

Nonlinear semigroups and evolution equations

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(Received May 2, 1967)

Introduction

This paper has been motivated by a recent paper by Y. Kōmura [3], in which a general theory of semigroups of nonlinear contraction operators in a Hilbert space is developed. Owing to the generality of the problem, Kōmura is led to consider multi-valued operators as the infinitesimal generators of such semigroups, which makes his theory appear somewhat complicated.

The object of the present paper is to restrict ourselves to single-valued operators in a Banach space X and to construct the semigroups generated by them in a more elementary fashion. Furthermore, we are able to treat, without essential modifications, time-dependent nonlinear equations of the form

$$(E) \quad du/dt + A(t)u = 0, \quad 0 \leq t \leq T,$$

where the unknown $u(t)$ is an X -valued function and where $\{A(t)\}$ is a family of nonlinear operators with domains and ranges in X . In particular we shall prove existence and uniqueness of the solution of (E) for a given initial condition.

The basic assumptions we make for (E) are that the adjoint space X^* is uniformly convex and that the $A(t)$ are m -monotonic operators (see below), together with some smoothness condition for $A(t)$ as a function of t . We make no explicit assumptions on the continuity of the operators $A(t)$.

Here an operator A with domain $D(A)$ and range $R(A)$ in an arbitrary Banach space X is said to be *monotonic* if

$$(M) \quad \|u - v + \alpha(Au - Av)\| \geq \|u - v\| \quad \text{for every } u, v \in D(A) \text{ and } \alpha > 0.$$

This implies that $(1 + \alpha A)^{-1}$ exists and is Lipschitz continuous provided $\alpha > 0$, where $1 + \alpha A$ is the operator with domain $D(A)$ which sends u into $u + \alpha Au$. It can be shown (see Lemma 2.1) that $(1 + \alpha A)^{-1}$ has domain X either for every $\alpha > 0$ or for no $\alpha > 0$; in the former case we say that A is *m-monotonic*.

The monotonicity thus defined can also be expressed in terms of the *duality*

* This work represents part of the results obtained while the author held a Miller Research Professorship.

map F from X to X^* . (Here X^* is defined to be the set of all bounded semi-linear forms on X , and the pairing between $x \in X$ and $f \in X^*$ is denoted by (x, f) , which is thus linear in x and semilinear in f . If X is a Hilbert space, X^* is identified with X and $(,)$ with the inner product in X .) F is in general a multi-valued operator; for each $x \in X$, Fx is by definition the (nonempty) set of all $f \in X^*$ such that $(x, f) = \|x\|^2 = \|f\|^2$. (Thus we employ a special "gauge function" for F .)

(M) is now equivalent to the following condition (see Lemma 1.1):

(M') For each $u, v \in D(A)$, there is $f \in F(u-v)$ such that

$$\operatorname{Re}(Au - Av, f) \geq 0.$$

Note that the inequality is not required to hold for every $f \in F(u-v)$. If X is a Hilbert space, (M') is equivalent to the monotonicity of A in the sense of Minty [4] and Browder [1].

The main results of this paper are stated in §3 and the proofs are given in §4. §§1 and 2 contain some preliminary results for the duality map F and for m -monotonic operators.

The crucial step in our existence proof is the proof of convergence for the approximate solutions $u_n(t)$ of (E), which is a straightforward generalization of an ingenious proof given in [3]. The author is indebted to Professor Y. Kōmura for having a chance to see his paper before publication and to Professor K. Yosida for many stimulating conversations.

1. The duality map

We first consider an arbitrary Banach space X . The duality map F from X to X^* was defined in Introduction.

LEMMA 1.1. *Let $x, y \in X$. Then $\|x\| \leq \|x + \alpha y\|$ for every $\alpha > 0$ if and only if there is $f \in Fx$ such that $\operatorname{Re}(y, f) \geq 0$.*

PROOF. The assertion is trivial if $x = 0$. So we shall assume $x \neq 0$ in the following. If $\operatorname{Re}(y, f) \geq 0$ for some $f \in Fx$, then $\|x\|^2 = (x, f) = \operatorname{Re}(x, f) \leq \operatorname{Re}(x + \alpha y, f) \leq \|x + \alpha y\| \|f\|$ for $\alpha > 0$. Since $\|f\| = \|x\|$, we obtain $\|x\| \leq \|x + \alpha y\|$.

Suppose, conversely, that $\|x\| \leq \|x + \alpha y\|$ for $\alpha > 0$. For each $\alpha > 0$ let $f_\alpha \in F(x + \alpha y)$ and $g_\alpha = f_\alpha / \|f_\alpha\|$ so that $\|g_\alpha\| = 1$. Then $\|x\| \leq \|x + \alpha y\| = (x + \alpha y, g_\alpha) = \operatorname{Re}(x, g_\alpha) + \alpha \operatorname{Re}(y, g_\alpha) \leq \|x\| + \alpha \operatorname{Re}(y, g_\alpha)$. Thus

$$(1.1) \quad \liminf_{\alpha \downarrow 0} \operatorname{Re}(x, g_\alpha) \geq \|x\| \quad \text{and} \quad \operatorname{Re}(y, g_\alpha) \geq 0.$$

Since the closed unit ball of X^* is compact in the weak* topology, the net $\{g_\alpha\}$ (with the index set $\{\alpha\}$ directed as $\alpha \downarrow 0$) has a cluster point $g \in X^*$ with $\|g\| \leq 1$. In view of (1.1), however, g satisfies $\operatorname{Re}(x, g) \geq \|x\|$ and $\operatorname{Re}(y, g) \geq 0$.

Hence we must have $\|g\| = 1$ and $(x, g) = \|x\|$. On setting $f := \|x\|g$, we see that $f \in Fx$ and $\operatorname{Re}(y, f) \geq 0$.

It is known (and is easy to prove) that F is single-valued if X^* is strictly convex. One would need somewhat stronger condition to ensure that F is continuous. A convenient sufficient condition is given by

LEMMA 1.2. *If X^* is uniformly convex, F is single-valued and is uniformly continuous on any bounded set of X . In other words, for each $\varepsilon > 0$ and $M > 0$, there is $\delta > 0$ such that $\|x\| < M$ and $\|x - y\| < \delta$ imply $\|Fx - Fy\| < \varepsilon$.*

PROOF. It suffices to show that the assumptions

$$\|x_n\| < M, \|x_n - y_n\| \rightarrow 0, \|Fx_n - Fy_n\| \geq \varepsilon_0 > 0, n = 1, 2, \dots,$$

lead to a contradiction. If $x_n \rightarrow 0$ (we denote by \rightarrow strong convergence), then $y_n \rightarrow 0$ and so $\|Fx_n\| = \|x_n\| \rightarrow 0$ and similarly $\|Fy_n\| \rightarrow 0$, hence $\|Fx_n - Fy_n\| \rightarrow 0$, a contradiction. Thus we may assume that $\|x_n\| \geq \alpha > 0$, replacing the given sequences by suitable subsequences if necessary. Then $\|y_n\| \geq \alpha/2$ for sufficiently large n . Set $u_n = x_n/\|x_n\|$ and $v_n = y_n/\|y_n\|$. Then $\|u_n\| = \|v_n\| = 1$ and $u_n - v_n = (x_n - y_n)/\|x_n\| + (\|x_n\|^{-1} - \|y_n\|^{-1})y_n$ so that $\|u_n - v_n\| \leq 2\|x_n - y_n\|/\|x_n\| \rightarrow 0$.

Since $\|Fu_n\| = \|u_n\| = 1$ and similarly $\|Fv_n\| = 1$, we thus obtain $\operatorname{Re}(u_n, Fu_n + Fv_n) = (u_n, Fu_n) + (v_n, Fv_n) + \operatorname{Re}(u_n - v_n, Fv_n) \geq 1 + 1 - \|u_n - v_n\| \rightarrow 2$. Hence $\liminf \|Fu_n + Fv_n\| \geq \liminf \operatorname{Re}(u_n, Fu_n + Fv_n) \geq 2$. Since $\|Fu_n\| = \|Fv_n\| = 1$ and X^* is uniformly convex, it follows that $Fu_n - Fv_n \rightarrow 0$.

Since $Fx_n = F(\|x_n\|u_n) = \|x_n\|Fu_n$ and similarly $Fy_n = \|y_n\|Fv_n$, we now obtain $Fx_n - Fy_n = \|x_n\|(Fu_n - Fv_n) + (\|x_n\| - \|y_n\|)Fv_n \rightarrow 0$ by $\|x_n\| < M$. Thus we have arrived at a contradiction again.

In this paper the usefulness of the duality map depends mainly on the following lemma.

LEMMA 1.3. *Let $x(t)$ be an X -valued function on an interval of real numbers. Suppose $x(t)$ has a weak derivative $x'(s) \in X$ at $t = s$ (that is, $d(x(t), g)/dt$ exists at $t = s$ and equals $(x'(s), g)$ for every $g \in X^*$). If $\|x(t)\|$ is also differentiable at $t = s$, then*

$$(1.2) \quad \|x(s)\|(d/ds)\|x(s)\| = \operatorname{Re}(x'(s), f)$$

for every $f \in Fx(s)$.

PROOF. Since $\operatorname{Re}(x(t), f) \leq \|x(t)\|\|f\| = \|x(t)\|\|x(s)\|$ and $\operatorname{Re}(x(s), f) = \|x(s)\|^2$, we have

$$\operatorname{Re}(x(t) - x(s), f) \leq \|x(s)\|(\|x(t)\| - \|x(s)\|).$$

Dividing both sides by $t - s$ and letting $t \rightarrow s$ from above and from below, we obtain $\operatorname{Re}(x'(s), f) \leq \|x(s)\|(d/ds)\|x(s)\|$. Thus we must have the equality (1.2).

2. Monotonic operators in X

Monotonic operators A in X have been defined by the equivalent conditions (M) and (M') given in Introduction. Their equivalence follows immediately from Lemma 1.1.

If X is a Hilbert space, the inverse of an invertible monotonic operator is also monotonic, but it might not be true in the general case.

If A is monotonic, $1+\alpha A$ is invertible for $\alpha > 0$ and the inverse operator $(1+\alpha A)^{-1}$ is Lipschitz continuous:

$$(2.1) \quad \|(1+\alpha A)^{-1}x - (1+\alpha A)^{-1}y\| \leq \|x - y\|, \quad x, y \in D((1+\alpha A)^{-1}).$$

This follows directly from (M).

LEMMA 2.1. *Let A be monotonic. If $D((1+\alpha A)^{-1}) = R(1+\alpha A)$ is the whole of X for some $\alpha > 0$, the same is true for all $\alpha > 0$.*

PROOF. $R(1+\alpha A) = X$ is equivalent to $R(A+\lambda) = X$ where $\lambda = 1/\alpha$. Thus it suffices to show that $R(A+\lambda) = X$ for all $\lambda > 0$ if it is true for some $\lambda > 0$. But this is proved essentially in [3].

As stated in Introduction, we say that A is m -monotonic if the conditions of Lemma 2.1 are satisfied. Here we do not assume that $D(A)$ is dense in X . If A is a linear operator in a Hilbert space, the m -monotonicity of A implies that $D(A)$ is dense, but we do not know whether or not the same is true in the general case.

For an m -monotonic operator A , we introduce the following sequences of operators ($n = 1, 2, \dots$):

$$(2.2) \quad J_n = (1+n^{-1}A)^{-1},$$

$$(2.3) \quad A_n = AJ_n = n(1-J_n),$$

where AJ_n denotes the composition of the two maps A and J_n . The J_n and A_n are defined everywhere on X . The identity given by (2.3), which is easy to verify, is rather important in the following arguments.

LEMMA 2.2. *Let A be m -monotonic. J_n and A_n are uniformly Lipschitz continuous, with*

$$(2.4) \quad \|J_n x - J_n y\| \leq \|x - y\|, \quad \|A_n x - A_n y\| \leq 2n\|x - y\|,$$

where $2n$ may be replaced by n if X is a Hilbert space.

PROOF. The first inequality of (2.4) is a special case of (2.1). The second then follows from (2.3). The assertion about the case of X a Hilbert space is easy to prove and the proof is omitted (it is not used in the following).

LEMMA 2.3. *Let A be m -monotonic. The A_n are also monotonic. Furthermore, we have*

$$(2.5) \quad \|A_n u\| \leq \|Au\| \quad \text{for } u \in D(A).$$

PROOF. Let $x, y \in X$ and $f \in F(x-y)$. Then

$$\begin{aligned} \operatorname{Re}(A_n x - A_n y, f) &= n \operatorname{Re}(x-y, f) - n \operatorname{Re}(J_n x - J_n y, f) \\ &\geq n\|x-y\|^2 - n\|J_n x - J_n y\|\|f\| \geq n\|x-y\|^2 - n\|x-y\|^2 = 0, \end{aligned}$$

where we have used (2.3) and (2.4). Thus A_n is monotonic by (M'). If $u \in D(A)$, we have $A_n u = n(u - J_n u) = n[J_n(1 + n^{-1}A)u - J_n u]$ by (2.3) and so $\|A_n u\| \leq n\|u + n^{-1}Au - u\| = \|Au\|$ by (2.4).

LEMMA 2.4. If $u \in [D(A)]$ (the closure of $D(A)$ in X), $J_n u \rightarrow u$ as $n \rightarrow \infty$.

PROOF. If $u \in D(A)$, then $u - J_n u = n^{-1}A_n u \rightarrow 0$ since the $\|A_n u\|$ are bounded by (2.5). The result is extended to all $u \in [D(A)]$ since the J_n are Lipschitz continuous uniformly in n .

LEMMA 2.5. Let X^* be uniformly convex and let A be m -monotonic in X .

(a) If $u_n \in D(A)$, $n = 1, 2, \dots$, $u_n \rightarrow u \in X$ and if the $\|Au_n\|$ are bounded, then $u \in D(A)$ and $Au_n \rightarrow Au$ (we denote by \rightarrow weak convergence).

(b) If $x_n \in X$, $n = 1, 2, \dots$, $x_n \rightarrow u \in X$ and if the $\|A_n x_n\|$ are bounded, then $u \in D(A)$ and $A_n x_n \rightarrow Au$.

(c) $A_n u \rightarrow Au$ if $u \in D(A)$.

PROOF. In this case the duality map F is single-valued and is continuous (see Lemma 1.2).

(a) The monotonicity condition (M') gives

$$(2.6) \quad \operatorname{Re}(Av - Au_n, F(v - u_n)) \geq 0$$

for any $v \in D(A)$. Since X is reflexive with X^* and the $\|Au_n\|$ are bounded, there is a subsequence $\{u_{n'}\}$ of $\{u_n\}$ such that $Au_{n'} \rightarrow x \in X$. Since $v - u_{n'} \rightarrow v - u$ and hence $F(v - u_{n'}) \rightarrow F(v - u)$ by the continuity of F , we obtain from (2.6) the inequality $\operatorname{Re}(Av - x, F(v - u)) \geq 0$.

Using Lemma 1.1 with $\alpha = 1$, we then have $\|v - u + Av - x\| \geq \|v - u\|$. On setting $v = J_1(u + x)$ so that $v \in D(A)$ and $v + Av = u + x$, we see that $\|v - u\| \leq 0$, hence $u = v$ and $Au = x$. Thus $Au_{n'} \rightarrow x = Au$.

Since we could have started with any subsequence of $\{u_n\}$ instead of $\{u_n\}$ itself, the result obtained shows that Au_n converges weakly to Au .

(b) Set $u_n = J_n x_n \in D(A)$. Then $Au_n = A_n x_n$ and the $\|Au_n\|$ are bounded. Also $x_n - u_n = (1 - J_n)x_n = n^{-1}A_n x_n \rightarrow 0$ so that $u_n \rightarrow u$. Thus the result of (a) is applicable, with the result that $u \in D(A)$ and $A_n x_n = Au_n \rightarrow Au$.

(c) It suffices to set $x_n = u$ in (b); note that $\|A_n u\| \leq \|Au\|$ by Lemma 2.3.

3. The theorems

We now consider the Cauchy problem for the nonlinear evolution equation (E). We introduce the following conditions for the family $\{A(t)\}$.

- I. The domain D of $A(t)$ is independent of t .
- II. There is a constant L such that for all $v \in D$ and $s, t \in [0, T]$,

$$(3.1) \quad \|A(t)v - A(s)v\| \leq L|t-s|(1 + \|v\| + \|A(s)v\|).$$

- III. For each t , $A(t)$ is m -monotonic.

(3.1) implies that $A(t)v$ is continuous in t and hence is bounded. Then (3.1) shows that $A(t)v$ is uniformly Lipschitz continuous in t . It further shows that the Lipschitz continuity is uniform for $v \in D$ in a certain metric.

On the other hand, we do not make any assumptions on the continuity of the maps $v \rightarrow A(t)v$, except those implicitly contained in the m -monotonicity.

The main results of this paper are given by the following theorems.

THEOREM 1 (*existence theorem*). *Assume that X^* is uniformly convex and that the conditions I, II, III are satisfied. For each $a \in D$, there exists an X -valued function $u(t)$ on $[0, T]$ which satisfies (E) and the initial condition $u(0) = a$ in the following sense. (a) $u(t)$ is uniformly Lipschitz continuous on $[0, T]$, with $u(0) = a$. (b) $u(t) \in D$ for each $t \in [0, T]$ and $A(t)u(t)$ is weakly continuous on $[0, T]$. (c) The weak derivative of $u(t)$ exists for all $t \in [0, T]$ and equals $-A(t)u(t)$. (d) $u(t)$ is an indefinite integral of $-A(t)u(t)$, which is Bochner integrable, so that the strong derivative of $u(t)$ exists almost everywhere and equals $-A(t)u(t)$.*

THEOREM 2 (*uniqueness and continuous dependence on the initial value*). *Under the assumptions of Theorem 1, let $u(t)$ and $v(t)$ satisfy conditions (a), (b), (c) with the initial conditions $u(0) = a$ and $v(0) = b$, where $a, b \in D$. Then $\|u(t) - v(t)\| \leq \|a - b\|$ for all $t \in [0, T]$.*

THEOREM 3. *In addition to the assumptions of Theorem 1, assume that X is uniformly convex. Then the strong derivative $du/dt = -A(t)u(t)$ exists and is strongly continuous except at a countable number of values t .*

REMARKS. 1. Conditions (a) to (d) in Theorem 1 are not all independent. (a) follows from (b) and (c) (except, of course, $u(0) = a$).

2. When $A(t) = A$ is independent of t , these results give a partial generalization of the Hille-Yosida theorem to semigroups of nonlinear operators. Suppose X^* is uniformly convex and A is m -monotonic in X . Since $T > 0$ is arbitrary in this case, on setting $u(t) = U(t)a$ we obtain a family $\{U(t)\}$, $0 \leq t < \infty$, of nonlinear operators $U(t)$ on $D(A)$ to itself. Obviously $\{U(t)\}$ forms a semigroup generated by $-A$. It is a *contraction semigroup* on $D(A)$, for $\|U(t)a - U(t)b\| \leq \|a - b\|$, and it can be extended by continuity to a contraction semi-

group on $[D(A)]$ (the closure of $D(A)$ in X). It should be noted, however, that we have not been able to prove the strong differentiability of $U(t)a$ at $t=0$ for all $a \in D(A)$.

3. If the $A(t)$ are linear operators, the above theorems contain very little that is new. But their proofs are independent of the earlier ones, such as are given by [2], and are even simpler (of course under the restriction on X).

4. The assumptions I to III could be weakened to some extent. For example, it would suffice to assume, instead of III, that for each $t \in [0, T]$ there is a norm $\|\cdot\|_t$, equivalent to the given norm of X and depending on t "smoothly", with respect to which $A(t)$ is m -monotonic. I and II could be replaced by the condition that there is a function $Q(t)$, depending on t "smoothly", such that $Q(t)$ and $Q(t)^{-1}$ are bounded linear operators with domain and range X and that $\bar{A}(t) = Q(t)^{-1}A(t)Q(t)$ satisfies I and II. We want to deal with such generalizations in later publications.

5. III could also be weakened to the condition that $A(t) + \lambda$ be m -monotonic for some $\lambda > 0$. It should be noted that this is not a trivial generalization. If $A(t)$ were linear, the transformation $u(t) = e^{\lambda t}v(t)$ would change (E) into $dv/dt + (A(t) + \lambda)v = 0$. But the same transformation does not always work in the nonlinear case, for the transformed equation involves the operator $e^{-\lambda t}[A(t) + \lambda]e^{\lambda t}$, the domain of which may depend on t when $D(A(t))$ does not.

4. Proofs of the theorems

To construct a solution of (E), we introduce the operators

$$(4.1) \quad J_n(t) = (1 + n^{-1}A(t))^{-1}, \quad A_n(t) = A(t)J_n(t), \quad n = 1, 2, \dots,$$

for which the results of §2 are available, and consider the approximate equations

$$(E_n) \quad du_n/dt + A_n(t)u_n = 0, \quad u_n(0) = a.$$

To solve (E_n) and prove the convergence of $\{u_n(t)\}$, we need some estimates for the $A_n(t)$.

LEMMA 4.1. *For all n and $v \in D$, we have*

$$(4.2) \quad \|A_n(t)v - A_n(s)v\| \leq L|t-s|(1 + \|v\| + (1 + n^{-1})\|A_n(s)v\|).$$

PROOF. Since $A_n(t) = n(1 - J_n(t))$ by (2.3), we have

$$\begin{aligned} A_n(t)v - A_n(s)v &= nJ_n(s)v - nJ_n(t)v \\ &= nJ_n(t)[1 + n^{-1}A(t)]J_n(s)v - nJ_n(t)[1 + n^{-1}A(s)]J_n(s)v. \end{aligned}$$

Using the Lipschitz continuity (2.4) of the operator $J_n(t)$, we obtain

$$\begin{aligned} \|A_n(t)v - A_n(s)v\| &\leq n\|[1 + n^{-1}A(t)]J_n(s)v - [1 + n^{-1}A(s)]J_n(s)v\| \\ &= \|[A(t) - A(s)]J_n(s)v\|, \end{aligned}$$

and using (3.1),

$$(4.3) \quad \|A_n(t)v - A_n(s)v\| \leq L|t - s|(1 + \|J_n(s)v\| + \|A(s)J_n(s)v\|).$$

Here $\|J_n(s)v\|$ is estimated by (2.3) as $\|J_n(s)v\| \leq \|v\| + n^{-1}\|A_n(s)v\|$. Since $A(s)J_n(s) = A_n(s)$, (4.3) gives (4.2).

(4.2) shows that $A_n(t)v$ is Lipschitz continuous in t for each $v \in X$. On the other hand, the map $v \rightarrow A_n(t)v$ is Lipschitz continuous for fixed t , uniformly in v and t (see (2.4)). Thus (E_n) has a unique solution $u_n(t)$ for $t \in [0, T]$, for any initial condition $u_n(0) = a \in X$. We shall now deduce some estimates for $u_n(t)$.

LEMMA 4.2. *Let $a \in D$. Then there is a constant K such that $\|u_n(t)\| \leq K$, $\|u'_n(t)\| = \|A_n(t)u_n(t)\| \leq K$, for all $n = 1, 2, \dots$ and $t \in [0, T]$. (We write $du_n/dt = u'_n$.)*

PROOF. We apply Lemma 1.3 to $x_n(t) = u_n(t+h) - u_n(t)$, where $0 < h < T$. Since $x_n(t)$ is differentiable with $x'_n(t) = -[A_n(t+h)u_n(t+h) - A_n(t)u_n(t)]$, (1.2) gives

$$(4.4) \quad \|x_n(t)\|(d/dt)\|x_n(t)\| = -\operatorname{Re}(A_n(t+h)u_n(t+h) - A_n(t)u_n(t), Fx_n(t))$$

for each t where $\|x_n(t)\|$ is differentiable; note that the duality map F is single-valued because X^* is uniformly convex (see Lemma 1.2).

The first factor in the scalar product on the right of (4.4) can be written

$$[A_n(t+h)u_n(t+h) - A_n(t+h)u_n(t)] + [A_n(t+h)u_n(t) - A_n(t)u_n(t)],$$

of which the first term contributes to (4.4) a nonpositive value by the monotonicity of $A_n(t+h)$ (see Lemma 2.3). The second term can be estimated by (4.2); it is thus majorized in norm by $Lh(1 + \|u_n(t)\| + (1 + n^{-1})\|u'_n(t)\|)$, where we have used $A_n(t)u_n(t) = -u'_n(t)$. In this way we obtain from (4.4), using the Schwarz inequality and the norm-preserving property of F ,

$$(4.5) \quad \|x_n(t)\|(d/dt)\|x_n(t)\| \leq Lh(1 + \|u_n(t)\| + (1 + n^{-1})\|u'_n(t)\|)\|x_n(t)\|.$$

Since $\|x_n(t)\|$ is Lipschitz continuous with $x_n(t)$, it is differentiable almost everywhere, where (4.5) is true as shown above. Let N be the set of t for which $x_n(t) = 0$. If t is not in N , we can cancel $\|x_n(t)\|$ in (4.5) to obtain

$$(4.6) \quad (d/dt)\|x_n(t)\| \leq Lh(1 + \|u_n(t)\| + (1 + n^{-1})\|u'_n(t)\|).$$

If t is a cluster point of N , then $(d/dt)\|x_n(t)\| = 0$ as long as it exists, so that (4.6) is still true. Since there are only a countable number of isolated points of N , it follows that (4.6) is true almost everywhere. Since $\|x_n(t)\|$ is absolutely continuous, we obtain finally

$$(4.7) \quad \|x_n(t)\| \leq \|x_n(0)\| + Lh \int_0^t (1 + \|u_n(s)\| + (1+n^{-1})\|u'_n(s)\|) ds.$$

Since $x_n(t) = u_n(t+h) - u_n(t)$, by dividing (4.7) by h and letting $h \downarrow 0$ we obtain

$$(4.8) \quad \|u'_n(t)\| \leq \|u'_n(0)\| + Lt + L \int_0^t (\|u_n(s)\| + (1+n^{-1})\|u'_n(s)\|) ds.$$

Since $\|u'_n(0)\| = \|A_n(0)a\| \leq \|A(0)a\|$ by (2.5), we have

$$\|u'_n(t)\| \leq K + 2L \int_0^t (\|u_n(s)\| + \|u'_n(s)\|) ds,$$

where K is a constant independent of n . On the other hand, $u_n(t) = a + \int_0^t u'_n(s) ds$ so that

$$\|u_n(t)\| \leq \|a\| + \int_0^t \|u'_n(s)\| ds.$$

Adding the two inequalities, we obtain

$$\|u_n(t)\| + \|u'_n(t)\| \leq K + (2L+1) \int_0^t (\|u_n(s)\| + \|u'_n(s)\|) ds$$

with a different constant K . Solving this integral inequality, we see that $\|u_n(t)\| + \|u'_n(t)\|$ is bounded for all n and t .

LEMMA 4.3. *The strong limit $u(t) = \lim_{n \rightarrow \infty} u_n(t)$ exists uniformly for $t \in [0, T]$. $u(t)$ is Lipschitz continuous with $u(0) = a$.*

PROOF. We apply Lemma 1.3 to $x_{mn}(t) = u_m(t) - u_n(t)$. As above we obtain for almost all t

$$(4.9) \quad -\frac{1}{2} (d/dt) \|x_{mn}(t)\|^2 = -\operatorname{Re} (A_m(t)u_m(t) - A_n(t)u_n(t), Fx_{mn}(t)).$$

Since $A_m(t)u_m(t) = A(t)J_m(t)u_m(t)$ etc. and since $A(t)$ is monotonic, we have

$$(4.10) \quad 0 \leq \operatorname{Re} (A_m(t)u_m(t) - A_n(t)u_n(t), Fy_{mn}(t)),$$

where $y_{mn}(t) = J_m(t)u_m(t) - J_n(t)u_n(t)$. Addition of (4.9) and (4.10) gives

$$\begin{aligned} -\frac{1}{2} (d/dt) \|x_{mn}(t)\|^2 &\leq \operatorname{Re} (A_m(t)u_m(t) - A_n(t)u_n(t), Fy_{mn}(t) - Fx_{mn}(t)) \\ &\leq 2K \|Fy_{mn}(t) - Fx_{mn}(t)\| \quad \text{for almost all } t, \end{aligned}$$

where we have used Lemma 4.2.

Since $\|x_{mn}(t)\|^2$ is absolutely continuous and $x_{mn}(0) = a - a = 0$, we obtain

$$(4.11) \quad \|x_{mn}(t)\|^2 \leq 4K \int_0^t \|Fy_{mn}(s) - Fx_{mn}(s)\| ds.$$

We want to prove that $\|x_{mn}(t)\| \rightarrow 0$ uniformly in t , by showing that the irte-

grand in (4.11) tends to zero uniformly in s . Now $\|x_{mn}(s)\| = \|u_m(s) - u_n(s)\| \leq 2K$ by Lemma 4.2. Also

$$\begin{aligned} \|y_{mn}(s) - x_{mn}(s)\| &\leq \|J_m(s)u_m(s) - u_m(s)\| + \|J_n(s)u_n(s) - u_n(s)\| \\ &\leq m^{-1}\|A_m(s)u_m(s)\| + n^{-1}\|A_n(s)u_n(s)\| \leq (m^{-1} + n^{-1})K \rightarrow 0 \end{aligned}$$

as $m, n \rightarrow \infty$, where we have used (2.3) and Lemma 4.2. It follows from Lemma 1.2 that for any $\epsilon > 0$, we have $\|Fy_{mn}(s) - Fx_{mn}(s)\| < \epsilon, 0 \leq s \leq T$, for sufficiently large m, n , as we wished to show. Thus $u(t) = \lim u_n(t)$ exists uniformly in t .

Since $u_n(t)$ is Lipschitz continuous uniformly in t and n by $\|u'_n(t)\| \leq K$, the limit $u(t)$ is also Lipschitz continuous uniformly in t , with $u(0) = a$.

LEMMA 4.4. $u(t) \in D$ for all $t \in [0, T]$, and $A(t)u(t)$ is bounded and is weakly continuous.

PROOF. For each t we have $u_n(t) \rightarrow u(t)$ and $\|A_n(t)u_n(t)\| \leq K$. It follows from Lemma 2.5, (b), that $u(t) \in D(A(t)) = D$ and $A_n(t)u_n(t) \rightarrow A(t)u(t)$. Thus $\|A(t)u(t)\| \leq K$, too.

To prove the weak continuity of $A(t)u(t)$, let $t_k \rightarrow t$: we have to show that $A(t_k)u(t_k) \rightarrow A(t)u(t)$. Now

$$\begin{aligned} (4.12) \quad \|[A(t) - A(t_k)]u(t_k)\| &\leq L|t - t_k|(1 + \|u(t_k)\| + \|A(t_k)u(t_k)\|) \\ &\leq L|t - t_k|(1 + 2K) \rightarrow 0. \end{aligned}$$

This implies, in particular, that $\limsup \|A(t)u(t_k)\| = \limsup \|A(t_k)u(t_k)\| \leq K$. Since $u(t_k) \rightarrow u(t)$, it follows from Lemma 2.5, (a), that $A(t)u(t_k) \rightarrow A(t)u(t)$. Using (4.12) once more, we see that $A(t_k)u(t_k) \rightarrow A(t)u(t)$.

LEMMA 4.5. For each $f \in X^*$, $(u(t), f)$ is continuously differentiable on $[0, T]$, with $d(u(t), f)/dt = -(A(t)u(t), f)$.

PROOF. Since $u_n(t)$ satisfies (E_n), we have

$$(u_n(t), f) = (a, f) - \int_0^t (A_n(s)u_n(s), f) ds.$$

Since $u_n(t) \rightarrow u(t)$, $A_n(s)u_n(s) \rightarrow A(s)u(s)$, and $|(A_n(s)u_n(s), f)| \leq K\|f\|$ by Lemma 4.2, we obtain

$$(4.13) \quad (u(t), f) = (a, f) - \int_0^t (A(s)u(s), f) ds$$

by bounded convergence. Since the integrand is continuous in s by Lemma 4.4, the assertion follows.

LEMMA 4.6. $A(t)u(t)$ is Bochner integrable, and $u(t)$ is an indefinite integral of $-A(t)u(t)$. The strong derivative $du(t)/dt$ exists almost everywhere and equals $-A(t)u(t)$.

PROOF. Let X_0 be the smallest closed linear subspace of X containing all the values of the $A_n(t)u_n(t)$ for $t \in [0, T]$ and $n = 1, 2, \dots$. Since the $A_n(t)u_n(t)$

are continuous, X_0 is separable. Since $A_n(t)u_n(t) \rightarrow A(t)u(t)$ as shown above in the proof of Lemma 4.4 and since X_0 is weakly closed, $A(t)u(t) \in X_0$ too. Thus $A(t)u(t)$ is separably-valued. Since it is weakly continuous, it is strongly measurable (see e. g. Yosida [5], p. 131) and, being bounded, it is Bochner integrable (see [5], p. 133). Then (4.13) shows that $u(t)$ is an indefinite integral of $-A(t)u(t)$. The last statement of the lemma is a well-known result for Bochner integrals (see [5], p. 134).

LEMMA 4.7. *Let $u(t)$ and $v(t)$ be any functions satisfying the conditions of Lemma 4.5 and the initial conditions $u(0) = a \in D$, $v(0) = b \in D$. Then $\|u(t) - v(t)\| \leq \|a - b\|$.*

PROOF. $x(t) = u(t) - v(t)$ has weak derivative $-A(t)u(t) + A(t)v(t)$, which is weakly continuous and hence bounded. Thus $x(t)$ is Lipschitz continuous and so $\|x(t)\|$ is differentiable almost everywhere. It follows from Lemma 1.3 that

$$-\frac{1}{2}(d/dt)\|x(t)\|^2 = -\operatorname{Re}(A(t)u(t) - A(t)v(t), Fx(t)) \leq 0$$

almost everywhere. Since $\|x(t)\|^2$ is absolutely continuous, it follows that $\|x(t)\| \leq \|x(0)\| = \|a - b\|$.

The lemmas proved above give complete proof to Theorems 1 and 2. In particular we note that the solution $u(t)$ of the Cauchy problem is unique.

LEMMA 4.8. *For sufficiently large $M > 0$, $\|A(t)u(t)\| - Mt$ is monotonically decreasing in t . (Hence $\|A(t)u(t)\|$ is continuous except possibly at a countable number of points t .)*

PROOF. Returning to (4.8) and noting that the integrand is uniformly bounded by Lemma 4.2, we obtain

$$(4.14) \quad \|u'_n(t)\| \leq \|A(0)a\| + Mt,$$

where M is a constant independent of t and n (note that $\|u'_n(0)\| = \|A_n(0)a\| \leq \|A(0)a\|$ as shown before). Since $u'_n(t) = -A_n(t)u_n(t) \rightarrow -A(t)u(t)$, going to the limit $n \rightarrow \infty$ in (4.14) gives

$$(4.15) \quad \|A(t)u(t)\| \leq \|A(0)u(0)\| + Mt.$$

If we consider (E) on the interval $[s, T]$ with the initial value $u(s)$, the solution must coincide with our $u(t)$ on $[s, T]$ owing to the uniqueness of the solution. If we apply (4.15) to the new initial value problem, we see that $\|A(t)u(t)\| \leq \|A(s)u(s)\| + M(t-s)$ for $t > s$. Thus $\|A(t)u(t)\| - Mt$ is monotonically nonincreasing.

LEMMA 4.9. *If X is uniformly convex, then $A(t)u(t)$ is strongly continuous except possibly at a countable number of points t .*

PROOF. Since $A(t)u(t)$ is weakly continuous, it is strongly continuous at each point t where $\|A(t)u(t)\|$ is continuous. Thus the assertion follows from

Lemma 4.8.

Lemma 4.9 immediately leads to Theorem 3.

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(Notes added in proof) 1. In a recent paper by F. E. Browder, *Nonlinear accretive operators in Banach spaces*, *Bull. Amer. Math. Soc.*, **73** (1967), 470-476, the notion of accretive operators A in a Banach space X is introduced, which is almost identical with that of monotonic operators defined in the present paper. There is a slight difference that he requires $\text{Re}(Au - Av, f) \geq 0$ for every $f \in F(u - v)$ whereas we require it only for some $f \in F(u - v)$. Of course the two definitions coincide if F is single-valued.

2. Browder has called the attention of the writer to a paper by S. Ōharu, *Note on the representation of semi-groups of non-linear operators*, *Proc. Japan Acad.*, **42** (1967), 1149-1154, which contains, among others, a proof of Lemma 2.1.

3. Browder remarked also that the condition (3.1) can be weakened to

$$(3.1') \quad \|A(t)v - A(s)v\| \leq |t - s|L(\|v\|)(1 + \|A(s)v\|).$$

where $L(r)$ is a positive, nondecreasing function of $r > 0$. In this case the proof of the theorems needs a slight modification. First, it is easily seen that we have, instead of (4.2),

$$(4.2') \quad \|A_n(t)v - A_n(s)v\| \leq |t - s|L_1(\|v\|)(1 + \|A_n(s)v\|),$$

where $L_1(r) = L(r + K_1)$ for some constant $K_1 > 0$ (we may choose $K_1 = 2\|a\| + \sup_{0 \leq t \leq r} \|A(t)a\|$).

Lemma 4.2 is seen to remain true, but to prove it we first prove the uniform boundedness for $\|u_n(t)\|$, independently of $\|u_n'(t)\|$. This can be done easily by estimating $(d/dt)\|u_n(t) - a\|^2$ in the manner similar to the estimate for $\|x_n(t)\|$, with the result

$$\|u_n(t) - a\| \leq \int_0^t \|A_n(s)a\| ds \leq \int_0^t \|A(s)a\| ds \leq K_2.$$

Then the estimate for $\|u_n'(t)\|$ can be obtained from (the analogue of) (4.8) by solving an integral inequality for $\|u_n'(t)\|$. The proof of the remaining lemmas are unchanged.

4. The proof of Lemma 4.6 was unnecessarily long. It is sufficient to notice that a weakly continuous function of t is separably-valued.

5. Our theorems are rather weak when applied to *regular equations* (E), in which

the $A(t)$ are continuous operators defined everywhere on X , for it is known that we then need much less continuity of $A(t)$ as a function of t . The theorems could be strengthened by writing $A(t) = A_0(t) + B(t)$ in which $A_0(t)$ is assumed to satisfy Conditions I to III and $B(t)$ to be "regular" with a milder continuity condition as a function of t .
