Nonlinear Second-Order Elliptic Equations V. The Dirichlet Problem for Weingarten Hypersurfaces

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0. Introduction

In this paper we study the Dirichlet problem for a function u in a bounded domain Ω in $\mathbb{R}^n$ with smooth strictly convex boundary $\partial\Omega$. At any point x in Ω the principal curvatures $\kappa = (\kappa_1, \cdots, \kappa_n)$ of the graph $(x, u(x))$ are to satisfy a relation

$$f(\kappa_1, \cdots, \kappa_n) = \psi(x) > 0, \tag{1}$$

where ψ is a given smooth positive function on $\bar{\Omega}$. In addition, u is to satisfy the Dirichlet boundary condition

$$u = 0 \quad \text{on} \quad \partial\Omega. \tag{2}$$

The function f is of a special nature as in our papers [3] and [4] (though with somewhat different properties). It is a smooth symmetric (under interchange of any two κ_i) function satisfying

$$f_i = \frac{\partial f}{\partial \kappa_i} > 0 \quad \text{for all} \quad i, \qquad \sum \kappa_i f_i > 0; \tag{3}$$

furthermore,

$$f \text{ is a concave function} \tag{4}$$

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defined in an open convex cone $\Gamma \subsetneqq \mathbb{R}^n$ with vertex at the origin and containing the positive cone Γ^+. Γ is also supposed to be symmetric in the κ_i.

With

$$0 < \psi_0 = \min_{\bar{\Omega}} \psi \leqq \max_{\bar{\Omega}} \psi = \psi_1,$$

we assume that, for some $\bar{\psi}_0 < \psi_0$,

$$\overline{\lim_{\kappa \to \kappa_0}} f(\kappa) \leqq \bar{\psi}_0 \quad \text{for every} \quad \kappa_0 \in \partial\Gamma. \tag{5}$$

In addition we assume that for every $C > 0$ and every compact set K in Γ there is a number $R = R(C, K)$ such that

$$f(\kappa_1, \cdots, \kappa_{n-1}, \kappa_n + R) \geqq C \quad \text{for all} \quad \kappa \in K. \tag{6}$$

Furthermore we have the following conditions: for some constant $c_0 > 0$,

$$\sum f_i(\kappa) \geqq c_0 > 0 \quad \text{whenever} \quad f \geqq \psi_0, \tag{7}$$

$$\sum \kappa_i f_i(\kappa) \geqq c_0 \quad \text{whenever} \quad \psi_0 \leqq f \leqq \psi_1, \tag{8}$$

$$f(t\kappa) \leqq tf(\kappa) \quad \text{for} \quad \kappa \in \Gamma,\ t \geqq 1; \tag{9}$$

and, for some constant $c_1 > 0$, on the set

$$\begin{gathered} \{\kappa \in \Gamma | \psi_0 \leqq f(\kappa) \leqq \psi_1 \text{ and } \kappa_1 < 0\}, \\ f_1 \geqq c_1 > 0. \end{gathered} \tag{10}$$

In case f is non-negative, condition (9) follows from concavity of f, for we have: for $0 < \varepsilon < s < 1$, $\kappa \in \Gamma$,

$$\begin{aligned} f(s\kappa + (1-s)\varepsilon\kappa) &\geqq sf(\kappa) + (1-s)f(\varepsilon\kappa) \\ &\geqq sf(\kappa). \end{aligned}$$

Letting $\varepsilon \to 0$ we obtain $f(s\kappa) \geqq sf(\kappa)$ for $s < 1$, which is equivalent to (9).

DEFINITION. A function $u \in C^2(\bar{\Omega})$ is called *admissible* if, at every point of its graph, $\kappa \in \Gamma$.

We shall also assume the existence of a suitable admissible subsolution:

(11) there is an admissible $\underline{u}$, $\underline{u} = 0$ on $\partial\Omega$, such that the principal curvatures $\underline{\kappa}$ of its graph satisfy

$$f(\underline{\kappa}(x)) \geqq \psi(x) \quad \text{in} \quad \bar{\Omega}.$$

We can now state the main result of this paper.

THEOREM 1. *Under conditions* (3)–(11) *there exists a unique admissible smooth solution u of* (1), (2) *in* $\overline{\Omega}$.

EXAMPLE. The function $f(\kappa) = (\sigma^{(k)}(\kappa))^{1/k}$, where $\sigma^{(k)}$ is the k-th elementary symmetric function

$$\sigma^{(k)}(\kappa) = \sum_{i_1 < i_2 < \cdots < i_k} \kappa_{i_1} \cdots \kappa_{i_k},$$

satisfies all the conditions of the theorem. Indeed these have all been verified in [3] (or are obvious) except for condition (10). But that is also easily verified: if $\kappa_1 \leqq 0$, we have

$$\begin{aligned} \sigma_1^{(k)} &= \sigma^{(k-1)}(\kappa_2, \cdots, \kappa_n) \\ &\geqq \sigma^{(k-1)}(\kappa_1, \cdots, \kappa_n) \end{aligned}$$

since $\sigma^{(k-1)}$ is increasing in κ_1.

Now (see Section 1 in [3]) the connected component in $\mathbb{R}^n$ containing Γ^+ in which $\sigma^{(k)} > 0$ is a convex cone Γ with vertex at the origin. In Γ all the functions $\sigma^{(k-1)}, \cdots, \sigma^{(0)} = 1$ are positive. By inequality (6) on page 11 of [1] we have

$$\sigma^{(j-1)} \geqq \text{constant} \cdot \sqrt{\sigma^{(j)}\sigma^{(j-2)}} \quad \text{in} \quad \Gamma \quad \text{for} \quad j = 2, \cdots, k.$$

It follows easily that in Γ

$$\begin{aligned} \sigma^{(k-1)} &\geqq \text{constant} \cdot \left(\sigma^{(k)}\right)^{1-1/k} \\ &\geqq \text{constant} \cdot \psi_0^{k-1} \end{aligned}$$

and hence $\sigma_1^{(k)} \geqq \text{constant} \cdot \psi_0^{k-1}$, and (10) is proved.

In [6], N. Korevaar also uses and proves condition (10) for $(\sigma^{(k)})^{1/k}$. Up to now we have not been able to treat more general boundary values, or nonconvex domains Ω. We may perhaps return to these cases at a later time.

The uniqueness follows immediately from the following form of the maximum principle.

LEMMA A. *Let u be an admissible function; denote its principal curvature at x (of its graph) by* $\kappa(x)$. *Let* $v \in C^2(\Omega) \cap C(\overline{\Omega})$ *and assume that at every point x in* Ω *its principal curvatures* $\hat{\kappa}(x)$ *lie outside the set*

$$\tilde{\Gamma} = \{\lambda \in \Gamma | f(\lambda) \geqq f(\kappa(x))\}.$$

If $u \leqq v$ *on* $\partial\Omega$, *then* $u \leqq v$ *in* Ω.

Proof: If not, $v = u$ achieves a negative minimum at some point $x \in \Omega$. It follows that the ordered principal curvatures $\hat{\kappa}(x)$ at $(x, v(x))$ of the graph of v satisfy $\hat{\kappa}_i(x) \geqq \kappa_i(x)$. But then, by (3), $\hat{\kappa}$ lies in $\bar{\Gamma}$; contradiction.

We shall rely on some computations from earlier papers in this series. In Section 1 of [4] we showed that the principal curvatures κ of the graph of u are eigenvalues of the symmetric matrix (summation convention is used)

$$a_{il}(x) = \frac{1}{w}\left\{u_{il} - \frac{u_i u_j u_{jl}}{w(1+w)} - \frac{u_l u_k u_{ki}}{w(1+w)} + \frac{u_i u_l u_j u_k u_{jk}}{w^2(1+w)^2}\right\}, \tag{12}$$

where $w = (1 + |\nabla u|^2)^{1/2}$. At the beginning of Section 3 of [3], we remarked (rather, we left it to the reader to verify) that

$$F(a_{11}, \cdots, a_{nn}) = f(\kappa)$$

satisfies (for convenience we write the left-hand side as $F(a_{il}) = G(Du, D^2u)$):

$$\text{the matrix } \left\{\frac{\partial F}{\partial a_{il}}\right\} \text{ is positive definite,}$$

provided u is admissible. It follows easily that the equation

$$F(a_{il}) = G(Du, D^2u) = \psi(x) > 0 \tag{1$'$}$$

is elliptic at every admissible function u.

Furthermore, relying on the results of Section 3 of [3] we see that F is a concave function of the matrix (a_{il}) and hence G is a concave function in its dependence on the symmetric matrix D^2v.

As in all the preceding papers in this series, our proof of existence is based on the

Continuity Method. For $0 \leqq t \leqq 1$ set

$$\psi^t(x) = t\psi(x) + (1-t)\underline{\psi}(x),$$

where $\underline{\psi} = f(\underline{\kappa}(x))$. (Recall $\underline{\kappa}$ represents the principal curvatures of the graph of the subsolution $\underline{u}$.) For each t we wish to find an admissible solution u^t in $C^{2,\alpha}(\bar{\Omega})$, $0 < \alpha < 1$, of

$$f(\kappa(\text{graph of } u^t)) = \psi^t(x) \quad \text{in} \quad \bar{\Omega}, \quad 0 \leqq t \leqq 1, \qquad u^t = 0 \quad \text{on} \quad \partial\Omega,$$

starting with $u^0 = \underline{u}$. The function u^1 is then the desired solution u of (1).

As always one has to prove the openness and closedness of the set of t-values in $[0, 1]$ for which such a solution exists. The openness is proved directly with the aid of the implicit function theorem. To prove the closedness it suffices, since G

is concave in $\{D^2 u\}$, to obtain *a priori* estimates for the C^2 norms of the solutions u^t, as explained in the preceding papers [2]–[4]. The rest of this paper is thus taken up with the derivation of such estimates. For convenience we derive the estimates for $\psi^t = \psi$.

In Section 1 we derive an estimate for the C^1 norm of our (admissible) solution u. In Section 2 we show how to estimate the second derivatives of u if we have bounds for them at $\partial\Omega$. In Section 3 we study the effects on the solution surface of various conformal mappings in $\mathbb{R}^{n+1}$. Some of these will be used in the proof of the crucial Proposition 1 of Section 4. In Section 5 that proposition is proved while in Section 4 it is used to establish the desired bounds for the second derivatives of u at $\partial\Omega$. Section 5 contains the most delicate arguments of the paper.

1. The C^1 Estimate

We observe first, see (9) in [3], that there is a $\delta > 0$ such that

$$\Sigma\kappa_i \geqq \delta \quad \text{in} \quad \text{the set } T = \{\kappa \in \Gamma | f(\kappa) \geqq \psi_0\}.$$

It follows from Lemma A and the usual maximum principle that our solution u satisfies (see (11)) for each $x_0 \in \partial\Omega$,

$$\underline{u} \leqq u \leqq 0 \quad \text{in} \quad \Omega. \tag{1.1}$$

This implies

$$|\nabla u| \leqq C \quad \text{on} \quad \partial\Omega. \tag{1.2}$$

To estimate $|\nabla u|$ in Ω we shall obtain a bound for

$$z = |\nabla u| e^{Au},$$

where

$$A = \left[\frac{2}{c_1}\max|\nabla\psi|\right]^{1/2},$$

and c_1 is the constant in condition (10). If z achieves its maximum on $\partial\Omega$, then from (1.2) we have a bound and we are through.

Assume this is not the case; then it achieves its maximum at a point x in Ω. At that point we may suppose

$$|\nabla u| = u_1 > 0, \quad u_\alpha = 0, \quad \alpha > 1.$$

Then $\log u_1 + Au$ takes its maximum there. Consequently, at x,

$$\frac{u_{1i}}{u_1} + Au_i = 0. \tag{1.3}$$

So $u_{11} = -Au_1^2$ and $u_{1\alpha} = 0$ for $\alpha > 1$. After rotation of the coordinates $(x_2, \cdots, x_n)$ we may assume that $u_{ij}(x)$ is diagonal.

We also have, at x,

$$\frac{u_{1ii}}{u_1} - \frac{u_{1i}^2}{u_1^2} + Au_{ii} \leqq 0 \quad \text{for all } i. \tag{1.4}$$

Furthermore, from (12), we find that $a_{il}(x)$ is diagonal and

$$\begin{aligned} a_{11} &= \frac{1}{w}\left(u_{11} - \frac{2u_1^2(u_{11})}{w(1+w)} + \frac{u_1^4 u_{11}}{w^2(1+w)^2}\right) \\ &= \frac{u_{11}}{w}\left(1 - \frac{u_1^2}{w(1+w)}\right)^2 - \frac{u_{11}}{w^3}, \end{aligned} \tag{1.5}$$

while

$$a_{ii} = \frac{u_{ii}}{w} \quad \text{for} \quad i > 1. \tag{1.6}$$

Hence $F^{ij} = \partial F/\partial a_{ij}$ is diagonal and $F^{ii} = \partial f/\partial \kappa_i = f_i$.

Next we use the equation (1)′. Differentiate it with respect to x_1; we obtain (using summation convention),

$$\psi_1 = F^{ii}\frac{\partial}{\partial x_1}a_{ii}.$$

Now, for $i > 1$,

$$a_{ii,1} = \left(\frac{1}{w}\right)_1 u_{ii} + \frac{1}{w}u_{ii1} = -\frac{u_1 u_{11}}{w^2}a_{ii} + \frac{u_{ii1}}{w}$$

and

$$\begin{aligned} a_{11,1} &= \left(\frac{1}{w}\right)_1 \frac{u_{11}}{w^2} + \frac{1}{w}\left(u_{111} - \frac{2u_1^2 u_{111}}{w(1+w)} + \frac{u_1^4 u_{111}}{w^2(1+w)^2}\right) \\ &\quad + \frac{u_{11}}{w}\left[-2\left(\frac{u_1^2}{w(1+w)}\right)_1 + \left(\frac{u_1^4}{w^2(1+w)^2}\right)_1\right] \\ &= \left(\frac{1}{w}\right)_1 \frac{u_{11}}{w^2} + \frac{u_{111}}{w}\left(1 - \frac{w-1}{w}\right)^2 + \frac{u_{11}}{w}\left(\left(1 - \frac{w-1}{w}\right)^2\right)_1 \\ &= \frac{u_{111}}{w^3} + \frac{3}{w^2}\left(\frac{1}{w}\right)_1 u_{11}. \end{aligned}$$

Thus

$$\frac{2f_1 u_1 u_{11}^2}{w^5} + \frac{u_1 u_{11}}{w^2} f_i a_{ii} = -\psi_1 + \frac{1}{w}\sum_{\alpha > 1} f_\alpha u_{1\alpha\alpha} + \frac{1}{w^3} f_1 u_{111}.$$

Using (1.4) we find

$$\frac{2f_1u_1u_{11}^2}{w^5} + \frac{u_1u_{11}}{w^2}f_ia_{ii} \leqq -\psi_1 - \frac{Au_1}{w}\sum_{\alpha>1} f_\alpha u_{\alpha\alpha} + \frac{f_1}{w^3}\left(\frac{u_{11}^2}{u_1} - Au_1u_{11}\right).$$

Thus, by (1.5) and (1.6),

$$\frac{f_1u_{11}^2}{u_1w^5}(w^2-2) + \frac{u_1u_{11}}{w^2}f_ia_{ii} + Au_1f_ia_{ii} \leqq -\psi_1,$$

and using (1.3) we obtain

$$\frac{f_1A^2u_1^4}{u_1w^5}(w^2-2) + \frac{A}{w^2}u_1f_ia_{ii} \leqq -\psi_1.$$

Since $\Sigma f_i a_{ii} = \Sigma f_i \kappa_i > 0$, we see that

$$\frac{1}{w^5}f_1^2A^2u_1^3(u_1^2-1) \leq \max|\nabla\psi|.$$

By (9) it follows that

$$\frac{u_1^3(u_1^2-1)}{w^5} \leqq \frac{\max|\nabla\psi|}{c_1A^2} = \frac{1}{2},$$

by our choice of A. Since $w^2 = 1 + u_1^2$, this yields a bound for u_1 and hence for $\max|\nabla u|e^{Au}$.

We have derived the *a priori* estimate $|\nabla u| \leqq C$ in Ω and hence

$$|u|_{C^1} \leqq C. \tag{1.7}$$

A natural question is: given a solution u of our equation in Ω, with $|u| \leqq C$, can one estimate $|\nabla u|$ in every compact subset of Ω? Our method does not yield such an estimate. N. Korevaar [6] has derived such estimates for $f = [\sigma^{(k)}]^{1/k}$, $1 \leqq k \leqq n$. For $k = 1$ such estimates were proved many years ago.

In estimating second derivatives we will need an improvement of (1.1), namely: for some constant $a > 0$ under control,

$$ad(x) \leqq -u(x) \leqq \frac{1}{a}d(x), \tag{1.8}$$

where $d(x)$ represents the distance from x to $\partial\Omega$.

The right inequality follows immediately from (1.1). To derive the left one we use the fact, stated at the beginning of this section, that for the graph of u, $\Sigma\kappa_i \geqq \rho > 0$. There is a positive number δ depending just on Ω such that at every point $x_0 \in \partial\Omega$ there is a ball in Ω with radius δ touching $\partial\Omega$ only at x_0. Let S

be a sphere of radius $(n+1)/\rho$ lying in $\mathbb{R}^{n+1}$, i.e., x, u-space, with center above the hyperplane $u=0$, and such that its intersection with the hyperplane $u=0$ is a sphere $\tilde{S}$ of radius δ lying in $\bar{\Omega}$. Since each principal curvature of S is less than ρ/n, so that their sum is less than ρ, it follows that the sphere S lies above the graph of u—since it does so on $\tilde{S}$. This yields the left inequality of (1.8).

2. Estimates for Second Derivatives from their Bounds on the Boundary

In this section we shall show how to estimate the second derivatives of u in Ω if we know bounds for them on $\partial\Omega$. Our argument is similar to those we used in [4] and [5]. Let us assume we have a bound

$$|u_{ij}| \leqq \bar{J} \quad \text{on} \quad \partial\Omega. \tag{2.1}$$

To estimate the second derivatives in Ω it suffices, since $\Sigma\kappa_i \geqq \delta > 0$, to estimate $\max \kappa_i$, i.e., the maximum of the principal curvatures in $\bar{\Omega}$.

By (1.7) we have a bound for

$$k = 2\max_{\bar{\Omega}} w; \tag{2.2}$$

recall that $w = (1+|\nabla u|^2)^{1/2}$. Set

$$\begin{aligned} \tau &= 1/w, \\ a &= \frac{1}{k} = \frac{1}{2}\min_{\bar{\Omega}} \tau. \end{aligned} \tag{2.3}$$

Then

$$\frac{1}{\tau - a} \leqq \frac{1}{a} = k. \tag{2.4}$$

It suffices to estimate

$$M := \max_{\bar{\Omega}} \frac{1}{\tau - a}\kappa_i(x), \tag{2.5}$$

where the maximum is also taken over all principal curvatures κ_i. If M is assumed on $\partial\Omega$ we can estimate it in terms of $\bar{J}$ and we are through.

Thus suppose M is achieved at some point x^0 in Ω. Set

$$w(x^0) = W. \tag{2.6}$$

It is convenient to use new coordinates, describing the surface by $v(y)$, where y are tangential coordinates to the surface at the point $(x^0, u(x^0))$. Namely, let

$e_1, \cdots, e_{n+1}$ denote the unit vectors in the directions of the axes, and introduce new orthonormal vectors

$$\varepsilon_1, \cdots, \varepsilon_n, \varepsilon_{n+1},$$

ε_{n+1} being the normal at $x^0 = w^{-1}(-u_1, \cdots, -u_n, 1)$, and ε_1 corresponding to the tangential direction at x^0 with largest principal curvature. We represent the surface near $(x^0, u(x^0))$ by tangential coordinates $y_1, \cdots, y_n$ and $v(y)$ (summation is from 1 to n):

$$x_j e_j + u(x) e_{n+1} = x_j^0 e_j + u(x^0) e_{n+1} + y_j \varepsilon_j + v(y) \varepsilon_{n+1};$$

thus $\nabla v(0) = 0$. Set

$$\omega = \left(1 + |\nabla v|^2\right)^{1/2}.$$

Then the normal curvature in the ε_1 direction is

$$\kappa = \frac{v_{11}}{\left(1 + v_1^2\right)\omega}.$$

In the y coordinates we have the normal

$$N = -\frac{1}{\omega} v_j \varepsilon_j + \frac{1}{\omega} \varepsilon_{n+1},$$

and

$$\tau = \frac{1}{w} = N \cdot e_{n+1} = \frac{1}{\omega W} - \frac{1}{\omega} \sum a_j v_j, \tag{2.7}$$

where $a_j = \varepsilon_j \cdot e_{n+1}$, so $\Sigma a_j^2 \leqq 1$.

At the point $y = 0$ the function

$$\frac{1}{\tau - a} \frac{v_{11}}{\left(1 + v_1^2\right)\omega} \tag{2.8}$$

takes its maximum equal to M. At this point, since the y_1 direction is a direction of principal curvature, we have $v_{1j} = 0$ for $j > 1$. By rotating the $\varepsilon_2, \cdots, \varepsilon_n$, we may achieve that

$$v_{ij}(0) \text{ is diagonal.}$$

Now we begin to compute. At $y = 0$, the log of the function in (2.8) takes its maximum, and hence its first derivatives vanish:

$$\frac{v_{11i}}{v_{11}} - \frac{\tau_i}{\tau - a} - \frac{2 v_1 v_{1i}}{1 + v_1^2} - \frac{\omega_i}{\omega} = 0 \quad \text{for all} \quad i, \tag{2.9}$$

and also

$$0 \geqq \frac{v_{11ii}}{v_{11}} - \frac{v_{11i}^2}{v_{11}^2} - \left(\frac{\tau_i}{\tau - a}\right)_i - 2v_{1i}^2 - v_{ii}^2 \quad \text{for all} \quad i. \tag{2.10}$$

From (2.7) we also find, at $y = 0$, $i = 1, \cdots, n$,

$$\begin{aligned} \tau_i &= -a_i v_{ii}, \\ \tau_{ii} &= -a_j v_{jii} - \frac{v_{ii}^2}{W}. \end{aligned} \tag{2.11}$$

Next we must make use of our differential equation (1); here we rely on some computations from [4] and [5]. According to Lemma 1.1 of [4] the principal curvatures of the surface (in y-coordinates) are the eigenvalues of the symmetric matrix

$$a_{il} = \frac{1}{\omega}\left\{v_{il} - \frac{v_i v_j v_{jl}}{\omega(1+\omega)} - \frac{v_l v_k v_{ki}}{\omega(1+\omega)} + \frac{v_i v_l v_j v_k v_{jk}}{\omega^2(1+\omega)^2}\right\}.$$

At the origin we find therefore that

$$a_{il} = v_{il} \text{ is diagonal,}$$

and

$$\begin{aligned} \frac{\partial a_{il}}{\partial y_j} &= a_{il,j} = v_{ilj}, \\ \frac{\partial^2 a_{il}}{\partial y_1^2} &= a_{il,11} = v_{il11} - v_{11}^2(v_{il} + \delta_{i1}v_{1l} + \delta_{1l}v_{1i}). \end{aligned} \tag{2.12}$$

In the differential equation (1), the function f is a smooth concave function which is invariant on interchange of the κ_i, so $f(\kappa)$ can be written as a smooth function F of the symmetric matrix $A = \{a_{il}\}$. As indicated in [3], F is then also a concave function of its arguments. It is easy to verify that, at a matrix $A = (a_{il})$ which is diagonal,

$$\frac{\partial F}{\partial a_{il}} = \frac{\partial f}{\partial \kappa_i}\delta_{il} = f_i \delta_{il}. \tag{2.13}$$

We proceed to differentiate the equation

$$F(a_{il}) = \psi(x) = \tilde{\psi}(y),$$

first with respect to y_j, to obtain

$$\frac{\partial F}{\partial a_{il}} a_{il,j} = \tilde{\psi}_j.$$

Taking $j = 1$ and differentiating once more with respect to y_1 we find, using the concavity of F,

$$\tilde{\psi}_{11} \leqq \frac{\partial F}{\partial a_{il}} a_{il,11}.$$

Using (2.12) and (2.13) we have then, at $y = 0$,

$$f_i v_{iij} = \tilde{\psi}_j \quad \text{for all} \quad j, \tag{2.14}$$

and

$$\begin{aligned}
\tilde{\psi}_{ii} &\leqq f_i a_{ii,11} \\
&= f_i\left(v_{ii11} - v_{ii}v_{11}^2\right) - 2f_1 v_{11}^3 \\
&\leqq f_i\left(\frac{v_{i11}^2}{v_{i1}} + v_{11}\left(\frac{\tau_i}{\tau - a}\right)_i + 2v_{11}v_{1i}^2 + v_{11}v_{ii}^2 - v_{ii}v_{11}^2\right) - 2f_1 v_{11}^3 \\
&= v_{11} f_i\left(\frac{v_{i11}^2}{v_{11}^2} + \left(\frac{\tau_i}{\tau - a}\right)_i + v_{ii}^2 - v_{11}v_{ii}\right) \\
&= v_{11} f_i\left[\left(\frac{\tau_i}{\tau - a}\right)^2 + \left(\frac{\tau_i}{\tau - a}\right)_i + v_{ii}^2 - v_{11}v_{ii}\right] \\
&= v_{11} f_i\left[\frac{1}{\tau - a}\left(-a_j v_{jii} - \frac{v_{ii}^2}{w}\right) + v_{ii}^2 - v_{11}v_{ii}\right] \\
&= -v_{11} a_j \tilde{\psi}_j \frac{1}{\tau - a} + v_{11} f_i\left[\left(1 - \frac{\tau}{\tau - a}\right)v_{ii}^2 - v_{11}v_{ii}\right].
\end{aligned}$$

Therefore,

$$\begin{aligned}
v_{11} f_i v_{ii}^2 \frac{a}{\tau - a} + v_{11}^2 f_i v_{ii} &\leqq -v_{11} a_j \tilde{\psi}_j \frac{1}{\tau - a} - \tilde{\psi}_{ii} \\
&\leqq C(1 + v_{11})
\end{aligned}$$

and hence

$$M \cdot a \sum f_i v_{ii}^2 + M^2(\tau - a)^2 \sum f_i v_{ii} \leqq C(1 + M).$$

With the aid of (8) it follows that

$$M \leqq C$$

for a suitable constant C under control. Then if (2.1) holds we have established the estimate

$$|u_{ij}| \leqq C \quad \text{in} \quad \bar{\Omega}. \tag{2.15}$$

As in the preceding equation one may ask: under what conditions is it possible to establish purely interior estimates for $|u_{ij}|$, knowing a bound for the C^1 norm of u? This is possible under the additional condition (see (ii)′ in [5])

$$|\kappa|^2 \sum f_i(\kappa) \leqq B \sum f_i \kappa_i^2, \tag{2.16}$$

whenever $\psi_0 \leqq f(\kappa) \leqq \psi_1$. Indeed consider a solution of (1) in a ball $|x| < R$ and let ζ be a cutoff function as in Section 2 of [5]; in that section we considered

$$M := \max_{|x|<R} \zeta \frac{1}{\tau - a} \kappa_i(x) \geqq \frac{1}{\tau - a} \max \kappa_i(0),$$

and derived an inequality at the maximum point ((19) there):

$$\tilde{\psi}_{11} + \frac{1}{\tau - a} v_{11} a_j \tilde{\psi}_j \leqq v_{11} \sum f_i \left[\left(\frac{\zeta_i}{\zeta} \right)^2 - \left(\frac{\zeta_i}{\zeta} \right)_i - \frac{2\zeta_i}{\zeta} \frac{\tau_i}{\tau - a} - v_{ii}^2 \frac{a}{\tau - a} \right].$$

Multiplying this by ζ^2 we see that

$$v_{11} \frac{a\zeta^2}{\tau - a} \sum f_i v_{ii}^2 \leqq v_{11} \sum f_i \left(\frac{C}{R^2} + \frac{C\zeta}{R} \frac{|v_{ii}|}{\tau - a} \right) + C(1 + M).$$

It follows that (recall that $2a \leqq \tau \leqq 1$)

$$v_{11} \frac{a\zeta^2}{\tau - a} \sum f_i v_{ii}^2 \leqq v_{11} \frac{C}{R^2} \sum f_i + C(1 + M)$$

or, by (2.16),

$$v_{11}^3 \frac{a\zeta^2}{\tau - a} \frac{1}{B} \sum f_i \leqq v_{11} \frac{C}{R^2} \sum f_i + C(1 + M).$$

Consequently, with a different constant C,

$$M\left(M^2 - \frac{C}{R^2} \right) \sum f_i \leqq C(1 + M)$$

or, by (7),

$$M\left(M^2 - \frac{C}{R^2}\right)c_0 \leqq C(1 + M).$$

This yields a bound for M and hence the bound

$$|u_{ij}(0)| \leqq C.$$

Remark. Condition (2.16) is not satisfied by $f = [\sigma^{(k)}]^{1/k}$ for $k > 1$.

3. Effect on the Solution Surface of Conformal Mappings of Space

When we derive bounds on the second derivatives of u at $\partial\Omega$ we will make use of various barrier functions. Some are obtained by comparing the solution surface with its image under some conformal mapping in $\mathbb{R}^{n+1}$. In this section we first compute the effect of such mappings.

1. ***Infinitesimal rotation of the independent variables.*** Since $f(\kappa)$ is invariant under such a relation, it follows that for the operator $x_i\partial_j - x_j\partial_i$, $i \neq j$, which is the infinitesimal generator of a rotation, we have

$$L(x_iu_j - x_ju_i) = x_i\psi_j - x_j\psi_i. \tag{3.1}$$

Here L is the linearized operator of G (see (1)′ in the introduction) at u.

2. ***Infinitesimal stretching.*** If t is close to one, then the principal curvatures of the surface $(x/t, (1/t)u(x))$ are $t\kappa(x)$.

Thus if we consider the stretched surface (setting $y = x/t$): $(y, (1/t)u(ty))$, its principal curvatures at y are $t\kappa(ty)$. Hence for this surface we have (renaming y to be x), see (1)′ in the introduction,

$$f(t\kappa(tx)) = G(Dv, D^2v),$$

where $v(x) = (1/t)u(tx)$.

We find that, at $t = 1$,

$$\begin{aligned}\frac{d}{dt}G(Dv, D^2v) &= L\left(\frac{d}{dt}v\right)\\ &= L(ru_r - u).\end{aligned}$$

Here r is the polar coordinate $|x|$. Furthermore, at $t = 1$,

$$\begin{aligned}\frac{d}{dt} f(t\kappa(tx)) &= \sum f_{\kappa_i}(\kappa(x))\kappa_i(x) + \frac{d}{dt} f(\kappa(tx))\\ &= \sum \kappa_i(x) f_{\kappa_i} + \frac{d}{dt}\psi(tx)\\ &= \sum \kappa_i f_{\kappa_i} + r\psi_r(x).\end{aligned}$$

Thus

$$L(ru_r - u) \geqq c_0 + r\psi_r, \qquad \text{by (7)}. \tag{3.2}$$

3. ***Infinitesimal rotation in*** $\mathbb{R}^{n+1}$. Keeping the coordinates $x' = (x_1, \cdots, x_{n-1})$ fixed let us rotate by $d\theta$ in the (x_n, u) variables. To first order in $d\theta$ the image of $(x, u(x))$ under such a rotation is

$$(x', x_n - u(x)\, d\theta, u(x) + x_n\, d\theta).$$

The principal curvatures do not change under such a rotation. Thus to first order in $d\theta$, the image of

$$[x', x_n + u(x)\, d\theta, u(x', x_n + u(x)\, d\theta)]$$

is

$$(x', x_n, u(x', x_n + u(x)\, d\theta) + x_n\, d\theta).$$

Hence if

$$v(x) = u(x', x_n + u(x)\, d\theta) + x_n\, d\theta + \text{higher order in } d\theta,$$

we have, at x,

$$G(Dv, D^2v) = \psi(x', x_n + u(x)\, d\theta).$$

Consequently, if we compute the first-order term in $d\theta$ we find

$$L(x_n + u(x)u_n(x)) = u(x)\psi_n(x). \tag{3.3}$$

Next we take up the more interesting effect of

4. ***Reflection in a sphere in*** $\mathbb{R}^{n+1}$. Consider a hypersurface S in $\mathbb{R}^{n+1}$ with principal curvatures $\kappa_1, \cdots, \kappa_n$. Let us perform a reflection (or inversion) I in a sphere with center at the origin and radius R, $\tilde{S} = I(S)$. For convenience we suppose that the origin does not lie on S. The image of X on S is

$$I(X) = Y = \frac{X}{|X|^2}R^2, \quad \text{thus} \quad X = \frac{Y}{|Y|^2}R^2. \tag{3.4}$$

A useful observation is the following:

LEMMA 1. *The directions of principal curvature of S map into directions of principal curvature of $I(S)$ at the corresponding image. Furthermore, the principal curvature κ' of $I(S)$ at $I(X)$ corresponding to the principal curvature κ of S at X is given by*

$$\tilde{\kappa} = \frac{\kappa}{R^2}|X|^2 + \frac{2}{R^2}X \cdot \nu, \tag{3.5}$$

where ν is the normal to S at X.

Proof: Consider the surface S near X and let $\cdot$ denote differentiation with respect to a parameter on some tangent curve. According to the formula of Rodriguez this direction of the curve is a direction of principal curvature if and only if $\dot{\nu}$ is parallel to $\dot{X}$ as X moves on the curve. In that case, if κ is the corresponding principal curvature, then

$$\dot{\nu} + \kappa\dot{X} = 0.$$

The tangent curve through X maps into a tangent curve to $\tilde{S}$ at $I(X)$. We have only to establish the corresponding formula:

$$\dot{\tilde{\nu}} + \tilde{\kappa}\dot{Y} = 0 \tag{3.6}$$

at $I(X)$, where $\tilde{\nu}$ is the normal on $\tilde{S}$.

Since angles are preserved, the normal $\tilde{\nu}$ on $\tilde{S}$ is given by the direction of the tangent to the image (circle) at $s = 0$ of the line $X + s\nu$, s real. Since $X + s\nu$ maps into

$$Z = \frac{X + s\nu}{|X + s\nu|^2}R^2,$$

and, at $s = 0$,

$$Z_s = \frac{\nu}{|X|^2}R^2 - 2X\frac{(X \cdot \nu)}{|X|^4}R^2,$$

we see that

$$\begin{aligned}\tilde{\nu} &= Z_s/|Z_s| \\ &= \nu - 2(X \cdot \nu)\frac{X}{|x|^2}.\end{aligned}$$

Next we have

$$\dot{Y} = \frac{\dot{X}}{|X|^2}R^2 - 2X\frac{X \cdot \dot{X}}{|X|^4}R^2,$$

while (recall $\dot{X} \cdot \nu = 0$)

$$\dot{\tilde{\nu}} = \dot{\nu} - 2(X \cdot \dot{\nu})\frac{X}{|X|^2} - 2(X \cdot \nu)\frac{\dot{X}}{|X|^2} + 4(X \cdot \nu)(X \cdot \dot{X})\frac{X}{|X|^4}$$

$$= -\kappa\left(\dot{X} - 2(X \cdot \dot{X})\frac{X}{|X|^2}\right) - 2(X \cdot \nu)\left(\frac{\dot{X}}{|X|^2} - 2(X \cdot \dot{X})\frac{X}{|X|^4}\right)$$

$$= -\frac{1}{R^2}(\kappa|X|^2 + 2X \cdot \nu)\dot{Y}.$$

Thus the lemma holds with $\tilde{\kappa}$ given by (3.5).

More generally, if we reflect S in a sphere of radius R with center W, then the new principal curvature of $\tilde{S}$ at $I(X)$ corresponding to κ is

$$\tilde{\kappa} = \frac{\kappa}{R^2}|X - W|^2 + \frac{2}{R^2}(X - W) \cdot \nu(X).$$

4. Bounds for Second Derivatives at the Boundary

In this section, assuming Proposition 1 below we shall establish the estimate (2.1):

$$|u_{ij}| \leqq C \quad \text{on} \quad \partial\Omega. \tag{4.1}$$

PROPOSITION 1. *Let u be the admissible solution of* (1), (2). *Given any $\varepsilon > 0$ there exists a number $\mu > 0$ depending only on ε, Ω and the functions f and ψ, such that in a μ-neighborhood of $\partial\Omega$ we have*

$$u_\nu \leqq \varepsilon,$$

where u_ν is the derivative of u in the interior normal direction.

This proposition will be proved in the next section. We shall now establish (4.1) at any point on $\partial\Omega$. We may suppose that the point is the origin and that the x_n-axis is interior normal there. We may assume that the boundary near 0 is represented by

$$x_n = \rho(x') = \frac{1}{2}\sum_1^{n-1}\lambda_\alpha x_\alpha^2 + O(|x'|^3). \tag{4.2}$$

Here $\lambda_\alpha > 0$ are the principal curvatures of $\partial\Omega$ at the origin.

We first establish the estimates

$$(4.3) \qquad |u_{\alpha n}(0)| \leqq C, \qquad \alpha < n.$$

For $\alpha < n$, let T_α be the operator

$$T_\alpha = \partial_\alpha + \lambda_\alpha(x_\alpha \partial_n - x_n \partial_\alpha).$$

Applying ∂_α to $u(x', \rho(x')) = 0$ we find

$$u_\alpha + u_n \rho_\alpha = 0.$$

Since $\rho = O(|x'|^2)$ and $\rho_\alpha = \lambda_\alpha x_\alpha + O(|x'|^2)$, it follows that

$$(4.4) \qquad |T_\alpha u| \leqq C|x|^2 \quad \text{on} \quad \partial\Omega \quad \text{near the origin.}$$

Since $Lu_\alpha = \psi_\alpha$ and since (3.1) holds, we find

$$(4.5) \qquad |LT_\alpha u| \leqq C_1,$$

where we recall that L is the linearization of G in (1)′ at u.

In the following, Ω_β will represent the small region $\Omega \cap \{x_n < \beta\}$. For δ, β small, set

$$(4.6) \qquad h = ru_r - u - \frac{\delta}{\beta}(x_n + uu_n).$$

LEMMA 4.1. *For suitable choice of δ small, then $\beta = \beta(\delta)$ small, and then A large, the function Ah satisfies in Ω_β the following conditions*:

$$(4.7) \qquad L(Ah) \geqq C_1 \quad \textit{in} \quad \Omega_\beta,$$

$$(4.8) \qquad Ah \leqq -C|x|^2 \quad \textit{on lower boundary of} \quad \Omega_\beta \quad (\textit{i.e., on } \partial\Omega \textit{ there}),$$

$$(4.9) \qquad Ah \leqq -|T_\alpha u| \quad \textit{on upper boundary of} \quad \Omega_\beta \quad (\textit{i.e., on } x_n = \beta).$$

Proof: First, we require $\beta < \mu$ of Proposition 1. By that proposition we have in Ω_β

$$\varepsilon \geqq u_\nu = \sum_1^{n-1} \nu_\alpha u_\alpha + \nu_n u_n.$$

For β small the ν_α are small and ν_n is close to 1; hence

$$(4.10) \qquad u_n \leqq \tfrac{3}{2}\varepsilon.$$

Now by (3.2) and (3.3) we have

$$Lh \geqq c_0 + r\psi_r - \frac{\delta}{\beta} u\psi_n$$

$$\geqq \frac{c_0}{2} - C\delta \text{ if } \beta \text{ is small,}$$

with C independent of β and δ—since $u = 0$ on $\partial\Omega$. Thus

$$Lh \geqq \frac{c_0}{3} \text{ if } \delta \text{ and } \beta \text{ are small.}$$

For A large it follows that (4.7) holds.

On the lower boundary of Ω_β, since $u = 0$ there, we have $|ru_r| \leqq C_2|x|^2$; also $x_n \geqq a|x|^2$, $a > 0$. Hence

$$h = ru_r - \frac{\delta}{\beta} x_n$$

$$\leqq \left(C_2 - \frac{\delta}{\beta} a\right)|x|^2.$$

For δ/β and A large we obtain (4.8).

Finally, on $x_n = \beta$ we have, using (4.10) and $0 \leqq -u \leqq C\beta$,

$$h = \beta u_n + \sum_1^{n-1} x_\alpha u_\alpha - u\left(1 + \frac{\delta}{\beta} u_n\right) - \delta$$

$$\leqq \tfrac{3}{2}\varepsilon\beta + C\beta^{1/2} - u\left(1 + \frac{\delta}{\beta} \cdot \tfrac{3}{2}\varepsilon\right) - \delta$$

$$\leqq \tfrac{3}{2}\varepsilon\beta + C\beta^{1/2} + C\beta + \tfrac{3}{2}C\delta\varepsilon - \delta$$

$$\leqq C\beta^{1/2} + \delta\left(\tfrac{3}{2}C\varepsilon - 1\right)$$

with (different) C independent of $\varepsilon, \beta, \delta$. Now choose ε and β so that

$$\tfrac{3}{2}C\varepsilon \leqq \frac{1}{2}, \qquad C\beta^{1/2} \leqq \frac{\delta}{4},$$

and δ/β large as required in the preceding paragraph. Then we obtain

$$h \leqq -\frac{\delta}{4} \quad \text{on} \quad x_n = \beta,$$

and so (4.9) follows for A large. The lemma is proved.

Using Lemma 2 and the maximum principle we see that

$$h \leqq \pm T_\alpha u \quad \text{in} \quad \Omega_\beta.$$

It follows that at the origin, where h and $T_\alpha u$ vanish,

$$|\partial_n T_\alpha u| \leqq -h_n$$

or

$$|u_{\alpha n}(0)| \leqq \frac{\delta}{\beta}\left(1 + u_n^2(0)\right) \leqq C;$$

(3.3) is proved.

To complete the proof of (4.1) we have to show that $|u_{nn}(0)| \leqq C$. Recall that the principal curvatures of the solution surface at the origin are the eigenvalues of the matrix $a_{il}(0)$ given by (12). Since $u_\alpha(0) = 0$ for $\alpha < n$, one finds easily that $a_{il}(0)$ has the block form

$$a_{il}(0) = \frac{1}{w} \begin{array}{c} \overbrace{\phantom{u_{\alpha\beta}}}^{n-1} \qquad\qquad \\ \begin{pmatrix} u_{\alpha\beta} & w^{-1} u_{\alpha n} \\ & \\ w^{-1} u_{n\beta} & w^{-2} u_{nn} \end{pmatrix} \end{array}.$$

Now since $u(x', \rho(x')) = 0$ we have at the origin

$$u_{\alpha\beta} + u_n \rho_{\alpha\beta} = 0, \qquad \alpha, \beta < n,$$

i.e.,

$$u_{\alpha\beta} + u_n \lambda_\alpha \delta_{\alpha\beta} = 0.$$

Hence from (1.8)

$$\sum_{\alpha,\beta=1}^{n-1} u_{\alpha\beta} \xi_\alpha \xi_\beta \geqq a \sum \lambda_\alpha \xi_\alpha^2 \geqq b|\xi|^2, \qquad b > 0.$$

Thus if $d_1 \leqq \cdots \leqq d_{n-1}$ are the eigenvalues of $u_{\alpha\beta}(0)$ we see that

$$b \leqq d_i \leqq C. \tag{4.11}$$

Suppose that $|u_{nn}(0)|$ can be arbitrarily large. Apply Lemma 1.2 of [3]. (It is only formulated for $a_{\alpha\beta}$ a diagonal $(n-1) \times (n-1)$ matrix, but ours may be diagonalized.) According to the lemma the eigenvalues $\kappa_1, \cdots, \kappa_n$ behave like

$$\kappa_\alpha = \frac{1}{w} d_\alpha + o(1), \qquad \alpha < n,$$

$$\kappa_n = \frac{1}{w^3} u_{nn}(0)\left(1 + O\left(\frac{1}{|u_{nn}(0)|}\right)\right)$$

as $|u_{nn}(0)| \to \infty$. Since Γ lies in the half-space $\Sigma \kappa_i > 0$ it follows that $u_{nn}(0)$ is arbitrarily large. But by (4.11) we see that at the origin $(\kappa_1, \cdots, \kappa_{n-1}, 1)$ lie in a compact subset K of Γ and so if κ_n is arbitrarily large we obtain a contradiction to (6) since $f(\kappa_1, \cdots, \kappa_n) \leqq \psi_1$.

The proof of (4.1) is complete except for that of Proposition 1.

5. Proof of Proposition 1

Suppose as in the preceding section that the origin belongs to $\partial\Omega$ and that the x_n-axis is interior normal there. Given ε we want to find a number $\mu > 0$ (which will work for any boundary point) such that

$$u_n(0, \cdots, 0, x_n) \leqq \varepsilon \quad \text{for} \quad 0 < x_n < \mu. \tag{5.1}$$

Without loss of generality (after a stretching) we may suppose that the graph S of u over Ω lies in the ball $B_{1/2}$ with center at $(0, 0, \cdots, \frac{1}{2}, 0)$ in $\mathbb{R}^{n+1}$.

From (1.8) we know that $u_n < 0$ at the origin and so this remains true in some undetermined neighbourhood. Consider the family of reflections I_δ depending on a parameter $\delta > 0$, in the boundary of the unit ball in $\mathbb{R}^{n+1}$: $B_1(e^\delta) = B^\delta$, with center $e_\delta = (0, \cdots, 0, 1 + \delta, \tilde{C}\delta)$, where $\tilde{C} > 1$ is a large constant to be chosen. S is contained in B^0. As δ becomes positive a portion of S near the origin in $\mathbb{R}^{n+1}$ lies outside B^δ. For very small δ, the reflection $I_\delta(S \cap \mathscr{C}B^\delta)$ does not touch $S \cap B^\delta$; furthermore at any point $X^0 \in S \cap \partial B^\delta$, $I(S \cap \mathscr{C}B^\delta)$ is not tangent to S. Suppose there is a first value of δ for which this statement fails, i.e., for which either

(a) $I_\delta(S \cap \mathscr{C}B^\delta)$ touches S at a point $I_\delta(X^0)$,

or

(b) $I(S \cap \mathscr{C}B^\delta)$ is tangent to S at some point $X^0 \in \partial B^\delta \cap S$.

We shall prove that there is a δ_0 (under control), with $\tilde{C}^2\delta_0 \leqq 1$, such that for $\delta \leqq \delta_0$ both cases are impossible. It then follows that, for $\delta \leqq \delta_0$, if a point $X \in S$ belongs to ∂B^δ, then

$$(X - e_\delta) \cdot \nu(X) < 0.$$

In particular, if we take $X = (0,\cdots,0, x_n, u(0, x_n))$, then

$$(x_n - 1 - \delta)\nu_n(X) + (u - \tilde{C}\delta)\nu_{n+1}(X) < 0.$$

Here

$$\nu_n = -u_n\left(1 + |\nabla u|^2\right)^{-1/2}, \qquad \nu_{n+1} = \left(1 + |\nabla u|^2\right)^{-1/2} \geqq a_0 > 0.$$

Thus for x_n and δ_0 small we have $1 + \delta - x_n > \frac{1}{2}$ and so

$$\begin{aligned} u_n(0, x_n) &< 2|\tilde{C}\delta - u| \\ &< \varepsilon \end{aligned}$$

if x_n, and so δ, are sufficiently small (under control).

Proposition 1 would then be proved.

Suppose case (a) first occurs for some $X^0 = (x, u(x))$. Since case (b) has not occurred for smaller δ' we know that for $X^0 \in S \cap \mathscr{C}B^\delta$ there is a $\delta' < \delta$ such that $X^0 \in \partial B^{\delta'}$ and

$$\left(X^0 - e_{\delta'}\right) \cdot \nu\left(X^0\right) < 0. \tag{5.2}$$

A principal curvature κ of S at X^0 has corresponding to it a principal curvature $\tilde{\kappa}$ of $I_\delta(S)$ at $I_\delta(X^0)$ given by Lemma 1 of Section 3:

$$\tilde{\kappa} = \kappa|X^0 - e_\delta|^2 + 2\left(X^0 - e_\delta\right) \cdot \nu\left(X^0\right). \tag{5.3}$$

Now, with A denoting various constants under control,

$$\begin{aligned} |X^0 - e_\delta|^2 &= |X^0 - e_{\delta'} + e_{\delta'} - e_\delta|^2 \\ &= 1 + 2\left(X^0 - e_{\delta'}\right) \cdot \left(e_{\delta'} - e_\delta\right) + |e_{\delta'} - e_\delta|^2 \\ &\leqq 1 + A(\delta - \delta') + \tilde{C}(\delta - \delta')\left(\tilde{C}\delta' - u(x)\right) + \tilde{C}^2\delta(\delta - \delta') \\ &\leqq 1 + A(\delta - \delta') + A\tilde{C}(\delta - \delta')x_n, \end{aligned} \tag{5.4}$$

since $\tilde{C}^2\delta \leqq 1$ and $-u(x) \leqq Cx_n$.

Since $|X^0 - e_{\delta'}| = 1$ we have, for $x = (x', x_n)$,

$$|x'|^2 + \left(1 + \delta' - x_n\right)^2 + |\tilde{C}\delta' - u(x)|^2 = 1.$$

Hence

$$|x'|^2 + \left(\tfrac{1}{2} - x_n\right)^2 + 2\left(\tfrac{1}{2} - x_n\right)\left(\tfrac{1}{2} + \delta'\right) + \left(\tfrac{1}{2} + \delta'\right)^2 + \left(\tilde{C}\delta' + Ax_n\right)^2 \geqq 1.$$

The sum of the first two terms is at most $\frac{1}{4}$, and it follows that

$$x_n(1 + 2\delta') \leqq 2\delta' + A\tilde{C}^2\delta'^2 + Ax_n^2$$
$$\leqq A\left(\delta' + x_n^2\right)$$

(the A keeps changing). It follows that

$$x_n \leqq A\delta'.$$

Inserting this in (5.4) we find

$$|X^0 - e_\delta|^2 \leqq 1 + A(\delta - \delta'). \tag{5.5}$$

Using this one finds also that

$$|X^0 - I_\delta(X^0)| \leqq A(\delta - \delta'). \tag{5.5$'$}$$

For the last term in (5.3) we have

$$\begin{aligned} 2\left(X^0 - e_\delta\right) \cdot \nu\left(X^0\right) &= 2\left(X^0 - e_{\delta'}\right) \cdot \nu\left(X^0\right) + 2(e_{\delta'} - e_\delta) \cdot \nu\left(X^0\right) \\ &\leqq 2(e_{\delta'} - e_\delta) \cdot \nu\left(X^0\right) \qquad \text{by (5.2)} \\ &= 2\nu_n(\delta' - \delta) + 2\nu_{n+1}\tilde{C}(\delta' - \delta) \\ &\leqq -c\tilde{C}(\delta - \delta'), \end{aligned} \tag{5.6}$$

with $c > 0$ (under control) since

$$\nu_{n+1} = \left(1 + |\nabla u|^2\right)^{-1/2} \geqq a_0 > 0.$$

Inserting (5.5) and (5.6) into (5.3) we infer that

$$\tilde{\kappa} \leqq \kappa(1 + A(\delta - \delta')) - c\tilde{C}(\delta - \delta'). \tag{5.7}$$

Now at the point $I_\delta(X^0)$ of contact, the principal curvatures $\tilde{\kappa}$ are not less than the principal curvatures of S at $I_\delta(X^0)$. Hence

$$\begin{aligned} f(\tilde{\kappa}_1, \cdots, \tilde{\kappa}_n) &\geqq f\left(\kappa\left(I_\delta(X^0)\right)\right) = \psi\left(I_\delta X^0\right) \\ &\geqq \psi(x) - B|X^0 - I_\delta X^0| \\ &\geqq \psi(x) - B(\delta - \delta'), \end{aligned} \tag{5.8}$$

by (5.5)$'$—for suitable constants B under control.

On the other hand, by (5.7), if we set $\delta - \delta' = \tau$,

$$f(\tilde{\kappa}) = f(\tilde{\kappa}_1, \cdots, \tilde{\kappa}_n) \leqq f\big(\kappa_1(1 + A\tau) - c\tilde{C}\tau, \cdots, \kappa_n(1 + A\tau) - c\tilde{C}\tau\big)$$
$$\leqq f\big(\kappa(1 + A\tau)\big) - c\tilde{C}\tau \sum f_{\kappa_i}\big(\kappa(1 + A\tau)\big),$$

by concavity of f. According to (9)

$$f(\kappa(1 + A\tau)) \leqq (1 + A\tau) f(\kappa)$$
$$= (1 + A\tau)\psi(x)$$

and so

$$f(\tilde{\kappa}) \leqq (1 + A\tau)\psi(x) - c\tilde{C}\tau \sum f_{\kappa_i}\big(\kappa(1 + A\tau)\big). \tag{5.9}$$

By (3), $f(\kappa(1 + A\tau) \geqq \psi(x)$ and hence, by (7), $\Sigma f_i(\kappa(1 + A\tau)) \geqq c_0$. Therefore,

$$f(\tilde{\kappa}) \leqq (1 + A\tau)\psi(x) - c_0 c\tilde{C}\tau. \tag{5.10}$$

Combining this with (5.8) we see that

$$\psi(x) - B\tau \leqq (1 + A\tau)\psi(x) - c_0 c\tilde{C}\tau$$

or

$$c_0 c\tilde{C} \leqq B + A\psi(x) \leqq B + A\psi_1.$$

If now $\tilde{C}$ were chosen so large that this cannot hold, it follows that for our corresponding small δ, case (a) cannot occur.

Turn now to case (b). For any $X \in (S \cap \mathscr{C}B^{\delta})$, $X \in \partial B^{\delta'}$, some of the computations above, in particular (5.10), hold. If $\tilde{X} = I_\delta(X) = (\tilde{x}, \tilde{x}_{n+1})$, then $\psi(x) \leqq \psi(\tilde{x}) + A(\delta - \delta')$ so that from (5.10) we find

$$\begin{aligned} f(\tilde{\kappa}) &\leqq \psi(\tilde{x}) + \big(A - c_0 c\tilde{C}\big)\tau \\ &\leqq \psi(\tilde{x}), \end{aligned} \tag{5.11}$$

for $\tilde{C}$ sufficiently large.

Thus the reflected surface with coordinates $(\tilde{x}, \tilde{x}_{n+1})$ lies above S, i.e., $\tilde{x}_{n+1} > u(\tilde{x})$. Setting $\tilde{x}_{n+1} = \tilde{u}(\tilde{x})$ we see that $\tilde{u}(\tilde{x}) > u(\tilde{x})$ and

$$G(D\tilde{u}, D^2\tilde{u}) \leqq G(Du, D^2u) \quad \text{at} \quad \tilde{x}.$$

The function $\tilde{u} - u$ has a minimum, namely zero, at $(X_1^0, \cdots, X_{n-1}^0) = x^0$. By the Hopf lemma the tangency there in case (b) cannot occur. Proposition 1 is proved and so therefore is Theorem 1.

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