Nonlinear stability analysis of the thin pseudoplastic liquid film flowing down along a vertical wall

Po-Jen Cheng
Department of Mechanical Engineering, Far-East College, Tainan, Taiwan, Republic of China

Cha'o-Kuang Chen[a] and Hsin-Yi Lai
Department of Mechanical Engineering, National Cheng-Kung University, Tainan, Taiwan, Republic of China

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This article investigates the weakly nonlinear stability theory of a thin pseudoplastic liquid film flowing down on a vertical wall. The long-wave perturbation method is employed to solve for generalized nonlinear kinematic equation with free film interface. The normal mode approach is used to compute the linear stability solution for the film flow. The method of multiple scales is then used to obtain the weak nonlinear dynamics of the film flow for stability analysis. It is shown that the necessary condition for the existence of such a solution is governed by the Ginzburg–Landau equation. The modeling results indicate that both subcritical instability and supercritical stability conditions are possible to occur in a pseudoplastic film flow system. The results also reveal that the pseudoplastic liquid film flows are less stable than Newtonian's as traveling down along the vertical wall. The degree of instability in the film flow is further intensified by decreasing the flow index n.
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I. INTRODUCTION

The stability of a film flow is a research subject of great importance commonly needed in mechanical, chemical, and nuclear engineering industries for various applications including the process of paint finishing, the process of laser cutting, and heavy casting production processes. It is known that macroscopic instabilities can cause disastrous conditions to fluid flow. It is thus highly desirable to understand the underlying flow characteristics and associated time-dependent properties so that suitable conditions for homogeneous film growth can be developed for various industrial applications.

The problem of the stability of the laminar flow of an ordinary viscous liquid film flowing down an inclined plane under gravity was first formulated and solved numerically by Yih.[1] The transition mechanism from laminar flow to turbulent flow was elegantly explained by the Landau equation.[2] That shed a light for later development on nonlinear film stability. The Landau equation was later rederived by Stuart[3] using the disturbed energy balance equation along with Reynolds stresses. Benjamin[4] and Yih[5] formulated the disturbed wave equation of free flow surface. The flow stability of the long disturbed wave was carefully studied and some characteristics of the flow stability on an inclined plane are observed. These include (1) the flow that is disturbed by a longer wave is less stable than that disturbed by a shorter wave; (2) the film flow is less stable as the inclined angle increases; (3) the film flowing down a vertical plate becomes unstable as the critical Reynolds number becomes nearly zero; (4) the film flow becomes relatively stable as the surface tension of the film increases; (5) the velocity of the unstable long disturbed wave is approximately twice of the wave velocity traveling on the free surface.

Benney[6] investigated the nonlinear evolution equation of free surface by using the method of small parameters. The solutions thus obtained can be used to predict nonlinear instability. However, the solutions cannot be used to predict supercritical stability since the influence of surface tension is not considered in the analysis of the small-parameter method. The effect of surface tension was realized by many researchers as one of the necessary conditions that will lead to the solution of supercritical stability. Lin,[7] Nakaya,[8] and Krishna and Lin[9] considered the significance of surface tension and treated it in terms of zeroth order terms in later studies. Pumir *et al.*[10] further included the effect of surface tension into the film flow model and solved for the solitary wave solutions. Hwang and Weng[11] showed that the conditions of both supercritical stability and subcritical instability are possible to occur for a liquid film flow system. An extensive review on the stability of falling liquid can be referred to Chang.[12] Renardy and Sun[13] and Tsai *et al.*[14] have done the work on both linear and nonlinear stability analysis of a fluid film flowing down along an inclined or vertical plate. Detailed flow analysis is indeed of great importance in the development of stability theory for characterizing various flow film conditions.

A vast majority of studies on thin-film flow problems were devoted to the stability analysis of Newtonian fluids. The film flow of non-Newtonian fluids attracted less attention in the past. The rheological behaviors of fluids during the plastic manufacture, the lubrication of bearings, or the glue in biological chemistry do not obey the Newtonian postulate. In recent years, the microstructure of fluid flows has emerged as a research subject of great interest to many re-

[a]Electronic mail: ckchen@mail.ncku.edu.tw

searchers. Hung *et al.*[15] employed the method of nonlinear analysis to study the stability of thin micropolar liquid films flowing down along a vertical plate. The results of their study indicated that the micropolar parameter plays an important role in stabilizing a film flow. The viscoelastic fluid, a subclass of microstructure flows, exhibits a great deal of influence on the normal and shear stresses in flow films. The stability problem of a falling film of viscoelastic fluid has been studied by Gupta[16] who considered the stability of a small-amplitude falling fluid of second order. The long wavelength disturbance is used in the article to conduct a linear stability analysis. After deriving the viscoelastic analog of the Orr–Sommerfeld equation with the requisite boundary condition, Gupta pointed out the viscoelastic effect can destabilize the film flow. Cheng *et al.*[17] further studied the nonlinear stability analysis of thin viscoelastic liquid film flowing down on a vertical wall. They also demonstrated that the viscoelastic property has a destabilizing effect on the nonlinear film flow system.

In practical applications, pseudoplastic fluids that show shear thinning are widely used in the analysis to characterize the molten polyethylene and polypropylene, and solutions of carboxymethylcellulose (CMC) in water, polyacrylamide in water and glycerin, and aluminum laurate in decalin and m-cresol in various industrial sectors. Almost all polymer solutions and melts that exhibit a shear-rate dependent viscosity are pseudoplastic. The polymeric fluid drains much more quickly than the Newtonian fluid when the fluids are allowed to flow out by gravity in the vertical tubes.[18] The viscosity of the macromolecular fluid appears to be lower in the higher shear rate part of the experiment. For many engineering applications, this is the most important characteristic of polymeric fluids. Ng and Mei[19] studied the roll waves on a shallow layer of mud modeled as a power-law fluid. The results indicated that longer roll waves, with dissipation at the discontinuous fronts, cannot be maintained if the uniform flow is linearly stable, when the fluid is slightly non-Newtonian. However, when the fluid is highly non-Newtonian, very long roll waves may still exist even if the corresponding uniform flow is stable to infinitesimal disturbances. Hwang *et al.*[20] studied the linear stability of power-law liquid film flows down an inclined plane by using the integral method. The results reveal that the system will be more unstable when power-law exponent *n* decreases.

The stability analysis of the pseudoplastic film flow is an interesting research area in both theoretical development and practical applications. So far the weakly nonlinear stability analysis of a thin pseudoplastic liquid film flowing down a vertical wall has not been seriously investigated. However, since the types of stability problems are of great importance in many practical applications, the behavior of a pseudoplastic liquid film traveling down along a vertical wall is carefully studied in this article by employing both linear and nonlinear stability analysis theories. The influence of pseudoplastic property on finite-amplitude equilibrium is studied and characterized mathematically. The sensitivity analysis of the flow index n is also carefully conducted. Several numerical examples are presented to verify the solutions and to demonstrate the effectiveness of the proposed modeling procedure.

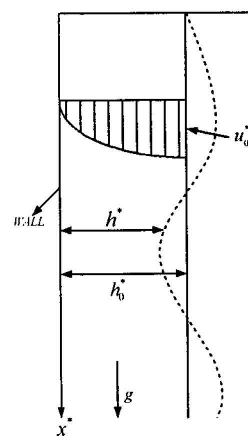


FIG. 1. Schematic diagram of a pseudoplastic thin film flow traveling down along a vertical wall.

II. GENERALIZED KINEMATIC EQUATIONS

Figure 1 shows the configuration of a thin pseudoplastic liquid film flowing down on a vertical wall. The pseudoplastic fluid that gives shear thinning to the Ostwald–de Waele model obeys the constitutive equation of state[21]

$$\tau_{ij} = -p^* \delta_{ij} + 2\mu_n \Theta e_{ij}, \tag{1}$$

where τ_{ij} is the stress tensor, e_{ij} is the rate-of-strain tensor, μ_n is the dynamic viscosity of pseudoplastic flow, p^* is the isotropic pressure, and

$$\Theta = [2(e^2_{x^*x^*} + e^2_{y^*y^*}) + 4e^2_{x^*y^*}]^{(n-1)/2} = \left\{ 2\left[\left(\frac{\partial u^*}{\partial x^*}\right)^2 + \left(\frac{\partial v^*}{\partial y^*}\right)^2 \right] + \left(\frac{\partial v^*}{\partial x^*} + \frac{\partial u^*}{\partial y^*}\right)^2 \right\}^{(n-1)/2}, \tag{2}$$

where u^* and v^* are the velocity components in x^* and y^* directions, respectively, and n is the flow index ($n<1$). The principles of mass and momentum conservation for an isothermal incompressible pseudoplastic flow configuration leads one to a set of system governing equations.[21] The governing equations can be expressed in terms of Cartesian coordinates (x^*, y^*) as

$$\frac{\partial u^*}{\partial x^*} + \frac{\partial v^*}{\partial y^*} = 0, \tag{3}$$

$$\rho\left(\frac{\partial u^*}{\partial t^*} + u^*\frac{\partial u^*}{\partial x^*} + v^*\frac{\partial u^*}{\partial y^*}\right) = \rho g + \frac{\partial \tau_{x^*x^*}}{\partial x^*} + \frac{\partial \tau_{y^*x^*}}{\partial y^*}, \tag{4}$$

$$\rho\left(\frac{\partial v^*}{\partial t^*} + u^*\frac{\partial v^*}{\partial x^*} + v^*\frac{\partial v^*}{\partial y^*}\right) = \frac{\partial \tau_{x^*y^*}}{\partial x^*} + \frac{\partial \tau_{y^*y^*}}{\partial y^*}, \tag{5}$$

where ρ is a constant density of the film flow, t^* is the time, g is the gravitational acceleration, and the individual stress components are given as

$$\tau_{x^*x^*} = -p^* + 2\mu_n \Theta e_{x^*x^*}$$
$$= -p^* + 2\mu_n \left\{ 2\left[\left(\frac{\partial u^*}{\partial x^*} \right)^2 + \left(\frac{\partial v^*}{\partial y^*} \right)^2 \right] + \left(\frac{\partial v^*}{\partial x^*} + \frac{\partial u^*}{\partial y^*} \right)^2 \right\}^{(n-1)/2} \frac{\partial u^*}{\partial x^*}, \tag{6}$$

$$\tau_{y^*y^*} = -p^* + 2\mu_n \Theta e_{y^*y^*}$$
$$= -p^* + 2\mu_n \left\{ 2\left[\left(\frac{\partial u^*}{\partial x^*} \right)^2 + \left(\frac{\partial v^*}{\partial y^*} \right)^2 \right] + \left(\frac{\partial v^*}{\partial x^*} + \frac{\partial u^*}{\partial y^*} \right)^2 \right\}^{(n-1)/2} \frac{\partial v^*}{\partial y^*}, \tag{7}$$

$$\tau_{x^*y^*} = \tau_{y^*x^*}$$
$$= 2\mu_n \Theta e_{x^*y^*}$$
$$= \mu_n \left\{ 2\left[\left(\frac{\partial u^*}{\partial x^*} \right)^2 + \left(\frac{\partial v^*}{\partial y^*} \right)^2 \right] + \left(\frac{\partial v^*}{\partial x^*} + \frac{\partial u^*}{\partial y^*} \right)^2 \right\}^{(n-1)/2} \left(\frac{\partial v^*}{\partial x^*} + \frac{\partial u^*}{\partial y^*} \right). \tag{8}$$

The no-slip boundary conditions on the wall surface at $y^* = 0$ are given as

$$u^* = 0, \tag{9}$$

$$v^* = 0. \tag{10}$$

The boundary conditions for free surface at $y^* = h^*$ are derived based on the results given by Edwards *et al.*[22] The vanishing of shear stress on free surface gives another boundary condition as

$$\frac{\partial h^*}{\partial x^*} \left[1 + \left(\frac{\partial h^*}{\partial x^*} \right)^2 \right]^{-1} (\tau_{y^*y^*} - \tau_{x^*x^*}) + \left[1 - \left(\frac{\partial h^*}{\partial x^*} \right)^2 \right] \left[1 + \left(\frac{\partial h^*}{\partial x^*} \right)^2 \right]^{-1} \tau_{x^*y^*} = 0. \tag{11}$$

By solving the balance equation in the direction normal to the free surface, the resulting normal stress condition can be expressed as

$$\left[1 + \left(\frac{\partial h^*}{\partial x^*} \right)^2 \right]^{-1} \left[2\tau_{x^*y^*} \frac{\partial h^*}{\partial x^*} - \tau_{y^*y^*} - \tau_{x^*x^*} \left(\frac{\partial h^*}{\partial x^*} \right)^2 \right] + S^* \left\{ \frac{\partial^2 h^*}{\partial x^{*2}} \left[1 + \left(\frac{\partial h^*}{\partial x^*} \right)^2 \right]^{-3/2} \right\} = p_a^*. \tag{12}$$

The kinematic condition that the flow does not travel across a free surface can be given as

$$\frac{\partial h^*}{\partial t^*} + \frac{\partial h^*}{\partial x^*} u^* - v^* = 0, \tag{13}$$

where h^* is the local film thickness, p_a^* is the atmosphere pressure, and S^* is the surface tension. The variable that is associated with a superscript ''*'' stands for a dimensional quantity. By introducing a stream function φ^*, the dimensional velocity components can be expressed as

$$u^* = \frac{\partial \varphi^*}{\partial y^*}, \quad v^* = -\frac{\partial \varphi^*}{\partial x^*}. \tag{14}$$

The dimensionless quantities can also be defined and given as

$$x = \frac{\alpha x^*}{h_0^*}, \quad y = \frac{y^*}{h_0^*}, \quad t = \frac{\alpha u_0^* t^*}{h_0^*}, \quad h = \frac{h^*}{h_0^*},$$
$$\varphi = \frac{\varphi^*}{u_0^* h_0^*}, \quad p = \frac{p^* - p_a^*}{\rho u_0^{*2}} \quad \mathrm{Re}_n = \frac{u_0^{*2-n} h_0^{*n}}{v_n}, \tag{15}$$
$$S = \frac{S^*}{(2^{-3n^2+3n+2} \rho^{n+2} v^4 g^{3n-2})^{1/(n+2)}}, \quad \alpha = \frac{2\pi h_0^*}{\lambda},$$

where Re_n is the Reynolds number of pseudoplastic flow, λ is the perturbed wave length, and α is the dimensionless wave number. h_0^* is the film thickness of local base flow and u_0^* is the reference velocity which can be expressed as

$$u_0^* = \frac{n}{n+1} \left(\frac{g}{v_n} \right)^{1/n} h_0^{*(n+1)/n}, \tag{16}$$

where v_n is the kinematic viscosity of pseudoplastic flow. Thus, the nondimensional governing equations and associated boundary conditions can now be given as

$$(\varphi_{yy})^{n-1} \varphi_{yyy} = -\frac{(n+1)^n}{n^{1+n}} + \alpha \frac{\mathrm{Re}_n}{n} (p_x + \varphi_{ty} + \varphi_y \varphi_{xy} - \varphi_x \varphi_{yy}) + O(\alpha^2), \tag{17}$$

$$p_y = \alpha \mathrm{Re}_n^{-1} [(n-2)(\varphi_{yy})^{n-1} \varphi_{xyy} - 2(n-1) \times (\varphi_{yy})^{n-2} \varphi_{xy} \varphi_{yyy}] + O(\alpha^2), \tag{18}$$

$$y = 0$$
$$\varphi = \varphi_x = \varphi_y = 0, \tag{19}$$

$$y = h$$
$$\varphi_{yy} = 0 + O(\alpha^2), \tag{20}$$

$$p = -2\alpha^2 S \mathrm{Re}_n^{3n^2-4n-4/(n+2)(2-n)} \left(\frac{n+1}{2n} \right)^{3n^2-2n/n+2} h_{xx} - \alpha\{2 \mathrm{Re}_n^{-1} [(\varphi_{yy})^n h_x + (\varphi_{yy})^{n-1} \varphi_{xy}]\} + O(\alpha^2), \tag{21}$$

$$h_t + \varphi_y h_x + \varphi_x = 0. \tag{22}$$

Subscripts of *x*, *y*, *xx*, *yy*, and *xy* are used to represent various partial derivatives of the associated underlying variable. In case of $n = 1$, the pseudoplastic film flow becomes a typical classical Newtonian flow.

Since the long perturbed wave may introduce flow instability, it is sometimes advantageous to employ the small wave to perturb the film flow. Mathematically, this can be done by expanding φ and p in terms of some small wave number α as

$$\varphi = \varphi_0 + \alpha \varphi_1 + O(\alpha^2), \tag{23}$$

$$p = p_0 + \alpha p_1 + O(\alpha^2). \tag{24}$$

By plugging the above two equations into Eqs. (17)–(22), we can solve system governing equations order by order. In practice, the nondimensional surface tension S is a large value, the term $\alpha^2 S$ can then be treated as a quantity of zeroth order.[11,14,15] After collecting all terms of zeroth order (α^0) in the governing equations, one has a set of zeroth order equations as

$$(\varphi_{0yy})^{n-1}\varphi_{0yyy} = -\frac{(n+1)^n}{n^{1+n}}, \tag{25}$$

$$p_{0y} = 0. \tag{26}$$

The boundary conditions associated with the equations of zeroth order are given as

$$\varphi_0 = \varphi_{0y} = 0, \quad \text{at} \quad y=0, \tag{27}$$

$$\varphi_{0yy} = 0,$$
$$p_0 = -2\alpha^2 S\,\mathrm{Re}_n^{3n^2-4n-4/(n+2)(2-n)} \times \left(\frac{n+1}{2n}\right)^{3n^2-2n/n+2} h_{xx}, \quad \text{at} \quad y=h. \tag{28}$$

The solutions for the equations of zeroth order can be given as

$$\varphi_0 = \frac{n(h-y)^{2+(1/n)} + (y+2ny-hn)h^{1+(1/n)}}{2+n}, \tag{29}$$

$$p_0 = -2\alpha^2 S\,\mathrm{Re}_n^{3n^2-4n-4/(n+2)(2-n)}\left(\frac{n+1}{2n}\right)^{3n^2-2n/n+2} h_{xx}. \tag{30}$$

After collecting all terms of first order (α^1) in the governing equations, one has a set of first-order equations as

$$\varphi_{1yyy} = \frac{\mathrm{Re}_n}{n}(p_{0x} + \varphi_{0ty} + \varphi_{0y}\varphi_{0xy} - \varphi_{0x}\varphi_{0yy})^{1-n}, \tag{31}$$

$$p_{1y} = \mathrm{Re}_n^{-1}[(n-2)(\varphi_{0yy})^{n-1}\varphi_{0xyy} - 2(n-1) \times (\varphi_{0yy})^{n-2}\varphi_{0xy}\varphi_{0yyy}]. \tag{32}$$

The boundary conditions associated with the equations of first order are given as

$$\varphi_1 = \varphi_{1y} = 0, \quad \text{at} \quad y=0, \tag{33}$$

$$\varphi_{1yy} = 0,$$
$$p_1 = 2\,\mathrm{Re}_n^{-1}[(\varphi_{0yy})^n h_x + (\varphi_{0yy})^{n-1}\varphi_{0xy}], \quad \text{at} \quad y=h. \tag{34}$$

The solutions for the equations of first order can be given as

$$\varphi_1 = \mathrm{Re}_n h_x n\left(1+\frac{1}{n}\right)^{3-n}\{2n(2+n)(2+3n)h^{1+(2/n)}(h-y)^{2+(1/n)} - n(1+2n)h^{1/n}(h-y)^{3+(2/n)} + h^{2+(3/n)}[-n(7h+14hn+6hn^2) + (1+2n)(3+2n)(2+3n)y]\}/[2(1+n)^2(2+n)(1+2n)(2+3n)] + 2\alpha^2 S\,\mathrm{Re}_n^{-2n/n+2}\left(\frac{n+1}{2n}\right)^{3n^2-2n/n+2} h_{xxx} n\left(1+\frac{1}{n}\right)^{1-n}[n(h-y)^{2+(1/n)} + h^{1+(1/n)}(y+2ny-hn)]/[(1+n)(1+2n)], \tag{35}$$

$$p_1 = \frac{h_x}{\mathrm{Re}_n}\left(1+\frac{1}{n}\right)^n[-2h^{1/n}(h-y)^{1-(1/n)} + h - y]. \tag{36}$$

By plugging the solutions for the equations of both the zeroth and the first orders into the dimensionless free surface kinematic equation of Eq. (22), the nonlinear generalized kinematic equation can be obtained and presented as

$$h_t + A(h)h_x + B(h)h_{xx} + C(h)h_{xxxx} + D(h)h_x^2 + E(h)h_x h_{xxx} = 0, \tag{37}$$

where

$$A(h) = h^{1+(1/n)}\left(1+\frac{1}{n}\right), \tag{38}$$

$$B(h) = \frac{3n^{n-2}(1+n)^{4-n}\alpha\,\mathrm{Re}_n\,h^{3+(3/n)}}{(2+n)(1+2n)(2+3n)}, \tag{39}$$

$$C(h) = \frac{2^{-3n^2+3n+2/n+2}n^{-2n^2+4n/n+2}(1+n)^{2n^2-3n+2/n+2}\alpha^3 S\,\mathrm{Re}_n^{-2n/n+2}\,h^{2+(1/n)}}{1+2n} \tag{40}$$

$$D(h) = \frac{9n^{n-2}(1+n)^{5-n}\alpha\,\mathrm{Re}_n h^{2+(3/n)}}{(2+n)(1+2n)(2+3n)}, \tag{41}$$

$$E(h) = 2^{-3n^2+3n+2/n+2}n^{-2n^2+3n-2/n+2}(1+n)^{2n^2-3n+2/n+2}\alpha^3 S\,\mathrm{Re}_n^{-2n/n+2}\,h^{1+1/n}. \tag{42}$$

In the case of $n=1$, Eq. (37) is reduced to the fluid of no viscoelastic effect. The reduced set of equations have been carefully derived and presented by Cheng *et al.*[17]

III. STABILITY ANALYSIS

The dimensionless film thickness when expressed in perturbed state can be given as

$$h(x,t)=1+\eta(x,t), \tag{43}$$

where η is a perturbed quantity to the stationary film thickness. After the above equation is inserted into Eq. (37) and all terms up to the order of η^3 are collected, the evolution equation of η can be obtained and given as

$$n_t+A\eta_x+B\eta_{xx}+C\eta_{xxxx}+D\eta_x^2+E\eta_x\eta_{xxx} = -\left[\left(A'\eta+\frac{A''}{2}\eta^2\right)\eta_x+\left(B'\eta+\frac{B''}{2}\eta^2\right)\eta_{xx} + \left(C'\eta+\frac{C''}{2}\eta^2\right)\eta_{xxxx}+(D+D'\eta)\eta_x^2 + (E+E'\eta)\eta_x\eta_{xxx}\right]+O(\eta^4), \tag{44}$$

where the values of A, B, C, D, E, and their derivatives are all evaluated at the dimensionless height of the film $h=1$.

A. Linear stability analysis

As the nonlinear terms of Eq. (44) are neglected, the linearized equation is obtained and given as

$$\eta_t+A\eta_x+B\eta_{xx}+C\eta_{xxxx}=0. \tag{45}$$

In order to use the normal mode analysis we assume that

$$\eta=a\exp[i(x-dt)]+\text{c.c.}, \tag{46}$$

where a is the perturbation amplitude, and c.c. is the complex conjugate counterpart. The complex wave celerity, d, is given as

$$d=d_r+id_i=A+i(B-C), \tag{47}$$

where d_r is linear wave speed, and d_i is linear growth rate of the amplitudes. The flow is linearly unstable supercritical condition for $d_i>0$, and is linearly stable subcritical condition for $d_i<0$.

B. Nonlinear stability analysis

The method of multiple scales can be used to study the nonlinear stability using the notions given as

$$\frac{\partial}{\partial t}\to\frac{\partial}{\partial t}+\varepsilon\frac{\partial}{\partial t_1}+\varepsilon^2\frac{\partial}{\partial t_2}, \tag{48}$$

$$\frac{\partial}{\partial x}\to\frac{\partial}{\partial x}+\varepsilon\frac{\partial}{\partial x_1}, \tag{49}$$

$$\eta(\varepsilon,x,x_1,t,t_1,t_2)=\varepsilon\eta_1+\varepsilon^2\eta_2+\varepsilon^3\eta_3, \tag{50}$$

where ε is a small perturbation parameter, $t_1=\varepsilon t$, $t_2=\varepsilon^2 t$, $x_1=\varepsilon x$. By plugging the above expressions into Eq. (44) and after expansion, one has

$$(L_0+\varepsilon L_1+\varepsilon^2 L_2)(\varepsilon\eta_1+\varepsilon^2\eta_2+\varepsilon^3\eta_3)=-\varepsilon^2 N_2-\varepsilon^3 N_3, \tag{51}$$

where

$$L_0=\frac{\partial}{\partial t}+A\frac{\partial}{\partial x}+B\frac{\partial^2}{\partial x^2}+C\frac{\partial^4}{\partial x^4}, \tag{52}$$

$$L_1=\frac{\partial}{\partial t_1}+A\frac{\partial}{\partial x_1}+2B\frac{\partial}{\partial x}\frac{\partial}{\partial x_1}+4C\frac{\partial^3}{\partial x^3}\frac{\partial}{\partial x_1}, \tag{53}$$

$$L_2=\frac{\partial}{\partial t_2}+B\frac{\partial^2}{\partial x_1^2}+6C\frac{\partial^2}{\partial x^2}\frac{\partial^2}{\partial x_1^2}, \tag{54}$$

$$N_2=A'\eta_1\eta_{1x}+B'\eta_1\eta_{1xx}+C'\eta_1\eta_{1xxxx}+D\eta_{1x}^2 +E\eta_{1x}\eta_{1xxx} \tag{55}$$

$$N_3=A'(\eta_1\eta_{2x}+\eta_{1x}\eta_2+\eta_1\eta_{1x_1})+B'(\eta_1\eta_{2xx}+2n_1\eta_{1xx_1}+\eta_{1xx}\eta_2)+C'(\eta_1\eta_{2xxxx}+4\eta_1\eta_{1xxxx_1}+\eta_{1xxxx}\eta_2) +D(2\eta_{1x}\eta_{2x}+2\eta_{1x}\eta_{1x_1})+E(\eta_{1x}\eta_{2xxx}+3\eta_{1x}\eta_{1xxx_1}+\eta_{1xxx}\eta_{2x}+\eta_{1xxx}\eta_{1x_1})+\tfrac{1}{2}A''\eta_1^2\eta_{1x}+\tfrac{1}{2}B''\eta_1^2\eta_{1xx} +\tfrac{1}{2}C''\eta_1^2\eta_{1xxxx}+D'\eta_1\eta_{1x}^2+E'\eta_1\eta_{1x}\eta_{1xxx}. \tag{56}$$

Equation (51) can now be solved order by order. The equation of the order $O(\varepsilon)$ is given as $L_0\eta_1=0$, and the corresponding solution can be given as

$$\eta_1=a(x_1,t_1,t_2)\exp[i(x-d_r t)]+\text{c.c.} \tag{57}$$

The solution of η_2, after solving the secular equation of order $O(\varepsilon^2)$, gives

$$\eta_2=ea^2\exp[2i(x-d_r t)]+\text{c.c.} \tag{58}$$

By plugging both η_1 and η_2 into the equation of order $O(\varepsilon^3)$, the resulting equation becomes

$$\frac{\partial a}{\partial t_2}+D_1\frac{\partial^2 a}{\partial x_1^2}-\varepsilon^{-2}d_i a+(E_1+iF_1)a^2\bar{a}=0, \tag{59}$$

where

$$e=e_r+ie_i=\frac{(B'-C'+D-E)}{16C-4B}+i\frac{-A'}{16C-4B}, \tag{60}$$

$$D_1=B-6C, \tag{61}$$

$$E_1=(-5\bar{B}'+17\bar{C}'+4D-10E)e_r-A'e_i$$
$$+(-\tfrac{3}{2}\bar{B}''+\tfrac{3}{2}\bar{C}''+D'-E'), \qquad (62)$$

$$F_1=(-5\bar{B}'+17\bar{C}'+4D-10E)e_i+A'e_r+\tfrac{1}{2}A''. \qquad (63)$$

In the above expressions, the overhead bar denotes the complex conjugate. Equation (59) is generally referred to as the Ginzburg–Landau equation.[23] It can be used to investigate the weak nonlinear behavior of the fluid film flow. In order to solve for Eq. (59), the solution for a filtered wave in which the spatial modulation does not exist and the diffusion term in Eq. (59) vanishes is used to simplify the equation and to obtain a solution of the form

$$a=a_0\exp[-ib(t_2)t_2]. \qquad (64)$$

By substituting the solution of a filtered wave into Eq. (59) and dropping out the second term, one can obtain the expressions as

$$\frac{\partial a_0}{\partial t_2}=(\varepsilon^{-2}d_i-E_1a_0^2)a_0, \qquad (65)$$

$$\frac{\partial[b(t_2)t_2]}{\partial t_2}=F_1a_0^2. \qquad (66)$$

Of course, if E_1 becomes zero, Eq. (65) is reduced to a linear equation. The second term on the right-hand side of Eq. (65) is induced by the effect of nonlinearity. It can either decelerate or accelerate the exponential growth of the linear disturbance based on the signs of d_i and E_1. Equation (66) can be used to modify the perturbed wave speed caused by infinitesimal disturbances and appeared in the nonlinear system. In the linear unstable region $(d_i>0)$, the condition for a supercritical stable region to exist is given as $E_1>0$. The threshold amplitude, εa_0, is given as

$$\varepsilon a_0=\sqrt{\frac{d_i}{E_1}}, \qquad (67)$$

and the nonlinear wave speed is given as

$$Nc_r=d_r+\varepsilon^2b=d_r+d_i\left(\frac{F_1}{E_1}\right). \qquad (68)$$

On the other hand, in the linearly stable region $(d_i<0)$, if $E_1<0$, the film flow presents the behavior of subcritical instability, and εa_0 is the threshold amplitude. The condition for a subcritical stable region to exist is given as $E_1>0$. Also, the condition for a neutral stability curve to exist is $E_1=0$. Based upon the discussion presented above, various characteristic states of the Landau equation can be summarized and presented in Table I.

TABLE I. Various states of the Landau equation.

Linearly stable (subcritical region) $d_i<0$	Subcritical instability $E_1<0$	$\varepsilon a_0<\left(\frac{d_i}{E_1}\right)^{1/2}$	$a_0\to0$	Conditional stability
		$\varepsilon a_0>\left(\frac{d_i}{E_1}\right)^{1/2}$	$a_0\uparrow$	Subcritical explosive state
	Subcritical (absolute) stability $E_1>0$	$a_0\to0$		
Linearly unstable (supercritical region) $d_i>0$	Supercritical explosive state $E_1<0$	$a_0\uparrow$		
	Supercritical stability $E_1>0$	$\varepsilon a_0\to\left(\frac{d_i}{E_1}\right)^{1/2}$ $Nc_r\to d_r+d_i\frac{F_1}{E_1}$		

IV. NUMERICAL EXAMPLES

A numerical example is presented here to illustrate the effectiveness of the proposed modeling approach for dealing with the thin pseudoplastic liquid film flowing down on a vertical wall. In order to verify the result of theoretical derivation, a numerically generated finite amplitude perturbation apparatus is provided for linear and nonlinear stability modeling. Based on modeling results, the condition for thin-film flow stability can then be expressed as a function of the Reynolds number of the Newtonian flow, Re_1, flow index, n, and dimensionless perturbation wave number, α, respectively. Some important observations are concluded and compared with the results of theoretical derivation given in this article and many conclusive results appeared in the literature.

Figure 1 shows the schematic diagram of a pseudoplastic flow traveling down along a vertical plate. Physical parameters that are selected for study include (1) the Reynolds numbers of Newtonian flow ranging from 0 to 10; (2) the dimensionless perturbation wave numbers ranging from 0 to 0.12; (3) the values of flow index are 1.0, 0.98, and 0.95; and (4) the other quantities of physical properties are specified as $S^*=0.0726$ N/m, $\rho=997.1$ kg/m^3, and $\mu_n=8.94\times10^{-4}$ Pa s^n. The neutral stability curve is obtained by computing the conditions of linear stability for a linear amplitude growth rate of $d_i=0$. The stability of flow field ($\alpha-\mathrm{Re}_1$ plane) is separated into two different regions by the neutral curve. In a linearly stable subcritical region the perturbed small waves will decay as the perturbed time increases. However, in a linearly unstable supercritical region the perturbed small waves will grow as the perturbed time increases. In order to study the effect of flow index n on the stability of film flow, the same film thickness is used to show the influence of three different n values for all numerical computations. The results obtained by modeling a classical Newtonian flow (i.e., setting $n=1$) agree well with those data given in Cheng *et al.*[17]

A. Linear stability solutions

The linear neutral stability curve can be obtained by setting $d_i=0$ for Eq. (47). Figure 2 shows the linear neutral stability curves of pseudoplastic film flow with different values on flow index n. The results indicate that the linearly unstable region $(d_i>0)$ becomes larger for a decreasing n. The temporal film growth rate is computed by using Eq.

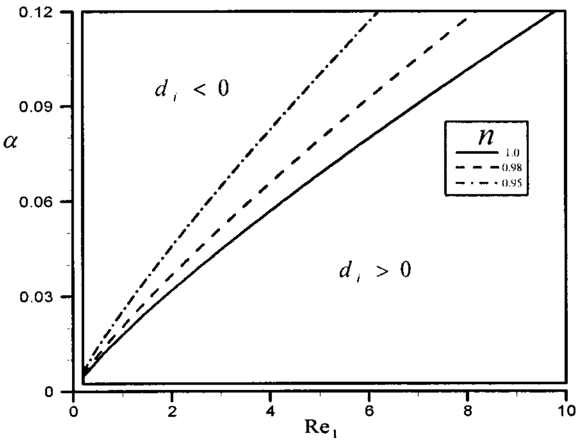
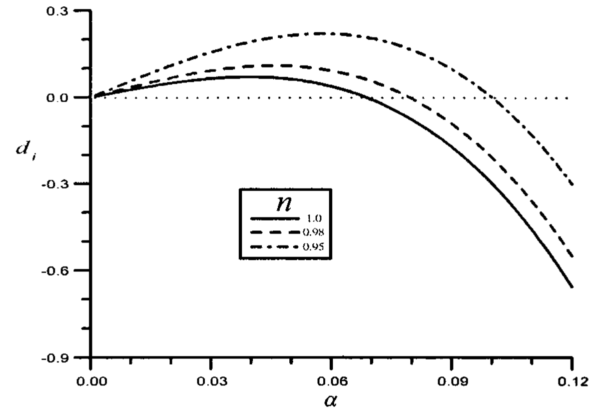
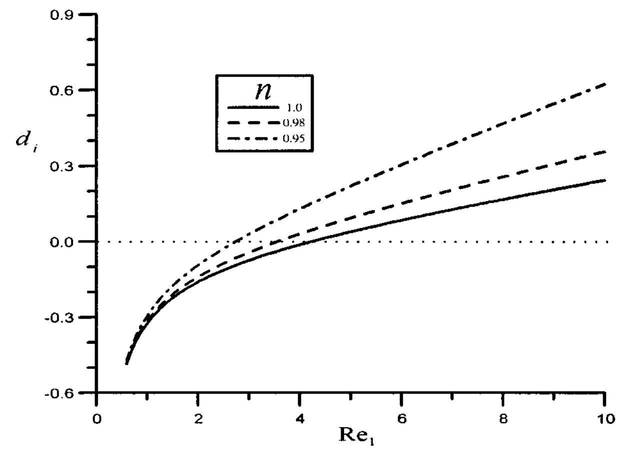


FIG. 2. Linear neutral stability curves for three different n.



(a)



(b)

FIG. 3. (a) Amplitude growth rate of disturbed waves in pseudoplastic flows for three different n at $\mathrm{Re}_1=5$. (b) Amplitude growth rate of disturbed waves in pseudoplastic flows for three different n at $\alpha=0.06$.

(47). Figures 3(a) and 3(b) show the temporal film growth rate of pseudoplastic fluid for $n=0.98$ and $n=0.95$ as compared to that of Newtonian flow (i.e., $n=1$). It is interesting to note that temporal film growth rate decreases for an increasing n and a decreasing Re_1. Furthermore, it is found that both the wave number of neutral mode and the maximum temporal film growth rate increase as the value of n decreases. In other words, the larger the value of flow index n is, the higher the stability of a liquid film should be.

B. Nonlinear stability solutions

As the perturbed wave grows to finite amplitude, the linear stability theory is no longer valid for accurate prediction of flow behavior. The theory of nonlinear stability should be used to study whether the disturbed wave amplitude in the linear stable region will become stable or unstable. The study also includes the subsequent nonlinear evolution of disturbance in the linear unstable region and will develop to a new equilibrium state with a finite amplitude (supercritical stability) or develop to an unstable situation. As previously discussed, a negative value of E_1 can cause the system to become unstable. Such a condition in the linear region is referred to as the subcritical instability. In other words, if the amplitude of disturbances is greater than the threshold amplitude, the amplitude of disturbed wave will increase. This is contradictory to the result predicted by using a linear theory. As a matter of fact such a condition in the subcritical unstable region can in some cases cause the system to become explosive.

The neutral stability curves can be obtained by setting $d_i=0$ for Eq. (47) and $E_1=0$ for Eq. (62). The hatched areas near the neutral stability curves in Figs. 4(a)–4(c) reveal that both the subcritical instability condition ($d_i<0, E_1<0$) and the explosive supercritical instability condition ($d_i>0, E_1<0$) are possible to occur for all values of n used in this study. It is interesting to note that the neutral stability curves of $d_i=0$ and $E_1=0$ are shifted as the values of n decrease. The area of shaded subcritical instability region decreases and the area of shaded supercritical instability region increases as the values of n decrease. The area of subcritical stability region ($d_i<0, E_1>0$) decreases as the values of n decrease.

The threshold amplitude in the subcritical unstable region is obtained by using Eq. (67). Figure 5 shows the threshold amplitude in the subcritical unstable region for various wave numbers with different n values at $\mathrm{Re}_1=5$. The result indicates that the threshold amplitude εa_0 becomes smaller as the value of flow index n decreases. In such a situation, the film flow will become unstable. That is to say, if the initial finite-amplitude disturbance is less than the threshold amplitude, the system will become conditionally stable. On the other hand, if the initial finite-amplitude disturbance is greater than the threshold amplitude, the system will become explosively unstable.

In the linearly unstable region, the linear amplification rate is positive, while the nonlinear amplification rate is negative. Therefore, a linear infinitesimal disturbance in the unstable region will reach such a finite equilibrium amplitude as given in Eq. (65) instead of going infinite. The threshold amplitude in the supercritical stable region is ob-

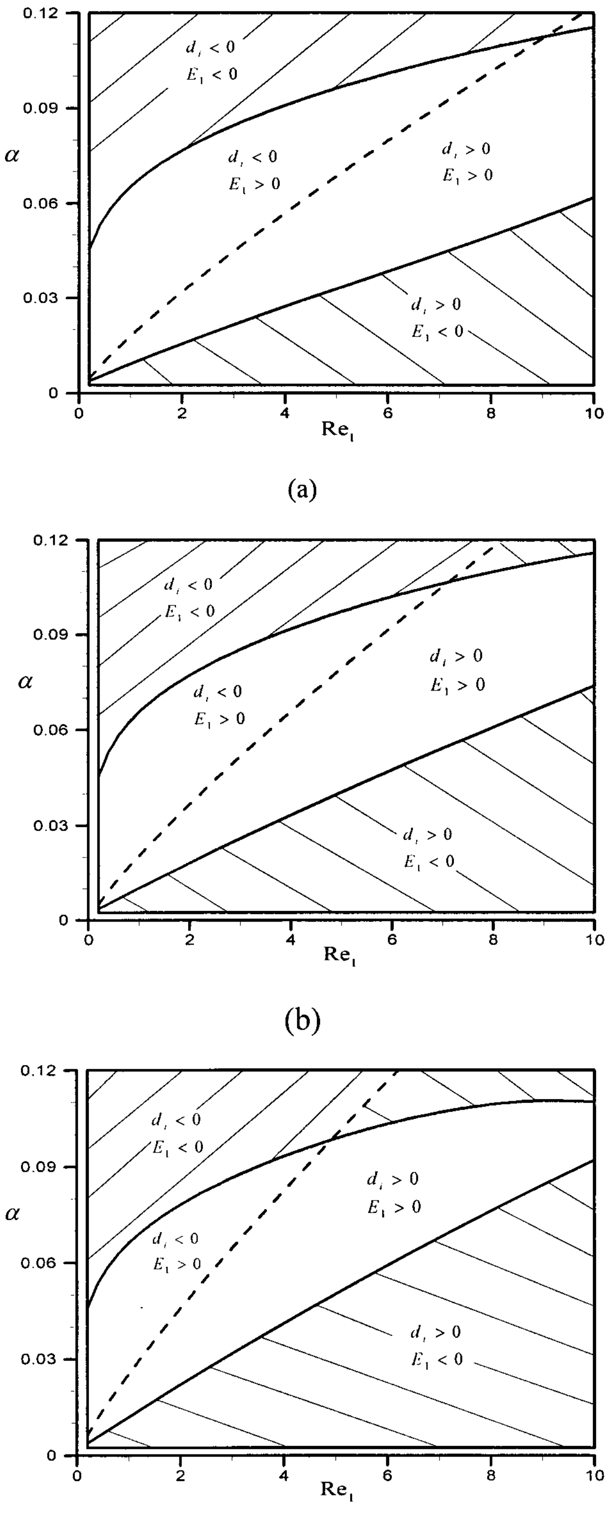


FIG. 4. (a) Neutral stability curves of Newtonian film flows for $n=1$; (b) neutral stability curves of pseudoplastic film flows for $n=0.98$; (c) neutral stability curves of pseudoplastic film flows for $n=0.95$.

tained by using Eq. (67). Figure 6 shows the threshold amplitude in the supercritical stable region for various wave numbers under different values of flow index n at $\mathrm{Re}_1=5$. It is found that the increase of n will lower the threshold am-

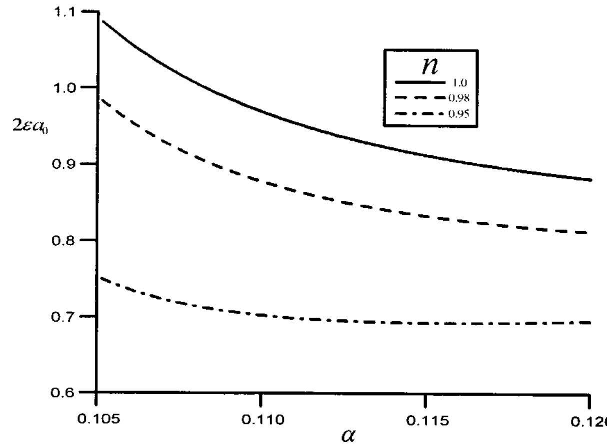


FIG. 5. Threshold amplitude in subcritical unstable region for three different n and various α at $\mathrm{Re}_1=5$.

plitude, and the flow will become relatively more stable. The wave speed of Eq. (47) predicted by using the linear theory is a constant value for all wave numbers. However, the wave speed of Eq. (68) predicted by using nonlinear theory is no longer a constant. It is actually a function of wave number, Reynolds number, and flow index n. The nonlinear wave speed is plotted in Fig. 7 for various wave numbers and n values. It is found that the nonlinear wave speed increases as the value of n decreases.

As discussed above, it becomes apparent that the stability characteristic of a film flow traveling down along a vertical plate is significantly affected by the value of flow index n. That is to say, in almost a full working range the stability of a pseudoplastic film flow gradually decreases as the value of n decreases. The pseudoplastic fluid ($n<1$) drains much more quickly than the Newtonian fluid ($n=1$) when the fluids are allowed to flow out by gravity in the vertical tubes. This phenomenon agrees well with the conclusion given by Bird *et al.*[18] By setting $n=1$, the results of a classical Newtonian flow are obtained. As compared to the modeling results given by Cheng *et al.*,[17] it is found that both solutions agree well with each other.

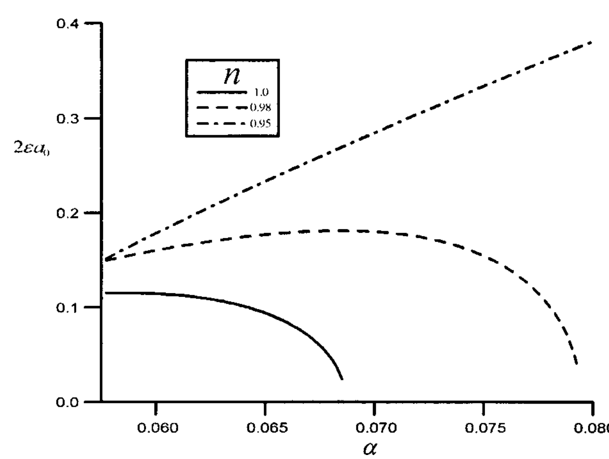


FIG. 6. Threshold amplitude in supercritical stable region for three different n and various α at $\mathrm{Re}_1=5$.

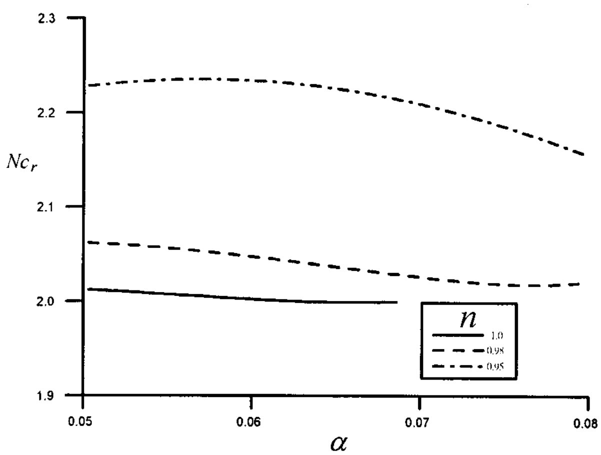


FIG. 7. Nonlinear wave speed in supercritical stable region for three different n and various α at $Re_1=5$.

V. CONCLUDING REMARKS

The stability of a thin pseudoplastic liquid film flowing down on a vertical wall is studied in this article by using the method of long wave perturbation. The generalized nonlinear kinematic equations of the fluid film on free surface of the wall is derived and numerically estimated to investigate the fields of flow stability associated with different n values of flow index. Based on the results of numerical modeling, several conclusions are given as follows:

(1) In the linear stability analysis the neutral stability curve that separates the flow field into two different regions was first computed for a linear amplitude growth rate of $d_i = 0$. The modeling results indicate the linearly unstable region becomes larger for a decreasing n. It is also noted that the increasing value of n and the decreasing value of Re_1 will reduce the growth rate of temporal film. In other words, it is interesting to note that the flow becomes relatively stable if it is perturbed by short waves at a low Reynolds number and a larger flow index.

(2) In the nonlinear stability analysis, it is noted that the area of shaded subcritical instability region decreases as the value of n decreases. On the other hand, the area of shaded supercritical instability region increases as the value of n decreases. It is also shown that the area of subcritical stability region decreases as the value of n decreases. It is noted that the threshold amplitude εa_0 in the subcritical instability region decreases as the value of flow index n decreases. If the initial finite-amplitude disturbance is greater than the threshold amplitude value, the system will become explosively unstable. The increase of the flow index n will decrease both the threshold amplitude and nonlinear wave speed in the supercritical stability region, therefore, the film flow will become relatively more stable.

(3) The stability of the pseudoplastic film flow decreases as the value of n decreases. When a pseudoplastic liquid film flow is modeled as a non-Newtonian flow, it possesses the characteristic of shear thinning effect. The smaller flow index n of the pseudoplastic fluid will tend to destabilize the flow in motion. Physically, the pseudoplastic fluid of thin film flow will decrease the effective viscosity as the flow travels down along a vertical plane, it can, therefore, increase the convective motion of flow. The decreasing flow index n indeed plays a significant role in destabilizing the flow and is thus of great practical importance.

ACKNOWLEDGMENT

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