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Nonlinear σ Models on Symmetric Spaces and Large N Limit

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Various nonlinear σ models on symmetric spaces are investigated in zero and one dimensional lattice systems. In the large N limit of such matrix models, the third order phase transitions are obtained for the compact case. Noncompact case has no third order transition in this limit. The renormalization group β-functions for one dimensional cases are evaluated.

§ 1. Introduction

Recently, the nonlinear σ models on symmetric spaces have been studied in renormalization group[1),2)] and in lattice calculation.[3),4)] The large N-limit of $O(N)/O(N-1)$ nonlinear σ model is known as a solvable model. The Grassmannian or chiral nonlinear σ model, which has the matrix form, has not been solved in the large N limit except for a special case. Gross and Witten[5)] have shown that there exists a third order phase transition in the large N limit of two dimensional lattice $U(N)$ gauge theory which becomes equivalent to one dimensional chiral nonlinear σ model of $N=\infty$.

In this paper, we investigate various lattice nonlinear σ models defined on symmetric spaces, which are coset spaces G/H where G is the Lie group and H is its maximum compact subgroup. We consider explicitly relevant invariant measure of our symmetric spaces. We consider the system in zero and one space dimension and calculate the energy and correlation function. It will be shown that the large N limit of these models has third order phase transition for the compact case. It will be also shown that for anisotropic Grassmannian model like CP^{N-1} model, there exists a phase transition which has a discontinuity of specific heat in the large N limit. These zero and one dimensional studies may give basis of further investigations in higher dimensions.

This article will be divided as follows: In § 2, we express the action by the angle variables and determine the Haar measure for various symmetric spaces. In § 3, we calculate energy of the one link and we discuss the large N behavior. In § 4, we consider CP^{N-1} and RP^{N-1} model as anisotropic large N cases. In § 5, two point correlation function in one dimension is considered. In § 6, β-function is derived. Section 7 is devoted to discussion. In the Appendix, we present the large N calculation for d-dimensional lattice RP^{N-1} model.

§ 2. Angle variable representations of the nonlinear σ models on symmetric spaces

The symmetric space is a coset space G/H, where G is a Lie group and H is its maximum compact subgroup. We consider orthogonal $O(N)$, unitary $U(N)$ and symplectic $Sp(N)$ Lie groups. We have various symmetric spaces G/H and we present them in Table I.

The Haar measure of the group G is noted by $d\mu(g)$ and it is given by

$$d\mu(g)=\prod_{a}\omega^{a}\,, \tag{2·1}$$

where ω^a is defined by the generator Λ^a of the group G as

$$g^{-1}dg=\sum_{a}\omega^{a}\Lambda^{a}\,,\qquad g\in G\,. \tag{2·2}$$

When a and a' belong to the group H, and f and f' are given by the following relation,

$$f=af'a'\,, \tag{2·3}$$

Table I. The group G represents orthogonal, unitary and symplectic group. These cases are represented by a parameter α. The value of α becomes one, two and four for orthogonal, unitary and symplectic case, respectively. For the Grassmannian model, we assume $n\leq m$ for convenience. Gauss's notation $[N/2]$ represents the maximum integer I which is $I\leq N/2$.

symmetric space G/H	action density	measure $J(\theta)$
$\frac{G(n+m)}{G(n)\times G(m)}$	$\frac{1}{t}\sum_{i=1}^{n}\cos^2\theta_i$	$\prod_{1\leq i<j\leq n}[\sin(\theta_i-\theta_j)\sin(\theta_i+\theta_j)]^{\alpha}\prod_{i=1}^{n}(\sin\theta_i)^{\alpha(m-n)}\prod_{i=1}^{n}(\sin 2\theta_i)^{\alpha-1}$
$\frac{G(n,\,m)}{G(n)\times G(m)}$	$-\frac{1}{t}\sum_{i=1}^{n}\cosh^2\theta_i$	$\prod_{1\leq i<j\leq n}[\sinh(\theta_i-\theta_j)\sinh(\theta_i+\theta_j)]^{\alpha}\prod_{i=1}^{n}(\sinh\theta_i)^{\alpha(m-n)}\prod_{i=1}^{n}(\sinh 2\theta_i)^{\alpha-1}$
$O(N)$	$\frac{4}{t}\sum_{i=1}^{[N/2]}\cos\theta_i$	$\prod_{i}^{[N/2]}\left(\sin\frac{\theta_i-\theta_j}{2}\sin\frac{\theta_i+\theta_j}{2}\right)^2\left(\prod_{i}^{[N/2]}\sin^2\frac{\theta_i}{2}\right)^{\varepsilon}$ $\varepsilon=1$ if N is odd, $\varepsilon=0$ if N is even
$U(N)$	$\frac{1}{t}\sum_{i=1}^{N}\cos\theta_i$	$\prod_{i<j}\left(\sin\frac{\theta_i-\theta_j}{2}\right)^2$
$Sp(N)$	$\frac{4}{t}\sum_{i=1}^{N}\cos\theta_i$	$\prod_{i<j}^{N}\left[\sin^2\frac{(\theta_i-\theta_j)}{2}\sin^2\frac{(\theta_i+\theta_j)}{2}\right]\prod_{i}\sin^2\theta_i$
$U(N)/O(N)$	$\frac{1}{t}\sum_{i=1}^{N}\cos 2\theta_i$	$\prod_{i<j}^{N}\sin(\theta_i-\theta_j)$
$U(2N)/Sp(N)$	$\frac{2}{t}\sum_{i=1}^{N}\cos 2\theta_i$.	$\prod_{i<j}^{N}\sin^4(\theta_i-\theta_j)$
$O(2N)/U(N)$	$\frac{4}{t}\sum_{i=1}^{N}\cos 2\theta_i$	$\prod_{i<j}^{[N/2]}[\sin(\theta_i-\theta_j)\sin(\theta_i+\theta_j)]^4\prod_{i}^{[N/2]}(\sin 2\theta_i)\prod_{i}^{[N/2]}(\sin\theta_i)^{4\varepsilon}$ $\varepsilon=1$ if N is odd, $\varepsilon=0$ if N is even
$Sp(N)/U(N)$	$\frac{1}{t}\sum_{i=1}^{N}\cos 2\theta_i$	$\prod_{i<j}[\sin(\theta_i-\theta_j)\sin(\theta_i+\theta_j)]^2\prod_{i}(\sin 2\theta_i)$

we have an equivalent relation between f and f'. This leads to the classification. If we take k as representative of $G/\sim$, the element g of the group G is expressed by

$$g = aka' , \tag{2·4}$$

where a belongs to H/H_0 and a' belongs to H. H_0 is called as centralizer,

$$H_0 = \{a : aka^{-1} = k,\ a \in H\}. \tag{2·5}$$

The Haar measure of the group G is written as

$$d\mu_G(g) = d\mu_{H/H_0}(a)\, d\mu_H(a')\, d\mu(k) J(k), \tag{2·6}$$

where $d\mu(k)$ is a measure of a set $G/\sim$. If we integrate $d\mu_H(a')$, we obtain

$$d\mu_{G/H}(g) = d\mu_{H/H_0}(a)\, d\mu(k) J(k). \tag{2·7}$$

The G invariant action on coset space G/H becomes independent of H_0. We neglect the difference between $d\mu_{H/H_0}$ and $d\mu_H$.

The G-invariant action on a coset G/H is not uniquely determined. We consider here the simplest form of the action which is derived on the basis of fundamental representation. According to this derivation, the action density is given in Table I. Here we discuss the Grassmannian nonlinear σ model $U(n+m)/U(n)\times U(m)$ as an example. The Grassmannian nonlinear σ model is described by a projection matrix P,[1)]

$$P = g\eta g^* ,$$

where

$$\eta_{ij} = \begin{cases} \delta_{ij} , & 1 \le i, j \le n \\ 0 , & n < i, j \le n+m \end{cases} \tag{2·8}$$

and g is the group element. The projection matrix P satisfies

$$P^2 = P . \tag{2·9}$$

We consider the following one dimensional action with length L

$$A = \frac{1}{t} \sum_{l=1}^{L} \mathrm{tr}(g_l \eta g_l{}^* g_{l+1} \eta g_{l+1}^*). \tag{2·10}$$

We take the link variable f_l as

$$f_1 = g_1 , \qquad f_{l+1} = g_l g_{l+1}^* . \quad (l = 1, \cdots, L) \tag{2·11}$$

The action of (2·9) becomes

$$A=\frac{1}{t}\sum_{l=1}^{L}\mathrm{tr}\;\eta f_l\eta f_l^*\;. \tag{2·12}$$

The group element f_l is expressed as

$$f_l=\begin{pmatrix} f_1 & \\ & f_2\end{pmatrix}\exp\begin{pmatrix} & \theta(l) \\ -{}^t\theta(l) & \end{pmatrix}\begin{pmatrix} f_1' & \\ & f_2'\end{pmatrix}, \tag{2·13}$$

where f_1 and f_1' are elements of $U(m)$, f_2 and f_2' are elements of $U(n)$ matrix. The matrix $\theta(l)$ is given by

$$\theta(l)=\underbrace{\begin{bmatrix} \theta_1(l) & & \\ & \ddots & \\ & & \theta_n(l) \\ & 0 & \end{bmatrix}}_{n}\Big\updownarrow m\,.$$

From (2·12), we have

$$A=\frac{1}{t}\sum_{i=1}^{n}\sum_{l=1}^{L}\cos^2\theta_i(l). \tag{2·14}$$

The change of variable from g_l to f_l does not effect the measure,

$$\prod_l d\mu(g_l)=\prod_l d\mu(f_l). \tag{2·15}$$

For the noncompact $U(n, m)/U(n)\times U(m)$ model, we write the group element f as

$$f=\begin{pmatrix} f_1 & \\ & f_2\end{pmatrix}\exp\begin{pmatrix} & \theta \\ {}^t\theta & \end{pmatrix}\begin{pmatrix} f_1' & \\ & f_2'\end{pmatrix}. \tag{2·16}$$

The action defined by (2·11) is

$$A=\frac{1}{t}\sum_{i=1}^{n}\sum_{l=1}^{L}\cosh^2\theta_i(l). \tag{2·17}$$

Other nonlinear σ models are not described by the projection matrix. However, similar representations to (2·13) are possible for all elements of symmetric spaces. For the invariant action of the chiral model $U(N)$, we have a following expression,

$$A=\frac{1}{2t}\sum_{l=1}^{L}(\mathrm{tr}\,f(l)+\mathrm{tr}\,f^*(l))$$

$$=\frac{1}{t}\sum_{i}^{N}\sum_{l=1}^{L}\cos\,\theta_i(l), \tag{2·18}$$

$$f(l)=f_0^*(l)e^{i\theta(l)}f_0(l)\,, \qquad (f_0(l)\in U(N)) \tag{2·19}$$

$$\theta(l)=\begin{bmatrix}\theta_1(l) & & & \\ & \theta_2(l) & & \\ & & \ddots & \\ & & & \theta_N(l)\end{bmatrix}. \tag{2·20}$$

The derivation of the measure $J(\theta)$ in Table I will be discussed in detail by a separate article.

§ 3. Calculation of energy and large N-limit

Using the measure $J(\theta)$ in Table I, we are able to evaluate the thermodynamic quantities, the partition function and the energy. For the one-link integration, it is easy to perform the high temperature expansion or the low temperature expansion. For example, the high temperature expansion of the partition function for the compact $U(2N)/U(N)\times U(N)$ model becomes in the case of one link (zero dimension) as

$$Z=\int\prod_{1\le i<j\le n}[\sin(\theta_i-\theta_j)\sin(\theta_i+\theta_j)]^2\prod_{i=1}^{N}(\sin 2\theta_i)\exp\left(\frac{1}{t}\sum_{i=1}^{N}\cos^2\theta_i\right)\prod d\theta_i$$

$$=\frac{e^{N/2t}N\,!}{2^{N^2-N}\left[\prod_{l=0}^{N-1}\frac{(2l-1)!\,!}{l\,!}\right]^2(2l-1)!\,!}\left[1+\frac{1}{8t^2}\frac{N^2}{(4N^2-1)}\right.$$

$$\left.+\frac{1}{4\,!}\left(\frac{1}{2t}\right)^4\frac{3N^2(N^2-2)}{(4N^2-1)(4N^2-9)}+\cdots\right]. \tag{3·1}$$

This expression is generalized to the orthogonal $O(2N)/O(N)\times O(N)$ and the symplectic $Sp(2N)/Sp(N)\times Sp(N)$ cases with a parameter α, which takes one, two and four for orthogonal, unitary and symplectic cases, respectively as

$$Z=ce^{N/2t}\left\{1+\frac{1}{2\,!}\left(\frac{1}{2t}\right)^2\frac{N^2}{2\alpha N^2+(2-\alpha)N-1}+\cdots\right\}. \tag{3·2}$$

Although these expansions are not written by a simple function, it is possible to be expressed by Vandermonde's determinant. For $U(2N)/U(N)\times U(N)$ case, the partition function becomes

$$Z=\int_0^1\cdots\int_0^1\prod_{i<j}|y_i-y_j|^2e^{(1/t)\Sigma y_i}\prod dy_i \tag{3·3}$$

$$=N\,!\begin{vmatrix}J_0 & J_1\cdots J_{N-1}\\ J_1 & J_2\cdots J_N\\ \vdots & \\ J_{N-1}\cdots & J_{2N-2}\end{vmatrix}, \tag{3·4}$$

where

$$J_N=\int_0^1 y^N e^{(1/t)y}dy=te^{1/t}-tNJ_{N-1}\,. \tag{3·5}$$

Using this expression, we have for $N=1$, 2 and 3,

$$Z=\begin{cases} te^{1/t}(1-e^{-(1/t)}), & (N=1)\\ t^2e^{2/t}[t^2(1-e^{-(1/t)})^2-e^{-(1/t)}], & (N=2)\\ 6(t^3e^{3/t})[4t^6(1-e^{-(1/t)})^3+(t^2+4t^3+12t^4)e^{-(2/t)} & \\ \qquad -(t^2-4t^3+12t^4)e^{-(1/t)}]. & (N=3)\end{cases} \tag{3·6}$$

It is shown that in the low temperature region, Z is given as

$$Z=c\frac{e^{N/t}}{t^{N^2}}[1+O(e^{-(1/t)})]. \tag{3·7}$$

In a general case, the dominant term in the low temperature region is given by

$$Z=ce^{N/t}/t^{(\alpha/2)N^2}\,. \tag{3·8}$$

The large N limit of two dimensional lattice gauge theory[5)~7)] and one dimensional matrix ϕ^4 model[8)] have been studied. In lattice gauge theory, interesting third order phase transition has been obtained. We consider the large N limit of our various nonlinear σ models. For one-link problem, the partition function is obtained in the closed form in the large N limit. For the appropriate large N limit, we assume the coupling constant t as a quantity of order $1/N$. We note t by g/N. The partition function for the $U(2N)/U(N)\times U(N)$ model is written by the change of variable $y'=1-y$ in (3·3) as

$$A=ce^{N/t}\int_0^1\cdots\int\prod_{i<j}[y_i-y_j]^2e^{-(1/t)\Sigma y_i}\prod dy_i\,. \tag{3·9}$$

By exponentiating the integrand of (3·9), we have

$$Z=ce^{N/t}\int_0^1\cdots\int\exp\{\sum_{i\neq j}\ln|y_i-y_j|-\frac{1}{t}\Sigma\, y_i\}\prod dy_i\,. \tag{3·10}$$

The saddle point equation in the large N limit becomes

$$-\frac{1}{g}+\frac{2}{N}\sum_{i\neq j}\frac{1}{y_i-y_j}=0\,. \tag{3·11}$$

In the large N limit, the summation becomes the integral as

$$\frac{1}{g}=2\int_0^1\frac{dx'}{y(x)-y(x')}\,. \tag{3·12}$$

Introducing the density of eigenvalue $\rho(\beta)$ as

$$dx'=\rho(y)dy\,, \tag{3·13}$$

we have

$$\frac{1}{g}=2\int_0^a\frac{\rho(\beta)}{\alpha-\beta}d\beta \tag{3·14}$$

with conditions $0\leq\alpha\leq a$, $0\leq\beta\leq a$ and $a\leq1$. The integral in (3·14) is singular, therefore, the principal part is taken. The density of eigenvalue $\rho(\beta)$ is normalized as

$$\int_0^1 dx=\int_0^a\rho(\beta)d\beta=1\,. \tag{3·15}$$

The integral equation (3·14) is satisfied by

$$\rho(\beta)=\frac{1}{2\pi g}\sqrt{\frac{a-\beta}{\beta}} \tag{3·16}$$

and the normalization condition becomes

$$\int_0^a\rho(\beta)d\beta=\frac{a}{2\pi g}\int_0^1\sqrt{\frac{1-x}{x}}dx=\frac{a}{4g}=1\,. \tag{3·17}$$

The condition $a\leq1$ is satisfied for $g>\frac{1}{4}$. However for $g<\frac{1}{4}$, the normalization condition contradicts with $a\leq1$. Therefore at $g=\frac{1}{4}$ the phase transition occurs. For $g>\frac{1}{4}$, the density of eigenvalue becomes different as

$$\rho(\beta)=\frac{1}{\pi}\left[1-\frac{1}{4g}\right]\sqrt{\frac{\beta}{1-\beta}}+\frac{1}{\pi}\left[1+\frac{1}{4g}\right]\sqrt{\frac{1-\beta}{\beta}}\,. \tag{3·18}$$

This $\rho(\beta)$ satisfies the normalization condition,

$$\int_0^1 dx=\int_0^1\rho(\beta)d\beta=1\,. \tag{3·19}$$

With these two different expressions for the density of state, the free energy becomes

$$\lim_{N\to\infty}\frac{1}{N^2}\ln Z=\frac{1}{g}-\frac{1}{g}\int_0^a\rho(\alpha)\alpha d\alpha+\int_0^a\int\rho(\alpha)\rho(\beta)\ln|\alpha-\beta|d\alpha d\beta\,. \tag{3·20}$$

By the integration of the saddle point equation, we have

$$\frac{\alpha}{g}=2\int_0^a\rho(\beta)[\ln|\alpha-\beta|-\ln\beta]d\beta\,. \tag{3·21}$$

With this expression, the free energy (3·20) is given by

$$\lim_{N\to\infty}\frac{1}{N^2}\ln Z=\frac{1}{g}-\frac{1}{2g}\int_0^a\rho(\alpha)\alpha d\alpha+\int_0^a\rho(\beta)\ln\beta d\beta\,. \tag{3·22}$$

The energy E becomes

$$E=g^2\frac{\partial}{\partial g}\left\{\lim_{N\to\infty}\frac{1}{N^2}\ln Z\right\} \tag{3·23}$$

$$=\begin{cases}-1+g\,, & g<\dfrac{1}{4}\,,\\ -\dfrac{1}{2}-\dfrac{1}{16g}\,, & g>\dfrac{1}{4}\,.\end{cases} \tag{3·24}$$

It can be seen that the energy is continuous, and the specific heat shows a kink at $g=\frac{1}{4}$. This leads to the third order phase transition. For other Grassmannian nonlinear σ models, the density of eigenvalue becomes similar to (4·8) and (4·10) and the energy is evaluated ($\alpha=1$, 2 and 4 for orthogonal, unitary and symplectic cases, respectively) as

$$E=\begin{cases}-1+\dfrac{\alpha}{2}g\,, & g<\dfrac{1}{2\alpha}\,,\\ -\dfrac{1}{2}-\dfrac{1}{8\alpha g}\,, & g>\dfrac{1}{2\alpha}\end{cases} \tag{3·25}$$

and the third order phase transition occurs at $g_c=(1/2\alpha)$. These transitions are different from the usual critical phenomena which have infinite correlation length at transition point.

These results of compact Grassmannian models are almost identical with two dimensional lattice gauge theory which has been discussed by Gross and Witten. The chiral $U(N)$ model, which becomes equivalent to two dimensional $U(N)$ lattice gauge theory, has the following saddle point equation:

$$\frac{2}{g}=\int_0^1\frac{dy}{\cos\theta(y)-\cos\theta(x)} \tag{3·26}$$

and using the density of eigenvalue $\rho(\beta)$, we write

$$\frac{4}{g}=\rho\int_0^a\frac{\rho(\beta)}{\alpha-\beta}d\beta\,. \tag{3·27}$$

Following a similar procedure to (3·14)~(3·22), we have the following expression for the energy:

$$E=\begin{cases}-\dfrac{1}{2g}\,, & g>1\,,\\ -1+\dfrac{1}{2}g\,, & g<1\,.\end{cases} \tag{3·28}$$

Other symmetric space models have the same third order phase transitions. For example, $Sp(N)/U(N)$ model has the following expression for the energy,

$$E=\begin{cases} -\dfrac{1}{4g}\,, & g>\dfrac{1}{2}\,, \\ -1+g\,, & g<\dfrac{1}{2}\,. \end{cases} \tag{3·29}$$

In the large N limit, all models which we consider in this paper (Table I) become very similar. Previously, one of the authors discussed the universal behavior of β-function in two dimensions in the large N limit.[2] The universal nature seems to persist in higher dimensions.

For the noncompact model, these third order transitions disappear. We consider here $U(N, N)/U(N)\times U(N)$ model for example. Using the measure in Table I, we have the following saddle point equation similar to the compact case ($t=g/N$),

$$\frac{1}{g}=2\int_0^a \frac{\rho(\beta)}{\alpha-\beta}d\beta\,. \tag{3·30}$$

For the noncompact case, the variable takes 0 to ∞ and the normalization condition corresponding to (3·15) becomes

$$\int_0^\infty dx=\int_0^a \rho(\beta)\,d\beta=1\,. \tag{3·31}$$

Thus the value of a takes 0 to ∞ and no phase transition occurs. The energy becomes

$$E=-1-g\,. \tag{3·32}$$

For the unitary Grassmannian case, the integration for finite N can be performed exactly and it is shown that the expression is identical to the compact case in the low temperature phase except for the sign of g.

§ 4. CP^{N-1} and RP^{N-1} models

In the previous section, we have considered the large N limits of nonlinear σ models defined on symmetric spaces. For the Grassmannian case, we have investigated $G(2N)/G(N)\times G(N)$ model. However, the large N limit of anisotropic Grassmann model $U(N+M)/U(N)\times U(M)$ for fixed value of M ($M \ll N$) may become different. In this section, we consider CP^{N-1} and RP^{N-1} models in the large N limit as a typical anisotropic case.

For the CP^{N-1} model, we have (Table I) the following partition function for one-link problem,

$$Z=\int_0^{\pi/2}(\sin\theta)^{2(N-2)}(\sin 2\theta)\exp\left\{\frac{1}{t}\cos^2\theta\right\}d\theta\,. \tag{4·1}$$

In the large N limit ($t=g/N$), we have the following saddle point equation as

$$\cos\theta\left[\frac{1}{\sin\theta}-\frac{1}{g}\sin\theta\right]=0\,. \tag{4·2}$$

The solution of this equation is

$$\cos\theta=0 \qquad \text{or} \qquad g=\sin^2\theta\,. \tag{4·3}$$

Since $\sin\theta$ is less than one, the solution of (4·2) becomes $\cos\theta=0$ when g becomes greater than one. The free energy becomes

$$\begin{aligned}\lim_{N\to\infty}\frac{1}{N}\ln Z&=\frac{1}{g}\cos^2\theta+2\ln(\sin\theta)\\&=\begin{cases}0\,, & g>1\\ -1+\dfrac{1}{g}+\ln g\,, & g<1\end{cases}\end{aligned} \tag{4·4}$$

and the energy becomes ($g=kT/J$, k is the Boltzmann factor, T is temperature and J is interaction)

$$\begin{aligned}E/J&=\frac{kT^2}{J}\frac{\partial}{\partial T}\left\{\lim_{N\to\infty}\frac{1}{N}\ln Z\right\}\\&=\begin{cases}0\,, & g>1\,,\\ -1+g\,, & g<1\,.\end{cases}\end{aligned} \tag{4·5}$$

The specific heat becomes discontinuous at $g=1$, and this behavior is different from the third order phase transition discussed in the previous section. In the Appendix, we solve the large N limit of RP^{N-1} model in d-dimensions. The result is essentially the same as zero dimensional CP^{N-1} model. The specific heat shows discontinuity and the energy becomes zero for high temperature phase.

§ 5. Two point correlation function in one dimension

The two point correlation function χ is defined by

$$\chi=\langle\mathrm{tr}\, g_a g_b\rangle\,, \tag{5·1}$$

where $g_l\in G/H$ and the suffices a and b represent the sites on a line. As discussed in (2·11), we make a change of variable from g_l to the link variable f_l for the diagonalization of the action. Using this link variable f_l, the two point correlation function of $U(2N)/U(N)\times U(N)$ model becomes

$$\chi=\int\prod_a d\mu(f_a)\mathrm{Tr}(\eta f_{a+1}\cdots f_b\eta f_b{}^*\cdots f_{a+1}^*)$$

$$\times \exp\left[\frac{1}{t}\sum \mathrm{tr}\ \eta f_l \eta f_l^*\right] \Big/ \int \prod_a d\mu(f_a) \exp\left[\frac{1}{t}\sum \mathrm{tr}\ \eta f_l \eta f_l^*\right]. \tag{5·2}$$

By the representation of f_l as (2·13), and using the following identity,

$$\int d\mu(f_1)d\mu(f_2)\begin{pmatrix} f_1 & \\ & f_2 \end{pmatrix}_{ab}\begin{pmatrix} f_1^* & \\ & f_2^* \end{pmatrix}_{cd}$$

$$=\frac{1}{N}[\eta_{ad}\eta_{cb}+(1-\eta)_{ad}(1-\eta)_{cb}] \tag{5·3}$$

we have

$$\chi=\frac{N}{2}\left[1+\left(\frac{C-S}{N}\right)^L\right] \tag{5·4}$$

with

$$C=\langle\sum_{i=1}^{N}\cos^2 h_i\rangle\ , \tag{5·5}$$

$$S=\langle\sum_{i=1}^{N}\sin^2 h_i\rangle=N-C\ . \tag{5·6}$$

We have defined χ as (5·1). For the proper correlation function, which becomes zero at $|a-b|\to\infty$ in disordered phase, we redefine as

$$\tilde{\chi}=\langle\mathrm{tr}\left(g_a g_b-\frac{1}{2}\right)\rangle=\chi-\frac{N}{2}\ . \tag{5·7}$$

From (5·4), $\tilde{\chi}$ is shown to be factorized. This factorization is common to all one dimensional models.

§ 6. Renormalization group β function

The renormalization group β function is derived from the expression for the two point function $\tilde{\chi}$. Various models show the exponential decay as

$$\tilde{\chi}\sim e^{-maL}\ , \tag{6·1}$$

where a is a lattice constant and aL is the length between two points. When the mass m is fixed, the β-function is obtained from the change of the coupling constant g according to the change of the lattice constant a. The β-function is defined as

$$\beta(g)=-a\frac{\partial g}{\partial a}\ . \tag{6·2}$$

For the Grassmannian model of $U(2N)/U(N)\times U(N)$, the two point function $\tilde{\chi}$ is evaluated explicitly in the large N limit by the following expression,

$$(C-S)/N=\begin{cases} 1-2g\,, & g<\frac{1}{4}\,, \\ \frac{1}{8g}\,, & g>\frac{1}{4}\,. \end{cases} \tag{6·3}$$

The β-function becomes

$$\beta(g)=\begin{cases} \frac{1}{2}(1-2g)\ln(1-2g), & g<\frac{1}{4}\,, \\ -8g^2\ln(8g), & g>\frac{1}{4}\,. \end{cases} \tag{6·4}$$

We see that the β-function has a kink at $g=\frac{1}{4}$ and two solutions in (6·4) are completely different from each other. For finite N, β- function is smooth and no third order transition occurs. The large N behaviors of the β-functions of the compact symmetric spaces are almost identical with each other and they are equivalent to the result of two dimensional lattice gauge theory studied by Gross and Witten.[5]

For the noncompact case, the two point correlation function $\tilde{\chi}$ is described by the power law of the following quantity,

$$\frac{\tilde{C}+\tilde{S}}{N}=1+2g\,, \tag{6·5}$$

$$\tilde{C}=\langle\sum_{i=1}^{N}\cosh^2\theta_i\rangle=1+\tilde{S}\,. \tag{6·6}$$

From the expression for $\tilde{C}$(3·33), we obtain

$$\beta(g)=-\frac{1}{2}(1+2g)\ln(1+2g). \tag{6·7}$$

This expression shows that there is no phase transition. We notice that this result becomes the same as the expression (6·4) for the compact case in the small coupling region when we interchange the sign of g.

§ 7. Discussion

We have investigated various nonlinear σ models and we have found that the third order phase transition takes place in the large N limit. For CP^{N-1} and RP^{N-1} models, the large N behavior leads to the phase transition which has a discontinuity in the specific heat. In the Appendix, we evaluate the energy of RP^{N-1} model in the large N limit in d-dimension by using the saddle point method. The behavior is similar to the result which we have derived for one-link problem.

Recently, Monte Carlo calculation has been performed for CP^{N-1} and RP^{N-1} models[4),9)] and it has been suggested that there exists a first order phase transition for $N>3$. By employing a mean field approximation we are able to show that a first order phase transition occurs for CP^{N-1} model or $U(N+M)/U(N)\times U(M)$ $(N\ll M)$ model. The mean field results are easily obtained by our angle variable representation. These results and the relation between mean field solution and the third order transition in the large N limit will be discussed in a separate article.

The applications of the nonlinear σ models on symmetric spaces to the critical phenomena are interesting. Recently, noncompact Grassmannian models are considered for Anderson localization problem.[10),11)] For one dimensional case, we notice that a similar β-function to (6·7) has been discussed in the problem of electron conduction in impurity potential.[12)]

Appendix

We discuss the large N limit of a real projection matrix RP^{N-1} model which is defined on $O(N)/O(1)\times O(N-1)$ symmetric space. The large N limit of this model gives a different behavior from the case of $O(2N)/O(N)\times O(N)$ model. We consider N-component vector spin $\boldsymbol{S}_i=(\sigma_i(1),\cdots,\sigma_i(N))$ where i represents the space coordinate. The length of spin $\boldsymbol{S}_i$ is fixed as N.

The partition function becomes

$$Z=\int_{-\infty}^{\infty}\prod d\boldsymbol{S}_i \exp(\sum_{<i,j>}K_{ij}(\boldsymbol{S}_i\cdot\boldsymbol{S}_j)^2)\prod_{i=1}^{M}\delta[N-\boldsymbol{S}_i^2], \tag{A·1}$$

where K_{ij} is J/kT for the nearest neighbor coupling (J is interaction and k is the Boltzmann constant, T is temperature). We introduce two parameters λ_{ij} and t_i and we write the partition function as

$$Z=\frac{1}{(2\pi i)^M}\int_{a-i\infty}^{a+i\infty}\int\prod dt_i\int\prod d\lambda_{ij}\exp(-K\sum_{<i,j>}\lambda_{ij}^2$$
$$+2\sum_{<i,j>}\lambda_{ij}K_{ij}\sum_{m=1}^{N}\sigma_i(m)\sigma_j(m)+N\sum t_j-\sum t_j\sigma_j^2(m))\prod d\sigma_j(m), \tag{A·2}$$

where M is the number of sites.

We take $\lambda_{ij}=\mu_{ij}N$ and $K=\tilde{K}/N$. In the large N limit we apply the saddle point method by choosing t_j and λ_{ij} as

$$t_j=t \quad \text{and} \quad \lambda_{ij}=\mu N\,. \tag{A·3}$$

The partition function becomes

$$Z=\int\prod dt\int\prod d\mu\exp(-2d\tilde{K}\mu^2NM+NMt+N\ln f) \tag{A·4}$$

with

$$\ln f=c-\frac{1}{2}\sum\ln[t-2\mu\tilde{K}(q)],$$

$$\tilde{K}(q)=2K[\cos q_1+\cdots+\cos q_d]. \tag{A·5}$$

In (A·4) and (A·5), d is a space dimensionality. The saddle point equations for variables μ and t are obtained as

$$\frac{1}{M}\sum_q\frac{1}{2t-8\mu\tilde{K}\sum_{i=1}^d\cos q_i}=1\,, \tag{A·6}$$

$$\frac{1}{M}\sum_q\frac{\sum_{i=1}^d\cos q_i}{2t-8\mu\tilde{K}\sum_{i=1}^d\cos q_i}=d\mu\,. \tag{A·7}$$

From (A·6) and (A·7), we have

$$-1+2t=8d\tilde{K}\mu^2\,. \tag{A·8}$$

The saddle point values t and μ are obtained as the solution of (A·6) and (A·7). Expanding small μ in (A·6), we have the following equation,

$$t=\frac{1}{2}+16d\tilde{K}^2\mu^2+\cdots\,. \tag{A·9}$$

Therefore, in comparison with (A·8), we obtain the critical value $\tilde{K}_c$ as

$$\tilde{K}_c=\frac{1}{4}\,. \tag{A·10}$$

For $\tilde{K}>\frac{1}{4}$, we have nonvanishing solution for μ. For $\tilde{K}<\frac{1}{4}$, the value of μ becomes zero. The energy is obtained as

$$E=-2d\tilde{J}\mu^2\,, \tag{A·11}$$

where we put $\tilde{K}=\tilde{J}/kT$. The energy becomes zero for $\tilde{K}<\frac{1}{4}$ and the specific heat becomes discontinuous at $\tilde{K}_c$. For the sufficient low temperature region, μ is almost one and the solution is the same as the usual spherical model. We note this phase transition is a similar one which we have discussed in the zero dimensional case in (4·5).

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