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Nonlinear stability of centered rarefaction waves of the Jin-Xin relaxation model for 2×2 conservation laws *

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Abstract

We study the asymptotic equivalence of the Jin-Xin relaxation model and its formal limit for genuinely nonlinear 2×2 conservation laws. The initial data is allowed to have jump discontinuities corresponding to centered rarefaction waves, which includes Riemann data connected by rarefaction curves. We show that, as long as the initial data is a small perturbation of a constant state, the solution for the relaxation system exists globally in time and converges, in the zero relaxation limit, to the solution of the corresponding conservation law uniformly except for an initial layer.

1 Introduction

In this paper, we study the asymptotic behavior of the Jin-Xin model for a semilinear hyperbolic system with relaxation:

$$\begin{aligned} \mathbf{u}_t + \mathbf{v}_x &= \mathbf{0} \\ \mathbf{v}_t + \alpha^2 \mathbf{u}_x &= \frac{1}{\epsilon}(\mathbf{f}(\mathbf{u}) - \mathbf{v}) \end{aligned} \tag{1.1}$$

where $\mathbf{u}, \mathbf{v}, \mathbf{f} \in \mathbb{R}^2$, $x \in \mathbb{R}$, $t > 0$. In the formal limit as $\epsilon \to 0$, we expect the second equation of (1.1) to be well approximated by the local equilibrium

$$\mathbf{v} = \mathbf{f}(\mathbf{u}) \tag{1.2}$$

and the relaxation system reduces to

$$\mathbf{u}_t + \mathbf{f}(\mathbf{u})_x = \mathbf{0}. \tag{1.3}$$

The relaxation limit for 2×2 nonlinear hyperbolic systems was first studied by Liu in [11], where the stability criterion and asymptotic limit were established

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for some basic wave patterns such as diffusion waves, expansion waves and traveling waves. In Chen, Levermore and Liu [2], further mathematical theories were developed including the entropic extension and asymptotic expansions.

Distinguished by the special structure of its nonlinear terms, (1.1) was proposed by Jin and Xin in [7] with an interesting numerical origin. It is used as an approximation for general conservation laws and the authors developed a new class of numerical methods for (1.3) called relaxation schemes based on discretizing (1.1). Due to the simplicity of the relaxation scheme and its outstanding performance, the hyperbolic system (1.1) has stimulated much research activities on rigorous justification of the asymptotic equivalence between (1.1) and (1.3).

When $\mathbf{u}$ and $\mathbf{f}$ are scalars, (1.1) admits compact invariant regions. With this a priori L^∞ bound, Natalini [6], established the asymptotic equivalence between the Cauchy problems of (1.1) and (1.3) within the class of BV data. Teng [18] gave an optimal L^1 error estimate between (1.1) and (1.3) using the matching method introduced in [4]. An optimal pointwise estimate was derived in Tadmor and Tang [17] based the optimal L^1 error estimate and the Lip^+ analysis. For initial boundary value problems, the stiff well-posedness for (1.1), its asymptotic equivalence to (1.3) and the boundary layer structure was obtained in Wang and Xin [20].

For 2×2 conservation laws (1.3), a priori L^∞ bound for (1.1) is not available in general. Tzavaras [21], Gosse and Tzavaras [3] established the strong dissipation estimate for $\mathbf{u}$ with growth assumption on $\mathbf{f}$. A typical example is the elastodynamics equation

$$\partial_t u_1 + \partial_x u_2 = 0, \quad \partial_t u_2 + \partial_x g(u_1) = 0, \tag{1.4}$$

with growth assumption on the stress-strain function g. The convergence result then follows from the L^p theory of compensated compactness.

With a slightly different approach, Serre [15] showed that if (1.3) has a convex characteristic set whose boundary is stable under $\mathbf{f}'(\mathbf{u})$, and if the following sub-characteristic condition

$$\alpha > \rho(\mathbf{f}'(\mathbf{u})) \tag{1.5}$$

holds, one can find an invariant region for (1.1) and hence establish convergence results using compensated compactness. Examples of 2×2 conservation laws satisfying the assumption above include the Temple system and the elastodynamics equation (1.4) with g satisfying $g' > 0$ and $sg''(s) > 0$ for $s \neq 0$. It should be noted that although the result in [21, 3, 15] cover quite a wide range of equations, the isentropic Euler equation, for example, is not included.

In this paper, we take an alternative approach to establish the convergence result for general 2×2 conservation laws equipped with a convex entropy. We consider the Cauchy problem with initial data allowed to have a jump discontinuity corresponding to a centered rarefaction wave. We show that, as long as the initial data is a small perturbation of a non-vacuum constant state, the solution of (1.1) exists globally in time and converges, as $\epsilon \to 0$, to the solution of (1.3) uniformly except for an initial layer. This is done by approximating the solution of (1.3) with a smooth rarefaction wave followed by a nonlinear stability analysis of the smooth rarefaction wave under discontinuous initial perturbations for (1.1). The a priori sup norm control then follows from the Sobolev embedding. The result here can be easily extended to the case of two weak centered rarefaction waves of different families. The result in this paper was announced in [19] for the special case of the isentropic p system.

Other related works include Luo [12], where the author studied the stability of rarefaction wave in the scalar, multidimensional setting of the Jin-Xin model; Luo and Xin [13] showed nonlinear stability of the traveling wave solution also in the scalar, multidimensional case. Several discrete velocity kinetic models have also been proposed as relaxation approximations for (1.3), see [1, 6, 10]. In the 1-D case, the 2 speed case of these models are equivalent to the Jin-Xin model. However, they are genuinely different in the multidimensional case. The relaxation approximation was later generalized to the Hamilton-Jacobi equation [8] and to curvature dependent front propagation [9].

The rest of the paper is organized as follows: In section 2, we construct the smooth approximate solution and list some preliminary estimates. We then state the main theorem (Theorem 2.4). In sections 3, we review the entropic extension property for the Jin-Xin system, which is the foundation for our energy estimate. In sections 4, we proceed to prove the main theorem by treating the Riemann initial data as a discontinuous perturbation of the smooth approximation. We then proceed by a piecewise H^1 estimate on the error. To this end, we first study how the initial jump propagates and decays along the characteristics in (1.1). We then finish the proof by piecewise energy estimate and the Sobolev inequality.

2 Smooth approximations

We first rewrite (1.1) in a simpler form as

$$\begin{gathered}\mathbf{U}^\epsilon_t + \mathbf{A}\mathbf{U}^\epsilon_x = \frac{1}{\epsilon}\mathbf{N}(\mathbf{U}^\epsilon) \quad t \geq 0, \quad x \in \mathbb{R} \\ \mathbf{U}^\epsilon(x,0) = \begin{cases} \mathbf{U}_r & \text{if } x > 0 \\ \mathbf{U}_l & \text{if } x < 0 \end{cases}\end{gathered} \tag{2.1}$$

and consider also the scaled version of (2.1):

$$\begin{gathered}\mathbf{U}_t + \mathbf{A}\mathbf{U}_x = \mathbf{N}(\mathbf{U}) \quad t \geq 0, \quad x \in \mathbb{R} \\ \mathbf{U}(x,0) = \begin{cases} \mathbf{U}_r & \text{if } x > 0 \\ \mathbf{U}_l & \text{if } x < 0, \end{cases}\end{gathered} \tag{2.2}$$

where

$$\mathbf{U} = \begin{pmatrix} \mathbf{u} \\ \mathbf{v} \end{pmatrix}, \quad \mathbf{N}(\mathbf{U}) = \begin{pmatrix} \mathbf{0} \\ \mathbf{f}(\mathbf{u}) - \mathbf{v} \end{pmatrix}, \quad \mathbf{A} = \begin{pmatrix} \mathbf{0} & \mathbf{I}_2 \\ \alpha^2\mathbf{I}_2 & \mathbf{0} \end{pmatrix} \tag{2.3}$$

It is clear that $\mathbf{U}(x,t)$ is a solution of (2.2) if and only if

$$\mathbf{U}^\epsilon(x,t) \stackrel{\text{def}}{=} \mathbf{U}(\frac{x}{\epsilon},\frac{t}{\epsilon}) \tag{2.4}$$

is a solution of (2.1).

We further assume that the Riemann Cauchy data $(\mathbf{U}_l,\mathbf{U}_r)$ for (2.2) are in local equilibrium

$$\mathbf{U}_l = \begin{pmatrix} \mathbf{u}_l \\ \mathbf{f}(\mathbf{u}_l) \end{pmatrix}, \quad \mathbf{U}_r = \begin{pmatrix} \mathbf{u}_r \\ \mathbf{f}(\mathbf{u}_r) \end{pmatrix}, \tag{2.5}$$

and that $\mathbf{u}_l$ and $\mathbf{u}_r$ are connected by a rarefaction curve in the phase space. Thus the solution of (1.3) is a self similar centered rarefaction wave:

$$\mathbf{u}(x,t) = \check{\mathbf{u}}(\frac{x}{t}) \tag{2.6}$$

(For an introduction on Cauchy problems with Riemann initial data and rarefaction curves, see [16].)

We want to study the time asymptotic/small mean free path limit of (2.2)/(2.1). We will show that if $|\mathbf{u}_l-\mathbf{u}_r|$ is small enough and α satisfies (1.5), then (2.2) has a unique global in time solution and this solution is asymptotically equivalent to a self similar function $\check{\mathbf{U}}$:

$$\lim_{t\to\infty}\sup_{x\in R} |\mathbf{U}(x,t) - \check{\mathbf{U}}(\frac{x}{t})| = 0, \tag{2.7}$$

where $\check{\mathbf{U}}$ is obtained by imposing local equilibrium (1.2) together with (2.6):

$$\check{\mathbf{U}} = \begin{pmatrix} \check{\mathbf{u}} \\ \mathbf{f}(\check{\mathbf{u}}) \end{pmatrix} \tag{2.8}$$

In view of (2.4), we conclude that (2.1) has a unique solution satisfying

$$\lim_{\epsilon\to 0}\sup_{x\in R,\ t\geq\epsilon^\delta} |\mathbf{U}^\epsilon(x,t) - \check{\mathbf{U}}(\frac{x}{t})| = 0 \tag{2.9}$$

for any $\delta < 1$ (one can replace ϵ^δ by any $g(\epsilon)$ such that $\epsilon = o(g(\epsilon))$ as $\epsilon \to 0$).

To prove (2.7), we will construct $\widetilde{\mathbf{U}}(x,t)$, a smooth approximation of $\check{\mathbf{U}}$ and show that both $\mathbf{U}$ and $\check{\mathbf{U}}$ are asymptotically equivalent to $\widetilde{\mathbf{U}}$ (Lemma 2.3 (b) and (2.20) below).

We proceed by constructing $\widetilde{\mathbf{U}}$ as follows: Let $\widetilde{w}(x,t)$ be the solution of the Cauchy problem

$$\begin{gathered} \widetilde{w}_t + \widetilde{w}\widetilde{w}_x = 0 \\ \widetilde{w}(x,0) = \frac{1}{2}\{(w_r+w_l) + (w_r-w_l)\tanh x\} \end{gathered} \tag{2.10}$$

Lemma 2.1 ([5]) *Suppose $w_r > w_l$, then (2.10) has a unique global smooth solution satisfying* *(a)* $w_l < \widetilde{w}(x,t) < w_r$, $\widetilde{w}_x(x,t) > 0$ *for* $t \geq 0$, $x \in R$.

(b) For any $p \in [1,\infty]$, *there is a positive constant* C *such that for* $t \geq 0$,

$$\begin{aligned} \|\widetilde{w}_x(t)\|_{L^p} &\leq C \min(|w_r - w_l|, |w_r - w_l|(1+t)^{-1+\frac{1}{p}}), \\ \|\widetilde{w}_{xx}(t)\|_{L^p}, \|\widetilde{w}_{xxx}(t)\|_{L^p} &\leq C \min(|w_r - w_l|, (1+t)^{-1}). \end{aligned} \tag{2.11}$$

(c) $\lim_{t\to\infty} \sup_{x\in R} |\widetilde{w}(x,t) - \check{w}(\frac{x}{t})| = 0$.

Since the rarefaction wave solution written in the Riemann invariant coordinate reduces to a rarefaction wave for the Burgers' equation up to a nonlinear change of variables, we can thus construct the corresponding solutions of the Euler equation by inverting this change of variables. The corresponding solution of (1.3), $\widetilde{\mathbf{u}}$ satisfies

Lemma 2.2 *For each* $\mathbf{u}_l$ *satisfying the sub-characteristic condition (1.5), there exists a* $\delta_0 > 0$ *such that if* $\mathbf{u}_r$ *can be connected to* $\mathbf{u}_l$ *by a centered rarefaction wave* $\check{\mathbf{u}}$ *and* $|\mathbf{u}_l - \mathbf{u}_r| < \delta_0$, *then the corresponding smooth approximation* $\widetilde{\mathbf{u}}$ *satisfies the following:*

(a) $\widetilde{\mathbf{u}}$ *is a smooth global solution of*

$$\widetilde{\mathbf{u}}_t - \mathbf{f}(\widetilde{\mathbf{u}})_x = \mathbf{0}. \tag{2.12}$$

(b) For any $p \in [1,\infty]$, *there is a positive constant* C *such that for* $t \geq 0$,

$$\begin{aligned} \|\widetilde{\mathbf{u}}_x(t), \widetilde{\mathbf{u}}_t(t)\|_{L^p} &\leq C \min(|\mathbf{u}_r - \mathbf{u}_l|, |\mathbf{u}_r - \mathbf{u}_l|^{1/p}(1+t)^{-1+\frac{1}{p}}), \\ \|\widetilde{\mathbf{u}}_{xx}(t), \widetilde{\mathbf{u}}_{tx}(t), \widetilde{\mathbf{u}}_{xxx}(t), \widetilde{\mathbf{u}}_{txx}(t)\|_{L^p} &\leq C \min(|\mathbf{u}_r - \mathbf{u}_l|, (1+t)^{-1}). \end{aligned} \tag{2.13}$$

(c) $\lim_{t\to\infty} \sup_{x\in R} |\widetilde{\mathbf{u}}(x,t) - \check{\mathbf{u}}(\frac{x}{t})| = 0$.

Let us denote the zeroth order approximation by

$$\mathbf{U}^{(0)} = \begin{pmatrix} \mathbf{u}^{(0)} \\ \mathbf{v}^{(0)} \end{pmatrix}, \quad \mathbf{u}^{(0)} = \widetilde{\mathbf{u}}, \quad \mathbf{v}^{(0)} = \mathbf{f}(\mathbf{u}^{(0)}) \tag{2.14}$$

and following the Chapman-Enskog expansion, we get the first order correction

$$\mathbf{U}^{(1)} = \begin{pmatrix} \mathbf{0} \\ \mathbf{v}^{(1)} \end{pmatrix}, \mathbf{v}^{(1)} = -(\alpha^2 - \mathbf{f}'(\mathbf{u}^{(0)})^2)\mathbf{u}^{(0)}_x. \tag{2.15}$$

We then construct $\widetilde{\mathbf{U}}$, the smooth approximation of $\widetilde{\mathbf{U}}$, by

$$\widetilde{\mathbf{U}} = \mathbf{U}^{(0)} + \mathbf{U}^{(1)} = \begin{pmatrix} \widetilde{\mathbf{u}} \\ \widetilde{\mathbf{v}} \end{pmatrix} = \begin{pmatrix} \mathbf{u}^{(0)} \\ \mathbf{v}^{(0)} + \mathbf{v}^{(1)} \end{pmatrix} \tag{2.16}$$

Thus $\widetilde{\mathbf{U}}$ satisfies the following equation

$$\widetilde{\mathbf{U}}_t + \mathbf{A}\widetilde{\mathbf{U}}_x - \mathbf{N}(\widetilde{\mathbf{U}}) = \mathbf{U}^{(1)}_t + \mathbf{A}\mathbf{U}^{(1)}_x, \quad \mathbf{N}(\widetilde{\mathbf{U}}) = \begin{pmatrix} \mathbf{0} \\ -\mathbf{v}^{(1)} \end{pmatrix} \tag{2.17}$$

and we have the corresponding estimates for $\widetilde{\mathbf{U}}$.

Lemma 2.3 *Under the assumptions in Lemma 2.2, the smooth approximation* $\widetilde{\mathbf{U}}$ *satisfies (2.17) and*

$$\lim_{t\to\infty}\sup_{x\in R}|\widetilde{\mathbf{U}}(x,t)-\check{\mathbf{U}}(\frac{x}{t})|=0.$$

To show the asymptotic equivalence of $\mathbf{U}$ and $\widetilde{\mathbf{U}}$, we let $\mho=\mathbf{U}-\widetilde{\mathbf{U}}$. The equation satisfied by $\mho$ reads

$$\begin{gathered}\mho_t+\mathbf{A}\mho_x=\mathbf{N}(\mathbf{U})-\mathbf{N}(\widetilde{\mathbf{U}})-(\mathbf{U}_t^{(1)}+\mathbf{A}\mathbf{U}_x^{(1)})\\ \mho(x,0)=\mathbf{U}(x,0)-\widetilde{\mathbf{U}}(x,0)\end{gathered}\tag{2.18}$$

where

$$\begin{aligned}&\mathbf{N}(\mathbf{U})-\mathbf{N}(\widetilde{\mathbf{U}})\\ &=\begin{pmatrix}\mathbf{0}\\ \mathbf{f}(\mathbf{u})-\mathbf{v}+\mathbf{v}^{(1)}\end{pmatrix}=\begin{pmatrix}\mathbf{0}\\ \mathbf{f}(\mathbf{u})-\mathbf{f}(\tilde{\mathbf{u}})-\boldsymbol{\nu}\end{pmatrix}=-\begin{pmatrix}\mathbf{0}\\ \boldsymbol{\nu}-\bar{\mathbf{a}}\boldsymbol{\mu}+O(|\boldsymbol{\mu}|^3)\end{pmatrix}\end{aligned}\tag{2.19}$$

and $\bar{\mathbf{a}}=\mathbf{f}'(\bar{\mathbf{u}})=\mathbf{f}'(\tilde{\mathbf{u}}+\frac{\boldsymbol{\mu}}{2})$.

At this point, we briefly summarize the notations for readers' convenience: Lowercase bold letters denote 2-vectors or 2 by 2 matrices, like $\mathbf{u}$, $\mathbf{f}$, $\mathbf{a}$, etc. Uppercase letters are 4-vectors or 4 by 4 matrices, like $\mathbf{U}$ and $\mathbf{A}$. Tilded quantities are smooth approximations such as $\widetilde{w}$ and $\tilde{\mathbf{u}}$. Greek letters $\boldsymbol{\mu}=\mathbf{u}-\tilde{\mathbf{u}}$, $\boldsymbol{\nu}=\mathbf{v}-\tilde{\mathbf{v}}$ will denote perturbations. The barred quantities like $\bar{\mathbf{u}}=(\mathbf{u}+\tilde{\mathbf{u}})/2$ denote the average of the smooth approximation and the true solution.

(2.18), like (1.1), is a semilinear hyperbolic system. The discontinuities propagate along $x=\alpha t$ and $x=-\alpha t$. Thus we adopt the piecewise energy estimate. We introduce the following notations: Denote by Ω_k, $k=1,2,3$ the regions separated by $x=\alpha t$ and $x=-\alpha t$ in the upper half plane $t>0$, $\Omega_k^s=\Omega_k\cap\{t=s\}$ and for any interval $I\subset R$, $H^1(I)$ the usual Sobolev space with norm $\|\cdot\|_1=(\|\cdot\|_{L^2(I)}+\|\frac{\partial}{\partial x}\cdot\|_{L^2(I)})^{1/2}$.

Now we define the appropriate function space on which we will be working:

$$\begin{aligned}X(0,T)=\big\{&\mathbf{W}:R\times[0,T]\mapsto R^4:\\ &\mathbf{W}\in C^0([0,T),H^1(\Omega_k^s))\cap C^1([0,T),L^2(\Omega_k^s))\cap C^2(\overline{\Omega}_k),\ k=1,2,3\big\}\end{aligned}$$

For $\mathbf{W}\in X(0,T)$, we define

$$\begin{aligned}\|\mathbf{W}(\cdot,t)\|\!\!\!+^2&=\mkern-18mu\int|\mathbf{W}(x,t)|^2dx\\ &\stackrel{\text{def}}{=}\int_{-\infty}^{-\alpha t}|\mathbf{W}(x,t)|^2dx+\int_{-\alpha t}^{\alpha t}|\mathbf{W}(x,t)|^2dx+\int_{\alpha t}^{\infty}|\mathbf{W}(x,t)|^2dx\end{aligned}$$

$$
\begin{aligned}
\|\mathbf{W}(\cdot,t)\|_1^2 &= \|\mathbf{W}(\cdot,t)\|^2 + \|\mathbf{W}_x(\cdot,t)\|^2 \\
[\mathbf{W}]^+(t) &= \mathbf{W}(\alpha t+0,t) - \mathbf{W}(\alpha t-0,t) \\
[\mathbf{W}]^-(t) &= \mathbf{W}(-\alpha t+0,t) - \mathbf{W}(-\alpha t-0,t) \\
\langle \mathbf{W}\rangle^+(t) &= \frac{1}{2}(\mathbf{W}(\alpha t+0,t) + \mathbf{W}(\alpha t-0,t)) \\
\langle \mathbf{W}\rangle^-(t) &= \frac{1}{2}(\mathbf{W}(-\alpha t+0,t) + \mathbf{W}(-\alpha t-0,t)).
\end{aligned}
$$

We want to show that

$$\lim_{t\to\infty}\sup_{x\in R} |\mathbf{U}(x,t) - \widetilde{\mathbf{U}}(x,t)| = 0. \tag{2.20}$$

A key observation is the following: The a priori bound on $\sup_{t>0}\|\mho\|_{L^\infty(t)}$ implies exponential decay in time of the jumps (see Lemma 4.1 below). Thus in view of the Sobolev inequality

$$\|\mho\|^2_{L^\infty}(t) \le C\left(\|\mho\|\|\mho_x\|(t) + [\mho]^2_+(t) + [\mho]^2_-(t)\right), \tag{2.21}$$

our task remains to estimate $\|\mho\|_1$ (Theorem 2.4 below). During this process, terms involving line integrals of jumps across the discontinuities appear naturally. Therefor the exponential decay in time of the jumps implies the a priori bound on $\sup_{t\ge 0}\|\mho\|_1^2(t)$ (Lemma 4.4 and on). We close this bootstrapping argument by the local existence theorem (Theorem 2.5 below) to extend $\mho$ in $X(0,T+\Delta t)$ and conclude that $T=\infty$ (global in time existence).

Theorem 2.4 (A priori estimate) *There exists positive constants ϵ_1 and C_1 such that if $\mho \in X(0,T)$ is the solution of (2.18) in $0\le t\le T$ for some $T>0$ and*

$$\sup_{0\le t\le T}\|\mho(t)\|_1 + |\mathbf{u}_r - \mathbf{u}_l| < \epsilon_1,$$

then

$$\sup\nolimits_{0\le t\le T}\|\mho(t)\|_1^2 + \textstyle\int_0^T \|\mho_x(\tau)\|^2 \le C_1(\|\mho(0)\|_1^2 + |\mathbf{u}_r-\mathbf{u}_l|^{1/6}). \tag{2.22}$$

The proof of Theorem 2.4 will be given in section 4.

Theorem 2.5 (Local existence) *Let $T\ge 0$ and $g\in X(0,T)$ be a solution to (2.18) for $0\le t\le T$. Consider the initial value problem to (2.18) with the initial data*

$$\mho(T,x) = \mathbf{W}_T(x) \overset{\text{def}}{=} \mathbf{W}(T,x). \tag{2.23}$$

Then for any $M>0$, there exists a positive constant Δt depending only on M and $\sup_{x,t}|\widetilde{\mathbf{U}}(x,t)|$ such that if

$$\|\mathbf{W}_T\|_{C^0(-\infty,-\alpha T)} + \|\mathbf{W}_T\|_{C^0(-\alpha T,\alpha T)} + \|\mathbf{W}_T\|_{C^0(\alpha T,\infty)} < M$$

Then (2.18) together with (2.23) has a unique solution $\mho \in X(T, T+\Delta t)$ *satisfying*

$$\sup_{T\le t\le T+\Delta t}\big(\|\mho(t)\|_{C^0(-\infty,-\alpha t)}+\|\mho(t)\|_{C^0(-\alpha t,\alpha t)}+\|\mho(t)\|_{C^0(\alpha t,\infty)}\big)<2M.$$

As a consequence, $\mathbf{W}$ *can be extended to* $X(0, T+\Delta t)$.

The proof of Theorem 2.5 is standard, see [14] for the existence and uniqueness in the piecewise C^0 function class. The piecewise C^2 regularity is a direct consequence of the special structure of the nonlinearity (being the lower order term in (2.18)) and the C^2 regularity of the nonlinear functional $\mathbf{N}(\cdot)$.

From Theorem 2.4 and Theorem 2.5, we have the following.

Corollary 2.6 *For each* $\mathbf{u}_l$ *satisfying the sub-characteristic condition (1.5), there exists* ϵ_0 *and* C_0 *such that if* $\mathbf{u}_r$ *can be connected to* $\mathbf{u}_l$ *by a centered rarefaction wave and* $\|\mho(0)\|_1+|\mathbf{u}_r-\mathbf{u}_l|<\epsilon_0$, *then (2.18) has a unique solution* $\mho\in X(0,\infty)$ *satisfying*

$$\sup_{t\ge 0}\|\mho(t)\|_1^2+\int_0^\infty\|\mho_x(\tau)\|^2d\tau\le C_0(\|\mho(0)\|_1^2+|\mathbf{u}_r-\mathbf{u}_l|^{1/6}). \tag{2.24}$$

With Corollary 2.6, Lemma 4.1 below and the Sobolev inequality (2.21), it is easy to see that

$$\int_0^\infty\|\mho_x\|^2(\tau)+\Big|\frac{d}{d\tau}\|\mho_x\|^2\Big|(\tau)<\infty \tag{2.25}$$

and consequently (2.20) holds.

3 Entropic Extension

The Jin-Xin model has some nice mathematical properties. The entropic extension below follows from the idea outlined in [2]. The proof can also be found in [15]. The derived entropy $\Phi(\mathbf{u},\mathbf{v})$ and the functions $\mathbf{u}^{\pm}$ will be used in our main energy estimate so we briefly summarize the proof here for readers' convenience.

Proposition 3.1 *Let* $(\phi(\mathbf{u}),\psi(\mathbf{u}))$ *be a pair of entropy-flux functions of (1.3) with* ϕ *convex, then we can derive the corresponding entropy-flux pair* (Φ,Ψ) *for the system (1.1) from the following wave equation with Cauchy data*

$$\begin{aligned}\Phi_{\mathbf{u}}&=\Psi_{\mathbf{v}}\\ \Psi_{\mathbf{u}}&=\alpha^2\Phi_{\mathbf{v}}\\ \Phi(\mathbf{u},\mathbf{f}(\mathbf{u}))&=\phi(\mathbf{u})\\ \Phi_{\mathbf{v}}(\mathbf{u},\mathbf{f}(\mathbf{u}))&=\mathbf{0}.\end{aligned} \tag{3.1}$$

The solution (Φ,Ψ) *exists in a neighborhood of the equilibrium* $\mathbf{v}=\mathbf{f}(\mathbf{u})$ *and there it satisfies*

(a) Φ *is convex if* α *is sufficiently large.*

(b) $\Psi(\mathbf{u}, \mathbf{f}(\mathbf{u})) = \psi(\mathbf{u})$.

(c) $\Phi_t(\mathbf{u}, \mathbf{v}) + \Psi_x(\mathbf{u}, \mathbf{v}) = -\frac{1}{\epsilon}\Phi_{\mathbf{v}}(\mathbf{u}, \mathbf{v}) \cdot (\mathbf{v} - \mathbf{f}(\mathbf{u})) \leq 0$ *for any smooth solution of (1.1).*

Proof. From first two equation of (3.1), Φ satisfies the wave equation

$$\Phi_{\mathbf{uu}} = \alpha^2 \Phi_{\mathbf{vv}}$$

Therefore the general solution is given by

$$\Phi(\mathbf{u}, \mathbf{v}) = h_+(\mathbf{v} + \alpha\mathbf{u}) + h_-(\mathbf{v} - \alpha\mathbf{u}), \tag{3.2}$$

The last two equation in (3.1) gives:

$$\begin{gathered} h'_+(\mathbf{f}(\mathbf{u}) + \alpha\mathbf{u}) - h'_-(\mathbf{f}(\mathbf{u}) - \alpha\mathbf{u}) = \frac{\phi'(\mathbf{u})}{\alpha} \\ h'_+(\mathbf{f}(\mathbf{u}) + \alpha\mathbf{u}) + h'_-(\mathbf{f}(\mathbf{u}) - \alpha\mathbf{u}) = \mathbf{0}, \end{gathered} \tag{3.3}$$

Thus

$$h'_\pm(\mathbf{f}(\mathbf{u}) \pm \alpha\mathbf{u}) = \pm\frac{\phi'(\mathbf{u})}{2\alpha} \tag{3.4}$$

and

$$\frac{d}{d\mathbf{u}} h_\pm(\mathbf{f}(\mathbf{u}) \pm \alpha\mathbf{u}) = \pm\frac{\phi'(\mathbf{u})}{2\alpha}(\mathbf{f}'(\mathbf{u}) \pm \alpha I) = \frac{1}{2}\big(\phi'(\mathbf{u}) \pm \frac{\psi'(\mathbf{u})}{\alpha}\big) \tag{3.5}$$

Thus

$$h_\pm(\mathbf{f}(\mathbf{u}) \pm \alpha\mathbf{u}) = \frac{1}{2}\big(\phi(\mathbf{u}) \pm \frac{\psi(\mathbf{u})}{\alpha}\big) \tag{3.6}$$

and the solution to (3.1) is given by

$$\begin{gathered} \Phi(\mathbf{u}, \mathbf{v}) = h_+(\mathbf{v} + \alpha\mathbf{u}) + h_-(\mathbf{v} - \alpha\mathbf{u}) \\ \Psi(\mathbf{u}, \mathbf{v}) = \alpha\left(h_+(\mathbf{v} + \alpha\mathbf{u}) - h_-(\mathbf{v} - \alpha\mathbf{u})\right) \end{gathered} \tag{3.7}$$

and part (b) of the assertion follows.

From the Inverse Function Theorem, we can define two functions

$$\mathbf{u}^+ = \mathbf{u}^+(\mathbf{v} + \alpha\mathbf{u}), \quad \mathbf{u}^- = \mathbf{u}^-(\mathbf{v} - \alpha\mathbf{u})$$

in a neighborhood of $(\mathbf{u}_0, \mathbf{f}(\mathbf{u}_0))$ implicitly by

$$\mathbf{f}(\mathbf{u}^\pm) \pm \alpha\mathbf{u}^\pm = \mathbf{v} \pm \alpha\mathbf{u} \tag{3.8}$$

with $\mathbf{u}^\pm(\mathbf{f}(\mathbf{u}_0) \pm \alpha\mathbf{u}^\pm) = \mathbf{u}_0$ since

$$\frac{d}{d\mathbf{u}^\pm}(\mathbf{f}(\mathbf{u}^\pm) \pm \alpha\mathbf{u}^\pm)\,|_{\mathbf{u}_0} = \mathbf{f}'(\mathbf{u}_0) \pm \alpha$$

is nonsingular. In view of (3.6-3.7) and (3.8), we can write

$$\Phi(\mathbf{u},\mathbf{v}) = \frac{1}{2}\Big((\phi(\mathbf{u}^+) + \frac{\psi(\mathbf{u}^+)}{\alpha}) + (\phi(\mathbf{u}^-) - \frac{\psi(\mathbf{u}^-)}{\alpha})\Big) \tag{3.9}$$

and

$$\frac{\partial}{\partial \mathbf{u}}\mathbf{u}^{\pm} = \pm\alpha\left(\mathbf{f}'(\mathbf{u}^{\pm}) \pm \alpha\right)^{-1} \tag{3.10}$$

$$\frac{\partial}{\partial \mathbf{v}}\mathbf{u}^{\pm} = \left(\mathbf{f}'(\mathbf{u}^{\pm}) \pm \alpha\right)^{-1}. \tag{3.11}$$

Therefore

$$\begin{aligned}\Phi_{\mathbf{u}}(\mathbf{u},\mathbf{v}) =& \frac{1}{2}\Big((\phi'(\mathbf{u}^+) + \frac{\psi'(\mathbf{u}^+)}{\alpha})\frac{\partial}{\partial \mathbf{u}}\mathbf{u}^+ + (\phi'(\mathbf{u}^-) - \frac{\psi'(\mathbf{u}^-)}{\alpha})\frac{\partial}{\partial \mathbf{u}}\mathbf{u}^-\Big)\\ =& \frac{1}{2}\left(\phi'(\mathbf{u}^+) + \phi'(\mathbf{u}^-)\right),\end{aligned}$$

where we have used $\psi' = \phi'\mathbf{f}'$. Similarly

$$\begin{aligned}\Phi_{\mathbf{v}}(\mathbf{u},\mathbf{v}) &= \frac{1}{2\alpha}\left(\phi'(\mathbf{u}^+) - \phi'(\mathbf{u}^-)\right)\\ \Phi_{\mathbf{uu}}(\mathbf{u},\mathbf{v}) &= \alpha^2(h''_+(\mathbf{v}+\alpha\mathbf{u}) + h''_-(\mathbf{v}-\alpha\mathbf{u}))\\ \Phi_{\mathbf{uv}}(\mathbf{u},\mathbf{v}) &= \alpha(h''_+(\mathbf{v}+\alpha\mathbf{u}) - h''_-(\mathbf{v}-\alpha\mathbf{u}))\\ \Phi_{\mathbf{vv}}(\mathbf{u},\mathbf{v}) &= h''_+(\mathbf{v}+\alpha\mathbf{u}) + h''_-(\mathbf{v}-\alpha\mathbf{u})\\ h''_+(\mathbf{v}+\alpha\mathbf{u}) &= \phi''(\mathbf{u}^+)(\alpha+\mathbf{f}'(\mathbf{u}^+))^{-1}\\ h''_-(\mathbf{v}-\alpha\mathbf{u}) &= \phi''(\mathbf{u}^-)(\alpha-\mathbf{f}'(\mathbf{u}^-))^{-1}\end{aligned}$$

To show the convexity of Φ, we compute

$$\begin{aligned}(\mathbf{x}^T,\mathbf{y}^T)\begin{pmatrix}\Phi_{\mathbf{uu}} & \Phi_{\mathbf{uv}}\\ \Phi_{\mathbf{uv}} & \Phi_{\mathbf{vv}}\end{pmatrix}\begin{pmatrix}\mathbf{x}\\ \mathbf{y}\end{pmatrix} &= (\alpha\mathbf{x}+\mathbf{y})^T\phi''(\mathbf{u}^+)(\alpha+\mathbf{f}'(\mathbf{u}^+))^{-1}(\alpha\mathbf{x}+\mathbf{y})\\ &\quad + (\alpha\mathbf{x}-\mathbf{y})^T\phi''(\mathbf{u}^-)(\alpha-\mathbf{f}'(\mathbf{u}^-))^{-1}(\alpha\mathbf{x}-\mathbf{y}) \end{aligned}\tag{3.12}$$

Since $\phi''\mathbf{f}'$ is symmetric, so are $\phi''(\alpha\pm\mathbf{f}')^{-1}$. This makes $(\alpha\pm\mathbf{f}')^{-1}$ self adjoint operators on R^2 with respect to the inner product induced by ϕ'':

$$\langle \mathbf{y}, (\alpha\pm\mathbf{f}')^{-1}\mathbf{x}\rangle = \langle(\alpha\pm\mathbf{f}')^{-1}\mathbf{y},\mathbf{x}\rangle, \qquad \langle\mathbf{y},\mathbf{x}\rangle = \mathbf{y}^T\phi''\mathbf{x}$$

Therefore $\phi''(\alpha\pm\mathbf{f}')^{-1}$ are positive definite under the sub-characteristic condition (1.5). We conclude that Φ is convex in a neighborhood of $(\mathbf{u}_0, \mathbf{f}(\mathbf{u}_0))$ provided $\mathbf{u}_0$ satisfies (1.5). This proves (a).

Part (c) is a direct consequence of (a) since

$$\Phi_{\mathbf{v}}(\mathbf{u},\mathbf{v}) = \Phi_{\mathbf{v}}(\mathbf{u},\mathbf{v}) - \Phi_{\mathbf{v}}(\mathbf{u},\mathbf{f}(\mathbf{u})) = (\mathbf{v}-\mathbf{f}(\mathbf{u}))^T\int_0^1 \Phi_{\mathbf{vv}}(\mathbf{u},\mathbf{f}(\mathbf{u})+\theta(\mathbf{v}-\mathbf{f}(\mathbf{u})))d\theta$$

4 A priori Estimates

To proceed with the energy estimate, we first study how the singularity propagates. (1.1) is a semilinear hyperbolic system, the jump discontinuity in the initial data propagate along the characteristic curves and, due to the relaxation effect, decays exponentially in time. To show this, we write (1.1) in diagonal form in the characteristic variables $\mathbf{w}^{\pm} = \mathbf{v} \pm \alpha\mathbf{u}$,

$$\begin{aligned} \mathbf{w}_t^+ + \alpha\mathbf{w}_x^+ &= \mathbf{f}(\mathbf{u}) - \mathbf{v} \\ \mathbf{w}_t^- - \alpha\mathbf{w}_x^- &= \mathbf{f}(\mathbf{u}) - \mathbf{v} \end{aligned} \tag{4.1}$$

We denote by $[\cdot]^{\pm}$ the jumps along the characteristic line $dx/dt = \pm\alpha$. Taking the jump in the '+' family on both sides of the first equation in (4.1), we have

$$[\mathbf{w}_t^+ + \alpha\mathbf{w}_x^+]^+ = [\mathbf{f}(\frac{\mathbf{w}^+ - \mathbf{w}_r^-}{2\alpha})]^+ - [\frac{\mathbf{w}^+}{2}]^+ \tag{4.2}$$

where $\mathbf{w}_r^- = \mathbf{v}_r - \alpha\mathbf{u}_r$ is the corresponding Riemann data on $x > 0$. Since $\mathbf{U}$ has continuous first derivatives up to the boundary on either side of the jumps, we can interchange the tangential derivative with the jump to get an ODE along the characteristics

$$\begin{aligned} \frac{d}{dt}[\mathbf{w}^+]^+ =& [\mathbf{f}(\frac{\mathbf{w}^+ - \mathbf{w}_r^-}{2\alpha})]^+ - [\frac{\mathbf{w}^+}{2}]^+ \\ =& \mathbf{f}(\frac{\mathbf{w}_r^+ - \mathbf{w}_r^-}{2\alpha}) - \mathbf{f}(\frac{-[\mathbf{w}^+]^+ + \mathbf{w}_r^+ - \mathbf{w}_r^-}{2\alpha}) - \frac{[\mathbf{w}^+]^+}{2}. \end{aligned} \tag{4.3}$$

The equilibrium for (4.3) is given by $[\mathbf{w}^+]^+ = \mathbf{0}$ with $(\mathbf{u},\mathbf{v}) = (\mathbf{u}_r,\mathbf{v}_r)$ on either side of the characteristic. The linearization of the right hand side around this equilibrium state is thus

$$\frac{1}{2}\big(\frac{\mathbf{f}'(\mathbf{u}_r)}{\alpha} - \mathbf{I}\big)[\mathbf{w}^+]^+.$$

From (1.5), we conclude that the local equilibrium $(\mathbf{u}_r,\mathbf{v}_r)$ is a sink and $[\mathbf{w}^+]^+(t)$ decays exponentially in t. The same applies to $[\mathbf{w}_x^{\pm}]^+(t)$, $[\mathbf{w}^-]^-(t)$ and $[\mathbf{w}_x^{\pm}]^-(t)$. We therefore have the following

Lemma 4.1 *Let* $\beta = |\mathbf{u}_r - \mathbf{u}_l|$ *and*

$$E = \sup_{0\le t\le T} \left(\|\mho\|_1(t) + |[\mho]^+(t)| + |[\mho]^-(t)| \right),$$

then there exist positive constants ϵ_1, C, C_1 *such that if* $0 \le t \le T$, $E \le \epsilon_1$ *and* α *sufficiently large, we have*

$$|[\mho]^{\pm}(t)| + |[\mho_x]^{\pm}(t)| \le C\beta e^{-C_1 t} \tag{4.4}$$

To proceed, we define

$$
\begin{aligned}
\mathcal{E} &= \Phi(\widetilde{\mathbf{U}}+\mho)-\Phi(\widetilde{\mathbf{U}})-\Phi'(\widetilde{\mathbf{U}})\mho \\
\mathcal{E}_{\mho} &= \Phi'(\widetilde{\mathbf{U}}+\mho)-\Phi'(\widetilde{\mathbf{U}}) \\
\mathcal{E}_{\widetilde{U}} &= \Phi'(\widetilde{\mathbf{U}}+\mho)-\Phi'(\widetilde{\mathbf{U}})-\mho^T\Phi''(\widetilde{\mathbf{U}}) \\
\mathcal{J} &= \Psi(\widetilde{\mathbf{U}}+\mho)-\Psi(\widetilde{\mathbf{U}})-\Psi'(\widetilde{\mathbf{U}})\mho \\
\mathcal{J}_{\mho} &= \Psi'(\widetilde{\mathbf{U}}+\mho)-\Psi'(\widetilde{\mathbf{U}}) \\
\mathcal{J}_{\widetilde{U}} &= \Psi'(\widetilde{\mathbf{U}}+\mho)-\Psi'(\widetilde{\mathbf{U}})-\mho^T\Psi''(\widetilde{\mathbf{U}})
\end{aligned}
$$

It is easy to see that, for $E<\epsilon_2$, we have

$$
\begin{aligned}
c|\mho|^2 \le \mathcal{E} &\le C|\mho|^2 \\
\mathcal{E}_{\mho} &\le C|\mho| \\
\mathcal{E}_{\widetilde{U}} &\le C|\mho|^2
\end{aligned}
\tag{4.5}
$$

In addition to (4.5), a more refined estimate for $\mathcal{E}_{\mho}$ is needed due to the structure of the nonlinear term. Denote by

$$\mathcal{E}_{\mho}=(\mathcal{E}_{\mu},\mathcal{E}_{\nu})$$

In the following Lemma, we expand $\mathcal{E}_{\nu}$ in terms of the perturbation and identify the quadratic term, which will be of use in our energy estimate.

Lemma 4.2 *For* $\bar{\mathbf{u}}=\tilde{\mathbf{u}}+\boldsymbol{\mu}/2$ *and* $\bar{\mathbf{a}}=\mathbf{f}'(\bar{\mathbf{u}})$,

$$\mathcal{E}_{\nu}=\big((\alpha^2-\bar{\mathbf{a}}^2)^{-1}(\boldsymbol{\nu}-\bar{\mathbf{a}}\boldsymbol{\mu})\big)^T\phi''(\bar{\mathbf{u}})+O(|\mathbf{v}^{(1)}||\mho|+|\mho||\boldsymbol{\nu}-\bar{\mathbf{a}}\boldsymbol{\mu}|)+O(|\mathbf{v}^{(1)}|^3+|\mho|^3). \tag{4.6}$$

Proof. Since $\mathcal{E}_{\mho}=\Phi'(\mathbf{U})-\Phi'(\widetilde{\mathbf{U}})$ involves the difference of functions evaluated at $\mathbf{U}$ and $\widetilde{\mathbf{U}}$ respectively, we will, in the remainder of the proof, consider $(\bar{\mathbf{u}},\bar{\mathbf{v}})=\big(\frac{\tilde{\mathbf{u}}+\mathbf{u}}{2},\mathbf{f}(\frac{\tilde{\mathbf{u}}+\mathbf{u}}{2})\big)$ fixed as the base point for Taylor expansion. We take $(\mathbf{u}_0,\mathbf{f}(\mathbf{u}_0))=(\bar{\mathbf{u}},\bar{\mathbf{v}})$ and recall the two functions

$$\mathbf{u}^+=\mathbf{u}^+(\mathbf{v}+\alpha\mathbf{u}),\quad \mathbf{u}^-=\mathbf{u}^-(\mathbf{v}-\alpha\mathbf{u})$$

defined in (3.8) in a neighborhood of $(\bar{\mathbf{u}},\bar{\mathbf{v}})$. To estimate $\Phi_{\mathbf{v}}$, we let

$$\boldsymbol{\eta}^{\pm}=\mathbf{v}-\bar{\mathbf{v}}\pm\alpha(\mathbf{u}-\bar{\mathbf{u}}) \tag{4.7}$$

and expand $\mathbf{u}^{\pm}$ in Taylor series of $\boldsymbol{\eta}^{\pm}$

$$\mathbf{u}^{\pm}=\bar{\mathbf{u}}+\mathbf{u}_1^{\pm}+\mathbf{u}_2^{\pm}+O(|\boldsymbol{\eta}^{\pm}|^3) \tag{4.8}$$

with $\mathbf{u}_1^\pm = D\mathbf{u}^\pm(\bar{\mathbf{u}})\boldsymbol{\eta}^\pm$ and $\mathbf{u}_2^\pm = \frac{1}{2}D^2\mathbf{u}^\pm(\bar{\mathbf{u}})(\boldsymbol{\eta}^\pm, \boldsymbol{\eta}^\pm)$. Substitute (4.8) back to (3.8), expand in power series of $\boldsymbol{\eta}^\pm$ and equate the powers of $|\boldsymbol{\eta}^\pm|$, we get

$$\mathbf{u}_1^\pm = (\mathbf{f}'(\bar{\mathbf{u}}) \pm \alpha)^{-1}\boldsymbol{\eta}^\pm = (\bar{\mathbf{a}} \pm \alpha)^{-1}\boldsymbol{\eta}^\pm \tag{4.9}$$

$$\mathbf{u}_2^\pm = -(\bar{\mathbf{a}} \pm \alpha)^{-1}\frac{\mathbf{f}''(\bar{\mathbf{u}})}{2}(\mathbf{u}_1^\pm, \mathbf{u}_1^\pm) \tag{4.10}$$

Introduce the notation

$$\langle\cdot\rangle_{\tilde{U}}^U = \cdot|_U + \cdot|_{\tilde{U}}, \quad [\cdot]_{\tilde{U}}^U = \cdot|_U - \cdot|_{\tilde{U}}$$

we have

$$[\boldsymbol{\eta}^\pm]_{\tilde{U}}^U = [\mathbf{v} - \bar{\mathbf{v}}]_{\tilde{U}}^U \pm \alpha[\mathbf{u} - \bar{\mathbf{u}}]_{\tilde{U}}^U = [\mathbf{v}]_{\tilde{U}}^U \pm \alpha[\mathbf{u}]_{\tilde{U}}^U = \boldsymbol{\nu} \pm \alpha\boldsymbol{\mu}$$

and

$$\langle\boldsymbol{\eta}^\pm\rangle_{\tilde{U}}^U = \langle\mathbf{v} - \bar{\mathbf{v}}\rangle_{\tilde{U}}^U \pm \alpha\langle\mathbf{u} - \bar{\mathbf{u}}\rangle_{\tilde{U}}^U = \langle\mathbf{v} - \bar{\mathbf{v}}\rangle_{\tilde{U}}^U$$

Since

$$\begin{aligned}\mathbf{v} - \bar{\mathbf{v}}|_U = \mathbf{f}(\tilde{\mathbf{u}}) + \mathbf{v}^{(1)} + \boldsymbol{\nu} - \mathbf{f}(\bar{\mathbf{u}}) &= \mathbf{f}(\bar{\mathbf{u}} - \frac{\boldsymbol{\mu}}{2}) + \mathbf{v}^{(1)} + \boldsymbol{\nu} - \mathbf{f}(\bar{\mathbf{u}}) \\ &= -\bar{\mathbf{a}}\frac{\boldsymbol{\mu}}{2} + O(|\boldsymbol{\mu}|^2) + \mathbf{v}^{(1)} + \boldsymbol{\nu}\end{aligned} \tag{4.11}$$

and similarly

$$\mathbf{v} - \bar{\mathbf{v}}\,\big|_{\tilde{U}} = \mathbf{f}(\tilde{\mathbf{u}}) - \mathbf{f}(\bar{\mathbf{u}}) = -\bar{\mathbf{a}}\frac{\boldsymbol{\mu}}{2} + O(|\boldsymbol{\mu}|^2) + \mathbf{v}^{(1)} \tag{4.12}$$

we have

$$\langle\mathbf{v} - \bar{\mathbf{v}}\rangle_{\tilde{U}}^U = \boldsymbol{\nu} - \bar{\mathbf{a}}\boldsymbol{\mu} + 2\mathbf{v}^{(1)} + O(|\boldsymbol{\mu}|^2)$$

$$\begin{aligned}\Phi_{\mathbf{v}} =& \frac{1}{2\alpha}\left(\phi'(\mathbf{u}^+) - \phi'(\mathbf{u}^-)\right) \\ =& \frac{1}{2\alpha}\left(\phi'(\bar{\mathbf{u}} + \mathbf{u}_1^+ + \mathbf{u}_2^+ + O(|\boldsymbol{\eta}^+|^3)) - \phi'(\bar{\mathbf{u}} + \mathbf{u}_1^- + \mathbf{u}_2^- + O(|\boldsymbol{\eta}^-|^3))\right)\end{aligned}$$

Since

$$\phi'(\mathbf{u}^\pm) = \phi'(\bar{\mathbf{u}}) + (\mathbf{u}_1^\pm)^T\phi''(\bar{\mathbf{u}}) + \left((\mathbf{u}_2^\pm)^T\phi''(\bar{\mathbf{u}}) + \frac{\phi'''(\bar{\mathbf{u}})}{2}(\mathbf{u}_1^\pm, \mathbf{u}_1^\pm)\right) + O(|\boldsymbol{\eta}^\pm|^3)$$

we have the linear term for $\mathcal{E}_\nu$:

$$\left[\frac{1}{2\alpha}(\mathbf{u}_1^+ - \mathbf{u}_1^-)^T\phi''(\bar{\mathbf{u}})\right]_{\tilde{U}}^U = \left((\alpha^2 - \bar{\mathbf{a}}^2)^{-1}(\boldsymbol{\nu} - \bar{\mathbf{a}}\boldsymbol{\mu})\right)^T\phi''(\bar{\mathbf{u}}) \tag{4.13}$$

To estimate the quadratic term in $\mathcal{E}_\nu$, observe that from (4.9) and (4.10), both

$$(\mathbf{u}_2^\pm)^T\phi''(\bar{\mathbf{u}}) = -\Big((\bar{\mathbf{a}} \pm \alpha)^{-1}\frac{\mathbf{f}''(\bar{\mathbf{u}})}{2}\big((\bar{\mathbf{a}} \pm \alpha)^{-1}\bullet, (\bar{\mathbf{a}} \pm \alpha)^{-1}\bullet\big)\Big)^T\big|_{(\boldsymbol{\eta}^\pm, \boldsymbol{\eta}^\pm)}\phi''(\bar{\mathbf{u}})$$

and

$$\frac{\phi'''(\bar{\mathbf{u}})}{2}(\mathbf{u}_1^\pm, \mathbf{u}_1^\pm) = \frac{\phi'''(\bar{\mathbf{u}})}{2}\left((\bar{\mathbf{a}} \pm \alpha)^{-1}\bullet, (\bar{\mathbf{a}} \pm \alpha)^{-1}\bullet\right)|_{(\boldsymbol{\eta}^\pm, \boldsymbol{\eta}^\pm)}$$

are (vector valued) symmetric bilinear forms evaluated at $(\boldsymbol{\eta}^+, \boldsymbol{\eta}^+)$ and $(\boldsymbol{\eta}^-, \boldsymbol{\eta}^-)$ respectively. In addition, both are symmetric in their arguments. For any symmetric bilinear form $Q(\cdot, \cdot)$, we have

$$Q(\mathbf{p}, \mathbf{p}) - Q(\mathbf{q}, \mathbf{q}) = Q(\mathbf{p} + \mathbf{q}, \mathbf{p} - \mathbf{q}),$$

Thus

$$\left[(\mathbf{u}_2^\pm)^T \phi''(\bar{\mathbf{u}}) + \frac{\phi'''(\bar{\mathbf{u}})}{2}(\mathbf{u}_1^\pm, \mathbf{u}_1^\pm)\right]_{\widetilde{U}}^{U} = O(1)\left(\left|\langle\boldsymbol{\eta}^\pm\rangle_{\widetilde{U}}^{U}\right| \cdot \left|[\boldsymbol{\eta}^\pm]_{\widetilde{U}}^{U}\right|\right)$$

and (4.6) follows. □

We now proceed with the main energy estimate. Let β and E be defined as in Lemma 4.1.

Lemma 4.3 *There exist positive constants ϵ_2 and C such that for $0 \le t \le T$ and $E \le \epsilon_2$,*

$$\begin{aligned} &\|\mho(t)\|^2 + \int_0^t \|\boldsymbol{\nu} - \bar{\mathbf{a}}\boldsymbol{\mu}\|^2 d\tau \\ &\le \|\mho(0)\|^2 + C\int_0^t \fint \left(|\mathbf{v}^{(1)}|^4 + |\mathbf{v}_x^{(1)}|^2 + |\mho|^2|\mathbf{v}^{(1)}| + |\mho||\mathbf{v}_x^{(1)}| + |\mho|^6\right) dx d\tau + C\beta \end{aligned} \tag{4.14}$$

Proof. We multiply both sides of (2.18) by $\mathcal{E}_\mho$ from the left to get

$$\mathcal{E}_t + \mathcal{J}_x = \mathcal{E}_{\widetilde{U}}(\widetilde{\mathbf{U}}_t + \mathbf{A}\widetilde{\mathbf{U}}_x) + \mathcal{E}_\mho(\mathbf{N}(\mathbf{U}) - \mathbf{N}(\widetilde{\mathbf{U}})) - \mathcal{E}_\mho(\mathbf{U}_t^{(1)} + \mathbf{A}\mathbf{U}_x^{(1)}) \tag{4.15}$$

where we have used $\Phi'\mathbf{A} = \Psi'$. Next we calculate each term on the right hand side of (4.15):

$$\widetilde{\mathbf{U}}_t + \mathbf{A}\widetilde{\mathbf{U}}_x = \mathbf{N}(\widetilde{\mathbf{U}}) + \mathbf{U}_t^{(1)} + \mathbf{A}\mathbf{U}_x^{(1)} = O(\mathbf{v}^{(1)}, \mathbf{v}_t^{(1)}, \mathbf{v}_x^{(1)}), \tag{4.16}$$

$$\mathbf{N}(\mathbf{U}) - \mathbf{N}(\check{\mathbf{U}}) = -\begin{pmatrix} \mathbf{0} \\ \boldsymbol{\nu} - \bar{\mathbf{a}}\boldsymbol{\mu} + O(|\boldsymbol{\mu}|^3) \end{pmatrix} \tag{4.17}$$

Thus (4.14) follows after integrating (4.15) over $\mathbb{R} \times (0, T)$ and applying the Cauchy-Schwartz inequality and (4.4). □

Lemma 4.4 *There exist positive constants ϵ_2 and C such that if $E \le \epsilon_2$ and $0 \le t \le T$, then*

$$\begin{aligned} \|\mho_x(t)\|\!|^2 + \int_0^t \|\mho_x(\tau)\|\!|^2 d\tau \le &\|\mho_x(0)\|\!|^2 + C\int_0^t \|\boldsymbol{\nu} - \bar{\mathbf{a}}\boldsymbol{\mu}\|^2 \\ &+ C\int_0^t \fint \Big(|\mathbf{v}^{(1)}|^2|\mho|^2 + |\mho|^6 + |\mathbf{v}_x^{(1)}|^2 \\ &+ |\mathbf{v}_{xx}^{(1)}|^2 + |\mho_x|^2|\mathbf{v}^{(1)}|\Big) dx d\tau + C\beta \end{aligned} \tag{4.18}$$

Proof. We first derive the equation for $\mho_x$ by differentiating (2.18) with respect to x,

$$\begin{aligned}\mho_{xt} + \mathbf{A}\mho_{xx} &= -\begin{pmatrix} O(\mathbf{v}^{(1)}_{xx}) \\ \boldsymbol{\nu}_x - \mathbf{a}(\mathbf{u}^{(0)}_x + \boldsymbol{\mu}_x) + \widetilde{\mathbf{a}}\mathbf{u}^{(0)}_x + O(\mathbf{v}^{(1)}_{xx}) \end{pmatrix} \\ &= -\begin{pmatrix} O(\mathbf{v}^{(1)}_{xx}) \\ \boldsymbol{\nu}_x - \widetilde{\mathbf{a}}\boldsymbol{\mu}_x + O(|\mathbf{v}^{(1)}||\mho| + |\mho||\mho_x| + |\mathbf{v}^{(1)}_{xx}|) \end{pmatrix},\end{aligned} \tag{4.19}$$

where $\widetilde{\mathbf{a}} = \mathbf{f}'(\widetilde{\mathbf{u}})$ and $\mathbf{a} = \mathbf{f}'(\mathbf{u})$. Observe that

$$\begin{pmatrix} \alpha^2\widetilde{\phi}'' & -\widetilde{\phi}''\widetilde{\mathbf{a}} \\ -\widetilde{\phi}''\widetilde{\mathbf{a}} & \widetilde{\phi}'' \end{pmatrix}$$

is a symmetrizer for $\mathbf{A}$, we therefore multiply (4.19) from the left by

$$\mho_x^T \begin{pmatrix} \alpha^2\widetilde{\phi}'' & -\widetilde{\phi}''\widetilde{\mathbf{a}} \\ -\widetilde{\phi}''\widetilde{\mathbf{a}} & \widetilde{\phi}'' \end{pmatrix}, \quad \widetilde{\phi}'' = \phi''(\widetilde{\mathbf{u}}), \tag{4.20}$$

to get

$$\begin{aligned}&\widetilde{\mathcal{E}}(\mho_x,\mho_x)_t + \widetilde{\mathcal{J}}(\mho_x,\mho_x)_x \\ &= -O(1)|\boldsymbol{\nu}_x - \widetilde{\mathbf{a}}\boldsymbol{\mu}_x|^2 + O(1)\Big(|\alpha^2\boldsymbol{\mu}_x - \widetilde{\mathbf{a}}\boldsymbol{\nu}_x||\mathbf{v}^{(1)}_{xx}| \\ &\quad + |\boldsymbol{\nu}_x - \widetilde{\mathbf{a}}\boldsymbol{\mu}_x|\big(|\mathbf{v}^{(1)}||\mho| + |\mho||\mho_x| + |\mathbf{v}^{(1)}_{xx}|\big) + |\mho_x|^2|\mathbf{v}^{(1)}|\Big)\end{aligned} \tag{4.21}$$

where

$$\begin{aligned}\widetilde{\mathcal{E}}(\mho_x,\mho_x) &= \mho_x^T \begin{pmatrix} \alpha^2\widetilde{\phi}'' & -\widetilde{\phi}''\widetilde{\mathbf{a}} \\ -\widetilde{\phi}''\widetilde{\mathbf{a}} & \widetilde{\phi}'' \end{pmatrix}\mho_x, \\ \widetilde{\mathcal{J}}(\mho_x,\mho_x) &= \mho_x^T \begin{pmatrix} -\alpha^2\widetilde{\phi}''\widetilde{\mathbf{a}} & \alpha^2\widetilde{\phi}'' \\ \alpha^2\widetilde{\phi}'' & -\widetilde{\phi}''\widetilde{\mathbf{a}} \end{pmatrix}\mho_x.\end{aligned}$$

Here we have used the fact that $\widetilde{\phi}''\widetilde{\mathbf{a}}$ is a symmetric matrix and

$$\frac{\partial}{\partial x}\begin{pmatrix} \alpha^2\widetilde{\phi}'' & -\widetilde{\phi}''\widetilde{\mathbf{a}} \\ -\widetilde{\phi}''\widetilde{\mathbf{a}} & \widetilde{\phi}'' \end{pmatrix} = O(\mathbf{v}^{(1)}).$$

The symmetrizer

$$\begin{pmatrix} \alpha^2\widetilde{\phi}'' & -\widetilde{\phi}''\widetilde{\mathbf{a}} \\ -\widetilde{\phi}''\widetilde{\mathbf{a}} & \widetilde{\phi}'' \end{pmatrix}$$

is positive definite, since

$$\begin{aligned}&(\mathbf{x}^T,\mathbf{y}^T)\begin{pmatrix} \alpha^2\widetilde{\phi}'' & -\widetilde{\phi}''\widetilde{\mathbf{a}} \\ -\widetilde{\phi}''\widetilde{\mathbf{a}} & \widetilde{\phi}'' \end{pmatrix}\begin{pmatrix}\mathbf{x}\\ \mathbf{y}\end{pmatrix} \\ &= (\alpha\mathbf{x} - \mathbf{y})^T\widetilde{\phi}''(\alpha\mathbf{x} - \mathbf{y}) + \mathbf{x}^T(\alpha^2\widetilde{\phi} - \widetilde{\mathbf{a}}^T\widetilde{\phi}''\widetilde{\mathbf{a}})\mathbf{x}.\end{aligned}$$

We can similarly show that $\alpha^2\widetilde{\phi}'' - \widetilde{\mathbf{a}}^T\widetilde{\phi}''\widetilde{\mathbf{a}} = \widetilde{\phi}''(\alpha^2 - \widetilde{\mathbf{a}}^2)$ is positive definite under the sub-characteristic condition (1.5) using the same argument following (3.12). Thus

$$c|\mho_x|^2 \le \widetilde{\mathcal{E}}(\mho_x, \mho_x) \le C|\mho_x|^2$$

We then integrate (4.21) over $\mathbb{R} \times [0,t]$ to get, for E sufficiently small and any $m > 0$, there exists a $C > 0$ such that

$$\begin{aligned} \|\mho_x(t)\|^2 &+ \int_0^t \|\boldsymbol{\nu}_x - \widetilde{\mathbf{a}}\boldsymbol{\mu}_x\|^2 d\tau \\ \le& \|\mho_x(0)\|^2 + \int_0^t \int m|\alpha^2\boldsymbol{\mu}_x - \widetilde{\mathbf{a}}\boldsymbol{\nu}_x|^2 \\ &+ C\left(|\mathbf{v}^{(1)}|^2|\mho|^2 + |\mathbf{v}^{(1)}_{xx}|^2 + |\mho_x|^2|\mathbf{v}^{(1)}|\right) dx d\tau + C\beta \end{aligned} \tag{4.22}$$

where we have used (4.4).

We now proceed to estimate $\alpha^2\boldsymbol{\mu}_x - \widetilde{\mathbf{a}}\boldsymbol{\nu}_x$. First, let us rewrite (2.18) as

$$\begin{aligned} \boldsymbol{\mu}_t + \boldsymbol{\nu}_x &= -\mathbf{v}^{(1)}_x \\ \boldsymbol{\nu}_t + \alpha^2\boldsymbol{\mu}_x &= -(\boldsymbol{\nu} - \bar{\mathbf{a}}\boldsymbol{\mu}) + O(|\boldsymbol{\mu}|^3) + O(|\mathbf{v}^{(1)}_x|). \end{aligned} \tag{4.23}$$

We now multiply the first equation of (4.23) by $(-\widetilde{\mathbf{a}})$ from the left and add it to the second:

$$(\boldsymbol{\nu}_t - \widetilde{\mathbf{a}}\boldsymbol{\mu}_t) + (\alpha^2\boldsymbol{\mu}_x - \widetilde{\mathbf{a}}\boldsymbol{\nu}_x) = -(\boldsymbol{\nu} - \bar{\mathbf{a}}\boldsymbol{\mu}) + O(|\boldsymbol{\mu}|^3) + O(|\mathbf{v}^{(1)}_x|), \tag{4.24}$$

so

$$\begin{aligned} |\alpha^2\boldsymbol{\mu}_x - \widetilde{\mathbf{a}}\boldsymbol{\nu}_x|^2 =& -(\boldsymbol{\nu}_t - \widetilde{\mathbf{a}}\boldsymbol{\mu}_t)^T(\alpha^2\boldsymbol{\mu}_x - \widetilde{\mathbf{a}}\boldsymbol{\nu}_x) \\ & - \left(\boldsymbol{\nu} - \bar{\mathbf{a}}\boldsymbol{\mu} + O(|\boldsymbol{\mu}|^3) + O(\mathbf{v}^{(1)}_x)\right)^T(\alpha^2\boldsymbol{\mu}_x - \widetilde{\mathbf{a}}\boldsymbol{\nu}_x). \end{aligned} \tag{4.25}$$

The second term on the right hand side of (4.25) is bounded by

$$\begin{aligned} -&\left(\boldsymbol{\nu} - \bar{\mathbf{a}}\boldsymbol{\mu} + O(|\boldsymbol{\mu}|^3) + O(|\mathbf{v}^{(1)}_x|)\right)^T(\alpha^2\boldsymbol{\mu}_x - \widetilde{\mathbf{a}}\boldsymbol{\nu}_x) \\ &\le \frac{1}{4}|\alpha^2\boldsymbol{\mu}_x - \widetilde{\mathbf{a}}\boldsymbol{\nu}_x|^2 + C\left(|\boldsymbol{\nu} - \bar{\mathbf{a}}\boldsymbol{\mu}|^2 + |\boldsymbol{\mu}|^6 + |\mathbf{v}^{(1)}_x|^2\right) \end{aligned} \tag{4.26}$$

while the first term is bounded by

$$\begin{aligned} &\left((\boldsymbol{\nu} - \widetilde{\mathbf{a}}\boldsymbol{\mu})_t + O(|\mathbf{v}^{(1)}||\mho|)\right)^T\left((\alpha^2\boldsymbol{\mu} - \widetilde{\mathbf{a}}\boldsymbol{\nu})_x + O(|\mathbf{v}^{(1)}||\mho|)\right) \\ &\quad\le (\boldsymbol{\nu} - \widetilde{\mathbf{a}}\boldsymbol{\mu})_t^T(\alpha^2\boldsymbol{\mu} - \widetilde{\mathbf{a}}\boldsymbol{\nu})_x \\ &\qquad + C|\mathbf{v}^{(1)}|\,|\mho|\Big(|\boldsymbol{\nu}_t - \widetilde{\mathbf{a}}\boldsymbol{\mu}_t| + |\alpha^2\boldsymbol{\mu}_x - \widetilde{\mathbf{a}}\boldsymbol{\nu}_x| + O(|\mathbf{v}^{(1)}||\mho|)\Big) \end{aligned} \tag{4.27}$$

Substituting (4.24) into the second term on the right hand side of (4.27), we can estimate it by

$$\begin{aligned} &C|\mathbf{v}^{(1)}|\,|\mho|\left(|\boldsymbol{\nu}_t-\widetilde{\mathbf{a}}\boldsymbol{\mu}_t|+|\alpha^2\boldsymbol{\mu}_x-\widetilde{\mathbf{a}}\boldsymbol{\nu}_x|+|\mathbf{v}^{(1)}||\mho|\right)\\ \le&C|\mathbf{v}^{(1)}|\,|\mho|\left(2|\alpha^2\boldsymbol{\mu}_x-\widetilde{\mathbf{a}}\boldsymbol{\nu}_x|+|\boldsymbol{\nu}-\bar{\mathbf{a}}\boldsymbol{\mu}|+|\boldsymbol{\mu}|^3+|\mathbf{v}^{(1)}_x|+|\mathbf{v}^{(1)}||\mho|\right)\\ \le&\frac{1}{4}|\alpha^2\boldsymbol{\mu}_x-\widetilde{\mathbf{a}}\boldsymbol{\nu}_x|^2+C\left(|\mathbf{v}^{(1)}|^2|\mho|^2+|\boldsymbol{\nu}-\bar{\mathbf{a}}\boldsymbol{\mu}|^2+|\boldsymbol{\mu}|^6+|\mathbf{v}^{(1)}_x|^2\right) \end{aligned} \tag{4.28}$$

while

$$\begin{aligned} (\boldsymbol{\nu}-\widetilde{\mathbf{a}}\boldsymbol{\mu})_t^T(\alpha^2\boldsymbol{\mu}-\widetilde{\mathbf{a}}\boldsymbol{\nu})_x=&\{(\boldsymbol{\nu}-\widetilde{\mathbf{a}}\boldsymbol{\mu})^T(\alpha^2\boldsymbol{\mu}-\widetilde{\mathbf{a}}\boldsymbol{\nu})_x\}_t-\{(\boldsymbol{\nu}-\widetilde{\mathbf{a}}\boldsymbol{\mu})^T(\alpha^2\boldsymbol{\mu}-\widetilde{\mathbf{a}}\boldsymbol{\nu})_t\}_x\\ &+(\boldsymbol{\nu}-\widetilde{\mathbf{a}}\boldsymbol{\mu})_x^T(\alpha^2\boldsymbol{\mu}-\widetilde{\mathbf{a}}\boldsymbol{\nu})_t. \end{aligned}$$

The last term on the right hand side of the above equation can be treated similarly as (4.28):

$$(\boldsymbol{\nu}-\widetilde{\mathbf{a}}\boldsymbol{\mu})_x^T(\alpha^2\boldsymbol{\mu}-\widetilde{\mathbf{a}}\boldsymbol{\nu})_t\le C(|\boldsymbol{\nu}_x-\widetilde{\mathbf{a}}\boldsymbol{\mu}_x|^2+|\mathbf{v}^{(1)}|^2|\mho|^2). \tag{4.29}$$

After integrating (4.25) over $\mathbb{R}\times(0,t)$ and applying (4.25-4.29) and (4.4), we have

$$\begin{aligned} \int_0^t\|\alpha^2\boldsymbol{\mu}_x-\widetilde{\mathbf{a}}\boldsymbol{\nu}_x\|^2d\tau\le&C\Big\{\int_0^t\|\boldsymbol{\nu}-\bar{\mathbf{a}}\boldsymbol{\mu}\|^2d\tau+\int_0^t\|\boldsymbol{\nu}_x-\widetilde{\mathbf{a}}\boldsymbol{\mu}_x\|^2d\tau\\ &+\int_0^t\fint\left(|\mathbf{v}^{(1)}|^2|\mho|^2+|\mho|^6+|\mathbf{v}^{(1)}_x|^2\right)dxd\tau\Big\}+C\beta \end{aligned} \tag{4.30}$$

Since

$$c_1(|\boldsymbol{\nu}_x|^2+|\boldsymbol{\mu}_x|^2)\le|\alpha^2\boldsymbol{\mu}_x-\widetilde{\mathbf{a}}\boldsymbol{\nu}_x|^2+|\boldsymbol{\nu}_x-\widetilde{\mathbf{a}}\boldsymbol{\mu}_x|^2\le c_2(|\boldsymbol{\nu}_x|^2+|\boldsymbol{\mu}_x|^2), \tag{4.31}$$

we conclude with (4.18) from (4.22) and (4.29). □

Combining Lemma 4.3 and Lemma 4.4 with suitably chosen m, β and E, for $0\le t\le T$, we have

$$\begin{aligned} \|\mho(t)\|_1^2+\int_0^t\|\mho_x\|^2d\tau\le&\|\mho(0)\|_1^2+C_1\int_0^t\fint\Big(|\mathbf{v}^{(1)}|^4+|\mathbf{v}^{(1)}_x|^2+|\mho||\mathbf{v}^{(1)}_x|\\ &+|\mho|^6+|\mathbf{v}^{(1)}_{xx}|^2+|\mho_x|^2|\mathbf{v}^{(1)}|+|\mho|^2|\mathbf{v}^{(1)}|\Big)dxd\tau+C\beta \end{aligned} \tag{4.32}$$

Finally, we can estimate the right hand side of (4.32).

Lemma 4.5 *There exist a positive constant C such that for $E<\epsilon_2$ and $0\le t\le T$, the right hand side of (4.32)is bounded by*

$$C\Big(\|\mho(0)\|_1^2+\beta^{1/6}+(E+\beta)\int_0^t\|\mho_x\|^2d\tau\Big).$$

Proof. We estimate each term by virtue of Lemma 2.2 and the Sobolev inequality (2.21) as follows:

$$\int_0^t \fint |\mathbf{v}^{(1)}|^4 dx d\tau \leq C \int_0^t \|\mathbf{v}^{(1)}\|_{L^\infty}^2 \|\mathbf{v}^{(1)}\|^2 d\tau \leq C\beta$$

$$\int_0^t \fint |\mathbf{v}_x^{(1)}|^2 + |\mathbf{v}_{xx}^{(1)}|^2 dx d\tau \leq C\beta^{1/4}.$$

$$\begin{aligned}
\int_0^t \fint |\mho|\, |\mathbf{v}_x^{(1)}| dx d\tau \leq & C \int_0^t (\|\mho\|^{1/2} \|\mho_x\|^{1/2} + \beta e^{-C\tau}) \|\mathbf{v}_x^{(1)}\|_{L^1} d\tau \\
\leq & C \int_0^t (E^2 \|\mho_x\|^2 + \|\mathbf{v}_x^{(1)}\|_{L^1}^{4/3}) d\tau + C\beta \\
\leq & C \int_0^t (E^2 \|\mho_x\|^2 + \beta^{1/6} (1+\tau)^{\frac{-7}{6}}) d\tau + C\beta \\
\leq & CE \int_0^t \|\mho_x\|^2 d\tau + C\beta^{1/6}
\end{aligned}$$

$$\int_0^t \fint |\mho|^6 dx d\tau \leq \int_0^t \|\mho\|^4 \|\mho_x\|^2 d\tau + C\beta \leq E \int_0^t \|\mho_x\|^2 d\tau + C\beta$$

$$\int_0^t \fint |\mho_x|^2 |\mathbf{v}^{(1)}| dx d\tau \leq \beta \int_0^t \|\mho_x\|^2 d\tau$$

$$\begin{aligned}
\int_0^t \fint |\mho|^2 |\mathbf{v}^{(1)}| dx d\tau \leq & C \int_0^t \|\mho\|^{3/2} \|\mho_x\|^{1/2} \|\mathbf{v}^{(1)}\|_{L^\infty} d\tau + C\beta \\
\leq & C \int_0^t (E \|\mho_x\|^2 + \|\mathbf{v}^{(1)}\|_{L^\infty}^{4/3}) d\tau + C\beta \\
\leq & CE \int_0^t \|\mho_x\|^2 d\tau + C\beta^{1/6}
\end{aligned}$$

Which completes the proof.

References

[1] F. Bouchut *Construction of BGK models with a family of kinetic entropies for a given system of conservation laws,* J. Statist. Phys. **95** (1999) 113–170.

[2] G.-Q Chen, C. Levermore and T.-P. Liu *Hyperbolic conservation laws with stiff relaxation terms and entropy,* Comm. Pure Appl. Math., **47** (1994) 787-830.

[3] L. Gosse and A. E. Tzavaras *Convergence of relaxation schemes to the equations of elastodynamics,* Math. Comp. **70** (2001) 555–577.

[4] J. Goodman and Z. Xin, *Viscous limits for piecewise smooth solutions to systems of conservation laws,* Arch. Rat. Mech. and Anal. **121** (1992) 235–265.

[5] A. Matsumura and K. Nishihara *Asymptotics toward the rarefaction waves of the solutions of a one-dimensional model system for compressible viscous gas,* Japan J. Appl. Math, **3** (1983) 1-13.

[6] R. Natalini *Convergence to equilibrium for the relaxation approximations of conservation laws,* Comm. Pure Appl. Math., **49** (1996) 795-823.

[7] S. Jin and Z. Xin *The relaxation schemes for systems of conservation laws in arbitrary space dimensions,* Comm. Pure Appl. Math. **48** (1995) 235–276.

[8] S. Jin and Z. Xin *Numerical passage from systems of conservation laws to Hamilton-Jacobi equations, relaxation schemes,* SIAM J. Numer. Anal. **35** (1998) 2385–2404.

[9] S. Jin, M. A. Katsoulakis and Z. Xin *Relaxation schemes for curvature-dependent front propagation,* Comm. Pure Appl. Math. **52** (1999) 1587–1615.

[10] M. A. Katsoulakis and A. E. Tzavaras *Contractive relaxation systems and the scalar multidimensional conservation law,* Comm. Partial Differential Equations **22** (1997) 195–233.

[11] T.-P. Liu *Hyperbolic conservation laws with relaxation,* Comm. Math. Phys. **108** (1987) 153–175.

[12] T. Luo *Asymptotic Stability of Planar Rarefaction Waves for the Relaxation Approximation of Conservation Laws in Several Dimensions,* J. Diff. Eq., **133** (1997) 255-279.

[13] T. Luo and Z. Xin *Nonlinear stability of shock fronts for a relaxation system in several space dimensions,* J. Diff. Eq. **139** (1997) 365–408.

[14] J. Rauch and M. Reed *Jump discontinuities of semilinear, strictly hyperbolic systems in two variables: creation and propagation,* "Comm. Math. Phys" **81** (1981) 203-227.

[15] D. Serre *Relaxation semi-lineaire des systemes de lois de conservation,* Ann. Inst. H. Poincar Anal. Non Lineaire **17** (2000), 169–192.

[16] J. Smoller, *Shock Waves and Reaction-Diffusion Equations,* (1983), Springer–Verlag, New York.

[17] E. Tadmor and T. Tang *Pointwise error estimates for relaxation approximations to conservation laws,* SIAM J. Math. Anal. **32** (2000) 870–886.

[18] Z.-H. Teng *First-order L^1-convergence for relaxation approximations to conservation laws,* Comm. Pure Appl. Math. **51** (1998) 857–895.

[19] W.-C. Wang *Asymptotics Towards Rarefaction Wave of the Jin-Xin Relaxation Model for the p Systerm,* Proceedings of the IMS Conference on Differential Equations from Mechanics., Hong Kong, (1999), In press.

[20] W.-C. Wang and Z. Xin *Asymptotic limit of initial-boundary value problems for conservation laws with relaxational extensions,* Comm. Pure Appl. Math. **51** (1998) 505–535.

[21] A. E. Tzavaras *Materials with internal variables and relaxation to conservation laws,* Arch. Ration. Mech. Anal. **146** (1999) 129–155.

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