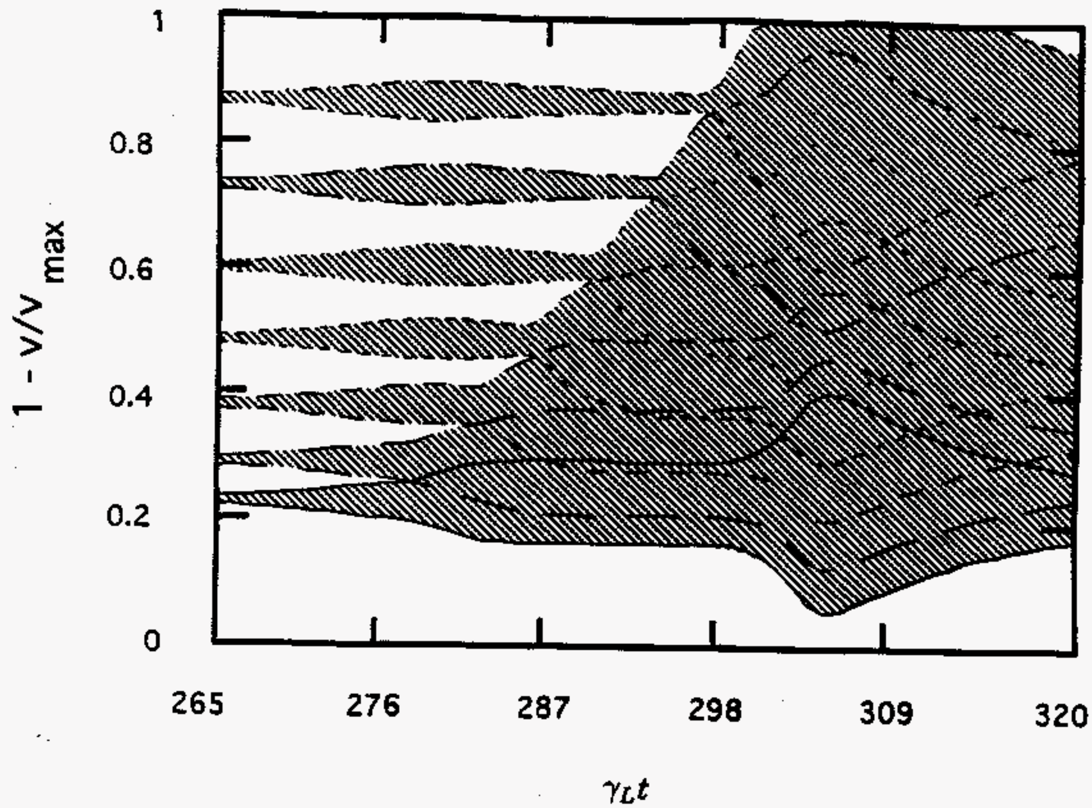
$\gamma/\omega=3\ 10^{-4}$ $\nu/\omega=2\ 10^{-7}$


f
No overlap
v
ω/k|n
ω/k|n+1

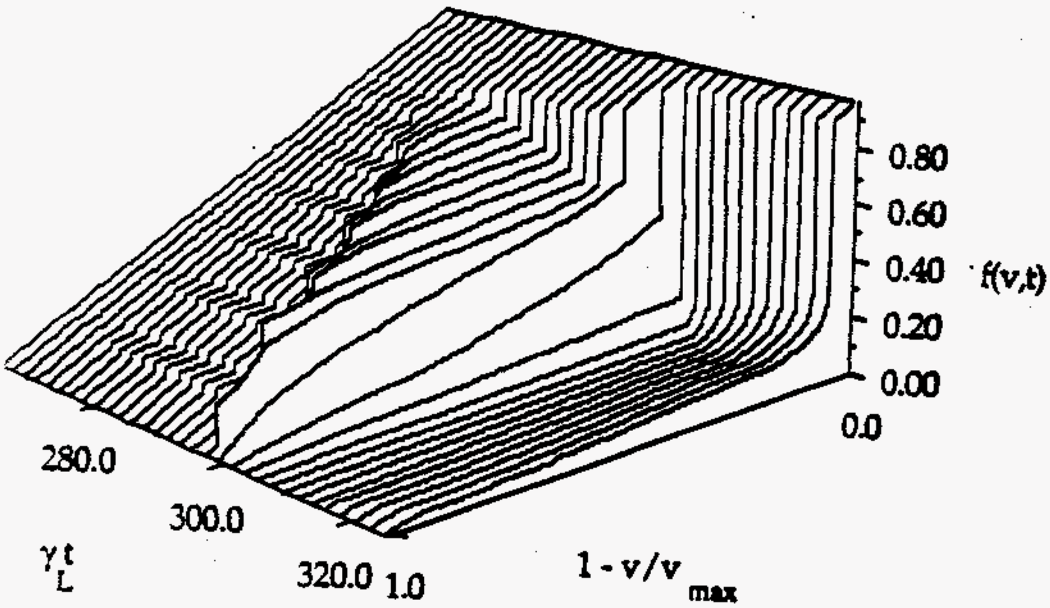

f
Overlap
v
ω/k|n
ω/k|n+1



f
v_pl
Δf
v
V̄

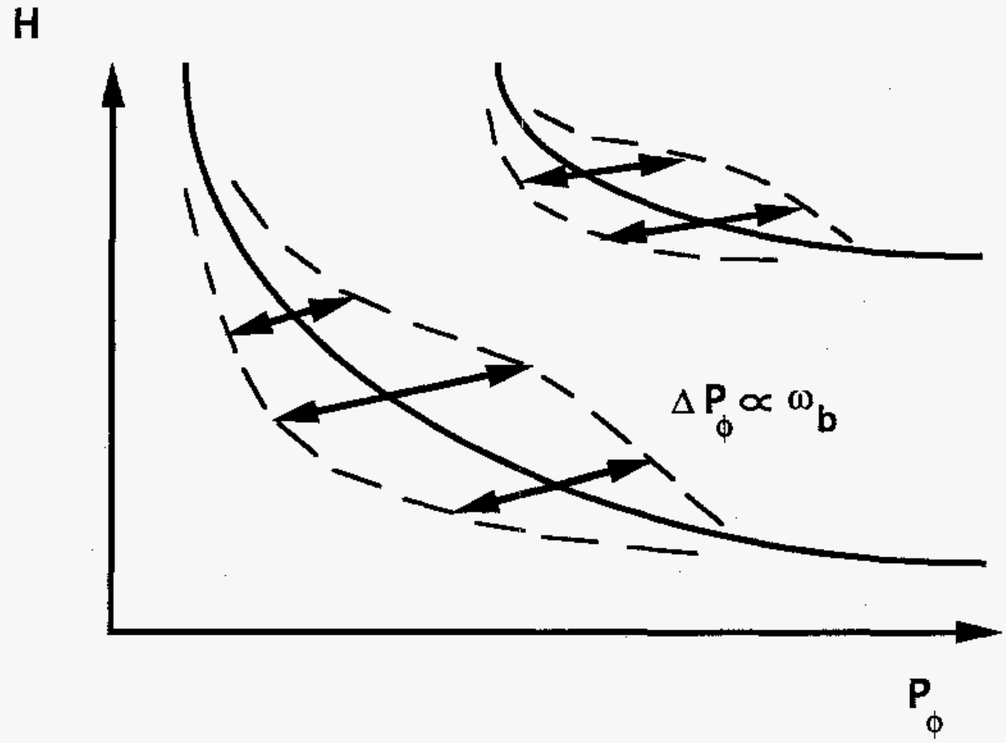


1
0.8
0.6
0.4
0.2
0
1 - v/v max
265
276
287
298
309
320
γLt

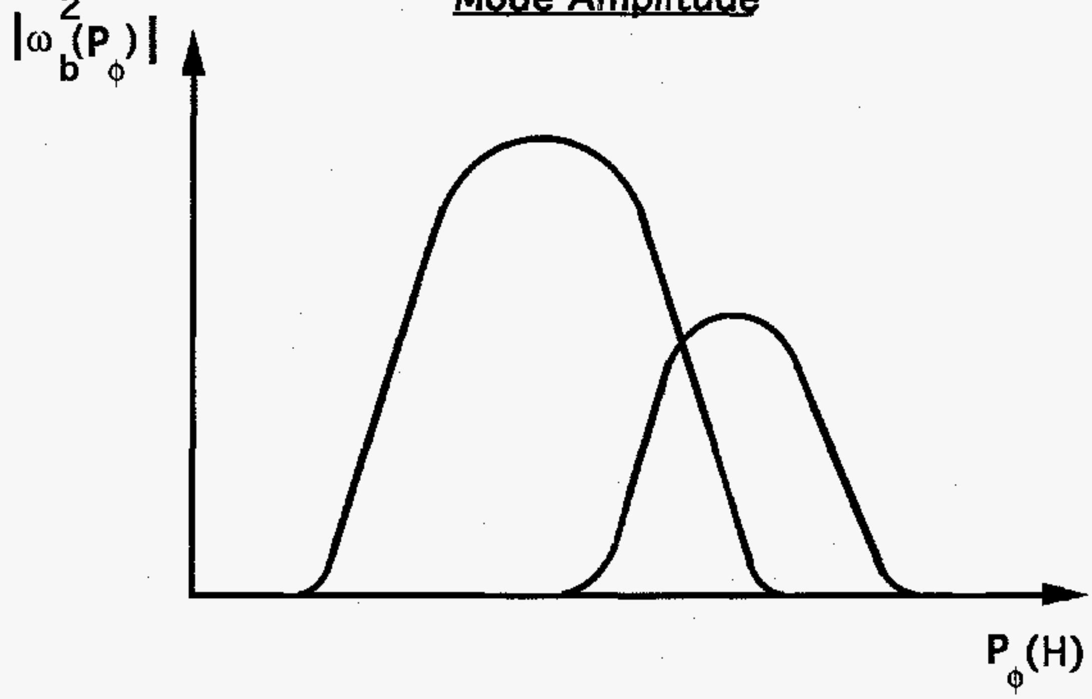


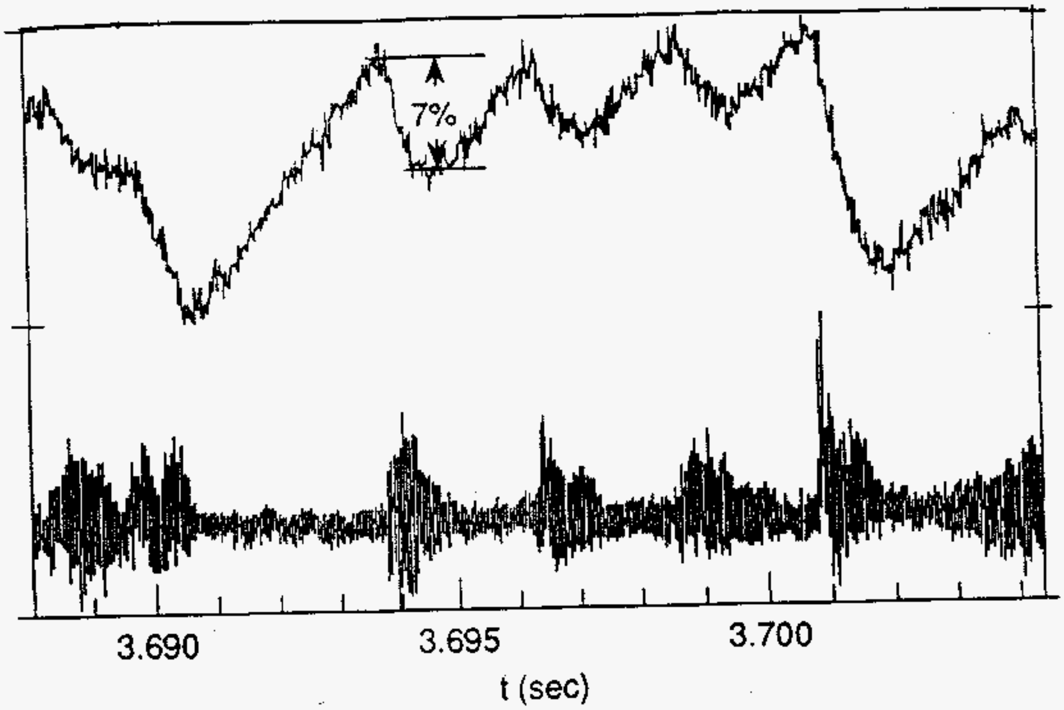
0.80
0.60
0.40
0.20
0.00
f(v,t)
0.0
280.0
300.0
320.0
1.0
γLt
1 - v/v max

Broadened Resonance

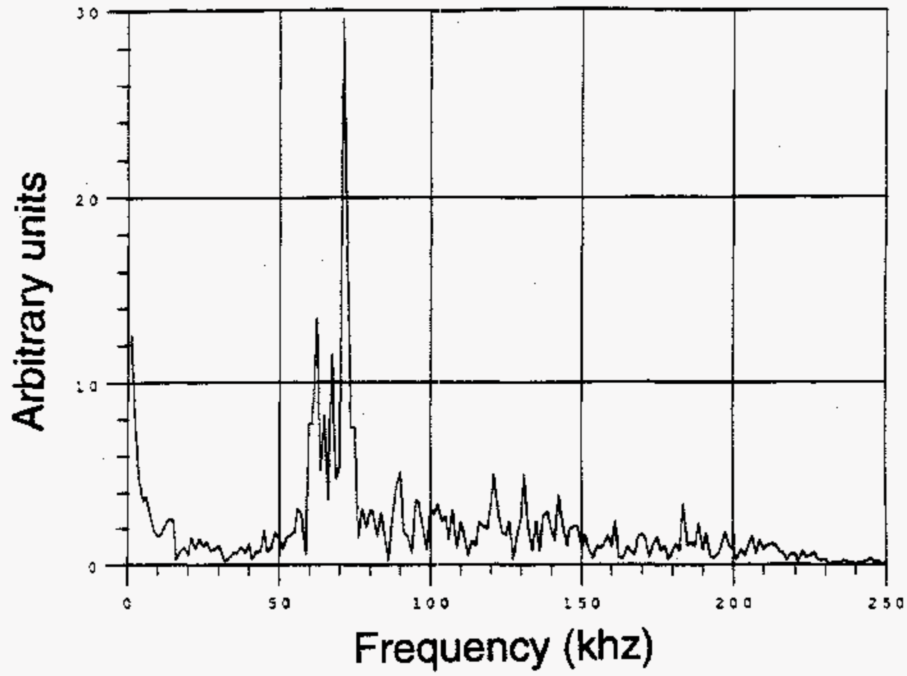


Mode Amplitude




7%
3.690
3.695
3.700
t (sec)

Shot 54742, DMM - IN - 08
time 3.69620 - 3.69700 sec



Shot 54742, DMM - IN - 08
time 3.70095 - 3.70180 sec