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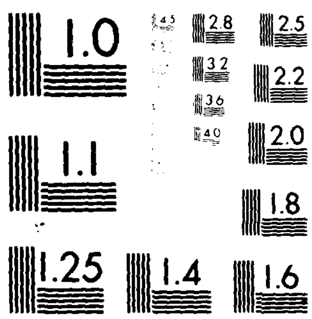

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NONLINEAR SEMIGROUP FOR CONTROLLED PARTIALLY OBSERVED DIFFUSIONS[+]

by

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NONLINEAR SEMIGROUP FOR CONTROLLED

PARTIALLY OBSERVED DIFFUSIONS

by

Wendell H. Fleming

Abstract: In this paper a "separated" control problem associated with controlled, partially observed diffusion processes is considered. The state in the separated problem is an unnormalized conditional distribution measure. The corresponding Nisio nonlinear semigroup associated with the separated problem is found.

NONLINEAR SEMIGROUP FOR CONTROLLED PARTIALLY OBSERVED DIFFUSIONS

Wendell H. Fleming

1. Introduction. In this paper we are concerned with stochastic control problems of the following kind. Let X_t denote the state of a process being controlled, Y_t the observation process, and U_t the control process, $t \geq 0$. The state and observation processes are governed by stochastic differential equations

$$
(1.1)\quad
\begin{aligned}
&(a)\quad dX_t = b(X_t,U_t)dt + \sigma(X_t)dW_t \\
&(b)\quad dY_t = h(X_t)dt + d\tilde{W}_t .
\end{aligned}
$$

X_t has values in N-dimensional $\mathbb{R}^N$, Y_t values in $\mathbb{R}^M$, and U_t values in $\mathcal{U} \subset \mathbb{R}^L$. X_0 has given distribution μ, and $Y_0 = 0$. In (1.1), W and $\tilde{W}$ are independent standard Wiener processes, with values in $\mathbb{R}^D, \mathbb{R}^M$ respectively. The problem is to find an admissible control minimizing some criterion J.

For instance, we may take $J = EG(X_{t_1})$ for some fixed time $t_1 > 0$. In case of completely observed, controlled diffusions (with $Y_t = X_t$ rather than Y_t as in (1.1b)), the problem can be treated using dynamic programming. Let $V(x,t_1)$ denote the minimum of J, for initial data $X_0 = x$. Under suitable asssumptions $V(x,t)$ has continuous partial derivatves

$\partial V/\partial t$, $\partial V/\partial x_i$, $\partial^2 V/\partial x_i \partial x_j$, $i, j = 1,\ldots,N$, $x = (x_1,\ldots,x_N)$. Among these assumptions is the condition that the symmetric matrix $a = \sigma\sigma'$ has a bounded inverse a^{-1}. The function V then satisfies the dynamic programming equation [4, Chap. VI.6]

$$(1.2) \qquad \frac{\partial V}{\partial t} = LV,$$

$$(1.3) \qquad LV = \min_{u\in\mathscr{U}} \Big[\frac{1}{2} \sum_{i,j=1}^{N} a_{ij}(x) \frac{\partial^2 V}{\partial x_i \partial x_j} + \sum_{i=1}^{N} b_i(x,u)\frac{\partial V}{\partial x_i}$$

The assumption that $a(x)$ has a bounded inverse can sometimes be weakened, by considering generalized solutions to the dynamic programming equation [4, p. 177].

In [6] Nisio introduced another treatment which is valid under much less restrictive conditions. Let $\mathscr{S}_t G(x) = V(x,t)$. Then Nisio showed that $\mathscr{S}_t$ is a nonlinear semigroup on the space $C_b(\mathbb{R}^N)$ of continuous bounded functions f on $\mathbb{R}^N$. Moreover, the operator L in (1.3) agrees with the generator of the semigroup $\mathscr{S}_t$ on the space $C_b^2(\mathbb{R}^N)$ of those f such that $f, f_{x_i}, f_{x_i x_j}$ are in $C_b(\mathbb{R}^N)$ for $i, j = 1,\ldots,N$. For another treatment of this nonlinear semigroup see [1, Chap. IV.5.1].

In this paper, we find a nonlinear semigroup $\mathscr{T}_t$ associated with the partially observed control problem. In this case, one should regard as the true "state" the conditional distribution of X_t given past data, or some quantity equivalent to the conditional distribution. For technical reasons, it is more convenient to consider an unnormalized conditional distribution Λ_t for X_t.

We have $\Lambda_t \in \mathcal{M}$, where $\mathcal{M}$ is the space of finite measures on $\mathbb{R}^N$. The problem we consider is to control the measure-valued process Λ_t such that a criterion of the form $J = E\Phi(\Lambda_{t_1})$ is minimized. The dynamics of the Λ_t-process are governed by the Zakai equation, written in a weak form as (3.1) below.

If one writes $V(\mu,t_1)$ for the minimum of J, given initial data $\Lambda_0 = \mu$, then $V(\mu,t)$ formally satisfies a dynamic programming equation of the form .

$$\frac{\partial V}{\partial t} = \mathscr{L}V, \tag{1.4}$$

where $\mathscr{L}V = \min\limits_{u\in\mathscr{U}} \mathscr{L}^u V$ and $\mathscr{L}^u$ is the generator of the linear semigroup $\mathscr{T}_t^u$ associated for a constant control u with the process Λ_t (for constant u, Λ_t is Markov). Equation (1.4) is called Mortensen's equation. However, (1.4) has been treated rigorously only in very special cases.

Following Nisio, we write $V(\mu,t) = \mathscr{T}_t\Phi(\mu)$. The purpose of this paper is to show that $\mathscr{T}_t$ is a nonlinear semigroup, on a space $C(\mathcal{M})$, with $\mathscr{T}_t\Phi$ continuous in t, and to describe the generator $\mathscr{L}$ on a dense subspace of $C(\mathcal{M})$. We rely heavily on results from [3]. In particular, it was shown in [3] that Λ_t can be defined pathwise, in such a way that Λ_t depends continuously on observation and control trajectories (Y,U) and on $\mu = \Lambda_0$. This and other results from [3] needed in this paper are summarized as 3.1-3.4 below.

For the case of a controlled Markov chain X_t, subject to observations Y_t of the form (1.1b) a corresponding nonlinear semigroup was constructed by Davis [2].

2. <u>The Spaces</u> $C_K(\mathcal{M})$, $C(\mathcal{M})$. Let $C_b(\mathbb{R}^N)$ denote the space of bounded, continuous f on $\mathbb{R}^N$, and $C_0(\mathbb{R}^N)$ the space of continuous f with compact support. Let $C_b^k(\mathbb{R}^N)$, $C_0^k(\mathbb{R}^N)$ be the spaces of f such that f together with all partial derivatives of orders $\leq k$ are in $C_b(\mathbb{R}^N)$, $C_0(\mathbb{R}^N)$ respectively. Similarly, for $\mathbb{R}^m$ valued functions on $\mathbb{R}^N$ we write $C_b^k(\mathbb{R}^N;\mathbb{R}^m)$, $C_0^k(\mathbb{R}^N;\mathbb{R}^m)$.

Let $\mathcal{B}(\mathbb{R}^N)$ denote the Borel σ-algebra of $\mathbb{R}^N$, and

$$(2.1) \qquad \mathcal{M} = \{\text{measures } \mu \geq 0 \text{ on } \mathcal{B}(\mathbb{R}^N) : \mu(\mathbb{R}^N) < \infty\}.$$

We write

$$\langle f,\mu\rangle = \int_{\mathbb{R}^N} f(x)\,d\mu(x)$$

for the scalar product and

$$||\mu|| = \langle 1,\mu\rangle = \mu(\mathbb{R}^N).$$

By convergence of sequences in $\mathcal{M}$ we mean w^*-convergence: $\mu_n \to \mu$ if and only if $\langle f,\mu_n\rangle \to \langle f,\mu\rangle$ as $n \to \infty$ for every $f \in C_b(\mathbb{R}^N)$ such that $f(x) \to 0$ as $|x| \to \infty$.

We denote real-valued functions on $\mathcal{M}$ by $\phi, \psi, \ldots$. For $K = 0,1,2,\ldots$ let

$$(2.2) \qquad ||\phi||_K = \sup_{\mu\in\mathcal{M}} \frac{|\phi(\mu)|}{1+||\mu||^K} .$$

By ϕ continuous on $\mathcal{M}$, we mean of course continuity of ϕ under w^*-sequential convergence. Let

(2.3) $$C_K(\mathcal{M}) = \{\phi \text{ continuous on } \mathcal{M}: ||\phi||_K < \infty\}.$$

Then, $||\ ||_K$ is a norm on $C_K(\mathcal{M})$. Let

(2.4) $$C(\mathcal{M}) = \bigcup_{K=0}^{\infty} C_K(\mathcal{M}).$$

For $r < \infty$, let

(2.5) $$\mathcal{M}_r = \{\mu \in \mathcal{M}: ||\mu|| \le r\}.$$

We give $C(\mathcal{M})$ the following metric

(2.6) $$d(\phi,\psi) = \sum_{\ell=1}^{\infty} 2^{-\ell}(\sup_{\mathcal{M}_\ell} |\phi(\mu) - \psi(\mu)| \wedge 1).$$

Thus d-convergence of ϕ_n to ϕ is equivalent to convergence of $\phi_n(\mu)$ to $\phi(\mu)$ uniformly on $\mathcal{M}_r$ for every $r < \infty$. For each K, $||\ ||_K$ is a lower semicontinuous function under d-convergence. Moreover, from (2.2), ϕ_n, $\phi \in C_K(\mathcal{M})$ and $||\phi_n - \phi||_K \to 0$ imply $d(\phi_n,\phi) \to 0$ as $n \to \infty$.

Let

$$\tilde{\mathcal{M}} = \{\mu \ge 0 \text{ on } \mathcal{B}(\mathbb{R}^N): \mu(B) < \infty \text{ for every compact } B\},$$

with the vague topology: $\mu_n \to \mu$ vaguely means $\langle f,\mu_n\rangle \to \langle f,\mu\rangle$

as $n \to \infty$ for every $f \in C_0(\mathbb{R}^N)$. $\tilde{\mathcal{M}}$ is a Polish space. In fact, one can choose a metric $\delta(\mu,\nu)$ for $\tilde{\mathcal{M}}$ of the form

$$(2.7) \qquad \delta(\mu,\nu) = \sum_{m=1}^{\infty} 2^{-m}(|\langle f_m,\mu\rangle - \langle f_m,\nu\rangle| \wedge 1)$$

for a suitably chosen sequence $f_m \in C_0(\mathbb{R}^N)$.

For each $r < \infty$, $\mathcal{M}_r$ is a compact subset of $\mathcal{M}$. For sequences in $\mathcal{M}_r$, vague convergence is equivalent to w^*-convergence. Moreover $\mu_n, \mu \in \mathcal{M}$ and $\mu_n \to \mu$ w^* imply $||\mu_n|| \le r$ for some some r. Thus, we have:

<u>Lemma 2.1</u>. ϕ <u>is continuous on</u> $\mathcal{M}$, <u>under w^*-sequential convergence, if and only if</u> $\phi|\mathcal{M}_r$ <u>is vaguely continuous for every</u> $r < \infty$.

This furnishes an alternate characterization of $C(\mathcal{M})$, in terms of the vague topology rather than in terms of w^*-sequntiel convergence.

A measure $\mu \in \mathcal{M}$ can be approximated by measures $\rho\mu$ with compact support, as follows. Let $\rho \in C_0(\mathbb{R}^N)$, $0 \le \rho \le 1$, and define $\rho\mu$ by $\langle f,\rho\mu\rangle = \langle \rho f,\mu\rangle$ for all $f \in C_b(\mathbb{R}^N)$. Define ϕ^ρ by

$$(2.8) \qquad \phi^\rho(\mu) = \phi(\rho\mu), \quad \mu \in \mathcal{M}.$$

Then $\phi \in C_K(\mathcal{M})$ implies $\phi^\rho \in C_K(\mathcal{M})$ and $||\phi^\rho||_K \le ||\phi||_K$. We write $\mu|B$ for the restriction of μ to a compact set B: $(\mu|B)(A) = \mu(A \cap B)$ for all $A \in \mathcal{B}(\mathbb{R}^N)$. Let

(2.9) $$C_K^0(\mathscr{M}) = \{\psi \in C_K(\mathscr{M}) : \text{there exists } B \text{ compact such that } \psi(\mu) = \psi(\mu|B) \text{ for all } \mu \in \mathscr{M}\}.$$

In particular, $\phi^\rho \in C_K^0(\mathscr{M})$ if $\phi \in C_K(\mathscr{M})$ and ϕ^ρ is defined by (2.8).

Lemma 2.2. For every $\phi \in C_K(\mathscr{M})$ there exists a sequence $\phi_n \in C_K^0(\mathscr{M})$ such that $||\phi_n||_K \leq ||\phi||_K$ and $d(\phi_n, \phi) \to 0$ as $n \to \infty$.

Proof. Let $\rho_n \in C_0(\mathbb{R}^N)$ with $0 \leq \rho_n \leq 1$, $\rho_n(x) = 1$ for $|x| \leq n$ and $\rho_n(x) = 0$ for $|x| \geq n + 1$. Let $\phi_n = \phi^{\rho_n}$. Then $||\phi_n||_K \leq ||\phi||_K$. Since $\phi_n(\mu) = \phi(\rho_n\mu)$ it suffices to show that $\phi(\rho_n\mu) - \phi(\mu)$ tends to 0 uniformly on $\mathscr{M}_r$ for every $r < \infty$. Let

$$\eta_n = \max_{\mathscr{M}_r} |\phi(\rho_n\mu) - \phi(\mu)| = |\phi(\rho_n\mu_n) - \phi(\mu_n)|$$

for some $\mu_n \in \mathscr{M}_r$ (recall that $\mathscr{M}_r$ is compact). We have $\rho_n\mu_n \in \mathscr{M}_r$. For each $f \in C_0(\mathbb{R}^N)$, $\langle f, \rho_n\mu_n\rangle = \langle f, \mu_n\rangle$ for all large enough n. Consider any subsequence such that μ_n tends to a limit μ. Then $\rho_n\mu_n$ also tends to μ for n in this subsequence. Since $\phi|\mathscr{M}_r$ is continuous, both $\phi(\rho_n\mu_n)$ and $\phi(\mu_n)$ tend to $\phi(\mu)$. If $\limsup_{n\to\infty} \eta_n > 0$, we could find some such subsequence for which $|\phi(\rho_n\mu_n) - \phi(\mu_n)|$ tends to a positive limit, a contradiction. This proves Lemma 2.2.

Lemma 2.3. *Let* $\psi \in C_K^0(\mathscr{M})$, *and* B *compact such that* $\psi(\mu) = \psi(\mu|B)$ *for all* $\mu \in \mathscr{M}$. *Then there exists a sequence* $\psi_n \in C_K^0(\mathscr{M})$ *such that* $||\psi_n||_K \leq ||\psi||_K$, $d(\psi_n,\psi) \to 0$ *as* $n \to \infty$ *and* $\psi_n(\mu) = 0$ *whenever* $\mu(B) \geq n$.

Proof. Choose $\rho \in C_0(\mathbb{R}^N)$ with $0 \leq \rho \leq 1$, $\rho(x) = 1$ for all $x \in B$. Let $g_n \in C_0(\mathbb{R}^1)$, with $0 \leq g_n \leq 1$, $g_n(s) = 1$ if $s \leq n - 1$, $g_n(s) = 0$ if $s \geq n$. Let

$$\psi_n(\mu) = g_n(\langle\rho,\mu\rangle)\psi(\mu).$$

Since $|\psi_n(\mu)| \leq |\psi(\mu)|$, $||\psi_n||_K \leq ||\psi||_K$. For $\mu \in \mathscr{M}_r$, $\langle\rho,\mu\rangle \leq r$. Hence $\psi_n(\mu) = \psi(\mu)$ if $n \geq r + 1$, which implies $\psi_n \to \psi$ uniformly on $\mathscr{M}_r$. Thus $d(\psi_n,\psi) \to 0$ as $n \to \infty$. Finally, $\mu(B) \geq n$ implies $\langle\rho,\mu\rangle \geq n$, and hence $\psi_n(\mu) = 0$. This proves Lemma 2.3.

The set $\mathscr{D}$ of "test functions". In §5 we shall define a "generator" for the nonlinear semigroup on the following set of functions ϕ, depending on finitely many scalar products:

$$(2.10) \qquad \mathscr{D} = \{\phi: \phi(\mu) = F(\langle f_1,\mu\rangle,\ldots,\langle f_J,\mu\rangle),$$
$$F \in C_b^\infty(\mathbb{R}^J),\ f_1,\ldots,f_J \in C_0^\infty(\mathbb{R}^N),\ J = 1,2,\ldots\}.$$

In §4, we shall weaken slightly the conditions on $F, f_1,\ldots,f_J$, to obtain certain sets $\mathscr{D}_m$ containing $\mathscr{D}$.

Lemma 2.4. For every $\phi \in C_K(\mathscr{M})$ there exists a sequence $\psi_n \in \mathscr{D}$ such that $\|\psi_n\|_K \leq \|\phi\|_K + n^{-1}$ and $d(\psi_n,\phi) \to 0$ as $n \to \infty$.

Proof. By Lemmas 2.2 and 2.3 it suffices to suppose that, in addition, there exist compact B and $a > 0$ such that $\phi(\mu) = \phi(\mu|B)$ for all μ and $\phi(\mu) = 0$ if $\mu(B) \geq a$. Following a similar construction in [5, §3], given $\varepsilon > 0$, we take $g_1,\ldots,g_J$, $x_1,\ldots,x_J$ with the following properties:

$$g_j \in C_0^\infty(\mathbb{R}^N),\ g_j \geq 0,\ \operatorname{diam}(\operatorname{spt} g_j) < \varepsilon$$

$$\sum_{j=1}^{J} g_j(x) \leq 1 \quad \text{for} \quad x \in \mathbb{R}^N, \quad \sum_{j=1}^{J} g_j(x) = 1, \quad x \in B,$$

$$x_j \in B \cap \operatorname{spt} g_j$$

Let

$$\mathbb{R}_+^J = \{z \in \mathbb{R}^J\colon\ z_j \geq 0 \quad \text{for} \quad j = 1,\ldots,J\},$$

$$\tilde{F}(z) = \phi\Big(\sum_{j=1}^{J} z_j \delta_{x_j}\Big)$$

where δ_x denotes the Dirac measure at x. Then $\tilde{F} \in C_0(\mathbb{R}_+^J)$. In fact, $\tilde{F}(z) = 0$ whenever

$$\sum_{j=1}^{J} z_j \delta_{x_j}(B) = \sum_{j=1}^{J} z_j \geq a.$$

By regularizing, there exists $F \in C_0^\infty(\mathbb{R}^J)$ such that $|F(z) - \tilde{F}(z)| \le \varepsilon$ for all $z \in \mathbb{R}_+^J$. Then

$$\psi(\mu) = F(<g_1,\mu>,\ldots,<g_J,\mu>)$$

is in $\mathscr{D}$. For all μ,

$$\begin{aligned} |\psi(\mu)| &\le |\phi(\sum_{j=1}^{J} <g_j,\mu>\delta_{x_j})| + \varepsilon \\ &\le ||\phi||_K(1 + ||\sum_{j=1}^{J} <g_j,\mu>\delta_{x_j}||^K) + \varepsilon \\ &\le ||\phi||_K(1 + ||\mu||^K) + \varepsilon. \end{aligned}$$

Therefore, $||\psi||_K \le ||\phi||_K + \varepsilon$.

We take $\varepsilon = \varepsilon_n = n^{-1}$, and corresponding g_{jn}, x_{jn}, $j = 1,\ldots,J_n$. The corresponding ψ_n obtained from the construction above has the properties required in Lemma 2.4. To show that $d(\psi_n,\phi) \to 0$, it suffices to show that $\psi_n(\mu) \to \phi(\mu)$ uniformly on $\mathscr{M}_r$ for any $r > 0$ as $n \to \infty$. Now

$$|\psi_n(\mu) - \phi[G_n(\mu)]| \le \varepsilon_n$$

$$G_n(\mu) = \sum_{j=1}^{J_n} <g_{jn},\mu>\delta_{x_{jn}}.$$

On $\mathscr{M}_r$, both vague and w^*-convergence of a sequence are equivalent to convergence in the metric δ in (2.7). For each m, $|<f_m,G_n(\mu)> - <f_m,\rho_n\mu>| \to 0$ as $n \to \infty$ uniformly for $\mu \in \mathscr{M}_r$, where $\rho_n = \sum_j g_{jn}$. Therefore, $\delta(G_n(\mu),\rho_n\mu) \to 0$ uniformly on $\mathcal{M}_r$ as $n \to \infty$. Since $\mathcal{M}_r$ is compact and ϕ continuous on $\mathcal{M}_r$, ϕ is uniformly continuous on $\mathcal{M}_r$. Thus, $|\phi[G_n(\mu)] - \phi(\rho_n\mu)| \to 0$ uniformly on $\mathcal{M}_r$. Since $\rho_n(x) = 1$ on B, $\phi(\rho_n\mu) = \phi(\mu) = \phi(\mu|B)$. This proves that $\psi_n(\mu) \to \phi(\mu)$ uniformly on $\mathcal{M}_r$, as required.

3. <u>The Control Problem for</u> Λ_t. We begin with a summary of assumptions and notations, together with a review of concepts from [3]. We make the same assumptions as in [3] about the coefficients in (1.1):

(A_1) σ is a bounded, Lipschitz $N \times D$ matrix-valued function on $\mathbb{R}^N$.

(A_2) $b(x,u) = b^0(x) + b^1(x)u$, where b^0, b^1 are bounded, Lipschitz functions on $\mathbb{R}^N$.

Note that b^0 has values in $\mathbb{R}^N$, and b^1 has $N \times L$ matrices as values. In §5, we shall impose additional smoothness conditions on σ, b^0, b^1.

(A_3) $h \in C_b^2(\mathbb{R}^N;\mathbb{R}^M)$.

(A_4) $\mathcal{U}$ is a convex, compact subset of $\mathbb{R}^L$.

We use Y to denote an $\mathbb{R}^M$-valued function, and U a $\mathcal{U}$-valued function, of time $t \geq 0$. Let Y_t, U_t denote their respective values at time t. Let

$$\Omega = \{(Y,U)\colon Y_0 = 0,\ Y \in C([0,\infty)\colon \mathbb{R}^M),\ U \in L^2([0,T];\mathcal{U}) \text{ for for each } T < \infty\}.$$

Let Ω_T denote the set of restrictions to $[0,T]$ of functions $(Y,U) \in \Omega$. As in [3], we give Ω_T a metric in which convergence of a sequence (Y_n,U_n) means uniform convergence on $[0,T]$ of Y_n and weak convergence of U_n in $L^2([0,T];\mathscr{U})$. We give Ω a metric in which convergence of (Y_n,U_n) is equivalent to convergence of (Y_n,U_n) restricted to $[0,T]$ for every $T < \infty$. Let

$$\mathscr{F}_t(Y) = \sigma\{Y_s,\ 0 \le s \le t\}$$

$$\mathscr{F}_t(U) = \sigma\{V_s,\ 0 \le s \le t\}, \qquad V_t = \int_0^t V_\theta \, d\theta,$$

$$\mathscr{G}_t = \mathscr{F}_t(Y) \times \mathscr{F}_t(U).$$

These are σ-algebras of subsets of Ω. However, if $t \le T$, they can also be regarded as σ-algebras of subsets of Ω_T. In [3], Ω_T was denoted by Ω^2 and $\mathscr{G}_t$ by $\mathscr{G}_t^2$.

Let $\mathscr{G}_\infty$ be the least σ-algebra containing $\mathscr{G}_t$ for all $t \ge 0$.

Definition. An admissible control on $[0,T]$ is a probability measure π_T on $(\Omega_T, \mathscr{G}_T)$, such that Y is a π_T, $\{\mathscr{G}_t\}$-Wiener process for $0 \le t \le T$.

An admissible control is a probability measure π on $(\Omega, \mathscr{G}_\infty)$ such that Y is a π, $\{\mathscr{G}_t\}$-Wiener process for $t \ge 0$.

The definition of admissible control on $[0,T]$ is exactly as in [3]. If π is an admissible control, then its restriction π_T to $\mathscr{G}_T$ is admissible on $[0,T]$.

Let $\mathscr{A}_T$ denote the set of all admissible controls π_T on $[0,T]$.

Then $\mathscr{A}_T$ is compact under weak sequential convergence of probability measures [3, Lemma 2.3]. Let $\mathscr{A}$ denote the set of all admissible controls with the weak sequential convergence topology. Then $\mathscr{A}$ is a compact metric space under (for instance) the Prokhorov metric. Moreover, $\pi_n \to \pi$ if and only if the restrictions $\pi_{n,T}$ tend to π_T as $n \to \infty$ for each T finite.

<u>The unnormalized conditional distribution measure</u> Λ_t. For every $\mu \in \mathscr{M}$, $(Y,U) \in \Omega$, and $t \geq 0$, we define $\Lambda_t = \Lambda_{t\mu}^{YU}$ by formula [3, (3.9)]. (In [3] we wrote Λ_t^{YU}, but now we wish to emphasize its dependence on the initial value $\mu = \Lambda_0$.) From its definition, $\Lambda_t \in \mathscr{M}$ and Λ_t is $\mathscr{G}_t$-measurable as a function of $(Y,U) \in \Omega$. In [3,§3] we interpreted Λ_t as an unnormalized conditional distribution of X_t in (1.1a) with respect to the σ-algebra $\mathscr{G}_t$ generated by the observation and control past up to t. The normalized conditional distribution of X_t is $||\Lambda_t||^{-1}\Lambda_t$. The intuitive reason for conditioning on $\mathscr{G}_t$, rather than on $\mathscr{F}_t(Y)$, is that U_t is not necessarily $\mathscr{F}_t(Y)$-measurable π-almost surely, when $\pi \in \mathscr{A}$. For the smaller class of strict-sense admissible controls [3, §6] one can condition on $\mathscr{F}_t(Y)$ instead of $\mathscr{G}_t$.

We shall need the following properties of Λ_t, proved in [3].

3.1. For each $t \geq 0$, $r < \infty$, $\Lambda_{t\mu}^{YU}$ is continuous on $\mathscr{M}_r \times \Omega$. See [3, Lemma 3.2].

3.2. For each finite T,r,a there exists $\rho = \rho(T,r,a)$ such that $0 \leq t \leq T$, $||\mu|| \leq r$, $||Y||_T \leq a$ imply $||\Lambda_{t\mu}^{YU}|| \leq \rho$. Here $||Y||_T = \max\limits_{0\leq t\leq T} |Y(t)|$. See [3, (3.6)]; since Λ_t depends linearly on $\mu = \Lambda_0$ it suffices to consider $||\mu|| = 1$.

3.3. The Zakai equation holds:

$$(3.1)\qquad d\langle f,\Lambda_t\rangle = \langle L^{U_t}f,\Lambda_t\rangle dt + \langle hf,\Lambda_t\rangle\cdot dY_t, \text{ all } f \in C_b^2(\mathbb{R}^N).$$

See [3, Thm. 5.2]. Here, for constant control $u \in \mathcal{U}$, L^u is the generator of the diffusion process in $\mathbb{R}^N$ corresponding to (1.1a):

$$(3.2)\qquad L^u f = \frac{1}{2}\sum_{i,j=1}^{N} a_{ij}(x)f_{x_i x_j} + (b^0(x) + b^1(x)u)\cdot\nabla f$$

with $a = \sigma\sigma'$.

3.4. For every $T < \infty$, $K = 1,2,\ldots$, there exists γ_{KT} such that

$$E_\pi||\Lambda_t||^K \le \gamma_{KT}||\mu||^K, \quad 0 \le t \le T,$$

for all $\pi \in \mathcal{A}$. See [3, Thm. 5.3] with $m = 0$.

For $t \ge 0$, $\mu \in \mathcal{M}$, $\pi \in \mathcal{A}$, $\phi \in C(\mathcal{M})$ let

$$(3.3)\qquad J(t,\mu,\pi,\phi) = E_\pi\phi(\Lambda_{t\mu}^{YU}).$$

Since $\phi \in C_K(\mathcal{M})$ for some K, the expectation exists by 3.1 and 3.4.

<u>Lemma 3.5</u>. <u>Let</u> $||\phi_n||_K \le C$ <u>and</u> $d(\phi_n,\phi) \to 0$ <u>as</u> $n \to \infty$. <u>Then</u>

$$J(t,\mu,\pi,\phi) = \lim_{n\to\infty} J(t,\mu,\pi,\phi_n)$$

<u>uniformly on</u> $[0,T] \times \mathcal{M}_r \times \mathcal{A}$, <u>for any finite</u> T,r.

<u>Proof</u>. Consider $\Gamma \subset \Omega$, and let $\Gamma_T \subset \Omega_T$ denote the set of restrictions to $[0,T]$ of $(Y,U) \in \Gamma$. Then

$$(*) \qquad |E_\pi\phi_n(\Lambda_t) - E_\pi\phi(\Lambda_t)| \le$$
$$\le \int_\Gamma |\phi_n(\Lambda_t) - \phi(\Lambda_t)|d\pi + \int_{\Gamma'} |\phi_n(\Lambda_t) - \phi(\Lambda_t)|d\pi$$

with $\Gamma' = \Omega - \Gamma$. If Γ_T is a compact subset of Ω_T, then $||Y||_T$ is bounded on Γ. By 3.2, $0 \le t \le T$, $(Y,U) \in \Gamma$, $\mu \in \mathcal{M}_r$ imply $\Lambda_t \in \mathcal{M}_\rho$ for some ρ. Since $d(\phi_n,\phi) \to 0$, $\phi_n \to \phi$ uniformly on $\mathcal{M}_\rho$. Therefore, the first term on the right side of (*) tends to 0 as $n \to \infty$, uniformly with respect to $(t,\mu,\pi) \in [0,T] \times \mathcal{M}_r \times \mathcal{A}$.

It remains to show that, given $\epsilon > 0$, Γ can be chosen such that the last term in (*) is less than ϵ, uniformly on $[0,T] \times \mathcal{M}_r \times \mathcal{A}$. Now

$$|\phi_n(\Lambda_t) - \phi(\Lambda_t)| \le (||\phi_n||_K + ||\phi||_K)(1 + ||\Lambda_t||^K)$$
$$\le 2C(1 + ||\Lambda_t||^K).$$

By Cauchy-Schwartz and 3.4

$$\int_{\Gamma'} ||\Lambda_t||^K d\pi \leq \pi(\Gamma')^{\frac{1}{2}} \left(\int_{\Gamma'} ||\Lambda_t||^{2K} d\pi\right)^{\frac{1}{2}}$$
$$\leq \pi(\Gamma')^{\frac{1}{2}} \gamma_{2K,T}^{\frac{1}{2}} ||\mu||^K.$$

Under $(Y,U) \to Y$, π projects onto Wiener measure w. Let $A \subset C([0,T];\mathbb{R}^N)$ be compact with $Y_0 = 0$ for all $Y \in A$ and

$$w[C([0,T];\mathbb{R}^N) - A] < \varepsilon^2 [2C(1 + \gamma_{2K,T}^{\frac{1}{2}} r^K)]^{-2}.$$

We choose Γ such that $\Gamma_T = A \times L^2([0,T];\mathscr{U})$. Since $L^2([0,T];\mathscr{U})$ is compact (weak topology), Γ_T is compact. We have

$$\int_{\Gamma'} |\phi_n(\Lambda_t) - \phi(\Lambda_t)| d\pi \leq 2C(\pi(\Gamma') + \pi(\Gamma')^{\frac{1}{2}} \gamma_{2K,T}^{\frac{1}{2}} r^K) < \varepsilon,$$

as required. This proves Lemma 3.5.

<u>Lemma 3.6</u>. <u>For each</u> $t \geq 0$, $\phi \in C(\mathscr{M})$, $r < \infty$, $J(t,\mu,\pi,\phi)$ <u>is continuous on</u> $\mathscr{M}_r \times \mathscr{A}$.

<u>Proof</u>. Let $g(\mu,Y,U) = \phi(\Lambda_{t,\mu}^{Y,U})$. By 3.1, 3.2, g is continuous on $\mathscr{M}_r \times \Omega$ (recall that $(Y_n,U_n) \to (Y,U)$ implies $||Y_n - Y||_t \to 0$, and hence $||Y_n||_t \leq a$ for some a.) Moreover, $g(\mu,\cdot,\cdot)$ is $\mathscr{G}_t$-measurable.

Suppose first that $\phi(\mu)$ is bounded on $\mathscr{M}$. Let $\mu_n \to \mu$, $\pi_n \to \pi$ with $\mu_n \in \mathscr{M}_r$. By definition of weak convergence

$$\lim_{n\to\infty} \int_\Omega g(\mu,Y,U)d\pi_n = \int_\Omega g(\mu,Y,U)d\pi.$$

Moreover, $|g(\mu_n,Y,U) - g(\mu,Y,U)| \to 0$ as $n \to \infty$, uniformly on any $\Gamma \subset \Omega$ such that the set Γ_t of restrictions to $[0,t]$ of $(Y,U) \in \Gamma$ is compact. As in the proof of Lemma 3.5, we can choose Γ such that $\pi_n(\Omega-\Gamma)$ is arbitrarily small, uniformly with respect to n. This proves Lemma 3.6 in case $\phi(\mu)$ is bounded on $\mathscr{M}$.

Now take any $\phi \in C_K(\mathscr{M})$. By Lemmas 2.2 and 2.3, there exist $\phi_n \in C_K(\mathscr{M})$ such that $|\phi_n(\mu)|$ is bounded on $\mathscr{M}$ for each n, $||\phi_n||_K$ is bounded, and $d(\phi_n,\phi) \to 0$ as $n \to \infty$. Lemma 3.6 now follows from Lemma 3.5.

<u>The control problem</u>. Given t,μ,ϕ, we consider the problem of minimizing $J(t,\mu,\pi,\mu) = E_\pi\phi(\Lambda_t)$ on the space $\mathscr{A}$ of admissible controls π. We can regard the Zakai equation (3.1) as governing the dynamics of the "state" process Λ_t for this control problem. Since Λ_t is an unnormalized conditional distribution measure for X_t in the partially-observed control system (1.1), we call the problem of minimizing $E_\pi\phi(\Lambda_t)$ a "separated" optimal control problem.

Following Nisio [6] let

$$\mathscr{T}_t\phi(\mu) = \min_{\pi\in\mathscr{A}} J(t,\mu,\pi,\phi). \tag{3.4}$$

The minimum is attained, by Lemma 3.6.

Since $\Phi(\Lambda_t)$ is $\mathcal{G}_t$-measurable, the minimum is the same taken in the class $\mathcal{A}_t$ of admissible controls on $[0,t]$:

$$(3.5) \qquad \mathcal{S}_t\Phi(\mu) = \min_{\pi_t \in \mathcal{A}_t} J(t,\mu,\pi_t,\Phi)$$

For the special case $\Phi(\mu) = \langle G,\mu\rangle$, $J = E_\pi\langle G,\Lambda_t\rangle$ which is of the form considered in the existence theorem [3 , Theorem 4.1]. However, if Φ has this special linear form, $\mathcal{S}_t\Phi(\mu)$ is not not linear in μ. Hence, we define $\mathcal{S}_t$ the bigger space $C(\mathcal{M})$, and not merely on the space of Φ of the form $\Phi(\mu) = \langle G,\mu\rangle$.

<u>Theorem 3.1</u>. $\Phi \in C_K(\mathcal{M})$ <u>implies</u> $\mathcal{S}_t\Phi \in C_K(\mathcal{M})$.

<u>Proof</u>. *By Lemma 3.6 and the fact that* $\mathcal{M}_r$ *and* $\mathcal{A}$ *are* compact, $||\mu_n|| \leq r$ and $\mu_n \to \mu$ imply $\mathcal{S}_t\Phi(\mu_n) \to \mathcal{S}_t\Phi(\mu)$. Since any w^*-convergent sequence μ_n has $||\mu_n||$ bounded, $\mathcal{S}_t\Phi$ is continuous on $\mathcal{M}$. From 3.4,

$$\begin{aligned}
|J(t,\mu,\pi,\Phi)| &\leq ||\Phi||_K \, E\int_\Omega (1+||\Lambda_t||^K)d\pi \\
&\leq ||\Phi||_K(1+\gamma_{Kt}||\mu||^K) \\
&\leq (1+\gamma_{Kt})||\Phi||_K(1+||\mu||^K).
\end{aligned}$$

Thus, $||\mathcal{S}_t\Phi||_K \leq (1+\gamma_{Kt})||\Phi||_K$, which proves Theorem 3.1.

In the next section we establish the semigroup property of $\mathcal{S}_t$.

4. The Semigroup Property. The purpose of this section is to prove the following two theorems.

Theorem 4.1. For every $\phi \in C(\mathscr{M})$, s, t ≥ 0,

$$\mathscr{S}_{s+t}\phi = \mathscr{S}_s \mathscr{S}_t\phi .$$

Theorems 3.1 and 4.1 imply that $\mathscr{S}_t$ is a (nonlinear) semigroup on $C(\mathscr{M})$. Let $C_b(\mathscr{M})$ denote the space of bounded continuous functions on $\mathscr{M}$ (it is the same as $C_K(\mathscr{M})$ when $K = 0$.) From (3.3), (3.4) $||\mathscr{T}_t\phi - \mathscr{T}_t\psi||_0 \leq ||\phi-\psi||_0$. Hence, when restricted to $C_b(\mathscr{M})$, $\mathscr{T}_t$ is a contracting semigroup on $C_b(\mathscr{M})$.

Theorem 4.2. For every $\phi \in C(\mathscr{M})$, $\mathscr{T}_t\phi$ is a continuous function of $t \in [0,\infty)$ in the d-metric on $C(\mathscr{M})$.

The proof of Theorem 4.1 will be based on a series of three lemmas. We begin by temporarily imposing rather stringent conditions on the coefficients in (1.1), and on Y,U,μ. We say that the coefficients are regular if σ, b^0, b^1, g are of class $C_b^\infty(\mathbb{R}^N;\mathbb{R}^\ell)$ for the appropriate ℓ = ND,N,NL,M, respectively. Let us denote by $C_e^{1,2}$ the class of functions q on $[0,\infty) \times \mathbb{R}^N$ with the following properties:

(i) q and the partial derivatives $q_t, q_{x_i}; q_{x_i x_j}$ are continuous, $i,j = 1,\ldots,N$.

(ii) For each $T > 0$, there exist $C,k > 0$ (depending perhaps on T) such that

$$|r(x,t)| \le C\exp(-k|x|),\quad 0 \le t \le T,$$

where r denotes any of the functions $q, q_{x_i}, q_{x_i x_j}$.

For brevity, we write $q(t) = q(t,\cdot)$.

Lemma 4.1. Assume that the coefficients in (1.1) are regular, and that $Y \in C^1([0,\infty);\mathbb{R}^N)$, $U \in C([0,\infty);\mathcal{U})$. Then:

(a) If μ has a density $p_0 \in C_0^\infty(\mathbb{R}^N)$, then $\Lambda_t (= \Lambda_{t\mu}^{Y,U})$ has a density $q \in C_e^{1,2}$, satisfying the partial differential equation

$$\text{(4.1)}\qquad \frac{dq}{dt} = (L^{U_t})^* q + hq\cdot\dot{Y}_t - \frac{1}{2}|h|^2 q,\quad t \ge 0$$

$$q(0) = p_0.$$

(b) If $q \in C_e^{1,2}$ is a solution of (4.1) with $q(0)$ the density of μ, then $q(t)$ is the density of Λ_t for all $t \ge 0$.

Here $(L^u)^*$ denotes the formal adjoint of the operator L^u in (3.2), and $\dot{Y}_t = dY/dt$. Note that part (a) of the Lemma, but not part (b), requires that $q(0)$ has compact support.

Proof of Lemma 4.1. To prove (a), we recall from [3, §5] that

$$\text{(4.2)}\qquad p(t) = q(t)\exp(-Y_t\cdot h)$$

is a solution in $C_e^{1,2}$ to the partial differential equation

$$\text{(4.3)} \qquad \frac{dp}{dt} = \check{L}_t^* p + e(t)p, \quad \text{where}$$

$$\check{L}_t = L_t - (aY_t \cdot \nabla h, \nabla), \ L_t = L^{U_t}$$

$$e(t) = \frac{1}{2}\,(aY_t \cdot \nabla h, Y_t \cdot \nabla h) - Y_t \cdot L_t h - \frac{1}{2}\,|h|^2$$

where $(a\xi,\eta) = \sum_{i,j=1}^{N} a_{ij}\xi_i\eta_j$ and $\cdot$ denotes the $\cdot$ product in $\mathbb{R}^M$. The operators $L_t^*, \check{L}_t^*$ are related by

$$\text{(4.4)} \qquad (\check{L}_t^* p)\exp(Y_t \cdot h) = L_t^* q - eq - \frac{1}{2}\,|h|^2 q.$$

Equation (4.4) follows upon multiplying both sides of (4.4) by $f \in C_0(\mathbb{R}^N)$, integrating by parts, and using the relation

$$\exp(Y_t \cdot h)L_t f = \check{L}_t[f \exp(Y_t \cdot h)] + e(t)f \exp(Y_t \cdot h)$$
$$+ \frac{1}{2}\,|h|^2 f \exp(Y_t \cdot h).$$

Then equation (4.1) follows from (4.3), (4.4) and the product rule applied to $\frac{d}{dt}\,[p \exp(Y_t \cdot h)]$.

To prove (b), if $q \in C_e^{1,2}$ satisfies (4.1), then the above calculation shows that $p(t)$ defined by (4.2) is a solution in $C_e^{1,2}$ to (4.3). It follows from [3, (5.5)] that $q(t)$ is the density of Λ_t. (In the derivation of [3, (5.5)] it was stated that $q(0) \in C_0(\mathbb{R}^N)$. However, the proof there is based on integrations by parts, and is the same if $q \in C_e^{1,2}$.) This proves Lemma 4.1.

For $s \geq 0$, let us introduce the notation

$$Y_\tau^s = Y_{s+\tau} - Y_s, \quad U_\tau^s = U_{s+\tau}, \quad \tau \geq 0.$$

In particular, $Y_0^s = 0$; and $(Y,U) \in \Omega$ implies $(Y^s,U^s) \in \Omega$.

Lemma 4.2. For every $(Y,U) \in \Omega$, $\mu \in \mathscr{M}$, s, $t \geq 0$,

$$\Lambda_{s+t,\mu}^{YU} = \Lambda_{t\Lambda_s}^{Y^sU^s}, \quad \text{where} \quad \Lambda_s = \Lambda_{s\mu}^{YU}. \tag{4.5}$$

Proof. Step 1. First assume the conditions of Lemma 4.1 on b^ℓ,σ,h,Y,U, and that $\mu = \Lambda_0$ has a density $p_0 \in C_0^\infty(\mathbb{R}^N)$. By Lemma 4.1(a), Λ_τ has a density $q(\tau) \in C_e^{1,2}$ satisfying (4.1) for $\tau \geq 0$. Let $q^s(\tau) = q(s+\tau)$. Then q^s is a solution in $C_e^{1,2}$ of (4.1), with (Y,U) replaced by (Y^s,U^s); note that $\dot{Y}_{s+\tau} = \dot{Y}_\tau^s$ and $q^s(0) = q(s)$. By Lemma 4.1(b), $q^s(t)$ is the density of $\Lambda_{t\Lambda_s}^{Y^sU^s}$. This proves (4.5) under these conditions.

Step 2. Again assume regular coefficients b^ℓ,σ,h, $\ell = 0,1$. Let $(Y,U) \in \Omega$, $\mu \in \mathscr{M}$. Let $(Y_n,U_n) \to (Y,U)$, $\mu_n \to \mu$, where Y_n,U_n,μ_n satisfy the conditions in Step 1 for each n. Write $\Lambda_s^n = \Lambda_{s\mu_n}^{Y_nU_n}$. By property 3.1, as $n \to \infty$

$$\Lambda_{s+t,\mu_n}^{Y_nU_n} \to \Lambda_{s+t,\mu}^{YU},$$

$$\Lambda_s^n \to \Lambda_s, \quad \Lambda_{t\Lambda_s^n}^{Y_n^sU_n^s} \to \Lambda_{t\Lambda_s}^{Y^sU^s} \quad .$$

At the last step we used the fact that $(Y_n^s,U_n^s) \to (Y^s,U^s)$. This implies (4.5).

<u>Step 3</u>. Fix $\mu \in \mathscr{M}$, $(Y,U) \in \Omega$. Let σ_n, b_n^ℓ, h_n be regular for each n, uniformly bounded together with their first order partial derivatives and tending uniformly to σ, b^ℓ, h as $n \to \infty$, $\ell = 0,1$. Write $\Lambda_{t\mu}^n = \Lambda_{t\mu}^{nYU}$ to indicate that the coefficients depend on n. The proof of [3, Theorem 5.1] shows the following: $\nu_n \to \nu$, $\nu_n \in \mathscr{M}_r$, implies $\Lambda_{\tau\nu_n}^n \to \Lambda_{\tau\nu}$ for any $\tau \geq 0$. We then have as $n \to \infty$

$$\Lambda_{s+t,\mu}^n \to \Lambda_{s+t,\mu}, \quad \Lambda_{s\mu}^n \to \Lambda_{s\mu}.$$

Similarly, if we write $\Lambda_{s\mu}^n = \Lambda_s^n$, then

$$\Lambda_{t\Lambda_s^n}^{nY^sU^s} \to \Lambda_{t\Lambda_s}^{Y^sU^s} .$$

This implies (4.5), and hence Lemma 4.2.

As in §3 let π_s denote the restriction to $\mathscr{G}_s$ of $\pi \in \mathscr{A}$. Let π_s^{YU} be a regular conditional distribution for (Y^s,U^s) given $\mathscr{G}_s$.

<u>Lemma 4.3</u>. <u>If</u> $\pi \in \mathscr{A}$, <u>then</u>:

(a) $\pi_s^{YU} \in \mathscr{A}$, π_s-<u>almost surely</u>;

(b) $J(s+t,\mu,\pi,\phi) = \int_\Omega J(s,\Lambda_{s\mu}^{YU},\pi_s^{YU},\phi)\,d\pi_s,$

<u>for any</u> $\phi \in C(\mathscr{M})$.

Proof. To prove (a) it suffices to verify that, for any $\mathscr{G}_s$-measurable $\Phi \in C_b(\Omega)$, $\mathscr{G}_t$-measurable $\Psi \in C_b(\Omega)$, $F \in C_b(\mathbb{R}^M)$, and $r > t$

$$E_\pi[\Psi(Y,U)\Phi(Y^s,U^s)F(Y^s_r - Y^s_t)] = E_\pi[\Psi(Y,U)\Phi(Y^s,U^s)]E_\pi F(Y^s_r - Y^s_t).$$

But this follows from independence under π of the random variables $\Psi(Y,U)\Phi(Y^s,U^s)$ and $F(Y^s_r - Y^s_t)$.

Part (b) is immediate from (3.3), Lemma 4.2 and properties of conditional expectations.

Proof of Theorem 4.1. For every $\pi \in \mathscr{A}$, Lemma 4.3, the definition (3.4) of $\mathscr{T}_t\phi$, and (3.5) imply

$$\begin{aligned} J(s+t,\mu,\pi,\phi) &= \int_\Omega J(s,\Lambda^{YU}_{s\mu},\pi^{YU}_s,\phi)\,d\pi_s \\ &\geq \int_\Omega \mathscr{T}_t\phi(\Lambda^{YU}_{s\mu})\,d\pi_s = E_{\pi_s}\mathscr{T}_t\phi(\Lambda^{YU}_{s\mu}) \\ &\geq \mathscr{T}_s\,\mathscr{T}_t\phi(\mu). \end{aligned}$$

Since this is true for every $\pi \in \mathscr{A}$,

$$\mathscr{T}_{s+t}\phi(\mu) \geq \mathscr{T}_s\,\mathscr{T}_t\phi(\mu).$$

To prove the opposite inequality, we make the following construction. Let $\rho > 0$, $\delta > 0$ to be chosen later. Let $A_0 = \mathscr{M} - \mathscr{M}_\rho$ and $A_1,\dots,A_m$ disjoint Borel subsets of $\mathscr{M}_\rho$, such that

$$\mathscr{M}_\rho = A_1 \cup \ldots \cup A_m,$$

and for $\nu, \nu' \in A_i$, $i = 1,\ldots,m$, $\pi \in \mathscr{A}$,

$$|J(t,\nu,\pi,\phi) - J(t,\nu',\pi,\phi)| < \delta.$$

This is possible by Lemma 3.6. Choose $\mu_i \in A_i$ and $\pi_i \in \mathscr{A}$ such that

$$J(t,\mu_i,\pi_i,\phi) < \mathscr{T}_t\phi(\mu_i) + \delta.$$

For all $\nu \in A_i$,

$$(*) \qquad J(t,\nu,\pi_i,\phi) < \mathscr{T}_t\phi(\nu) + 3\delta.$$

Let $\pi_0 \in \mathscr{A}$ be arbitrary. Let

$$\pi_s^{YU} = \pi_i \quad \text{if} \quad \Lambda_{s\mu}^{YU} \in A_i.$$

Given $\pi_s \in \mathscr{A}_s$, this defines $\pi \in \mathscr{A}$ such that π_s^{YU} is a regular conditional distribution for Y^s,U^s given $\mathscr{G}_s$ and $\pi|\mathscr{G}_s = \pi_s$. By Lemma 4.3 and (*), with $\nu = \Lambda_{s\mu}^{YU} = \Lambda_s$

$$\begin{aligned} J(s+t,\mu,\pi,\phi) &= \int_\Omega J(s,\Lambda_s,\pi_s^{YU}\ \phi)d\pi_s \\ &\leq \int_{\mathscr{M}_\rho} \mathscr{T}_t\phi(\Lambda_s)d\pi_s + \int_{\Lambda_0} J(s,\Lambda_s,\pi_0,\phi)d\pi_s + 3\delta. \end{aligned}$$

Since $\mathscr{T}_{s+t}\phi(\mu) \leq J(s+t,\mu,\pi,\phi)$, we have

$$\mathscr{T}_{s+t}\phi(\mu) \leq E_{\pi_s}\, \mathscr{T}_t\phi(\Lambda_s) + \int_{A_0} J(s,\Lambda_s,\pi_0,\phi)d\pi_s$$

$$+ \int_{A_0} |\mathscr{T}_t\phi(\Lambda_s)|d\pi_s + 3\delta.$$

Now $\phi \in C_K(\mathscr{M})$ for some K. We have, for some C_1,

$$|J(s,\Lambda_s,\pi_0,\phi)| \leq C_1(1 + ||\Lambda_s||^K)$$

$$|\mathscr{T}_t\phi(\Lambda_s)| \leq C_1(1 + ||\Lambda_s||^K),$$

while by 3.4 and the fact that $A_0 = \{\nu: ||\nu|| > \rho\}$

$$\int_{A_0}(1 + ||\Lambda_s||^K)d\pi_s \leq \rho^{-K}E_{\pi_s}[||\Lambda_s||^K + ||\Lambda_s||^{2K}]$$

$$\leq C_2\rho^{-K}(1 + r^{2K})$$

for $\mu \in \mathscr{M}_r$. Therefore, given $\varepsilon > 0$ we can choose ρ large enough and δ small enough that

$$\mathscr{T}_{s+t}\phi(\mu) \leq E_{\pi_s}\, \mathscr{T}_t\phi(\Lambda_s) + \varepsilon,$$

for all $\mu \in \mathscr{M}_r$ and $\pi_s \in \mathscr{A}_s$. Upon taking the inf over π_s (recall (3.5))

$$\mathscr{T}_{s+t}\phi(\mu) \leq \mathscr{T}_s \mathscr{T}_t\phi(\mu) + \varepsilon.$$

Since ε is arbitrary, we obtain Theorem 4.1.

In preparation for the proof of Theorem 4.2, and for §5, let us introduce the following family of operators $\mathscr{L}^u$, for constant controls $u \in \mathscr{U}$. Let

$$\tilde{\mathscr{D}} = \{\phi\colon \phi(\mu) = F(\langle f_1,\mu\rangle,\ldots,\langle f_J,\mu\rangle),$$
$$F \in C^2(\mathbb{R}^J),\ f_1,\ldots,f_J \in C_0^2(\mathbb{R}^J),\ J = 1,2,\ldots\},$$

and for each integer $m \geq 0$

(4.6) $$\mathscr{D}_m = \{\phi \in \tilde{\mathscr{D}}\colon |F_{z_j}(z)| \leq C(1+|z|^{m+1}),\ |F_{z_j z_k}(z)| \leq C(1+|z|^m),\quad j,\ k = 1,\ldots,J\}.$$

We have the inclusions $\mathscr{D} \subset \mathscr{D}_m \subset C_{m+2}(\mathscr{M})$.

For $\phi \in \mathscr{D}_m$ and $u \in \mathscr{U}$, let

(4.7) $$\mathscr{L}^u\phi(\mu) = \sum_{j=1}^{J} F_{z_j}(\ldots)\langle L^u f_j,\mu\rangle + \sum_{j,k=1}^{J} F_{z_j z_k}(\ldots)\langle hf_j,\mu\rangle\cdot\langle hf_k,\mu\rangle$$

where ... denotes that the partial derivatives $F_{z_j}, F_{z_j z_k}$ are evaluated at the vector $z = (\langle f_1,\mu\rangle,\ldots,\langle f_J,\mu\rangle)$. It might seem that $\mathscr{L}^u\phi$ depends not just on ϕ, but also on $F, f_1,\ldots,f_J$.

However, it follows from (4.13) below that this difficulty does not occur.

Lemma 4.4. Let $\phi \in \mathscr{D}_m$. Then there exists c such that:

(a) $\mathscr{L}^u\phi \in C_{m+2}(\mathscr{M})$, $||\mathscr{L}^u\phi||_{m+2} \leq c$ for all $u \in \mathscr{U}$.

(b) The mapping $(u,\mu) \to \mathscr{L}^u\phi(\mu)$ is continuous from $\mathscr{U} \times \mathscr{M}_r$ into $\mathbb{R}^1$ for every $r < \infty$.

This follows at once from (4.7).

Let us next apply the Itô differential rule to $\phi(\Lambda_t)$; for $\phi \in \mathscr{D}_m$,

$$\phi(\Lambda_t) = F(<f_1,\Lambda_t>,\ldots,<f_J,\Lambda_t>).$$

We get, using the Zakai equation (3.1),

$$(4.8) \qquad d\phi(\Lambda_t) = \mathscr{L}^{U_t}\phi(\Lambda_t)dt + \sum_{j=1}^{J} F_{z_j}(\ldots)<hf_j,\Lambda_t>\cdot dY_t,$$

where $\ldots$ denotes $(<f_1,\Lambda_t>,\ldots,<f_J,\Lambda_t>)$. Since $|F_{z_j}| \leq C(1+|z|^{m+1})$, the components of $F_{z_j}(<f_1,\mu>,\ldots,<f_J,\mu>)<hf_j,\mu>$ are in $C_{m+2}(\mathscr{M})$. From 3.4, the integral on $[0,t]$ of the last term in (4.8) is a square integrable π, $\{\mathscr{G}_t\}$ martingale for any $\pi \in \mathscr{A}$. By taking $E_\pi \int_0^t$ in (4.8) and using Lemma 4.4(a) we get

$$(4.9) \qquad E_\pi\phi(\Lambda_t) = \phi(\mu) + E_\pi \int_0^t \mathscr{L}^{U_\theta}\phi(\Lambda_\theta)d\theta,$$

for any $\phi \in \mathscr{D}_m$, $\pi \in \mathscr{A}$, and any initial data $\mu = \Lambda_0$.

Lemma 4.5. Let $\psi \in \mathscr{D}_m$, $0 \leq s \leq t \leq T$. Then there exists α (depending on ψ and T) such that

$$||\mathscr{T}_t \psi - \mathscr{T}_s \psi||_{m+2} \leq \alpha(t-s).$$

Proof. Consider any $\pi \in \mathscr{A}$. By (4.9)

$$|E_\pi \phi(\Lambda_t) - E_\pi \phi(\Lambda_s)| \leq E \int_s^t |\mathscr{L}^{U_\theta} \phi(\Lambda_\theta)| d\theta$$

$$\leq \max_{u \in \mathscr{U}} ||\mathscr{L}^u \phi||_{m+2} \int_s^t (1+E||\Lambda_\theta||^{m+2}) d\theta.$$

By Lemma 4.4a and 3.4

$$|E_\pi \phi(\Lambda_t) - E_\pi \phi(\Lambda_s)| \leq c(1+\gamma_{m+2,T})(t-s)(1+||\mu||^{m+2}).$$

Since this holds for all $\pi \in \mathscr{A}$, we get Lemma 4.5 with $\alpha = c(1+\gamma_{m+2,T})$.

Proof of Theorem 4.2. For some K, $\phi \in C_K(\mathscr{M})$. By Lemma 2.4 there exists $\psi_n \in \mathscr{D}$, $n = 1,2,\ldots$, such that $||\psi_n||_K$ is bounded and $d(\psi_n,\phi) \to 0$. Fix $T > 0$. For $0 \leq s < t \leq T$, we write

$$\mathscr{T}_t \phi - \mathscr{T}_s \phi = [\mathscr{T}_t \phi - \mathscr{T}_t \psi_n] + [\mathscr{T}_t \psi_n - \mathscr{T}_s \psi_n] + [\mathscr{T}_s \psi_n - \mathscr{T}_s \psi].$$

Lemma 3.5 implies that the first and third terms on the right side tend to 0 as $n \to \infty$, uniformly for $0 \le s < t \le T$ and $\mu \in \mathcal{M}_r$.

Lemma 4.5 with $m = 0$ implies, for $\mu \in \mathcal{M}_r$,

$$|\mathcal{T}_t\psi_n(\mu) - \mathcal{T}_s\psi_n(\mu)| \le \alpha_n(1+r^2)(t-s)$$

where α_n is some constant. Let

$$\eta(\varepsilon,r) = \sup\{|\mathcal{T}_t\phi(\mu) - \mathcal{T}_s\phi(\mu)|: \mu \in \mathcal{M}_r,\ 0 \le s < t \le T,\ t - s < \varepsilon\}.$$

For each r, $\eta(\varepsilon,r) \to 0$ as $\varepsilon \to 0$. This implies $d(\mathcal{T}_t\phi, \mathcal{T}_s\phi) \to 0$ as $t - s \to 0$, as required.

This proves Theorem 4.2.

Constant controls. In particular, let us consider a constant control u. In our formulation, this corresponds to taking $\pi = \pi^u = w \times \delta_u$, where w is Wiener measure on $C([0,\infty); \mathbb{R}^m)$ and δ_u is the Dirac measure on $L^2_{loc}([0,\infty); \mathcal{U})$ concentrated on the constant trajectory $U_t \equiv u$. We can then write $E(= E_w)$ instead of E_{π^u}, and obtain from

$$(4.10)\qquad E\phi(\Lambda_t) = \phi(\mu) + E\int_0^t \mathcal{L}^u\phi(\Lambda_\theta)d\theta, \quad \phi \in \mathcal{D}_m.$$

For constant u, we may regard Λ_t as defined on the sample space $C([0,\infty); \mathbb{R}^N)$ of Y-trajectories, endowed with the family $\{\mathcal{F}_t(Y)\}$ of σ-algebras and with Wiener measure w. It follows from Lemma 4.2 that $\Lambda_t = \Lambda^{Yu}_{t\mu}$ is a Markov process (u fixed), with which is associated the linear semigroup $\mathcal{T}^u_t$ on $C(\mathcal{M})$:

$$\mathscr{T}_t^u \phi(\mu) = E\eta(\Lambda_t), \tag{4.11}$$

where $E = E_w$.

From (4.10) we have, for $\phi \in \mathscr{D}_m$,

$$t^{-1}[\mathscr{T}_t^u \phi(\mu) - \phi(\mu) - t\mathscr{L}^u\phi(\mu)] = t^{-1}\int_0^t [\mathscr{T}_\theta^u(\mathscr{L}^u\phi)(\mu) - \mathscr{L}^u\phi(\mu)]d\theta. \tag{4.12}$$

Since $\mathscr{L}^u\phi \in C(\mathscr{M})$ the same proof as for Theorem 4.2 shows that $\mathscr{T}_\theta^u(\mathscr{L}^u\phi) \to \mathscr{L}^u\phi$ as $\theta \to 0^+$, uniformly on $\mathscr{M}_r$ for each $r < \infty$ (alternatively we could apply Theorem 4.2 with the control space $\mathscr{U}$ replaced by a new one-element control space $\{u\}$.) Hence the left side of (4.12) tends to 0 as $t \to 0^+$ uniformly on $\mathscr{M}_r$, which implies

$$\mathscr{L}^u\phi = \underset{t\to 0^+}{\text{d-lim}}\ t^{-1}[\mathscr{T}_t^u\phi - \phi], \qquad \phi \in \mathscr{D}_m. \tag{4.13}$$

This shows that for each $m = 0,1,2,\ldots$, $\mathscr{D}_m$ is contained in the domain of the generator of the linear semigroup $\mathscr{T}_t^u$ and that $\mathscr{L}^u$ agrees on $\mathscr{D}_m$ with the generator.

5. The Generator of the Semigroup $\mathscr{T}_t$. We define the operator $\mathscr{L}$ on the dense subset $\mathscr{D}$ of $C(\mathscr{M})$ by

$$\mathscr{L}\phi(\mu) = \min_{u\in\mathscr{U}} \mathscr{L}^u\phi(\mu), \qquad \phi \in \mathscr{D}. \tag{5.1}$$

Lemma 4.4 implies that $\mathscr{L}\phi \in C_2(\mathscr{M})$ for every $\phi \in \mathscr{D}$.

We need slightly stronger hypotheses on σ, b^0, b^1 than (A_1), (A_2) in §3:

(A_1') Condition (A_1) holds and, in addition, $a \in C_b^2(\mathbb{R}^N;\mathbb{R}^{N^2})$, where $a = \sigma\sigma'$.

(A_2') $b(x,u) = b^0(x) + b^1(x)u$, where $b^0 \in C_b^2(\mathbb{R}^N;\mathbb{R}^N)$ and $b^1 \in C_b^2(\mathbb{R}^N;\mathbb{R}^{NL})$.

When (A_1'), (A_2'), (A_3) hold, $f \in C_0^\infty(\mathbb{R}^N)$ implies $L^u f \in C_0^2(\mathbb{R}^N)$ and $hf \in C_0^2(\mathbb{R}^N;\mathbb{R}^M)$. From (4.7), $\phi \in \mathscr{D}$ implies $\mathscr{L}^u\phi \in \mathscr{D}_2$.

Theorem 5.1. For every $\phi \in \mathscr{D}$

$$\mathscr{L}\phi = \operatorname*{d-lim}_{t\to 0^+} t^{-1}(\mathscr{T}_t\phi - \phi). \tag{5.2}$$

This theorem justifies our calling $\mathscr{L}$ the generator of the nonlinear semigroup $\mathscr{T}_t$. Our proof of Theorem 5.1 follows the same general line of reasoning as Nisio [6].

The proof of Theorem 5.1 depends on the following estimates for the semigroups $\mathscr{T}_t^u$, for any constant control $u \in \mathscr{U}$. By the same calculation used in the proof of Theorem 3.1

$$||\mathscr{T}_t^u\phi||_K \le (1+\gamma_K t)||\phi||_K, \qquad \phi \in C_K(\mathscr{M}). \tag{5.3}$$

For $\phi \in \mathscr{D}_m$, 3.4 and (4.10) imply

$$||\mathscr{T}_t^u\phi - \phi||_{m+2} \le ||\mathscr{L}^u\phi||_{m+2}(1+\gamma_{m+2,t})t \tag{5.4}$$

Lemma 4.4(a) gives a bound for $||\mathscr{L}^u\phi||_{m+2}$.

Now consider $\phi \in \mathscr{D}$,

$$\phi(\mu) = F(<f_1,\mu>,\ldots,<f_J,\mu>)$$

with $F \in C_b^\infty(\mathbb{R}^J)$, $f_j \in C_0^\infty(\mathbb{R}^N)$. Then

$$\mathscr{L}^u\phi = \sum_{j=1}^{J}\phi_j + \sum_{j,k=1}^{J}\phi_{jk},$$

$$\phi_j(\mu) = F_{z_j}(<f_1,\mu>,\ldots,<f_J,\mu>)<L^u f_j,\mu>$$

$$\phi_{jk}(\mu) = F_{z_j z_k}(<f_1,\mu>,\ldots,<f_J,\mu>)<hf_j,\mu>\cdot<hf_k,\mu>$$

$$\mathscr{T}_t^u\phi - \phi - t\mathscr{L}^u\phi = \int_0^t[\mathscr{T}_\theta^u(\mathscr{L}^u\phi) - \mathscr{L}^u\phi]d\theta$$

$$= \sum_j \int_0^t[\mathscr{T}_\theta^u\phi_j - \phi_j]d\theta + \sum_{j,k}\int_0^t[\mathscr{T}_\theta^u\phi_{jk} - \phi_{jk}]d\theta.$$

Since $\phi_j,\phi_{jk} \in \mathscr{D}_2$, we can apply (5.4) to ϕ_j,ϕ_{jk} to get, for $0 \leq t \leq 1$,

$$(5.5) \qquad ||\mathscr{T}_t^u\phi - \phi - t\mathscr{L}^u\phi||_4 \leq \beta t^2$$

where the constant β depends on ϕ but not on $u \in \mathscr{U}$.

Lemma 5.1. For $\phi \in \mathscr{D}$

$$\mathscr{T}_t\phi - \phi \geq \int_0^t \mathscr{T}_\theta(\mathscr{L}\phi)d\theta.$$

Proof. By (4.9), for any $\pi \in \mathscr{A}$

$$E_\pi \phi(\Lambda_t) - \phi(\mu) = E_\pi \int_0^t \mathscr{L}^{U_\theta} \phi(\Lambda_\theta) d\theta$$

$$\geq E_\pi \int_0^t \mathscr{L}\phi(\Lambda_\theta) d\theta = \int_0^t E_\pi \mathscr{L}\phi(\Lambda_\theta) d\theta$$

$$\geq \int_0^t \mathscr{T}_\theta(\mathscr{L}\phi)(\mu) d\theta .$$

The minimum over $\mathscr{A}$ of the left side is $T_t\phi(\mu) - \phi(\mu)$. This proves Lemma 5.1.

Proof of Theorem 5.1. Observe that $\mathscr{T}_t\phi \leq \mathscr{T}_t^u\phi$ for all $u \in \mathscr{U}$ (constant controls are suboptimal) Then, for $\phi \in \mathscr{D}$, $0 < t \leq 1$,

$$t^{-1}[\mathscr{T}_t\phi - \phi - t\mathscr{L}^u\phi] \leq t^{-1}[\mathscr{T}_t^u\phi - \phi - t\mathscr{L}^u\phi].$$

In particular, given μ we take u such that $\mathscr{L}^u\phi(\mu) = \mathscr{L}\phi(\mu)$ [recall (5.1)]. By (5.5), when $0 < t \leq 1$,

$$t^{-1}[\mathscr{T}_t\phi(\mu) - \phi(\mu) - t\mathscr{L}\phi(\mu)] \leq \beta t(1+||\mu||^4).$$

Therefore, uniformly for $\mu \in \mathscr{M}_r$,

$$\limsup_{t\to 0^+} t^{-1}[\mathscr{T}_t\phi(\mu) - \phi(\mu)] \leq \mathscr{L}\phi(\mu).$$

On the other hand, by Lemma 5.1

$$\liminf_{t\to 0^+} t^{-1}[\mathcal{T}_t\phi(\mu) - \phi(\mu)] \geq \liminf_{t\to 0} t^{-1}\int_0^t \mathcal{T}_\theta(\mathcal{L}\phi)(\mu)d\theta.$$

Since $\mathcal{L}\phi \in C(\mathcal{M})$, Theorem 4.2 implies that $\mathcal{T}_\theta(\mathcal{L}\phi)(\mu) \to \mathcal{L}\phi(\mu)$ as $\theta \to 0^+$, uniformly on $\mathcal{M}_r$. Hence,

$$\lim_{t\to 0^+} t_+^{-1}[\mathcal{T}_t\phi(\mu) - \phi(\mu)) = \mathcal{L}\phi(\mu)$$

uniformly on $\mathcal{M}_r$, for each r. This proves Theorem 5.1.

Remark. The nonlinear semigroup $\mathcal{T}_t$ can be obtained from the family of linear semigroups $\mathcal{T}_t^u$, $u \in \mathcal{U}$, by the following procedure used in [6]. For $\Delta > 0$, let $\mathcal{J}_\Delta\phi(\mu) = \min_{u\in\mathcal{U}} \mathcal{T}_\Delta^u\phi(\mu)$. For $n = 1,2,\ldots$ and dyadic rational $t = m2^{-n}$ $(m = 1,2,\ldots)$ let

$$\mathcal{T}_t^n\phi = \mathcal{J}_{\Delta_n}^m\phi, \quad \Delta_n = 2^{-n}, \quad \phi \in C(\mathcal{M}).$$

It is easy to show that, for dyadic rational $t = m2^{-n}$,

$$\mathcal{T}_t^n\phi \geq \mathcal{T}_t^{n+1}\phi \geq \ldots \geq \mathcal{T}_t\phi.$$

By considering controls piecewise constant in time, one can show that $\mathcal{T}_t^n\phi \to \mathcal{T}_t\phi$ as $n \to \infty$, if t is dyadic rational. Choose n large enough such that $t = m2^{-n}$. Let $\tau_k = k2^{-n}$ and

$$\mathcal{A}_{nt} = \{\pi \in \mathcal{A}_t\colon \pi[U_\tau = U_{\tau_k} \text{ for } \tau \in [\tau_k,\tau_{k+1}),\ k = 0,1,\ldots,m-1] = 1\}.$$

By induction on m (for fixed n) and a construction like that in the proof of Theorem 4.1, it can be shown that

$$\mathcal{T}_t^n \phi(\mu) = \min_{\pi \in \mathcal{A}_{nt}} J(t,\mu,\pi,\phi).$$

By [3, Corollary 6.1], every $\pi \in \mathcal{A}_t$ is the limit of π_{nt} as $n \to \infty$, with $\pi_{nt} \in \mathcal{A}_{nt}$. Lemma 3.6 then implies that $\mathcal{T}_t^n \phi(\mu) \to \mathcal{T}_t \phi(\mu)$ as $n \to \infty$.

REFERENCES

[1] A. Bensoussan and J.L. Lions, Applications des inequations variationnelles en controle stochastique, Dunod 1978.

[2] M.H.A. Davis, Nonlinear semigroups in the control of partially-observable stochastic systems, in Measure Theory and Applications to Stochastic Analysis, Springer-Verlag Lecture Notes in Math. No. 695, 1978.

[3] W.H. Fleming and E. Pardoux, Existence of optimal controls for partially observed diffusions, submitted to SIAM J. on Control and Optimization.

[4] W.H. Fleming and R.W. Rishel, Deterministic and Stochastic Optimal Control, Springer-Verlag, 1975.

[5] W.H. Fleming and M. Viot, Some measure-valued Markov processes in population genetics theory, Indiana Univ. Math. J. 28(1979), 817-844.

[6] M. Nisio, On a nonlinear semigroup attached to optimal stochastic control. Publ. RIMS Kyoto Univ. 13(1976), 513-537.

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