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NONLINEAR SCALAR FIELD EQUATIONS

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Abstract. We prove existence results for a class of semilinear elliptic differential equations in $\mathbb{R}^N$ ($N \geq 3$). The nonlinearities contain sub- and supercritical exponents, and the assumptions for the coefficients are rather general. Moreover, we state some conditions so that the solutions decay exponentially.

1. Introduction and presentation of the results. In the present paper, we consider the nonlinear eigenvalue problem

$$-\Delta u - q(x)|u|^{\sigma_1}u + r(x)|u|^{\sigma_2}u = \lambda u \quad \text{in } \mathbb{R}^N, \tag{1.1}$$

where $N \geq 3$, $0 < \sigma_1 < 4/(N-2)$ and $\sigma_2 \geq 4/(N-2)$.

The nontrivial solutions of equation (1.1) supply standing waves for nonlinear Klein-Gordon and Schrödinger equations. In the case that q and r are positive constants, this equation has been studied by W.A. Strauss [11] (see Example 2) and by H. Berestycki and P.-L. Lions [3] (see also Example 2). These authors were motivated by a paper of D. Anderson [1] who considered the case $N = 3$, $\sigma_1 = 2$ and $\sigma_2 = 4$.

In the following, we require that the functions q and r satisfy the conditions (A)–(D) or (A_r)–(D_r).

(A) The functions $q, r : \mathbb{R}^N \to \mathbb{R}$ are measurable and r satisfies $r(x) \geq r_0$ almost everywhere in $\mathbb{R}^N$, where r_0 is a positive constant.

(B) There exist an open ball $B \subset \mathbb{R}^N$ with $B \neq \emptyset$ and $0 \notin \overline{B}$ and a sequence of real numbers (t_k) satisfying

$$1 = t_1 < t_2 < \cdots < t_k < t_{k+1} < \dots$$

and $t_k \to \infty$ ($k \to \infty$), so that

$$q(x) \geq f(x)|x|^{\sigma_1((N/2)-1)-2} \quad \text{holds for almost all } x \in \mathcal{B},$$

where $\mathcal{B} = \bigcup_{k=1}^{\infty} B_k$, $B_k = t_k B$ and $f : \mathcal{B} \to [0, \infty)$ is a measurable function satisfying

$$\gamma_k = \operatorname*{ess\,inf}_{x \in B_k} f(x) \to \infty \quad (k \to \infty).$$

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Furthermore, we assume that there exists a constant K such that

$$\int_{B_k} r(x)\,dx \le K t_k^{N-2+\sigma_2((N/2)-1)}$$

holds for all k.

(C) The functions q_- and r satisfy $q_-, r \in L^1_{\mathrm{loc}}$.

(D) The function q_+ can be written as $q_+ = q_1 + q_2$, where

(D1) $0 \le q_1 \in L^\infty$ and $q_1(x)$ tends uniformly to zero as $|x| \to \infty$

(D2) and $0 \le q_2 \in L^{p_0}$ holds for a constant

$$p_0 \in (2N/(4 - \sigma_1(N-2)), \infty).$$

Then we will prove the following theorem (see Lemma 3.1–Lemma 3.8):

Theorem 1.1. *Suppose that the functions q and r satisfy the assumptions* (A)–(D). *Then there exists a sequence (u_n) of pairwise distinct functions $u_n \in H^1 \cap L^\infty \setminus \{0\}$ and a sequence of real numbers $(\lambda(n))$ such that $u_n \ge 0$ and equation* (1.1) *holds in the generalized sense if $u = \pm u_n$ and $\lambda = \lambda(n)$.*

When the constant p_0 in condition (D2) *satisfies $p_0 \ge 2$, the functions u_n vanish at infinity.*

The conditions (A_r)–(D_r) read as follows:

(A_r) The functions q and r are radially symmetric and satisfy condition (A).

(B_r) There exists an annulus $A = \{x : a_1 < |x| < a_2\}$, with $0 < a_1 < a_2 < \infty$, and a sequence of real numbers (t_k) satisfying $1 = t_1 < t_2 < \cdots < t_k < t_{k+1} < \dots$ and $t_k \to \infty$ $(k \to \infty)$ so that

$$q(x) \ge f(x)|x|^{\sigma_1((N/2)-1)-2} \quad \text{holds for almost all } x \in \mathcal{A},$$

where $\mathcal{A} = \bigcup_{k=1}^\infty A_k$, $A_k = t_k A$ and $f\colon \mathcal{A} \to [0,\infty)$ is a measurable function satisfying

$$\gamma_k = \operatorname*{ess\,inf}_{x \in A_k} f(x) \to \infty \quad (k \to \infty).$$

Moreover, we assume that there exists a constant K such that

$$\int_{A_k} r(x)\,dx \le K t_k^{N-2+\sigma_2((N/2)-1)}$$

holds for all k.

(C_r) Is the same as (C).

(D_r) The function q_+ can be written as

$$q_+ = q_1 + q_2 + q_3,$$

where

(D_r1) $0 \le q_1 \in L^\infty$,

(D_r2) q_2 satisfies (D2)

(D_r3) and the function q_3 satisfies

$$0 \le q_3(x) \le g(x)|x|^{\sigma_1(N-1)/2} \quad \text{a.e. in } \mathbb{R}^N,$$

where $g \in L^\infty$ is a nonnegative function that vanishes at infinity.

Then, we will show that the following theorem holds true (see Lemmas 3.1–3.7).

Theorem 1.2. *Suppose that the functions q and r satisfy the assumptions* (A_r)–(D_r). *Then there exists a sequence $(u_{r,n})$ of pairwise distinct functions $u_{r,n} \in H_r^1 \cap L^\infty \setminus \{0\}$ and a sequence of real numbers $(\lambda_r(n))$ such that $u_{r,n} \geq 0$ and equation* (1.1) *holds in the generalized sense if $u = \pm u_{r,n}$ and $\lambda = \lambda_r(n)$.*

Remark 1.1. From inequality (2.2), it follows that the functions $u_{r,n}$ satisfy $u_{r,n}(x) = O(|x|^{(1-N)/2})$ $(|x| \to \infty)$.

Corollary 1.1. a) *Suppose that the functions q and r satisfy the assumptions* (A)–(D) *or* (A_r)–$(\mathrm{D}r)$. *Moreover, we assume tht q_-, $r \in L^p_{\mathrm{loc}}$ holds for some $p > N/2$. Then the functions u_n and $u_{r,n}$ are positive and locally Hölder continuous.*

b) *Suppose that the functions q and r satisfy the assumptions* (A)–(D) or (A_r)–(D_r). *Moreover, we assume that the functions q and r are locally Hölder continuous. Then, it follows that*

$$u_n, u_{r,n} \in C^{2,\delta} \quad \textit{holds for some } \delta \in (0,1)$$

and equation (1.1) *holds in the classical sense, provided that*

$$u = \pm u_n \quad (\textit{resp. } u = \pm u_{r,n}) \textit{ and } \lambda = \lambda(n) \quad (\textit{resp. } \lambda = \lambda_r(n)).$$

c) *Suppose that the functions q and r are continuous and satisfy the conditions* (A_r)–(D_r). *Then it follows that $u_{r,n} \in C^2$ and equation* (1.1) *holds in the classical sense if $u = \pm u_{r,n}$ and $\lambda = \lambda_r(n)$.*

Corollary 1.2. *Suppose that the functions q and r satisfy the conditions* (A)–(D). *Furthermore, we assume that $p_0 \geq 2$ and that there is a constant $R_0 > 0$ such that $q_+ \in L^\infty(\{x : |x| \geq R_0\})$. Then if $\lambda(n) < 0$ holds for some n, for each $c \in (0, -\lambda(n))$ we can find a constant A_c such that*

$$|u_n(x)| \leq A_c \exp(-(-\lambda(n) - c)^{1/2}|x|)$$

holds almost everywhere in $\mathbb{R}^N$.

Corollary 1.3. *Suppose that the functions q and r satisfy the conditions* (A_r)–(D_r). *Furthermore, we assume that there is a constant $R_0 > 0$ and a function $h \in L^\infty(\{x : |x| \geq R_0\})$, vanishing at infinity, so that $q_2(x) \leq h(x)|x|^{\sigma_1(N-1)/2}$ holds almost everywhere in $\{y : |y| \geq R_0\}$. Then if $\lambda_r(n) < 0$ holds for some n, for each $c \in (0, -\lambda_r(n))$ we can find a constant A_c so that*

$$|u_{r,n}(x)| \leq A_c \exp(-(-\lambda(n) - c)^{1/2}|x|)$$

holds almost everywhere in $\mathbb{R}^N$.

Corollary 1.2 and Corollary 1.3 show that it is an interesting problem to find conditions for the functions q and r so that $\lambda(n) < 0$ (resp. $\lambda_r(n) < 0$) holds for some n. In the following, we will present some of them. We start with the introduction of the assumptions (E) and (E_r).

(E) The functions q and r are differentiable almost everywhere in $\mathbb{R}^N$ and there exists a constant $\epsilon \in (0,1)$ such that

$$|q(\theta x) - q(x)||\theta - 1|^{-1} \leq f_1(x) + f_\infty(x)$$

and

$$|r(\theta x) - r(x)||\theta - 1|^{-1} \leq f_1(x) + f_\infty(x)$$

hold for all $\theta \in (1-\epsilon, 1+\epsilon)$ and almost all $x \in \mathbb{R}^N$, where $f_1 \in L^1$ and $f_\infty \in L^\infty$.

(E_r) The functions q and r are differentiable almost everywhere in $\mathbb{R}^N$ and there exists a constant $\epsilon \in (0,1)$ such that

$$|q(\theta x) - q(x)||\theta - 1|^{-1} \leq f_1(x) + f_\infty(x) + h_1(x)$$

and

$$|r(\theta x) - r(x)||\theta - 1|^{-1} \leq f_1(x) + f_\infty(x) + h_2(x)$$

hold for all $\theta \in (1-\epsilon, 1+\epsilon)$ and almost all $x \in \mathbb{R}^N$, where $f_1 \in L^1$, $f_\infty \in L^\infty$ and $h_i(\cdot)| \cdot |^{\sigma_i(1-N)/2} \in L^\infty$ $(i = 1, 2)$.

Example 1.1. Suppose that the functions q and r are continuously differentiable in $\mathbb{R}^N \setminus \{0\}$. Furthermore, we assume that there exist constants $C \geq 0$ and $\kappa \in [1, N+1)$ so that

$$|\nabla q(x)|, \quad |\nabla r(x)| \leq C|x|^{-1} + C|x|^{-\kappa}$$

holds for all $x \neq 0$. Then, by the mean value theorem, it is not difficult to verify that (E) holds true.

Example 1.2. Suppose again that the functions q and r are continuously differentiable in $\mathbb{R}^N \setminus \{0\}$. Moreover, we assume that there exist constants $C \geq 0$ and κ_1, $\kappa_2 \in [1, N+1)$ so that

$$|\nabla q(x)| \leq C|x|^{\sigma_1(N-1)-2)/2} + C|x|^{-\kappa_1}$$

and

$$|\nabla r(x)| \leq C|x|^{(\sigma_2(N-1)-2)/2} + C|x|^{-\kappa_2}$$

hold for all $x \neq 0$. Then it follows that (E_r) is satisfied.

In §4, we will prove the following theorem:

Theorem 1.3. *Suppose that the functions q and r satisfy the conditions* (A)–(E) *or* (A_r)–(E_r). *Moreover, we assume that one of the following two conditions is fulfilled:*

a) $((\sigma_1/N)(N-2)-2)q(x) \leq \nabla q(x) \cdot x$ *and* $((\sigma_2/N)(N-2)-2)r(x) \geq \nabla r(x) \cdot x$ *hold almost everywhere in* $\mathbb{R}^N$ *and one of these inequalities is strict;*

b) $0 \leq \nabla q(x) \cdot x$ *and* $((\sigma_2/N)(N-2)-2)r(x) \geq \nabla r(x) \cdot x$ *hold almost everywhere in* $\mathbb{R}^N$.

Then it follows that $\lambda(n) < 0$ *and* $\lambda_r(n) < 0$ *hold for all* n.

Using a device that we found in [2] (see Lemma 13), we will prove the following two theorems:

Theorem 1.4. *Suppose that* $N = 3$ *and that the functions* q *and* r *satisfy the conditions* (A)–(D). *Furthermore, we assume that there exist constants* $R_0 \geq 1$, $r_\infty > 0$ *and* $p > 3/2$ *such that*

$$r \in L^p(\{x : |x| \leq R_0\})$$

and $q(x) \geq 0$ *and* $r(x) \leq r_\infty$ *hold for almost all* $|x| \geq R_0$. *Then we have* $\lambda(n) < 0$ *for all* n.

Theorem 1.5. *Suppose that $N = 3$ and that the functions q and r satisfy the conditions* (A_r)–(D_r). *Moreover, we suppose that there exist constants $R_0 \geq 1$, $r_\infty > 0$, $p > 3/2$ and $\sigma \in [4/3, \sigma_2]$ such that $r \in L^p(\{x : |x| \leq R_0\})$ and $q(x) \geq 0$ and $r(x) \leq r_\infty |x|^{(\sigma_2 - \sigma)}$ hold for almost all $|x| \geq R_0$. Then it follows that $\lambda_r(n) < 0$ holds for all n.*

2. Some preliminaries. In the present paper, we only consider realvalued functions. By $L^p = L^p(\mathbb{R}^N)$ and $L^p_{\mathrm{loc}} = L^p_{\mathrm{loc}}(\mathbb{R}^N)$ $(1 \leq p \leq \infty)$ we denote the usual Lebesgue spaces and $\|\cdot\|_p$ is the norm on L^p. If $1 < p < \infty$, the dual index p' is defined by $p' = p/(p-1)$. Furthermore, $W^{k,p} = W^{k,p}(\mathbb{R}^N)$ $(k = 1, 2$ and $1 \leq p \leq \infty)$ is the usual Sobolev space and $H^1 = W^{1,2}$. By H^1_r, we denote the subspace of the radially symmetric functions in H^1. Finally, $C^1_0 = C^1_0(\mathbb{R}^N)$ is the space of all continuously differentiable functions with compact support and $C^\infty_0 = C^\infty_0(\mathbb{R}^N)$ is the set of all functions $\varphi \in C^1_0$ that have derivatives of any order.

The positive part φ_+ and the negative part φ_- of a function φ are defined by $\varphi_+ = \max(\varphi, 0)$ and $\varphi_- = \min(\varphi, 0)$. By 2^*, we denote the constant $2^* = 2N/(N-2)$. Then, from the Sobolev inequality, it follows that there is a constant C_0 so that

$$\|u\|_{2^*} \leq C_0 \|\nabla u\|_2 \quad \text{holds for all } u \in H^1. \tag{2.1}$$

Each function $u \in H^1_r$ can be identified with a continuous function on $\mathbb{R}^N \setminus \{0\}$, still denoted by u, such that

$$|u(x)| \leq (2/\omega_N)^{1/2} \|u\|_2^{1/2} \|\nabla u\|_2^{1/2} |x|^{(1-N)/2} \tag{2.2}$$

holds for all $x \neq 0$ (see [6, p. 317] and [9, p. 416]). Here ω_N is the surface area of the unit sphere in $\mathbb{R}^N$. A function $u \in H^1$ is called a generalized solution of equation (1.1) if and only if

$$\int \nabla u \nabla \varphi \, dx - \int q|u|^{\sigma_1} u\varphi \, dx + \int r|u|^{\sigma_2} u\varphi \, dx = \lambda \int u\varphi \, dx$$

holds for all $\varphi \in C^\infty_0$. When the domain of integration is not indicated, the integration extends over all of $\mathbb{R}^N$.

Proposition 2.1. *Suppose that $p \in (1, \infty)$, k_0 is a positive constant, $h \in L^p$ and u is a tempered distribution such that*

$$-\Delta u + k_0 u = h \quad \textit{holds in } \mathcal{D}'(\mathbb{R}^N).$$

Then it follows that $u \in W^{2,p}$.

Proof. See Proposition 27 in [4, p. 635].

3. Proof of the existence and regularity results. In the following, we consider the cases where the functions q and r satisfy the conditions (A)–(D) or (A_r)–(D_r) simultaneously.

Lemma 3.1. *Suppose that the functions q and r satisfy* (A)–(D) *(resp.* (A_r)–(D_r)*). Then there exist positive constants α and β and, for each $\epsilon > 0$, a constant K_ϵ such that*

$$(2+\sigma_1)^{-1}\int q_+|u|^{2+\sigma_1}\,dx \le \epsilon(\|u\|_{2+\sigma_2}^{2+\sigma_2} + \|\nabla u\|_2^2) + K_\epsilon(\|u\|_2^\alpha + \|u\|_2^\beta)$$

holds for all $u \in H^1$ (resp. $u \in H_r^1$).

Proof. We start with the case that q and r satisfy (A)–(D). Since $2 < 2+\sigma_1 < 2+\sigma_2$, we can find a constant $\nu \in (0,1)$ so that $2+\sigma_1 = \nu(2+\sigma_2) + (1-\nu)2$. Hence, by Hölder's inequality, we obtain

$$\int q_1|u|^{2+\sigma_1}\,dx \le \|q_1\|_\infty \Big(\int |u|^{2+\sigma_2}\,dx\Big)^\nu \Big(\int |u|^2\,dx\Big)^{1-\nu}. \tag{3.1}$$

Since $2 < (2+\sigma_1)p_0' < 2^* \le 2+\sigma_2$, there exists a constant $\tau \in (0,1)$ such that $(2+\sigma_1)p_0' = \tau(2+\sigma_2) + (1-\tau)2$. Now, we see that

$$\int q_2|u|^{2+\sigma_1}\,dx \le \|q_2\|_{p_0}\|u\|_{2+\sigma_2}^{\tau(2+\sigma_2)/p_0'}\|u\|_2^{(1-\tau)2/p_0'}. \tag{3.2}$$

Since $\sigma_2 \ge 4/(N-2) > \sigma_1$ and $p_0 > 2N/(4-\sigma_1(N-2))$, it follows that $\tau/p_0' < 1$. Then, by (3.1), (3.2) and Young's inequality, we get the assertion.

Next, we assume that the functions q and r satisfy the conditions (A_r)–(D_r). Then, from assumption (D_r3) and (2.2), we conclude that

$$\int q_3|u|^{2+\sigma_1}\,dx \le (2/\omega_N)^{\sigma_1/2}\|g\|_\infty\|\nabla u\|_2^{\sigma_1/2}\|u\|_2^{2+(\sigma_1/2)}$$

holds for all $u \in H_r^1$. Now, using the fact that $\sigma_1 < 4$, from Young's inequality and from what has already been proved, we again obtain the assertion. □

In the following, we always assume, without stating it explicitly each time, that the functions q and r satisfy the assumptions (A)–(D) (resp. (A_r)–(D_r)).

The nonlinear functional ξ may be defined by

$$\xi(u) = \frac{1}{2}\int |\nabla u|^2\,dx - (2+\sigma_1)^{-1}\int q|u|^{2+\sigma_1}\,dx + (2+\sigma_2)^{-1}\int r|u|^{2+\sigma_2}\,dx;$$

and by S_μ ($\mu > 0$) we denote the set

$$S_\mu = \Big\{u \in H^1 : \int |q_-|\,|u|^{2+\sigma_1}\,dx < \infty, \int r|u|^{2+\sigma_2}\,dx < \infty \text{ and } \|u\|_2 \le \mu\Big\}.$$

Furthermore, $S_{r,\mu}$ is defined by $S_{r,\mu} = \{u \in H_r^1 : u \in S_\mu\}$. Since $r(x) \ge r_0 > 0$ holds almost everywhere in $\mathbb{R}^N$, it follows from Lemma 3.1 that ξ is well defined on S_μ (resp. on $S_{r,\mu}$) and that

$$I(\mu) = \inf_{u\in S_\mu}\xi(u) \quad\text{and}\quad I_r(\mu) = \inf_{u\in S_{r,\mu}}\xi(u)$$

are well defined real numbers.

Lemma 3.2. *There exists a sequence* (μ_n) *(resp.* $(\mu_{r,n})$*) of real numbers such that*

$$1 \le \mu_1 < \mu_2 < \cdots < \mu_n < \mu_{n+1} < \dots$$
$$(\textit{resp. } 1 \le \mu_{r,1} < \mu_{r,2} < \cdots < \mu_{r,n} < \mu_{r,n+1} < \dots)$$

and

$$0 > I(\mu_1) > I(\mu_2) > \cdots > I(\mu_n) > I(\mu_{n+1}) > \dots$$
$$(\textit{resp. } 0 > I_r(\mu_{r,1}) > I_r(\mu_{r,2}) > \cdots > I_r(\mu_{r,n}) > \dots).$$

Proof. We only consider the case where the functions q and r satisfy the conditions (A)–(D). The proof for the radial case is nearly the same.

The ball B and the sequence (t_k) may be defined as in condition (B). Furthermore, the function $\varphi \in C_0^\infty$ may be chosen so that supp $\varphi \subset B$ and $\|\varphi\|_2 = 1$. For $k \in \mathbb{N}$, we define $\varphi_k(x) = t_k^{1-(N/2)}\varphi(t_k^{-1}x)$. Then we see that $\|\varphi_k\|_2 = t_k$ and

$$\begin{aligned} I(t_k) \le& \xi(\varphi_k) = \frac{1}{2}\|\nabla\varphi\|_2^2 - t_k^{2-\sigma_1((N/2)-1)}(2+\sigma_1)^{-1}\int_B q(t_kx)|\varphi(x)|^{2+\sigma_1}\,dx \\ &+ t_k^{2-\sigma_2((N/2)-1)}(2+\sigma_2)^{-1}\int_B r(t_kx)|\varphi(x)|^{2+\sigma_2}\,dx \\ \le& \frac{1}{2}\|\nabla\varphi\|_2^2 - \gamma_k(2+\sigma_1)^{-1}\int_B |x|^{\sigma_1((N/2)-1)-2}|\varphi(x)|^{2+\sigma_1}\,dx \\ &+ (2+\sigma_2)^{-1}K\|\varphi\|_\infty^{2+\sigma_2}. \end{aligned}$$

Since $\gamma_k \to \infty$ as $k \to \infty$, we obtain the assertion.

Lemma 3.3. *For* $n \in \mathbb{N}$*, the constants* μ_n *(resp.* $\mu_{r,n}$*) may be chosen as in Lemma 3.2. Then, for each* n*, there exists a function* $u_n \in S_{\mu_n}\backslash\{0\}$ *(resp.* $u_{r,n} \in S_{r,\mu_{r,n}}\backslash\{0\}$*) so that* u_n*,* $u_{r,n} \ge 0$*,* $u_n \ne u_m$ *(resp.* $u_{r,n} \ne u_{r,m}$*) if* $n \ne m$ *and* $\xi(u_n) = I(\mu_n)$ *(resp.* $\xi(\mu_{r,n}) = I_r(\mu_{r,n})$*).*

Proof. Let $n \in \mathbb{N}$ be fixed. Then, for the sake of convenience, we set $\mu = \mu_n$ (resp. $\mu_r = \mu_{r,n}$). The sequence $(v_m) \subset S_\mu$ (resp. $(v_m) \subset S_{r,\mu_r}$) may be chosen so that $\xi(v_m) \to I(\mu)$ (resp. $\xi(v_m) \to I_r(\mu_r)$) as $m \to \infty$. Since $I(\mu) < 0$ (resp. $I_r(\mu_r) < 0$), and $\|\nabla|v|\|_2 = \|\nabla v\|_2$ holds for all $v \in H^1$, we may assume without restriction that $\xi(v_m) \le 0$ and $v_m \ge 0$ holds for all m. Then, by Lemma 3.1 and the fact that $r(x) \ge r_0 > 0$ holds almost everywhere in $\mathbb{R}^N$, we can find a constant C so that

$$\begin{aligned} &\frac{1}{4}\|\nabla v_m\|_2^2 + (2+\sigma_1)^{-1}\int |q_-|\,|v_m|^{2+\sigma_1}\,dx \\ &+ \frac{1}{2}(2+\sigma_2)^{-1}\int r|v_m|^{2+\sigma_2}\,dx \le C(\mu^\alpha + \mu^\beta)\ (\text{resp. } \le C(\mu_r^\alpha + \mu_r^\beta)) \end{aligned} \tag{3.3}$$

holds for all m. Since (v_m) is bounded in H^1 (resp. in H_r^1), we can find a subsequence of (v_m), still denoted by (v_m), and a $u \in H^1$ (resp. $u_r \in H_r^1$) such that $v_m \underset{w}{\rightarrow} u$ in H^1 (resp. $v_m \underset{w}{\rightarrow} u_r$ in H_r^1) and $v_m(x) \to u(x)$ (resp. $v_m(x) \to u_r(x)$) for almost all

$x \in \mathbb{R}^N$. Hence, we obtain from Fatou's lemma, the uniform boundedness principle and (3.3) that $\|u\|_2 \le \mu$, $\|\nabla u\|_2^2 \le \liminf \|\nabla v_m\|_2^2$,

$$\int |q_-|\,|u|^{2+\sigma_1}\,dx \le \liminf \int |q_-|\,|v_m|^{2+\sigma_1}\,dx < \infty$$

and

$$\int r|u|^{2+\sigma_2}\,dx \le \liminf \int r|v_m|^{2+\sigma_2}\,dx < \infty.$$

Furthermore, we see that the corresponding estimates for the function u_r hold true. Since $(2+\sigma_1)p_0' < 2^*$, the imbedding $H^1(G) \to L^{(2+\sigma_1)p_0'}(G)$ is compact for all bounded balls G. Then, proceeding as in [8, p. 570] (resp. [9, p. 419–421]), it follows that

$$\int q_+|v_m|^{2+\sigma_1}\,dx \to \int q_+|u|^{2+\sigma_1}\,dx$$

$$\text{(resp. } \int q_+|v_m|^{2+\sigma_1}\,dx \to \int q_+|u_r|^{2+\sigma_1}\,dx)$$

and $\xi(u) = I(\mu)$ (resp. $\xi(u) = I_r(\mu)$). Since $I(\mu) < 0$ (resp. $I_r(\mu) < 0$), we see that $u \not\equiv 0$ (resp. $u_r \not\equiv 0$).

Now we define $u_n = u$ (resp. $u_{r,n} = u_r$). If $n \neq m$, it follows that $\xi(u_n) = I(\mu_n) \neq I(\mu_m) = \xi(u_m)$ (resp. $\xi(u_{r,n}) = I_r(\mu_{r,n}) \neq I_r(\mu_{r,m}) = \xi(u_{r,m})$). Hence, we see that $u_n \neq u_m$ (resp. $u_{r,n} \neq u_{r,m}$).

Lemma 3.4. *For each n, equation (1.1) holds in the generalized sense provided that $u = \pm u_n$ (resp. $u = \pm u_{r,n}$) and $\lambda = \lambda(n)$ (resp. $\lambda = \lambda_r(n)$). Here the constants $\lambda(n)$ and $\lambda_r(n)$ are defined by*

$$\lambda(n) = \|u_n\|_2^{-2}\Big[\|\nabla u_n\|_2^2 - \int q|u_n|^{2+\sigma_1}\,dx + \int r|u_n|^{2+\sigma_2}\,dx\Big]$$

and

$$\lambda_r(n) = \|u_{r,n}\|_2^{-2}\Big[\|\nabla u_{r,n}\|_2^2 - \int q|u_{r,n}|^{2+\sigma_1}\,dx + \int r|u_{r,n}|^{2+\sigma_2}\,dx\Big].$$

Proof. First we consider the case where the functions q and r satisfy the assumptions (A)–(D). Then, for each $\varphi \in C_0^1$ and $n \in \mathbb{N}$, there exists a positive constant $\epsilon_0 = \epsilon_0(\varphi, n)$ so that $\|u_n + \epsilon\varphi\|_2 > 0$ holds for all $\epsilon \in (-\epsilon_0, \epsilon_0)$. For these ϵ, we define $\Phi(\epsilon) = \xi(\|u_n\|_2\|u_n + \epsilon\varphi\|_2^{-1}(u_n + \epsilon\varphi))$. By Hölder's inequality, Lemma 3.1 and the fact that $u_n \in S_{\mu_n}$, it follows that

$$\int |q|\,|\varphi|\,|u_n|^{1+\sigma_1}\,dx < \infty \text{ and } \int r|\varphi|\,|u_n|^{1+\sigma_2}\,dx < \infty. \tag{3.4}$$

From (3.4), it is not difficult to conclude that $\Phi(\cdot)$ is differentiable at $\epsilon = 0$. But $d\Phi(\epsilon)/d\epsilon\,\big|_{\epsilon=0} = 0$ implies

$$\int \nabla u_n \nabla\varphi\,dx - \int q|u_n|^{\sigma_1}u_n\varphi\,dx + \int r|u_n|^{\sigma_2}u_n\varphi\,dx = \lambda(n)\int u_n\varphi\,dx. \tag{3.5}$$

Finally, we assume that q and r satisfy the conditions (A_r)–(D_r). Then, proceeding as above, we see that (3.5) holds for all radially symmetric functions $\varphi \in C_0^1$ when u_n is replaced by $u_{r,n}$ and $\lambda(n)$ is replaced by $\lambda_r(n)$. If $\varphi \in C_0^\infty$ is a general function, not necessarily radially symmetric, we define

$$P\varphi(x) = \omega_N^{-1} \int_{|z|=1} \varphi(|x|z)\, d\sigma(z).$$

Then $P\varphi$ is radially symmetric and satisfies $P\varphi \in C_0^1$. Inserting $P\varphi$ in (3.5) and using the fact that $u_{r,n}$, q and r are radially symmetric, we see that $\pm u_{r,n}$ solve equation (1.1) in the generalized sense if $\lambda = \lambda_r(n)$.

Lemma 3.5. *For each n, the function u_n (resp. $u_{r,n}$) and the constant $\lambda(n)$ (resp. $\lambda_r(n)$) may be defined as in Lemma 3.4. Then, for all nonnegative functions $v \in H^1$, we have*

$$\int \nabla u_n \nabla v\, dx \le \lambda(n) \int u_n v\, dx + \int q_+ u_n^{1+\sigma_1} v\, dx$$

and

$$\int \nabla u_{r,n} \nabla v\, dx \le \lambda_r(n) \int u_{r,n} v\, dx + \int q_+ u_{r,n}^{1+\sigma_1} v\, dx.$$

Proof. Clearly, the assertions hold true for all nonnegative functions $v \in C_0^\infty$. Now let v be a nonnegative function satisfying $v \in H^1$. Then, via regularization and truncation, one can find a sequence (v_k) of nonnegative functions $v_k \in C_0^\infty$ so that $v_k \to v$ in H^1. From condition (D) (resp. (D_r)), one easily concludes that

$$\int q_+ u_n^{1+\sigma_1} v_k\, dx \to \int q_+ u_n^{1+\sigma_1} v\, dx$$

and

$$\int q_+ u_{r,n}^{1+\sigma_1} v_k\, dx \to \int q_+ u_{r,n}^{1+\sigma_1} v\, dx.$$

Hence, we obtain the assertion.

Lemma 3.6. *For each n and all $p \in [2, \infty)$, we have u_n, $u_{r,n} \in L^p$.*

Proof. In the following, we will use an iteration technique which was introduced by J. Moser [7]. Let n be fixed, $u = u_n$ and $\lambda = \lambda(n)$. The constant p_0 may be chosen as in (D2). Then there exists a positive constant ϵ_0 such that $1/p_0' = (2+\sigma_1+2\epsilon_0)/2^*$. For all $k \in \mathbb{N} \cup \{0\}$, we define $r_k = 2^*(1-\epsilon_0)^k$ and

$$s_k = (r_k/p_0') - 1 - \sigma_1.$$

Now suppose that $u \in L^{r_k}$ holds for some k. Then, using the fact that

$$1 + s_k, \quad 1 + \sigma_1 + s_k, \quad p_0'(1 + \sigma_1 + s_k) \in [2, r_k],$$

we conclude that

$$\int u^{1+s_k}\, dx < \infty \quad \text{and} \quad \int q_+ u^{1+s_k+\sigma_1}\, dx < \infty.$$

For $t > 0$, we define $v_t = \min(u, t)$. Since $s_k > 1$, it follows that

$$v_t^{s_k} \in H^1 \cap L^\infty \quad \text{and} \quad \partial_i v_t^{s_k} = s_k v_t^{s_k - 1} \partial_i v_t \quad (i = 1, \dots, N).$$

Then, from Lemma 3.5, we conclude that

$$s_k \int \nabla u \nabla v_t v_t^{s_k - 1}\, dx \leq |\lambda| \int u v_t^{s_k}\, dx + \int q_+ u^{1+\sigma_1} v_t^{s_k}\, dx.$$

Since $\nabla v_t = \nabla u$ holds almost everywhere in $\{x : u(x) \leq t\}$ and $\nabla v_t = 0$ holds almost everywhere in $\{x : u(x) > t\}$, we see that

$$4s_k(s_k + 1)^{-2} \int |\nabla v_t^{(s_k+1)/2}|^2\, dx \leq |\lambda| \int u^{1+s_k}\, dx + \int q_+ u^{1+\sigma_1+s_k}\, dx. \tag{3.6}$$

Hence, by (2.1), it follows that $v_t^{(s_k+1)/2} \in L^{2^*}$ and that the norm $\|v_t^{(s_k+1)/2}\|_{2^*}$ can be estimated by the right hand side of (3.6) which is independent of t. Letting $t \to \infty$, we obtain by Fatou's lemma that $u \in L^{2^*(s_k+1)/2}$. Since $r_k \geq 2^*$, we see that

$$2^*(s_k + 1)/2 = r_k(2 + \sigma_1 + 2\epsilon_0)/2 - (2^*\sigma_1/2) \geq r_k(1 + \epsilon_0) = r_{k+1}.$$

Hence, by induction, it follows that $u \in L^{r_k}$ holds for all k. Furthermore, preceding as above and making some obvious changes, one can show that $u_{r,n} \in L^p$ holds for all $p \in [2, \infty)$.

Lemma 3.7. *For each n we have u_n, $u_{r,n} \in L^\infty$.*

Proof. In this proof, we use techniques which were developed by G. Stampacchia (see [10]). First we assume that the functions q and r satisfy the assumptions (A)–(D). For the sake of convenience, we define $u = u_n$ and $\lambda = \lambda(n)$. For each $k > 0$, the set $A(k)$ and the function U_k may be defined by $A(k) = \{x : u(x) \geq k\}$ and $U_k = (u - k)_+$. Then it is well known (see Lemma 1.1 in [10] and Theorem 7.8 in [5]) that $U_k \in H^1$, $\partial_i U_k = \partial_i u$ holds on $A(k)$ and $\partial_i U_k = 0$ holds on $\mathbb{R}^N \setminus A(k)$. Hence, it follows from Lemma 3.5 that

$$\int \nabla u \nabla U_k\, dx \leq |\lambda| \int_{A(k)} u^2\, dx + \int_{A(k)} q_+ u^{2+\sigma_1}\, dx. \tag{3.7}$$

The constant p_0 may be defined as in (D2) and by p_1 we denote the constant $p_1 = 2N/(4 - \sigma_1(N-2))$. Since $p_0 > p_1$, we can find a constant $p_2 \in (1, \infty)$ so that $1/p_0' \cdot 1/p_2' = 1/p_1'$. Thus, inequality (3.7) implies

$$\begin{aligned} \int |\nabla U_k|^2\, dx \leq{}& |\lambda| \Big[\int u^{2p_1}\, dx \Big]^{1/p_1} (\text{ meas } A(k))^{1/p_1'} \\ &+ \|q_1\|_\infty \Big[\int u^{(2+\sigma_1)p_1}\, dx \Big]^{1/p_1} (\text{ meas } A(k))^{1/p_1'} \\ &+ \|q_2\|_{p_0} \Big[\int u^{(2+\sigma_1)p_0' p_2}\, dx \Big]^{1/(p_0' p_2)} (\text{ meas } A(k))^{1/p_1'}. \end{aligned}$$

Hence, there exists a constant C, independent of k, such that

$$\left[\int_{A(k)} (u-k)^{2^*}\,dx\right]^{2/2^*} \le C(\text{ meas } A(k))^{1/p_1'}. \tag{3.8}$$

Moreover, for each $h > k > 0$, it follows that

$$\left[\int_{A(k)} (u-k)^{2^*}\,dx\right]^{2/2^*} \ge \left[\int_{A(h)} (u-k)^{2^*}\,dx\right]^{2/2^*} \ge (h-k)^2(\text{ meas } A(h))^{2/2^*}. \tag{3.9}$$

Combining (3.8) and (3.9), we see that

$$\text{meas } A(h) \le C^{2^*/2}(h-k)^{-2^*}\ (\text{meas } A(k))^{2^*/(2p_1')}$$

holds for all $h > k > 0$. Since $2^*/(2p_1') = 1 + (\sigma_1/2) > 1$, we conclude from part i) of Lemma 4.1 in [10, p. 212] that u is essentially bounded.

In the case where the functions q and r satisfy (A_r)–(D_r), we conclude from (2.2) that

$$\begin{aligned}\int_{A(k)} q_3 u_{r,n}^{2+\sigma_1}\,dx &\le \left[\int q_3^{p_1} u_{r,n}^{(2+\sigma_1)p_1}\,dx\right]^{1/p_1} (\text{meas } A(k))^{1/p_1'} \\ &\le C\left[\int u_{r,n}^{2p_1}\,dx\right]^{1/p_1} (\text{meas } A(k))^{1/p_1'},\end{aligned}$$

where $C = (2/\omega_N)^{\sigma_1/2}\|g\|_\infty \|u_{r,n}\|_2^{\sigma_1/2}\|\nabla u_{r,n}\|_2^{\sigma_1/2}$. Then, proceeding as above, we see that $u_{r,n}$ is essentially bounded.

Lemma 3.8. *Suppose that the constant p_0 in condition* (D2) *satisfies $p_0 \ge 2$. Then, for each n, the function u_n vanishes at infinity.*

Proof. Let n be fixed and define $u = u_n$ and $\lambda = \lambda(n)$. Then, from Lemma 3.5, we conclude that

$$\int \nabla u \nabla w\,dx + \int uw\,dx \le (|\lambda|+1)\int uw\,dx + \int q_+ u^{1+\sigma_1} w\,dx \tag{3.10}$$

holds for all nonnegative functions $w \in H^1$. The linear functional $L\colon H^1 \to \mathbb{R}$ may be defined by

$$L(w) = (|\lambda|+1)\int uw\,dx + \int q_+ u^{1+\sigma_1} w\,dx.$$

Since $u \in L^p$ holds for all $p \in [2,\infty]$, one easily verifies that L is continuous. Hence, there exists a function $v \in H^1$ so that

$$\int \nabla v \nabla w\,dx + \int vw\,dx = L(w) \tag{3.11}$$

holds for all $w \in H^1$. Since $p_0 \ge 2$, it follows that

$$(|\lambda|+1)u + q_+ u^{1+\sigma_1} \in L^{p_0}.$$

Now, from (3.11) and Proposition 2.1, we conclude that $v \in W^{2,p_0}$. Since $p_0 > (N/2)$, by the Sobolev imbedding theorem it follows that the imbedding $W^{2,p_0} \to L^\infty$ is continuous. Now let $(\varphi_k) \subset C_0^\infty$ be a sequence so that $\varphi_k \to v$ in W^{2,p_0}. Then we see that $\varphi_k \to v$ in L_∞. But this shows that v is a continuous function that vanishes at infinity. From (3.10) and (3.11), we conclude that

$$\int \nabla(u-v)\nabla w\,dx + \int (u-v)w\,dx \leq 0 \tag{3.12}$$

holds for all nonnegative functions $w \in H^1$. Inserting $w = (u-v)_+$ in (3.12) implies that $u \leq v$ holds almost everywhere in $\mathbb{R}^N$. Since u is nonnegative, we obtain the assertion.

Proof of Corollary 1.1 (part (a)). Let n be fixed, $u = u_n$ (resp. $u = u_{r,n}$) and $\lambda = \lambda(n)$ (resp. $\lambda = \lambda_r(n)$). Then it follows from Lemma 3.4 that $-\Delta u + c(x)u = 0$ holds in the generalized sense, where $c(x) = -q(x)u^{\sigma_1}(x) + r(x)u^{\sigma_2}(x) - \lambda$. Since $p_0 > N/2$ and $u \in L^\infty$, we see that $c \in L^{p_1}_{\mathrm{loc}}$, where $p_1 = \min(p_0, p) > N/2$. Now the assertions follow from Theorem 7.1 and Corollary 8.1 in [10].

Proof of Corollary 1.1 (part (b)). From part (a),it follows that u_n and $u_{r,n}$ are locally Hölder continuous. Hence, the distributions Δu_n and $\Delta u_{r,n}$ can be represented by a locally Hölder continuous function. Now the assertion follows by a well known result from the regularity theory of elliptic differential equations.

Proof of Corollary 1.1 (part (c)). From the assumptions and part (a), it follows that the distribution $\Delta u_{r,n}$ can be represented by a continuous function. Now the assertion follows from Proposition 7 in [4, p. 287].

Proof of Corollary 1.2. Suppose that $\lambda(n) < 0$ holds for some n and that $c \in (0, -\lambda(n))$. Then, since $q_+ \in L^\infty(\{x : |x| > R_0\})$ and u_n vanishes at infinity (see Lemma 3.8), we can find a constant $R_c > R_0$ so that

$$q_+(x)|u_n(x)|^{\sigma_1} \leq c \text{ holds a.e. in } \{x : |x| > R_c\}. \tag{3.13}$$

The function ψ may be defined by

$$\psi(x) = A_c \exp(-(-\lambda(n) - c)^{1/2})|x|) \quad (x \in \mathbb{R}^N),$$

where the constant A_c is chosen so that

$$\psi(x) \geq u_n(x) \text{ holds a.e. in } \{x : |x| \leq R_c + 1\}. \tag{3.14}$$

Then, it is well known that $\psi \in H^1$ and that

$$\Delta\psi = (-\lambda(n) - c)\psi - (N-1)(-\lambda(n) - c)^{1/2}|x|^{-1}\psi$$

holds in the generalized sense. Hence, we obtain from Lemma 3.5, (3.13) and (3.14),

$$\begin{aligned}
&\|\nabla(u_n - \psi)_+\|_2^2 = \int \nabla(u_n - \psi)\nabla(u_n - \psi)_+\,dx \\
&\leq \lambda(n) \int u_n(u_n - \psi)_+\,dx + c\int u_n(u_n - \psi)_+ dx - (\lambda(n) + c)\int \psi(u_n - \psi)_+ dx \\
&= (\lambda(n) + c)\|u_n - \psi)_+\|_2^2.
\end{aligned}$$

Since $\lambda(n) + c < 0$, we obtain the assertion.

Proof of Corollary 1.3. Suppose that $\lambda_r(n) < 0$ holds for some n and that $c \in (0, -\lambda_r(n))$. Then, from (2.2) and the assumptions, one easily concludes that there is a constant $R_c > R_0$ such that

$$q_+(x)|u_{r,n}(x)|^{\sigma_1} \leq c \text{ holds a.e. in } \{x : |x| > R_c\}.$$

Then, proceeding as in the proof of Corollary 1.2, we obtain the assertion.

4. Sufficient conditions for $\lambda(n) < 0$ and $\lambda_r(n) < 0$. In this paragraph, we will prove Theorem 1.3–Theorem 1.5.

Proof of Theorem 1.3. In the following, we consider n as fixed and define $u = u_n$ (resp. $u = u_{r,n}$), $\lambda = \lambda(n)$ (resp. $\lambda = \lambda_r(n)$) and $\mu = \mu_n$ (resp. $\mu = \mu_{r,n}$). The constant ϵ may be chosen as in condition (E) (resp. (E_r)). Then, according to Lemma 3.6, Lemma 3.7, (2.2) and (E) (resp. (E_r)), it follows that

$$\int |q(\theta x)|\,|u(x)|^{2+\sigma_1}\,dx < \infty$$

holds for all $\theta \in (1-\epsilon, 1+\epsilon)$. Hence, by the change of variable theorem, we see that

$$\int |q(x)|\,|u(\theta^{-1}x)|^{2+\sigma_1}\,dx < \infty. \tag{4.1}$$

Quite similarly, one can show that

$$\int r(x)|u(\theta^{-1}x)|^{2+\sigma_2}\,dx < \infty. \tag{4.2}$$

The function v_θ may be defined by $v_\theta(x) = \theta^{-N/2}u(\theta^{-1}x)$. Since $\|v_\theta\|_2 = \|u\|_2 \leq \mu$, we conclude from (4.1) and (4.2) that $v_\theta \in S_\mu$ holds for all $\theta \in (1-\epsilon, 1+\epsilon)$. Hence, we obtain that

$$\begin{aligned} I(\mu) \leq & \theta^{-2}\frac{1}{2}\|\nabla u\|_2^2 - \theta^{-\sigma_1 N/2}(2+\sigma_1)^{-1}\int q(\theta x)|u(x)|^{2+\sigma_1}\,dx \\ & + \theta^{-\sigma_2 N/2}(2+\sigma_2)^{-1}\int r(\theta x)|u(x)|^{2+\sigma_2}\,dx. \end{aligned} \tag{4.3}$$

Then, by the dominated convergence theorem, it follows that the right hand side of (4.3) defines a function that is differentiable at $\theta = 1$. But $d\xi(v_\theta)/d\theta\,\big|_{\theta=1} = 0$ implies

$$\begin{aligned} 0 = & -\|\nabla u\|_2^2 + (\sigma_1 N)/(2(2+\sigma_1))\int q|u|^{2+\sigma_1}\,dx \\ & -(2+\sigma_1)^{-1}\int \nabla q(x)\cdot x|u|^{2+\sigma_1}\,dx \\ & -(\sigma_2 N)/(2(2+\sigma_2))\int r|u|^{2+\sigma_2}\,dx + (2+\sigma_2)^{-1}\int \nabla r(x)\cdot x|u|^{2+\sigma_2}\,dx. \end{aligned} \tag{4.4}$$

From Lemma 3.4 and (4.4), it follows that

$$\begin{aligned}\lambda\|u\|_2^2 =&(2+\sigma_1)^{-1}\int[((\sigma_1/2)(N-2)-2)q(x)-\nabla q(x)\cdot x]|u|^{2+\sigma_1}\,dx\\&+(2+\sigma_2)^{-1}\int[\nabla r(x)\cdot x-((\sigma_2/2)(N-2)-2)r(x)]|u|^{2+\sigma_1}\,dx.\end{aligned}\tag{4.5}$$

Hence, we obtain the assertion of part a).

Finally, we suppose that the assumptions of part b) are fulfilled. Then, from (4.5), it follows that

$$\lambda\|u\|_2^2 \le (2+\sigma_1)^{-1}((\sigma_1/2)(N-2)-2)\int q|u|^{2+\sigma_1}\,dx. \tag{4.6}$$

Since $\xi(u) = I(\mu) < 0$, we see that

$$\int q|u|^{2+\sigma_1}\,dx > 0. \tag{4.7}$$

But (4.6), (4.7) and the fact that $\sigma_1 < 4/(N-2)$ imply the assertion.

Proof of Theorem 1.4. We again assume that n is fixed, that $u = u_n$ and $\lambda = \lambda(n)$. Then it follows from the assumptions that u is positive and continuous in $\mathbb{R}^3$ (see Corollary 1.1).

Now suppose that $\lambda \ge 0$. Then, from Lemma 3.4 and the assumptions, it follows that

$$\int \nabla u \nabla v\,dx + r_\infty \int u^{1+\sigma_2} v\,dx \ge 0 \tag{4.8}$$

holds for all nonnegative functions $v \in H_0^{1,2}(\{x : |x| > R_0\})$. The function ψ may be defined by

$$\psi(x) = C|x|^{-3/2} \quad (x \ne 0),$$

where C is a positive constant such that

$$\begin{gathered}C \le (3/4r_\infty)^{1/\sigma_2} \quad \text{and}\\ \psi(x) \le u(x) \quad \text{holds for all } x \text{ satisfying } R_0 \le |x| \le R_0 + 1.\end{gathered}\tag{4.9}$$

Then, for $x \ne 0$, we obtain

$$-\Delta\psi(x) + r_\infty \psi^{1+\sigma_2}(x) = -(3C/4)|x|^{-7/2} + r_\infty C^{1+\sigma_2}|x|^{-3(1+\sigma_2)/2}.$$

Since $\sigma_2 \ge 4/3$ and $R_0 \ge 1$, we see that

$$-\Delta\psi(x) + r_\infty\psi^{1+\sigma_2}(x) \le 0 \tag{4.10}$$

holds for all x satisfying $|x| > R_0$. The fact that σ_2 is positive implies that

$$\psi^{1+\sigma_2} \in L^2(\{x : |x| > R_0\}).$$

Furthermore, we have $|\nabla\psi| \in L^2(\{x : |x| > R_0\})$. Hence, by (4.10), we obtain that

$$\int \nabla\psi\nabla v\, dx + r_\infty \int \psi^{1+\sigma_2} v\, dx \leq 0 \tag{4.11}$$

holds for all nonnegative functions $v \in H_0^{1,2}(\{x : |x| > R_0\})$.

The function $\zeta \in C_0^\infty$ may be chosen so that $0 \leq \zeta \leq 1$, $\zeta \equiv 1$ on the unit ball and $\zeta(x) = 0$ holds for $|x| \geq 2$. Moreover, for $k \in \mathbb{N}$, we define the function ζ_k by $\zeta_k(x) = \zeta(k^{-1}x)$. For all x satisfying $|x| > R_0$ and all $k \in \mathbb{N}$, we set

$$v_k(x) = (\psi - u)_+(x)\zeta_k(x).$$

Then, according to (4.9), we see that $v_k \in H_0^{1,2}(\{x : |x| > R_0\})$. Inserting v_k in (4.8) and in (4.11) yields

$$\begin{aligned}
&\int_{|x|>R_0} |\nabla(\psi - u)_+|^2\zeta_k\, dx + r_\infty \int_{|x|>R_0} (\psi^{1+\sigma_2} - u^{1+\sigma_2})(\psi - u)_+\zeta_k\, dx \\
&\leq -\int_{|x|>R_0} \nabla(\psi - u)(\psi - u)_+\nabla\zeta_k\, dx \\
&\leq \frac{1}{k}\|\nabla(\psi - u)\|_{L^2(\{x\,:\,|x|>R_0\})}\|\nabla\zeta\|_\infty \Big[\int_{R_0<|x|\leq 2k} \psi^2\, dx\Big]^{1/2} \\
&\leq C(4\pi)^{1/2}\|\nabla(\psi - u)\|_{L^2(\{x\,:\,|x|>R_0\})}\|\nabla\zeta\|_\infty k^{-1}\log^{1/2}(2k)
\end{aligned}$$

holds for all $k > R_0/2$. Letting $k \to \infty$, we see that

$$\psi(x) \leq u(x) \quad \text{holds for all } x \text{ satisfying } |x| > R_0. \tag{4.12}$$

But (4.12) contradicts $u \in L^2$.

Proof of Theorem 1.5. As in the proof of Theorem 1.4, we assume that $\lambda \geq 0$. Then, from Lemma 3.4, (2.2) and the assumptions, we conclude that

$$\int \nabla u\nabla v\, dx \geq -R_\infty \int u^{1+\sigma} v\, dx$$

holds for all nonnegative functions $v \in H_0^{1,2}(\{x : |x| > R_0\})$, where R_∞ is defined by

$$R_\infty = [(2\pi)^{-1}\|u\|_2\|\nabla u\|_2]^{(\sigma_2-\sigma)/2} r_\infty.$$

Since $\sigma \geq 4/3$, we can precede as in the proof of Theorem 1.4 to get a contradiction.

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