J. Math. Soc. Japan
Vol. 19, No. 4, 1967

Nonlinear semi-groups in Hilbert space

By Yukio KŌMURA

(Received April 3, 1967)

In this paper we intend to study the theorem of Hille-Yosida in case of semi-groups of nonlinear contraction operators in Hilbert spaces. Our main results are the following: a nonlinear dissipative operator A in a Hilbert space H generates uniquely a nonlinear semi-group if $(I-A)^{-1}$ is defined on H (Theorem 4), and conversely, the infinitesimal generator A_0 of a nonlinear contraction semi-group in H has an (dissipative) extension A such that $(I-A)^{-1}$ is defined on H and A generates the given nonlinear semi-groups (Theorem 5). Our nonlinear dissipative operator A is in general multi-valued. Hence the set $\{(x, -Ax) | x \in D(A)\}$ is monotone in the sense of Minty. Some part of our results (e.g. Theorem 2) is obtained more easily from the theory of monotone operators and of monotone sets by Minty [7]. But we shall make use of the method of semi-groups.

In recent years there appeared many works on nonlinear evolution equations in Hilbert or Banach spaces, for instance see Browder [1], Kato [2], Segal [9] and Sobolevskii [10]. But most of them are concerned with the semilinear case: $\frac{d}{dt} u(t) = A(t)u(t) + f(t, u)$, where $A(t)$ is a linear unbounded operator, and $f(t, \cdot)$ is a nonlinear perturbation.

From the view points of pure theory and its application both, it should be desirable to solve more general nonlinear evolution equations. The author's intension here is to treat the case of not necessarily semilinear (or such) evolution equations.

I would express my hearty thanks to Professor Jörgens who suggested me this problem, and also to Professor Tanabe who gave me kind advices.

1. Nonlinear semi-groups and infinitesimal generators.

Let H be a Hilbert space. A continuous one-parameter semi-group $\{T_t | 0 \leqq t < \infty\}$ of nonlinear contraction operators on H is defined by the following conditions:

1) For any fixed $t \geqq 0$, T_t is a continuous (nonlinear) operator defined on H into H.

2) For any fixed $x \in H$, $T_t x$ is strongly continuous in t.

3) $T_{t+s}=T_tT_s$ for $t, s\geqq 0$, and $T_0=I$ (I means the identity mapping).

4) $\|T_tx-T_ty\|\leqq\|x-y\|$ for every $x, y\in H$.

We call such a family $\{T_t\}$ simply nonlinear contraction semigroup. The *strict infinitesimal generator* A_0 of a nonlinear semigroup $\{T_t\}$ is defined by

$$A_0x=\lim_{h\downarrow 0}\frac{T_hx-x}{h}\qquad x\in H,$$

if the right side exists in H. (We use the notation "lim" in the sense of norm topology, unless otherwise stated.) By virtue of the contraction condition 4), A_0 satisfies the following

5) $$\mathrm{Re}\langle A_0x-A_0y, x-y\rangle\leqq 0,\quad\text{for every}\quad x, y\in D(A_0),$$

where $D(A_0)$ means the definition domain of A_0. In fact, the inequality 5) follows from the following

$$\mathrm{Re}\left\langle\frac{T_hx-x}{h}-\frac{T_hy-y}{h}, x-y\right\rangle=\frac{1}{h}(\mathrm{Re}\langle T_hx-T_hy, x-y\rangle-\|x-y\|^2)$$
$$\leqq\frac{1}{h}\|x-y\|(\|T_hx-T_hy\|-\|x-y\|)\leqq 0.$$

In general, a mapping A satisfying the condition 5) is called dissipative. For a multi-valued mapping A (i.e. for an A which maps an element $x\in D(A)$ to a subset Ax of H), we also say that A is *dissipative*, if the following condition is satisfied:

$$\mathrm{Re}\langle x'-y', x-y\rangle\leqq 0\qquad\text{for any } x'\in Ax, y'\in Ay.$$

For a linear semi-group $\{T_t\}$, it holds that $A_0T_t\supset T_tA_0$. But for a nonlinear semi-group, it is not the case. So we cannot tell whether $T_tx\in D(A_0)$ for every $x\in D(A_0)$. Hence we shall introduce the notion of *Φ-infinitesimal generator* A_Φ of a nonlinear semi-group $\{T_t\}$. Let $\Phi=\{\varphi\}$ be an ultra-filter of sets $\varphi\subset(0,\infty)$, which converges to 0. Then A_Φ is defined by

$$A_\Phi x=\underset{h\in\varphi\in\Phi}{\text{w-lim}}\frac{T_hx-x}{h},\qquad x\in H,$$

if the right side exists in H. (We use "w-lim" for the limit pertaining to the weak topology.) Evidently the Φ-infinitesimal generator A_Φ is an extension of the strict infinitesimal generator A_0, and in case of a linear semi-group, we have $A_\Phi=A_0$.

THEOREM 1. *The domain $D(A_\Phi)$ of a Φ-infinitesimal generator A_Φ of a nonlinear contraction semi-group $\{T_t\}$ does not depend on Φ. Moreover it holds good that for $x\in D(A_\Phi)$*

6) $$T_tx\in D(A_\Phi)\qquad\textit{for all } t\geqq 0,$$

7) $\quad T_t x \in D(A_0) \quad$ *for a.e.* $t \geqq 0$, *and* $T_t x = x + \int_0^t A_0 T_s x ds$.

PROOF. First we shall prove that $T_t x$ is strongly absolutely continuous in t for every fixed $x \in D(A_\Phi)$, more precisely,

8) $$\sup \{\sum \|T_{t_{i+1}}x - T_{t_i}x\| \,|\, t' = t_0 < t_1 < \cdots < t_k = t''\} \leqq (t'' - t') \varliminf_{\Phi} \left\| \frac{T_h x - x}{h} \right\|$$

for any interval $[t', t''] \subset [0, \infty)$. We fix an arbitrary division $t_1' = t_0 < t_1 < \cdots < t_k = t''$. Let ε be an arbitrary positive constant. Since $T_t x$ is continuous in t, there exists a constant $\delta > 0$ such that

$$|t - t_j| < \delta \quad \text{implies} \quad \|T_t x - T_{t_j} x\| < \varepsilon.$$

We pick up $h_0 \in \varphi \in \Phi$ such that

$$0 < h_0 < \delta, \qquad \left\| \frac{T_{h_0} x - x}{h_0} \right\| \leqq \varliminf_{\Phi} \left\| \frac{T_h x - x}{h} \right\| + \varepsilon.$$

From the relation $0 < h_0 < \delta$, it follows that $|n_j h_0 - t_j| < \delta$ for a suitable integer n_j. Then we have

$$\begin{aligned}
&\sum_{j=1}^{k} \| T_{t_j} x - T_{t_{j-1}} x \| \\
&\leqq \sum_{j=1}^{k} (\| T_{t_j} x - T_{n_j h_0} x \| + \| T_{n_j h_0} x - T_{n_{j-1} h_0} x \| + \| T_{n_{j-1} h_0} x - T_{t_{j-1}} x \|) \\
&\leqq 2\varepsilon k + \sum_{j=1}^{k} \| T_{n_j h_0} x - T_{n_{j-1} h_0} x \| \\
&\leqq 2\varepsilon k + \sum_{n=1}^{\left[\frac{t''-t'}{h_0}+1\right]} \| T_{n h_0} x - T_{(n-1) h_0} x \| \\
&\leqq 2\varepsilon k + \sum_{n=1}^{\left[\frac{t''-t'}{h_0}+1\right]} \| T_{h_0} x - x \| \\
&\leqq 2\varepsilon k + \sum_{n=1}^{\left[\frac{t''-t'}{h_0}+1\right]} h_0 \Big(\varliminf_{\Phi} \left\| \frac{T_h x - x}{h} \right\| + \varepsilon \Big) \\
&\leqq 2\varepsilon k + (t'' - t' + \delta) \Big(\varliminf_{\Phi} \left\| \frac{T_h x - x}{h} \right\| + \varepsilon \Big).
\end{aligned}$$

Since ε and δ can be chosen arbitrarily small, the inequality 8) is proved.

The independence of $D(A_\Phi)$ on Φ follows from 8). In fact, by 8) $x \in D(A_\Phi)$ implies $\varlimsup_{h \downarrow 0} \left\| \frac{T_h x - x}{h} \right\| < \infty$, and conversely, $\varlimsup_{h \downarrow 0} \left\| \frac{T_h x - x}{h} \right\| < \infty$ implies the existence of w-$\lim_{\Phi} \frac{T_h x - x}{h}$, since a bounded set in H is relatively compact in the weak topology. That is,

$$D(A_\phi)=\left\{x\,\middle|\,\overline{\lim_{h\downarrow 0}}\left\|\frac{T_h x-x}{h}\right\|<\infty\right\}.$$

Thus we obtain 6) also, since $\overline{\lim_{h\downarrow 0}}\left\|\frac{T_{t+h}x-T_t x}{h}\right\|\leqq\overline{\lim_{h\downarrow 0}}\left\|\frac{T_h x-x}{h}\right\|$.

Now we can show the last assertion 7). By virtue of Lemma in the Appendix, $T_t x$ has the strong derivative $\frac{d}{dt}T_t x$ for a. e. t. The definition of A_0 implies $\frac{d}{dt}T_t x=A_0 T_t x$. Thus we have

$$T_t x-x=\int_0^t\frac{d}{ds}T_s x ds=\int_0^t A_0 T_s x ds.$$

Corollary. *The closure of $D(A_0)$ = the closure of $D(A_\phi)$.*

Proof. For any $x\in D(A_\phi)$, there exists a sequence $t_k\downarrow 0$ such than $T_{t_k}x\in D(A_0)$. Since $T_{t_k}x\to x$, we have $x\in$ the closure of $D(A_0)$. Hence the closure of $D(A_\phi)\subset$ the closure of $D(A_0)$. The converse including relation is obvious.

Q. E. D.

We cannot tell whether the strict infinitesimal generator A_0 of every nonlinear contraction semi-group is densely defined in H or not. (It seems that every strict infinitesimal generator is nontrivial, that is, its domain is not empty.) Further for positive λ, $(I-\lambda A_\phi)^{-1}$ is not necessarily defined on H (cf. example 2 in §4). However we have the following

Theorem 2. *Let $\{T_t\}$ be a nonlinear contraction semi-group with a nontrivial strict infinitesimal generator A_0. Then A_0 has a dissipative (multi-valued) extension A such that $(I-A)^{-1}$ is (one-valued,) continuous and defined on H.*

Proof. Let $A_h=\frac{T_h-I}{h}$, for $h>0$. First we shall show that $(I-A_h)^{-1}$ is continuous and defined on H. For $x, y\in H$, we have seen already

$$\operatorname{Re}\langle A_h x-A_h y, x-y\rangle\leqq 0.$$

Thus A_h is dissipative, which implies in general the continuity of $(I-\lambda A_h)^{-1}$ for fixed $\lambda>0$. In fact, $(I-\lambda A_h)^{-1}$ is a contraction, which we see as follows. For $x, y\in R((I-\lambda A_h))$ (= the range of $(I-\lambda A_h)$) we put $x'=(I-\lambda A_h)^{-1}x$ and $y'=(I-\lambda A_h)^{-1}y$. Then we have

$$9)\qquad \begin{aligned}\|x-y\|^2&=\|x'-\lambda A_h x'-y'+\lambda A_h y'\|^2\\&=\|x'-y'\|^2+\|\lambda A_h x'-\lambda A_h y'\|^2-2\lambda\operatorname{Re}\langle A_h x'-A_h y', x'-y'\rangle\\&\geqq\|x'-y'\|^2+\|\lambda A_h x'-\lambda A_h y'\|^2,\end{aligned}$$

which implies $\|(I-\lambda A_h)^{-1}x-(I-\lambda A_h)^{-1}y\|\leqq\|x-y\|$.

Now let us prove that $(I-A_h)^{-1}$ is defined on H. Let x be an arbitrary element of H. It holds that $y=(I-A_h)^{-1}x$ for $y\in H$ if and only if $y-A_h y=x$, or equivalently, $y=\frac{h}{1+h}x+\frac{1}{1+h}T_h y$. The mapping $P: z\to\frac{h}{1+h}x+\frac{1}{1+h}T_h z$ satisfies $\|Pz-Pz'\|\leqq\frac{1}{1+h}\|z-z'\|$, hence the equation $y=Py$ has a solution.

We shall construct a dissipative operator A, an extension of A_0, such that $(I-A)^{-1}$ is defined on H. Let $y\in D(A_0)$. Then $x=y-\lim_{h\downarrow 0}A_h y$ exists. We put $y_h=(I-A_h)^{-1}x$ and $x_h=y-A_h y$. Then $\|x-x_h\|\to 0$ as $h\downarrow 0$. By 9) we have $\|x-x_h\|\geqq\|y-y_h\|$, hence $\|y-y_h\|\to 0$ as $h\downarrow 0$. That is to say,

10) $$\lim_{h\downarrow 0}(I-A_h)^{-1}x=y \qquad \text{for } y\in D(A_0),\ x=y-A_0 y\,.$$

By the assumption of nontriviality of $D(A_0)$ we can pick up an element $y_0\in D(A_0)$ and put $x_0=y_0-A_0 y_0$. For an arbitrary fixed element $x\in H$ the inequality

$$\|(I-A_h)^{-1}x-(I-A_h)^{-1}x_0\|\leqq\|x-x_0\|$$

implies the boundedness of $\{(I-A_h)^{-1}x\,|\,0<h\leqq h_0\}$. Let $\Phi=\{\varphi\}$ be an ultrafilter of sets $\varphi\subset(0,\infty)$ converging to 0. Then

$$y=\underset{h\in\varphi\in\Phi}{\text{w-lim}}\,(I-A_h)^{-1}x$$

exists, since a bounded set in H is relatively compact in the weak topology. Now we define a (multi-valued) mapping A as follows:

$$Ay=\{y-x\,|\,x\in H,\ y=\underset{\Phi}{\text{w-lim}}\,(I-A_h)^{-1}x\}\,.$$

Evidently $(I-A)^{-1}x=\underset{\Phi}{\text{w-lim}}\,(I-A_h)^{-1}x$, so $(I-A)^{-1}$ is defined on H. Moreover, by 10) we see that $Ax\ni A_0x$ if $x\in D(A_0)$. This means that A is an extension of A_0. Let $y, y'\in D(A)$ and $x\in y-Ay$, $x'\in y'-Ay'$. We let $y_h=(I-A_h)^{-1}x$ and $y_h'=(I-A_h)^{-1}x'$. Then $\underset{\Phi}{\text{w-lim}}\,y_h=y$ and $\underset{\Phi}{\text{w-lim}}\,y_h'=y'$. Notice that $y-x-(y'-x')$ may be an arbitrary element of $Ay-Ay'$. The dissipativity of A follows from the following inequality:

$$\begin{aligned}\operatorname{Re}\langle y-x-(y'-x'),\,y-y'\rangle&=\|y-y'\|^2-\operatorname{Re}\langle x-x',\,y-y'\rangle\\&\leqq\lim_{\Phi}\|y_h-y_h'\|^2-\lim_{\Phi}\operatorname{Re}\langle x-x',\,y_h-y_h'\rangle\\&=\lim_{\Phi}\operatorname{Re}\langle y_h-x-(y_h'-x'),\,y_h-y_h'\rangle\\&=\lim_{\Phi}\operatorname{Re}\langle A_hy_h-A_hy_h',\,y_h-y_h'\rangle\leqq 0\,.\end{aligned}$$

2. Weak solutions of abstract differential equations.

In this section we shall discuss a generalization of solutions of the Cauchy problem in H

$$11)\quad \begin{cases} \dfrac{d}{dt}f(t)\in Af(t), & 0\leqq t\leqq t_0, \\ f(0)=x. \end{cases}$$

for a multi-valued mapping $A:H\to H$. Let $f(t)$ be absolutely continuous in $[0, t_0]$. If $f(t)$ is contained in $D(A)$ for a.e. t and satisfies 11), then $f(t)$ is called a *genuine solution* of 11).

We denote by $C_H[0, t]$ the space of all H-valued strongly continuous functions on the interval $[0, t]$, and $L_H^1[0, t]$ the space of all H-valued strongly summable functions on $[0, t]$. These spaces $C_H[0, t]$ and $L_H^1[0, t]$ are Banach spaces with respect to the norm $\|f\|_\infty=\sup_{0\leqq s\leqq t}\|f(s)\|$ and the norm $\|f\|_1=\int_0^t\|f(s)\|ds$ respectively. The scalar product of $f\in C_H[0, t]$ and $g\in L_H[0, t]$ is defined by

$$(f, g)_t=\int_0^t\langle f(s), g(s)\rangle ds.$$

The mapping A is considered in itself as a (multi-valued) mapping from $C_H[0, t]$ to $L_H^1[0, t]: f\to\{g\in L_H^1[0, t]\,|\,g(s)\in Af(s)$ for a.e. $s\}$, which we denote again by A.

Now we introduce the following (multi-valued) mapping

$$\tilde{A}: C_H[0, t]\ni f\to\{g\in L_H^1[0, t]\,|\,{}^\exists f_n\in D(A)\subset C_H[0, t],$$
$$g_n\in Af_n,\ \lim_{n\to\infty}f_n=f,\ \sigma\text{-}\lim_{n\to\infty}g_n=g\},$$

where $\sigma=\sigma(L_H^1[0, t], C_H[0, t])$ is the weak topology of $L_H^1[0, t]$ with respect to $C_H[0, t]$. The mapping $\tilde{A}$ is called the *associated extension* of A.

DEFINITION. A function $f\in C_H[0, t_0]$ is called a *weak solution* of the equation 11), if there exists a sequence of absolutely continuous solutions $\{f_n\}$ of

$$12)\quad \frac{d}{dt}f_n(t)\in\tilde{A}f_n(t) \qquad \text{for a.e. } t\ \ 0\leqq t\leqq t_0,$$

such that $f_n\to f$ in $C_H[0, t_0]$.

Notice that a genuine solution of 11) is necessarily a weak solution. The associated extension $\tilde{A}$ of A may be multi-valued, even if A is one-valued. Nevertheless we have the following

THEOREM 3. *Let A be a dissipative (multi-valued) mapping. Then weak solutions of the Cauchy problem* 11) *are unique.*

PROOF. Let $f^1, f^2\in D(\tilde{A})$ and $g^1\in\tilde{A}f^1, g^2\in\tilde{A}f^2$. We shall show that

$$13)\quad \mathrm{Re}\,(g^1-g^2, f^1-f^2)_t\leqq 0, \qquad 0\leqq t\leqq t_0.$$

By the definition of $\tilde{A}$, there exist two sequences $\{f_n^{(j)}\in D(A)\}$, $\{g_n^{(j)}\in Af_n^{(j)}\}$, $j=1, 2$, such that $\lim_{n\to\infty}f_n^{(j)}=f^j$ in the norm of $C_H[0, t]$ and $\sigma\text{-}\lim_{n\to\infty}g_n^{(j)}=g^j$.

where $\sigma=\sigma(L_H^1[0, t], C_H[0, t])$. Notice that $\mathrm{Re}\langle g^{(1)}(s)-g^{(2)}(s), f^{(1)}(s)-f^{(2)}(s)\rangle\leqq 0$ for a. e. s. Hence we have $\mathrm{Re}(g^{(1)}-g^{(2)}, f^{(1)}-f^{(2)})_t=\lim_{n\to\infty}\mathrm{Re}(g_n^{(1)}-g_n^{(2)}, f^{(1)}-f^{(2)})_t$ $=\lim_{n\to\infty}\mathrm{Re}(g_n^{(1)}-g_n^{(2)}, f_n^{(1)}-f_n^{(2)})_t\leqq 0$, since the restrictions of $f_n^{(j)}$ to $C_H[0, t]$ and of $g_n^{(j)}$ to $L_H^1[0, t]$ are respectively in the norm and in $\sigma(L_H^1[0, t], C_H[0, t])$ convergent to the restrictions of $f^{(j)}$ and $g^{(j)}$.

Now let f^1 and f^2 be two genuine solutions of 12) with Cauchy data $f^1(0)=x^1$ and $f^2(0)=x^2$. Then we have by 13)

$$\begin{aligned}&\|f^1(t)-f^2(t)\|^2-\|f^1(0)-f^2(0)\|^2\\&=\int_0^t\frac{d}{ds}\|f^1(s)-f^2(s)\|^2ds\\&=\int_0^t 2\,\mathrm{Re}\left\langle\frac{d}{ds}f^1(s)-\frac{d}{ds}f^2(s), f^1(s)-f^2(s)\right\rangle ds\\&=2\,\mathrm{Re}\left(\frac{d}{ds}f^1-\frac{d}{ds}f^2, f^1-f^2\right)_t\leqq 0,\end{aligned}$$

since $\frac{d}{ds}f^1\in\tilde{A}f^1$, $\frac{d}{ds}f^2\in\tilde{A}f^2$. Thus we have

$$\|f^1(t)-f^2(t)\|\leqq\|f^1(0)-f^2(0)\|,\qquad 0\leqq t\leqq t_0.$$

By the definition of weak solutions, the above inequality holds good for two weak solutions f^1 and f^2. Hence the uniqueness of our Cauchy problem is established.

3. Generation of nonlinear semi-groups.

We shall state our main theorem as follows.

THEOREM 4. *Let A be a densely defined dissipative (multi-valued) operator in H. If $(I-A)^{-1}$ is defined on H, then A generates a nonlinear contraction semi-group $\{T_t\}$, that is, there exists a uniquely determined nonlinear contraction semi-group $\{T_t\}$ such that for each $x\in H$ T_tx is a weak solution of* 11).

PROOF. We divide the proof in several steps.

I. First we shall show that

$$D((I-\lambda A)^{-1})=H,\qquad 0<\lambda\leqq 1.$$

Notice that $(I-\lambda A)^{-1}$ is a contraction (see 9)) and hence $(I-\lambda A)^{-1}$ is one-valued. Let x be an arbitrary fixed element of H, and μ be a number such that $\frac{1}{2}<\mu\leqq 1$. The equality $y=(I-\mu A)^{-1}x$ means $y-\mu Ay=x$, or equivalently, $y=(I-A)^{-1}\left(\frac{1}{\mu}x-\frac{1-\mu}{\mu}y\right)$. The mapping $P_x: z\to(I-A)^{-1}\left(\frac{1}{\mu}x-\frac{1-\mu}{\mu}z\right)$

satisfies $\|P_x z - P_x z'\| \leqq \frac{1-\mu}{\mu} \|z-z'\|$. Hence the equation $P_x y = y$ has a solution, since $(I-A)^{-1}$ and hence P_x also, are defined on H. Repeating this process, we see that $(I-\mu^k A)^{-1}$ is defined on H for $k=1, 2, \cdots$. For a number λ, $0<\lambda \leqq 1$, we may put $\lambda = \mu^k$ for a suitable μ and k.

II. Let A_n be a mapping: $x - \frac{1}{n} y \to y$ for $x \in D(A)$, $y \in Ax$. Then A_n is one-valued. In fact, let $x - \frac{1}{n} y = x' - \frac{1}{n} y'$ for $y \in Ax$, $y' \in Ax'$. Then $x = \left(I - \frac{1}{n} A\right)^{-1} \left(x - \frac{1}{n} y\right) = \left(I - \frac{1}{n} A\right)^{-1} \left(x' - \frac{1}{n} y'\right) = x'$, since $\left(I - \frac{1}{n} A\right)^{-1}$ is one-valued. This implies $y = y'$. Notice that $A_n = A\left(I - \frac{1}{n} A\right)^{-1}$, if A is one-valued.

We shall show that A_n is dissipative and generates a nonlinear contraction semi-group $\{T_t^{(n)}\}$ satisfying

14) $$\|A_n T_t^{(n)} x\| \leqq \|A_n x\| .$$

Let x, x' be two arbitrary elements of H and let $y = \left(I - \frac{1}{n} A\right)^{-1} x$, $y' = \left(I - \frac{1}{n} A\right)^{-1} x'$. For suitable $y_A \in Ay$ and $y'_A \in Ay'$ we have $x = y - \frac{1}{n} y_A$ and $x' = y - \frac{1}{n} y'_A$. Hence $A_n x = y_A$, $A_n x' = y'_A$. It holds that

$$\begin{aligned} \operatorname{Re} \langle A_n x - A_n x', x - x' \rangle &= \operatorname{Re} \langle y_A - y'_A, y - \frac{1}{n} y_A - y' + \frac{1}{n} y'_A \rangle \\ &= \operatorname{Re} \langle y_A - y'_A, y - y' \rangle - \frac{1}{n} \langle y_A - y'_A, y_A - y'_A \rangle \leqq 0 . \end{aligned}$$

Thus A_n is dissipative. Moreover in a similar way as in 9) we have

$$\begin{aligned} \|x - x'\|^2 &\geqq \|y - y'\|^2 + \frac{1}{n^2} \|y_A - y'_A\|^2 \\ &\geqq \frac{1}{n^2} \|A_n x - A_n x'\|^2 . \end{aligned}$$

Therefore the dissipative operator A_n satisfies two conditions: a) *A_n is continuous*, and b) *A_n maps bounded sets into bounded sets.* This implies that the equation

$$\begin{cases} \frac{d}{dt} f(t) = A_n f(t) \\ f(0) = x, \qquad x \in H, \end{cases}$$

has a unique solution (cf. Brodwer [1] or Kato [4]). Since A_n is independent of t, we can write $f(t) = T_t^{(n)} x$, and as is easily seen, $\{T_t^{(n)}\}$ is a nonlinear contraction semi-group. Notice that $T_t^{(n)}x$ is continuously differentiable in the strong topology for any $x \in H$. Since $T_t^{(n)}$ is a contraction mapping, we have $\| T_h^{(n)}x - x \| \geqq \| T_{t+h}^{(n)}x - T_t^{(n)}x \|$, which implies

$$\| A_n x \| = \| \lim_{h \downarrow 0} \frac{1}{h}(T_h^{(n)}x - x) \| \geqq \| \lim_{h \downarrow 0} \frac{1}{h}(T_{t+h}^{(n)}x - T_t^{(n)}x) \|$$
$$= \| A_n T_t^{(n)} x \| .$$

III. We fix an arbitrary element x of $D(A)$. We shall prove that $T_t^{(n)}x$ is convergent uniformly in t in every finite interval. Let $y \in Ax$, $x_n = x - \frac{1}{n}y$, $n = 1, 2, \cdots$. We introduce a conventional notation A': $A'\left(I - \frac{1}{n}A\right)^{-1} T_s^{(n)} x_n$ denotes a suitable element of $A\left(I - \frac{1}{n}A\right)^{-1} T_s^{(n)} x_n$, that is, $A_n T_s^{(n)} x_n$. Strictly speaking, A' depends on s and n. But there are no confusion. We consider A' as a restriction of A, and so A' is dissipative. Then we have

15)
$$\| T_t^{(m)}x_m - T_t^{(n)}x_n \|^2 - \| x_m - x_n \|^2 = \int_0^t \frac{d}{ds} \| T_s^{(m)}x_m - T_s^{(n)}x_n \|^2 ds$$
$$= 2\int_0^t \mathrm{Re} \left\langle \frac{d}{ds} T_s^{(m)}x_m - \frac{d}{ds} T_s^{(n)}x_n,\ T_s^{(m)}x_m - T_s^{(n)}x_n \right\rangle ds$$
$$= 2\int_0^t \mathrm{Re} \langle A_m T_s^{(m)}x_m - A_n T_s^{(n)}x_n,\ T_s^{(m)}x_m - T_s^{(n)}x_n \rangle ds$$
$$= 2\int_0^t \mathrm{Re} \left\langle A'\left(I - \frac{1}{m}A\right)^{-1} T_s^{(m)}x_m - A'\left(I - \frac{1}{n}A\right)^{-1} T_s^{(n)}x_n, \right.$$
$$\left. \left(I - \frac{1}{m}A\right)^{-1} T_s^{(m)}x_m - \left(I - \frac{1}{n}A\right)^{-1} T_s^{(n)}x_n \right\rangle ds$$
$$+ 2\int_0^t \mathrm{Re} \left\langle A_m T_s^{(m)}x_m - A_n T_s^{(n)}x_n, \right.$$
$$\left. \left(I - \left(I - \frac{1}{m}A\right)^{-1}\right) T_s^{(m)}x_m - \left(I - \left(I - \frac{1}{n}A\right)^{-1} T_s^{(n)}x_n \right\rangle ds$$
$$\leqq 2\int_0^t \mathrm{Re} \left\langle A_m T_s^{(m)}x_m - A_n T_s^{(n)}x_n, \right.$$
$$\left. \left(I - \left(I - \frac{1}{m}A\right)^{-1}\right) T_s^{(m)}x_m - \left(I - \left(I - \frac{1}{n}A\right)^{-1}\right) T_s^{(n)}x_n \right\rangle ds .$$

Recalling that $y \in Ax$, $x_n = x - \frac{1}{n}y$, we have by 14)

$$\| A_m T_s^{(m)} x_m \| \leqq \| A_m x_m \| = \| y \| ,$$
$$\| A_n T_s^{(n)} x_n \| \leqq \| A_n x_n \| = \| y \| .$$

It holds moreover that

$$\Big(I-\Big(I-\frac{1}{m}A\Big)^{-1}\Big)T_s^{(m)}x_m=\Big(I-\frac{1}{m}A'\Big)\Big(I-\frac{1}{m}A\Big)^{-1}T_s^{(m)}x_m$$

$$-\Big(I-\frac{1}{m}A\Big)^{-1}T_s^{(m)}x_m$$

$$=-\frac{1}{m}A_mT_s^{(m)}x_m,$$

$$\Big(I-\Big(I-\frac{1}{n}A\Big)^{-1}\Big)T_s^{(n)}x_n=-\frac{1}{n}A_nT_s^{(n)}x_n.$$

From this it follows that

16) $$\Big\|\Big(I-\Big(I-\frac{1}{m}A\Big)^{-1}\Big)T_s^{(m)}x_m\Big\|=\frac{1}{m}\|A_mT_s^{(m)}x_m\|\leqq\frac{1}{m}\|A_mx_m\|=\frac{1}{m}\|y\|,$$

$$\Big\|\Big(I-\Big(I-\frac{1}{n}A\Big)^{-1}\Big)T_s^{(n)}x_n\Big\|\leqq\frac{1}{n}\|y\|.$$

Hence the last term of 15) is evaluated as follows:

17) $$2\int_0^t\mathrm{Re}\Big\langle A_mT_s^{(m)}x_m-A_nT_s^{(n)}x_n,$$

$$\Big(I-\Big(I-\frac{1}{m}A\Big)^{-1}\Big)T_s^{(m)}x_m-\Big(I-\Big(I-\frac{1}{n}A\Big)^{-1}T_s^{(n)}x_n\Big\rangle ds$$

$$\leqq 2\int_0^t(\|y\|+\|y\|)\Big(\frac{1}{m}\|y\|+\frac{1}{n}\|y\|\Big)ds=4\Big(\frac{1}{m}+\frac{1}{n}\Big)\|y\|^2t.$$

For any fixed $t_0>0$ we have by 15) and 17)

$$\sup_{0\leqq t\leqq t_0}\|T_t^{(m)}x-T_t^{(n)}x\|$$

$$\leqq\sup_{0\leqq t\leqq t_0}(\|T_t^{(m)}x-T_t^{(m)}x_m\|+\|T_t^{(n)}x_n-T_t^{(n)}x\|+\|T_t^{(m)}x_m-T_t^{(n)}x_n\|)$$

$$\leqq\|x-x_m\|+\|x_n-x\|+\sup_{0\leqq t\leqq t_0}\sqrt{\|x_m-x_n\|^2+4\Big(\frac{1}{m}+\frac{1}{n}\Big)\|y\|^2t}$$

$$\to 0,\ \text{as } m,\ n\to\infty.$$

Since H is complete, $T_t^{(n)}x$ converges uniformly in t in every finite interval.

IV. Since $\|T_t^{(n)}x-T_t^{(n)}x'\|\leqq\|x-x'\|$, and since $T_t^{(n)}x'$ is convergent uniformly in t in every finite interval for any x' in the dense set $D(A)$, we conclude that $T_t^{(n)}x$ itself is convergent uniformly in t in every finite interval for any $x\in H$. Le $T_tx=\lim_{n\to\infty}T_t^{(n)}x$. We shall show that $\{T_t\}$ is a nonlinear semi-group satisfying 12). It is clear that $\{T_t\}$ satisfies the conditions 1)—4), since each $\{T_t^{(n)}\}$ satisfies those. Hence it suffices to verify 12).

Let $x \in D(A)$, and $x_n = x - \frac{1}{n} y$ for a fixed $y \in Ax$. Then by 16) we have

$$\left(I - \frac{1}{n} A\right)^{-1} T_t^{(n)} x_n \to T_t x$$

uniformly in t in every finite interval. Moreover, the set $\left\{ A'\left(I - \frac{1}{n} A\right)^{-1} T_t^{(n)} x_n \,\middle|\, 0 \leqq t \leqq t_0,\ n = 1, 2, \cdots \right\}$ is bounded in H, so we can choose a subsequence $\left\{ A'\left(I - \frac{1}{n_k} A\right)^{-1} T_t^{(n_k)} x_{n_k} \right\}$ convergent in the topology $\sigma = \sigma(L_H^1[0, t_0], C_H[0, t_0])$. Thus we have

$$\tilde{A} T_t x \in \sigma\text{-lim}\, A_{n_k} T_t^{(n_k)} x_{n_k}.$$

On the other hand, $A_n T_t^{(n)} x_n = \frac{d}{dt} T_t^{(n)} x_n$ converges to $\frac{d}{dt} T_t x$ in the topology of H-valued distrbutions, i.e., in the topology $\sigma_1 = \sigma(L_H^1[0, t_0], \mathscr{D}_H[0, t_0])$, where $\mathscr{D}_H[0, t_0] =$ the space of C^∞-functions with carrier in $(0, t_0)$. Since $\sigma_1 < \sigma$, we have thus

$$\frac{d}{dt} T_t x = \sigma_1\text{-}\lim_{k \to \infty} A_{n_k} T_t^{(n_k)} x_{n_k} \in \tilde{A} T_t x.$$

Q. E. D.

The semi-group $\{T_t\}$ generated by A in the above theorem has a densely defined strict infinitesimal generator A_0. However we don't know whether $A_0 \subset A$. As a partial converse we have

THEOREM 5. *Let $\{T_t\}$ be a nonlinear contraction semi-group with a densely defined strict infinitesimal generator A_0. Then A_0 has an extension A which generates a nonlinear contraction semi-group. The semi-group generated by A is equal to $\{T_t\}$.*

PROOF. By Theorem 2, A_0 has an extension A such that: a) A is dissipative, and b) $D((I-A)^{-1}) = H$. Then by Theorem 4, A generates a semi-group $\{\tilde{T}_t\}$. Thus we have only to show that $T_t = \tilde{T}_t$. Let $x \in D(A_0)$. Then by Theorem 1, $T_t x$ is absolutely continuous and

$$\frac{d}{dt} T_t x = A_0 T_t x \qquad \text{for a.e. } t.$$

Hence $T_t x$ satisfies the equation

$$\frac{d}{dt} T_t x \in \tilde{A} T_t x.$$

Recalling that $\tilde{T}_t x$ satisfies this equation also, we see that $\tilde{T}_t x = T_t x$, since the solutions are unique by Theorem 3. For an arbitrary element x of the dense set $D(A_0)$ $\tilde{T}_t x$ is equal to $T_t x$, so it holds good that

$$\tilde{T}_t x = T_t x \qquad \text{for all } x \in H,$$

since $\tilde{T}_t$ and T_t are contractions.

4. Nonlinear semi-groups in R^n.

We shall explain our theory in the simplest case, in the n-dimensional real space R^n. The norm in R^n is defined by $\|(x_i)\|=\sqrt{\Sigma x_i^2}$.

THEOREM 6. *For a nonlinear contraction semi-group* $\{T_t\}$ *in* R^n *we have* $D(A_\phi)=R^n$.

PROOF. Assume that for some $x_0\in R^n$

$$\overline{\lim_{h\downarrow 0}}\|A_h x_0\|=\infty, \qquad A_h=\frac{T_h-I}{h}.$$

We pick up a sequence $h_k\downarrow 0$ such that $y_k=\dfrac{A_{h_k}x_0}{\|A_{h_k}x_0\|}$ converges to some element y_∞ and $\|A_{h_k}x_0\|\to\infty$. By a suitable transformation of coordinates, we may put $y_\infty=(1,0,\cdots,0)$ and $x_0=(0,\cdots,0)$. Let $S=\left\{x\,\middle|\,\|x+y_\infty\|\leqq\frac{1}{2}\right\}$. Recalling $y_k\to y_\infty$, we see that for a suitable $\kappa>0$ and k_0,

$$\|y_k\|>2\kappa,\ \|y_k-y_\infty\|<\kappa, \qquad \text{for } k>k_0,$$

and for any $\lambda>0$,

$$\text{18)} \qquad x,\ z\in S,\ \|z-\lambda y_k\|\leqq\|x\| \Rightarrow \|x\|-\|z\|>\kappa\lambda.$$

Let $\varepsilon=\inf\{t\mid T_t(-y_\infty)\notin S\}$. It is easily seen that $0<\varepsilon<\infty$. On the other hand, for suitable $k_1>k_0$ we have

$$\lambda_1=h_{k_1}\|A_{h_{k_1}}x_0\|>\frac{h_{k_1}}{\kappa\varepsilon} \quad \text{and} \quad \left[\frac{\varepsilon}{h_{k_1}}\right]>\frac{\varepsilon}{2h_{k_1}}.$$

We put $z_{j+1}=T_{h_{k_1}}z_j$, $j=0,1,\cdots$ and $z_0=-y_\infty$. For $j\leqq\dfrac{\varepsilon}{h_{k_1}}$, we have $z_j\in S$. Thus we have

$$\|z_{j+1}-\lambda_1 y_{k_1}\|=\|z_{j+1}-T_{h_{k_1}}x_0\|\leqq\|z_j-x_0\|=\|z_j\|.$$

Hence by 18)

$$\|z_j\|-\|z_{j+1}\|>\kappa\lambda_1, \qquad j=0,1,\cdots,$$

which implies $\|z_j\|\leqq -jk\lambda_1+1<\dfrac{-h_{k_1}}{\varepsilon}j+1$, $j=0,1,\cdots\left[\dfrac{\varepsilon}{h_{k_1}}\right]$. For $j=\left[\dfrac{\varepsilon}{h_{k_1}}\right]$, we have $\|z_j\|<\dfrac{1}{2}$. Hence $z_j\notin S$, which is a contradiction.

COROLLARY. *The strict infinitesimal generator of a nonlinear contraction semi-group in* R^n *is densely defined.*

This does not hold in general for a nonlinear noncotraction semigroup, as may be seen by the following

EXAMPLE 1. Let $f(t)$ be a continuous function not differentiable at every

$t \geqq 0$, and $f(t)=0$ for $t \leqq 0$. We define a nonlinear $\{T_t\}$ in R^2 so that

$$T_t(x, y)=(x+t, y+f(x+t)-f(x)), \qquad (x, y) \in R^2,\ t \geqq 0.$$

Then we have $D(A_0)=\{(x, y) \mid x<0\}$. Thus $(x, y) \in D(A_{\emptyset})$ does not imply $T_t(x, y) \in D(A_{\emptyset})$, that is, Theorem 1 also does not hold in this case.

EXAMPLE 2. Let

$$T_t x=\begin{cases} \max(0, x-t) & \text{for } x>0, \\ x & \text{for } x \leqq 0. \end{cases}$$

Then $\{T_t\}$ is a nonlinear contraction semi-group in R^1. In this case the strict infinitesimal generator A_0 is defined on R^1, and hence $A_0=A_{\emptyset}$. In fact we have

$$A_0 x=\begin{cases} -1 & \text{for } x>0, \\ 0 & \text{for } x \leqq 0. \end{cases}$$

The extension of A_0 in Theorem 2 is given by

$$A x=\begin{cases} -1 & \text{for } x>0 \\ [-1, 0] & \text{for } x=0 \\ 0 & \text{for } x<0, \end{cases}$$

since we have

$$(I-A)^{-1} x=\lim_{h \downarrow 0}(I-A_h)^{-1} x=\begin{cases} \max(x-1, 0) & \text{for } x \geqq 0 \\ x & \text{for } x<0 \end{cases}$$

by the equality

$$y_h=(I-A_h)^{-1} x \qquad \text{for } 0 \leqq x \leqq 1,\ h>0 \text{ and } y_h=\frac{h}{1+h} x.$$

Appendix

Since the following lemma is not found in elementary textbooks, we shall prove it for the completeness of our work.

LEMMA. *Let E be a reflexive Banach space. Then every E-valued strongly absolutely continuous function $f(t)$, $0 \leqq t \leqq t_0$, has the strong derivative $\frac{d}{dt} f(t)$ for a.e. t and is expressed as the indefinite integral of the derivative, i.e., $f(t)=\int_0^t \frac{d}{ds} f(s) ds+f(0)$.*

PROOF. The set $\{f(t) \mid t \in[0, t_0]\}$ is compact in E, hence it is separable. Therefore we may assume without loss of generality that E itself is separable. The strong absolute continuity of $f(t)$ means that for any $\varepsilon>0$ there exists some $\delta >$ such that

$$\sum_{i=1}^{n} |s_i - t_i| < \delta, \quad s_i, t_i \in [0, t_0] \Rightarrow \sum_{i=1}^{n} \|f(s_i) - f(t_i)\| < \varepsilon .$$

Thus $f(t)$ is of strongly bounded variation.

We put $f_h(t) = \frac{1}{h}(f(t+h) - f(t))$ for $h \neq 0$, and $F_+(t) = \overline{\lim}_{h \downarrow 0} \|f_h(t)\|$, $F_-(t) = \overline{\lim}_{h \uparrow 0} \|f_h(t)\|$. Since $f(t)$ is strongly continuous, so is $f_h(t)$. Hence $F_+(t)$ and $F_-(t)$ are measurable. We can show moreover that $F_+(t)$ and $F_-(t)$ ($\geqq 0$) are finite for a.e. t. In fact, suppose that $\lambda =$ the measure of $\{t \mid F_+(t) = \infty\}$ be >0. Let $E_n = \left\{t \,\middle|\, \sup\left\{\left\|\frac{1}{h}(f(t+h) - f(t))\right\| \,\middle|\, h \geqq \frac{1}{n}, 0 \leqq t < t+h \leqq t_0\right\} \geqq \frac{2}{\lambda} \text{var.} f\right\}$. Then each E_n is a closed set and $\cup E_n \supset \{t \mid F_+(t) = \infty\}$. Since $\{E_n\}$ is an increasing sequence, there exists such an E_n that $m(E_n) > \frac{\lambda}{2}$. We define $\{t_i\}$ and $\{h_i\}$ as follows: $t_1 = \inf\{t \mid t \in E_n\}$, $t_{i+1} = \inf\{t \in E_n \mid t \geqq t_i + h_i\}$ and $h_i = \sup\left\{h \,\middle|\, \left\|\frac{1}{h}(f(t_i+h) - f(t_i))\right\| \geqq \frac{2}{\lambda} \text{var.} f\right\}$. Then we have evidently $\cup [t_i, t_i + h_i] \supset E_n$. Hence

$$\sum \|f(t_i + h_i) - f(t_i)\| \geqq \frac{2}{\lambda} \text{var.} f \sum h_i \geqq \frac{2}{\lambda} \text{var.} f \; m(E_n) > \text{var.} f .$$

Thus we obtain a contradiction. In the same way we see that $\{t \mid F_-(t) = \infty\}$ is of measure zero. That is to say, there exists a null set $N_0 \subset [0, t_0]$ such that $\{f_h(t) \mid h \neq 0\}$ is bounded for any fixed $t \in [0, t_0] - N_0$.

Since E is separable by assumption, the dual space E' has a countable weakly dense subset $\{x_k\}$. Each function $g_k(t) = \langle f(t), x_k \rangle$ is absolutely continuous, hence its derivative $g'_k(t)$ exists except at a point of a null set N_k. Recalling that for any fixed $t \in [0, t_0] - N_0$ $\{f_h(t) \mid h \neq 0\}$ is bounded and hence weakly relatively compact, we see that

$$f'(t) = \text{w-}\lim_{h \to 0} f_h(t)$$

exists for any $t \in [0, t_0] - \bigcup_{k=0}^{\infty} N_k$, since $\{x_k\}$ is total. The weak derivative $f'(t)$ is weakly measurable, hence strongly measurable by the separability of E.

Now we define

$$f'_n(t) = 2^n\left[f\left(\frac{k}{2^n}\right) - f\left(\frac{k-1}{2^n}\right)\right], \quad \frac{k-1}{2^n} \leqq t < \frac{k}{2^n},$$

for $k = t_0, 2t_0, \cdots, 2^n t_0$. Then $\int_0^{t_0} \|f'_n(t)\| dt \leqq \text{var.} f$. Since $f'_n(t)$ converges weakly to $f'(t)$ for $t \in [0, t_0] - \cup N_k$, we have $\|f'(t)\| \leqq \underline{\lim} \|f'_n(t)\|$ for $t \in [0, t_0] - \cup N_k$. Hence by Fatou's lemma,

$$\int_0^{t_0} \|f'(t)\| dt \leqq \underline{\lim} \int_0^{t_0} \|f'_n(t)\| dt \leqq \text{var.} f .$$

Thus $f'(t)$ is strongly integrable. We put $\tilde{f}(t)=\int_0^t f'(s)ds+f(0)$. Since $\langle \tilde{f}(t), x_k\rangle =\langle f(t), x_k\rangle$ for a.e. t for $k=1, 2, \cdots$, we have $\tilde{f}(t)=f(t)$ for a.e. t. By Bochner's theorem (see for instance [**12**, p. 133]) $f(t)$ is strongly differentiable for a.e. t and the strong derivative $\frac{d}{dt}f(t)$ is equal to $f'(t)$ for a.e. t.

Waseda University

References

[1] F. E. Browder, Nonlinear equations of evolution, Ann. of Math., **80** (1964), 485-523.

[2] H. Fujita, On the existence and regularity of the steady-state solutions of the Navier-Stokes equation, J. Fac. Sci. Univ. Tokyo, Sect. I, **9** (1961), 59-102.

[3] E. Hille and R. S. Phillips, Functional analysis and semi-groups, Amer. Math. Soc. Colloq. Publ. 31, 1957.

[4] T. Kato, Nonlinear evolution equations in Banach spaces, Proc. Symp. Appl. Math., Vol. XVII, 50-67, Amer. Math. Soc., Providence, R. I., 1965.

[5] T. Kōmura, Semi-groups of operators in locally convex spaces, to appear.

[6] J. L. Lions, Équation différentielles opérationelles, Springer, 1961.

[7] G. J. Minty, Monotone (nonlinear) operators in Hilbert space, Duke Math. J., **29** (1962), 541-546.

[8] J. W. Neuberger, An exponential formula one-parameter semi-groups of nonlinear transformations, J. Math. Soc. Japan, **18** (1966), 154-157.

[9] I. E. Segal, Nonlinear semi-groups, Ann. of Math., **78** (1963), 339-364.

[10] P. E. Sobolevskii. On the use of fractional powers of self-adjoint operators for the investigation of some nonlinear differential equations in Hilbert space (Russian), Dokl. Akad. Nauk SSSR, **130** (1960), 272-275.

[11] H. Tanabe, On the equations of evolution in a Banach space, Osaka Math. J., **12** (1960), 363-376.

[12] K. Yosida, Functional analysis, Springer, 1965.