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Nonlinear Stability of Circular Vortex Patches*

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Abstract. This paper proves that circular vortex patches in the plane are stable for the nonlinear dynamical system generated by the Euler equations of incompressible fluids. This is achieved by establishing a relative variational principle in terms of either energy or angular momentum. Thus, we exploit and extend Arnold's idea in (1965, 1969) to a nonsmooth setting as well.

1. Introduction

This paper proves that circular vortex patches in the plane (the vorticity is one inside a circle and is zero outside it) are stable for the nonlinear dynamical system generated by the two dimensional Euler equations of incompressible hydrodynamics. The stability is of Liapunov type: it is global in time in the L^1 norm on the vorticities. A consequence of the *a priori* estimates used to establish this stability is the following: a nearly circular vortex patch must evolve in such a way that the area of the region of deviation from circularity it uniformly bounded, globally in time. Our vortex patch is enclosed in a (large) circular disk, and the flow is parallel to the boundary of this containing disk.

Our results are stated precisely in Sect. 2. Here we comment on some of the relevant literature and the significance of the results. Kelvin (1880) showed that a small perturbation of the circular vortex patch which is proportional to cos $m\theta$ rotates uniformly with angular velocity $\Omega_m = \frac{1}{2}(m-1)/m$ (see Lamb, 1945, Sect. 158). In particular his result established the linearized stability of the circular patch in the plane.

In Arnold (1965), (1969), a method for proving a nonlinear version of the classical Rayleigh inflection point criterion for linearized stability of two dimensional shear flows is presented. The argument involves a combination of a geometric setting for the fluid variational principle and convexity arguments. The geometric setting has been exploited by a number of authors such as Ebin and Marsden (1970), Benjamin

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(1976), McKee (1981), Marsden and Weinstein (1983) and Turkington (1983). The convexity argument has also been exploited to prove stability for related Hamiltonian systems; see Holm, et al. (1983) and references therein.

Unfortunately, Arnold's method does not, as formulated, apply to the circular vortex patch. The discontinuity in the vorticity forces one to work in function spaces for which the differential calculus ideas in Arnold require careful interpretation. Moreover, the convexity argument in Arnold (1969) does not apply directly either (formally one is in the case of his more delicate second theorem). However, these ideas, together with direct estimates were important to the formulation and execution of the results of this paper.

Some hint of why the stability analysis is relatively delicate can be gained from numerical and theoretical work. On the numerical side, Deem and Zabusky (1978) show that nearly circular vortex patches are numerically stable, but they allow the possibility of long thin filaments moving relatively far from the original circular patch. Our theorem shows that although these filaments could become long and complex, their area remains uniformly small, globally in time. This picture is consistent with the results of Marsden and Weinstein (1983) who show that the (Hamiltonian) evolution equation for nearly circular waves does not have strong enough dispersion to prevent breaking or singularities, but yet the system is formally stable. On the other hand, for vorticities in L^∞, there is a satisfactory global existence theory in two dimensions (Wolibner, 1933, Yudovich, 1963). We shall not explicitly require estimates from this existence theory since our *a priori* estimates are established by other methods.

Because of symmetry, energy and angular momentum fail to be independent at circular patches. In this situation, one can obtain a (relative) variational principle in terms of either energy or angular momentum. *A priori* estimates will be obtained for both the energy function and the angular momentum. However it is much easier to establish *a priori* estimates for the latter. The ideas of using angular momentum were developed from Marchioro and Pulvirenti (1983).

We expect that the methods based on energy estimates will be applicable to other vortex patches. In particular we note that linearized stability of the rotating Kirchhoff elliptical vortex patches has been proved by Love (1893). As was pointed out to us by N. Zabusky, it would be of interest to use the methods herein to prove their nonlinear stability; indeed, computations of Zabusky and his coworkers suggest that the area of the filaments that form in perturbed solutions remain bounded and may even go to zero. The theoretical setting needed to prove this is expected to be similar to the present paper with the addition of Hamiltonian reduction by an S^1 symmetry group which realizes the Kirchhoff solution as a relative equilibrium (a fixed point for a reduced system on a surface of constant angular momentum). To pave a way for such an investigation, we like to carry out an energy estimate in its simplest situation in detail, i.e. for a circular vortex patch. It goes without saying that these energy estimates are intrinsically interesting in themselves.

Finally, we note that the motion of vortex patches are not only of interest in fluid dynamics, but arise as beams in the study of guiding center plasmas; see Morrison (1982, p. 31).

2. Statement of the Main Result

Let $D \subseteq R^2$ be a disk of radius R centered at the origin. Consider the motion of an incompressible inviscid flow with unit density in this fixed domain D, in the absence of external forces. At any instant, the velocity field $\underset{\sim}{u} = (u, v)$ can be described by a stream function $\phi, \underset{\sim}{u} = (\phi_y, -\phi_x)$ and its vorticity $\omega = v_x - u_y$ given by $\omega = -\phi_{xx} - \phi_{yy} = -\Delta\phi$. Given a vorticity ω, denote by $G\omega$ the stream function associated to ω which is defined by $\Delta(G\omega) = -\omega$ and $G\omega|_{\partial D} = 0$ (the restriction of $G\omega$ to the boundary ∂D of D is zero). The vorticity evolves according to the vorticity equation: $\omega_t + u\omega_x + v\omega_y = 0$. Denote by $\Phi_t(\omega)$, $t \geqq 0$, the vorticity at the time t, with initial vorticity ω.

A vortex patch ω is a vorticity distribution in the form $\lambda\chi_A, \lambda_\varepsilon R$, where χ_A denotes the characteristic function of a subset A in D. Here λ and the area of A are called the *strength* and the *size* of this patch respectively. From Yudovich (1963), the evolution operator Φ_t is implemented by area preserving homeomorphisms (of a certain Hölder class); it follows that Φ_t preserves vortex patches together with their strength and sizes. Denote by ω_0 the *circular vortex patch* of radius γ $(0 < \gamma < R)$, with unit strength, i.e. $\omega_0 = \chi_B$, where $B = \{x \| x| \leqq \gamma\}$. By circular invariance, one sees immediately that ω_0 is a *stationary* solution.

Now, we can state the main result of this paper as follows.

The Stability Theorem. *Let ω_0 be the circular vortex patch of unit strength and radius γ in a disk of radius R. For any $\eta > 0$, there is a $\delta > 0$ such that if ω is any vortex patch satisfying $|\omega - \omega_0|_{L^1} < \delta$, then $|\Phi_t(\omega) - \omega_0|_{L^1} < \eta$ for all $t \geqq 0$.*

Additional results on stability relative to vorticity distributions other than patches are given at the end of the paper in Sect. 7.

Translating the L^1 norm into areas, we have the following.

Corollary. *Let B be the circle of radius γ centered at the origin in $\mathbb{R}^2$ and let $\omega_0 = \chi_B$, as above, and $\omega = \lambda\chi_A$. Let $\eta > 0$. Then there is $\delta > 0$ such that if*

(i) $|\lambda - 1| < \delta$
and (ii) area $[(B \backslash A) \cup (A \backslash B)] < \delta$,
then area $[(A_t \backslash B) \cup (B \backslash A_t)] < \eta$ for all $t \geqq 0$, where $\Phi_t(\omega) = \lambda\chi_{A_t}$.

Using integration by parts, we express the total energy E as a function of the vorticity ω as follows:

$$E = \tfrac{1}{2}\int_D |u|^2 dx dy = \tfrac{1}{2}\int_D \omega G\omega \, dx dy = \tfrac{1}{2}\langle \omega, G\omega \rangle.$$

The total energy E is a first integral of our vortex flow (i.e. $E(\phi_t(\omega)) = \text{constant}$).

We note that deviations from the circular vortex patch are also limited by conservation of "angular momentum" due to S^1 invariance, namely of

$$J_\theta = \int_D r^2 \omega(x, y) dx dy.$$

In the geometric setting of Arnold, one needs to show that the total energy E has a "non-degenerate" local maximum on the coadjoint orbit through ω_0. This coadjoint orbit consists of vortex patches having the same strength and size as that of ω_0. In analytic terms, the above idea is implemented through the following *a priori estimate*:

There are constants $c_3 > 0$, $C_3 > 0$ *and* $\varepsilon > 0$ *such that*

$$\text{(E)} \qquad c_3|\omega_c - \hat{\omega}|_{L^1}^2 \leqq E(\omega_c) - E(\hat{\omega}) \leqq C_3|\omega_c - \hat{\omega}|_{L^1},$$

where ω_c *and* $\hat{\omega}$ *are vortex patches of the same size and strength with* ω_c *circular and satisfying* $|\omega_c - \omega_0|_{L^1} < \varepsilon$ and $|\omega - \omega_0|_{L^1} < \varepsilon$.

Alternatively, one can also show that the total angular momentum J has a "non-degenerate" (local) minimum on the coadjoint orbit through ω_0. In analytic terms, one wants to establish the following *a priori* estimate:

There are constants $c_4 > 0$, $C_4 > 0$, *and* $\varepsilon > 0$ *such that*

$$\text{(J)} \qquad c_4|\omega_c - \hat{\omega}|_{L^1}^2 \leqq J(\hat{\omega}) - J(\omega_c) \leqq C_4|\omega_c - \omega|_{L^1},$$

where $\hat{\omega}$ *is a vortex patch of the same size and strength as the circular patch* ω_c, *and satisfying* $|\omega_c - \omega_0|_{L^1} < \varepsilon$.

As we shall see in the later sections, the left-hand inequality (E) is the difficult one; the right-hand inequality (E) will be easily proved. The inequality (J) can be established without any difficulty. The proofs of our stability theorem based on either inequality (E) or inequality (J) are essentially the same. Thus, we present the arguments only for the former one.

Proof of the Stability Theorem Assuming (E). Let ω^* be the circular patch with the same size and strength as ω. Write $\omega = \lambda\omega_1 = \lambda\chi_A$, $\omega^* = \lambda\omega_1^*$, and $\omega_0 = \chi_B$. One may assume $|\omega - \omega_0|_{L^1} < \delta \leqq \frac{1}{2}\,|\omega_0|_{L^1}$, which imply $2|\chi_{B\setminus A}|_{L^1} \geqq |\omega_0|_{L^1}$ and $|\lambda - 1|\,|\omega_0|_{L^1} \leqq 2|\omega - \omega_0|_{L^1}(< 2\delta)$. Since ω_1^* and ω_0 are both supported on concentric disks,

$$|\omega_1^* - \omega_0|_{L^1} = \int_D \omega_1^* - \int_D \omega_0| = |\int_D \omega_1 - \int_D \omega_0| \leqq \int_D |\omega_1 - \omega_0| = |\omega_1 - \omega_0|_{L^1}.$$

Combining the above two inequalities, we have

$$\begin{aligned}|\omega^* - \omega_0|_{L^1} = |\lambda\omega_1^* - \omega_0|_{L^1} &\leqq |\lambda|\,|\omega_1^* - \omega_0|_{L^1} + |\lambda - 1|\,|\omega_0|_{L^1}\\ &\leqq |\lambda|\,|\omega_1 - \omega_0|_{L^1} + |\lambda - 1|\,|\omega_0|_{L^1}\\ &\leqq |\omega - \omega_0|_{L^1} + 2|\lambda - 1|\,|\omega_0|_{L^1} \leqq 5\delta.\end{aligned}$$

By conservation of circulation, $\Phi_t(\omega)$ is the same size and strength as ω and hence as ω^*. Suppose $|\Phi_t(\omega) - \omega_0|_{L^1} < \varepsilon/5$ on some interval $0 \leqq t \leqq t_1$, so by (E), applied to $\omega^* = \omega_c$ and $\hat{\omega} = \Phi_t(\omega)$, we get

$$|\Phi_t(\omega) - \omega^*|_{L^1}^2 \leqq c_3^{-1}[E(\omega^*) - E(\Phi_t(\omega))] = c_3^{-1}[E(\omega^*) - E(\omega)] \leqq C_3c_3^{-1}|\omega^* - \omega|_{L^1}$$

by conservation of energy. Thus, for $0 \leqq t \leqq t_1$,

$$|\Phi_t(\omega) - \omega_0|_{L^1} \leqq |\Phi_t(\omega) - \omega^*|_{L^1} + |\omega^* - \omega|_{L^1} \leqq (C_3c_3^{-1}|\omega^* - \omega|_{L^1})^{1/2} + |\omega^* - \omega_0|_{L^1}.$$

Now $|\omega^* - \omega|_{L^1} \leqq |\omega^* - \omega_0|_{L^1} + |\omega_0 - \omega|_{L^1} \leqq 6\delta$, so if $|\omega_0 - \omega|_{L^1} < \delta$, then

$$|\Phi_t(\omega) - \omega_0|_{L^1} \leqq (C_3c_3^{-1})^{1/2}\sqrt{6\delta} + 5\delta.$$

Now, choose δ small so that

$$\sqrt{6}(C_3c_3^{-1})^{1/2}\sqrt{\delta}+5\delta<\min\left(\eta,\frac{\varepsilon}{5}\right)$$

Thus, if $|\omega-\omega_0|_{L^1}<\delta$, then $|\Phi_t(\omega)-\omega_0|_{L^1}<\varepsilon/5$ for all $t\geqq 0$. Hence, $|\Phi_t(\omega)-\omega_0| \leqq\sqrt{6}(C_3c_3^{-1})^{1/2}\sqrt{\delta}+5\delta<\eta$ for all $t\geqq 0$ ■

The inequality (J) is justified in Sect. 3. So, we obtain a short and elegant proof of our main result. The rest of this paper establishes the inequality (E). In Sect. 4, we reduce the justification of the inequality (E) from a general L^1 small perturbation to a C^1 small radial one (Proposition 1). In Sect. 5, by using potential theory, one has a neat expression for the Taylor expansion of E (Proposition 2). The second order term is shown to be negative definite (Proposition 3). Consequently, we establish the inequality (E), and hence our main stability result follows.

3. The Proof of the Inequality (*J*)

Denote by $|S|$ the measure of a set $S\subset R^2$. First, let us establish the following inequality

$$\text{(J}'\text{)}\qquad \frac{1}{4\pi}|\chi_A-\chi_C|_{L^1}^2\leqq J(\chi_A)-J(\chi_C)\leqq\frac{R^2}{2}|\chi_A-\chi_C|_{L^1},$$

where $C=\{x|\ |x|\leqq r_0\}$ is a disk in D, and $|A|=|C|$. Putting $|A\backslash C|=|C\backslash A| =\frac{1}{2}|\chi_A-\chi_C|_{L^1}=a$,

$$J(\chi_A)-J(\chi_C)=\int_A r^2dxdy-\int_C r^2dxdy=\int_{A\backslash C} r^2dxdy-\int_{C\backslash A} r^2dxdy$$
$$\leqq R^2|A\backslash C|=\frac{R^2}{2}|\chi_A-\chi_C|_{L^1}.$$

On the other hand,

$$J(\chi_A)-J(\chi_C)=\int_{A\backslash C} r^2dxdy-\int_{C\backslash A} r^2dxdy\geqq\int_{\Sigma_1} r^2dxdy-\int_{\Sigma_2} r^2dxdy,$$

where $\Sigma_1=\{x|r_0\leqq|x_1|\leqq r_1\}$, and $\Sigma_2=\{x|r_2\leqq|x|\leqq r_0\}$. The values of r_1, r_2 are determined by the condition $|\Sigma_1|=|\Sigma_2|=a$. Clearly, one has $r_1^2=a/\pi+r_0^2$, $r_2^2= r_0^2-a/\pi$. Therefore,

$$J(\chi_A)-J(\chi_C)\geqq\frac{2\pi}{4}\left[(r_0^4-r_1^4)-(r_2^4-r_0^4)\right]=\frac{a^2}{\pi}=\frac{1}{4\pi}|\chi_A-\chi_C|_{L^1}^2.$$

Here, one uses the simple facts that $r^2=$ constant are circles, and r^2 in an increasing function of the radius r.

Write $\hat{\omega}=\lambda\chi_A$, $\omega_c=\lambda\chi_C$, and take $\varepsilon=|\omega_0|_L$. Thus, $|\omega_c-\omega_0|_{L^1}<\varepsilon$ implies $(0<)$ $\lambda\leqq 2$. Therefore, we have by the inequality (J)′,

$$\frac{1}{4\pi\lambda}|\hat{\omega}-\omega_c|_{L^1}^2\leqq J(\hat{\omega})-J(\omega_c)\leqq\frac{R^2}{2}|\hat{\omega}-\omega_c|_{L^1}.$$

For $1/8\pi\ |\hat{\omega}-\omega_c|_{L^1}^2 \leqq 1/4\pi\lambda\ |\hat{\omega}-\omega_c|_{L^1}^2$, the inequality (J) follows by choosing $c_4 = 1/8\pi$, and $C_4 = R^2/2$.

The ideas of our main stability theorem based on an angular momentum estimate can easily be adapted to recover the case where our circular patch lies in R^2 instead of a bounded disk.

If one is only interested in the stability of circular patches, the proof based on conservation of angular momentum appears to be satisfactory. However, in order to handle the non-linear stability questions in general, it seems natural to establish a (relative) variational principle in terms of an *energy function* subject to certain natural constraints. These natural constraints may come from the symmetry of the system. Conservation of angular momentum can be one of them.

At circular patches, the energy function and the angular momentum are not independent. Indeed, a relative variational principle can be formulated in terms of the energy function or the angular momentum alone. As the simplest model example for establishing an energy estimate, we aim to give its lengthy proof throughout the rest of this paper. This is the price that one may have to pay for important generalizations in return.

4. Reduction to the Radial Case

Recall that $G\omega$ solves the Dirichlet problem:

$$\begin{cases} -\Delta\phi = \omega & \text{in } D \\ \phi = 0 & \text{on } \partial D. \end{cases}$$

For the circular patch ω_0 of radius γ, with $r = \sqrt{x^2+y^2}$,

$$4G\omega_0 = \begin{cases} \gamma^2 \ln\dfrac{R^2}{r^2}, & r \geqq \gamma, \\[2ex] (\gamma^2 - r^2) + \gamma^2 \ln\dfrac{R^2}{\gamma^2}, & r < \gamma. \end{cases}$$

Thus

$$G\omega_0 = \tfrac{1}{4}\gamma^2 \ln\frac{R^2}{\gamma^2}, \text{and} \frac{\partial(G\omega_0)}{\partial r} = -\frac{1}{2}\gamma \quad \text{on} \quad r = \gamma.$$

As we shall see in the proof, it will be enough to prove (E) for the special case $\omega_c = \omega_0$ and $\hat{\omega} = \omega_1$, where ω_1 is a vortex patch of unit strength, $\int_D \omega_1 = \int_D \omega_0$, and ω_1 is L^1-close to ω_0. From the Calderon-Zygmund (1952) L^p-estimates for the Dirichlet problem (see also Morrey, 1966) and the Sobolev inequality $W^{2,P} \subset C^1$ if $p > 2$, one knows that $\zeta = G\omega_1$ is C^1-close to $G\omega_0$. Thus, $G\omega_1$, $(G\omega_1)_x$, $(G\omega_1)_y$ are uniformly close to $G\omega_0$, $(G\omega_0)_x$, $(G\omega_0)_y$ in D. Fix a number a, $0 < a < \gamma$, and set $\zeta^* = G\omega_0|_{|x|=a/2}$. For $|\omega_1 - \omega_0|_{L^1}$ sufficiently small, (ζ, θ) defines a C^1-coordinate system with $\partial\zeta/\partial r < 0$ on the annulus-like region $f^*(\theta) \leqq r \leqq R$, with $f^*(\theta) \leqq a$. Here, (r, θ) denotes polar coordinates and $r = f^*(\theta)$ describes the closed curve $\zeta = \zeta^*$. Note that $\zeta = 0$ when $r = R$. Hence, there exists a unique vortex patch of unit strength $\bar{\omega}_1$ which is the characteristic function of the set $\{(x,y)\in D|G\omega_1(x,y)\geqq\zeta_0\}$, for some constant ζ_0 close to $G\omega_0|_{r=\gamma}$ such that $\int_D\bar{\omega}_1=\int_D\omega_1$. Note that $\bar{\omega}_1$ is a vortex patch whose boundary is a stream-line for ω_1; it plays a key role in "completing the square" in Proposition 1 below. Furthermore, the boundary $G\omega_1=\zeta_0$ of the vortex patch $\bar{\omega}_1$ can be represented as $r=f(\theta)$. Indeed, $f(\theta)\to\gamma$ (the function $r(\theta)=\gamma$) in C^1-norm as $\omega_1\to\omega_0$ in L^1-norm.

Let us first show that

$$|E(\omega_0)-E(\omega_1)|\leqq C_3|\omega_0-\omega_1|_{L^1},$$

which will prove the right-hand inequality of (E), assuming the left-hand one. Indeed,

$$\begin{aligned}E(\omega_0)-E(\omega_1)&=\tfrac{1}{2}\langle\omega_0,G\omega_0\rangle-\tfrac{1}{2}\langle\omega_1,G\omega_1\rangle\\&=\tfrac{1}{2}\langle\omega_0-\omega_1,G\omega_1\rangle-\tfrac{1}{2}\langle\omega_0,G\omega_1-G\omega_0\rangle.\end{aligned}$$

Since $G\omega_1$ is C^1 close to $G\omega_0$ and G is bounded in L^1,

$$\begin{aligned}|E(\omega_0)-E(\omega_1)|&\leqq\tfrac{1}{2}\sup(G\omega_1)|\omega_0-\omega_1|_{L^1}+\tfrac{1}{2}(\sup\omega_0)|G\omega_1-G\omega_0|_{L^1}\\&\leqq C_3|\omega_1-\omega_0|_{L^1}.\end{aligned}$$

Now we turn our attention to the left-hand inequality in (E) and shall reduce its proof to the radial case.

The next two lemmas will be needed in carrying out the reduction.

Lemma 1. *Let* ζ_1, ζ_2 *be two constants,* $0<\zeta_2<\zeta_0<\zeta_1<\zeta^*$, *such that* area $\{(x,y)|\zeta_0\geqq\zeta(x,y)\geqq\zeta_2\}=$ area $\{(x,y)|\ \zeta_1\geqq\zeta(x,y)\geqq\zeta_0\}$. *Then,*

$$\begin{aligned}\int\limits_{\zeta_0\geqq\zeta\geqq\zeta_2}\zeta\,dx\,dy-\int\limits_{\zeta_1\geqq\zeta\geqq\zeta_0}\delta\,dx\,dy&=\frac{1}{A}\text{area}^2\{(x,y)|\zeta_0\geqq\zeta(x,y)\geqq\zeta_2\}\\&\quad+o(\text{area}^2\{(x,y)|\zeta_0\geqq\zeta(x,y)\geqq\zeta_2\}),\end{aligned}$$

for some constant A depending on ω_1, *with* $1/A\to-1/4\pi$ *as* $|\omega_1-\omega_0|_{L^1}\to 0$.

Proof. Set $A=\int_0^{2\pi}r(\zeta_0,\theta)\ \partial r/\partial\zeta(\zeta_0,\theta)\ d\theta$.

Express the closed curves $\zeta=\zeta_2$, $\zeta=\zeta_1$ in polar coordinates as $r=f_2(\theta)$, and $r=f_1(\theta)$.

(i) area $\{x|\zeta_0\geqq\zeta(x)\geqq\zeta_2\}=\int_0^{2\pi}\int_{f(\theta)}^{f_2(\theta)}r\,dr\,d\theta=\int_0^{2\pi}\int_{\zeta_0}^{\zeta_2}r\frac{\partial r}{\partial\zeta}d\zeta\,d\theta$

$$=\int_{\zeta_0}^{\zeta_2}\int_0^{2\pi}r\frac{\partial r}{\partial\zeta}d\theta\,d\zeta=A(\zeta_2-\zeta_0)+o(|\zeta_2-\zeta_0|).$$

Similarly, area $\{\zeta_1\geqq\zeta\geqq\zeta_0\}=A(\zeta_0-\zeta_1)+o(|\zeta_1-\zeta_0|)$.

(ii) $$\int_0^{2\pi}\int_{\zeta_0}^{\zeta_2} \zeta r\frac{\partial r}{\partial \zeta}d\zeta\, d\theta = \zeta_0 \int_0^{2\pi}\int_{\zeta_0}^{\zeta_2} r\frac{\partial r}{\partial \zeta}d\zeta\, d\theta + \int_0^{2\pi}\int_{\zeta_0}^{\zeta_2} (\zeta-\zeta_0) r\frac{\partial r}{\partial \zeta}d\zeta\, d\theta$$
$$= \zeta_0 \,\text{area}\{\zeta_0 \geqq \zeta \geqq \zeta_2\} + \tfrac{1}{2}A(\zeta_2-\zeta_0)^2 + o(|\zeta_2-\zeta_0|^2).$$

Similarly,

$$\int_0^{2\pi}\int_{\zeta_1}^{\zeta_0} \zeta r\frac{\partial r}{\partial \zeta}d\zeta\, d\theta = \zeta_0 \,\text{area}\{\zeta_1 \geqq \zeta \geqq \zeta_0\} - \tfrac{1}{2}A(\zeta_1-\zeta_0)^2 + o(|\zeta_1-\zeta_0|^2).$$

(iii) Since $(\partial r/\partial \zeta)(\zeta_0,\theta) = ((\partial\zeta/\partial r)(\gamma,\theta))^{-1} \to -2/\gamma$ as $|\omega_1-\omega_0|_{L^1}\to 0$, we have

$$A = \int_0^{2\pi} r(\zeta_0,\theta)\frac{\partial r}{\partial \zeta}(\zeta_0,\theta)\, d\theta \to -4\pi \qquad \text{as } |\omega_1-\omega_0|_{L^1}\to 0.$$

Hence, Lemma 1 follows from (i), (ii) and (iii). ■

Lemma 2. *Let c_1 be a constant such that $0 < c_1 < 1/16\pi$. Then $\langle \bar\omega_1 - \omega_1, G\omega_1\rangle \geqq c_1|\omega_1-\bar\omega_1|^2_{L^1}$ if $|\omega_1-\omega_0|_{L^1}$ is sufficiently small.*

Proof. Write $\bar\omega_1 - \omega_1 = \chi_V - \chi_U$ as a difference of characteristic functions. For $|\omega_1-\omega_0|_{L^1}$ sufficiently small, one can choose $\zeta_2 < \zeta_0$, and $\zeta_1 > \zeta_0$ such that $\text{area}\{\zeta_0 \geqq \zeta \geqq \zeta_2\} = \text{area}\, U = \text{area}\{\zeta_1 \geqq \zeta \geqq \zeta_0\} = \text{area}\, V$, and so the hypotheses of Lemma 1 are fulfilled. It is not hard to see from $\partial\zeta/\partial r < 0$ that

$$\langle \chi_U, G\omega_1\rangle \leqq \int_{\zeta_0 \geqq \zeta \geqq \zeta_2} \zeta\, dx\, dy,$$
$$\langle \chi_V, G\omega_1\rangle \geqq \int_{\zeta_1 \geqq \zeta \geqq \zeta_0} \zeta\, dx\, dy.$$

The desired inequality now follows from Lemma 1. ■

Now, we present the reduction proposition:

Proposition 1. *Suppose $E(\omega_0) - E(\bar\omega_1) \geqq c_2|\bar\omega_1-\omega_0|^2_{L^1}$ for some constant $c_2 > 0$, when $f(\theta)$ is C^1-close to γ. Then, $E(\omega_0)-E(\omega_1) \geqq c_3|\omega_1-\omega_0|^2_{L^1}$ with $c_3 < \frac{1}{2}\min(c_1, c_2)$ when $|\omega_1-\omega_0|_{L^1}$ is sufficiently small.*

Proof. $$E(\omega_0)-E(\omega_1) = E(\omega_0)-E(\bar\omega_1)+E(\bar\omega_1)-E(\omega_1)$$
$$= E(\omega_0)-E(\bar\omega_1) + \left\langle \bar\omega_1-\omega_1, G\left(\frac{\omega_1+\bar\omega_1}{2}\right)\right\rangle$$
$$= E(\omega_0)-E(\bar\omega_1) + \langle \bar\omega_1-\omega_1, G\omega_1\rangle + \tfrac{1}{2}\langle \bar\omega_1-\omega_1, G(\bar\omega_1-\omega_1)\rangle$$
$$\geqq E(\omega_0)-E(\bar\omega_1) + \langle \bar\omega_1-\omega_1, G\omega_1\rangle \text{(since } \langle \bar\omega_1 - \omega_1, G(\bar\omega_1-\omega_1)\rangle \geqq 0)$$
$$\geqq E(\omega_0)-E(\bar\omega_1) + c_1|\bar\omega_1-\omega_1|^2_{L^1} \quad \text{(by Lemma 2)}$$
$$\geqq c_1|\bar\omega_1-\omega_1|^2_{L^1} + c_2|\bar\omega_1-\omega_0|^2_{L^1} \quad \text{(by hypothesis)}$$
$$\geqq 2c_3(|\bar\omega_1-\omega_1|^2_{L^1} + |\bar\omega_1-\omega_0|^2_{L^1})$$

$$\geqq c_3(|\bar{\omega}_1 - \omega_1|_{L^1} + |\bar{\omega}_1 - \omega_0|_{L^1})^2$$
$$\geqq c_3|\omega_1 - \omega_0|_{L^1}^2.$$

This completes the proof of Proposition 1. ■

Consequently, by Proposition 1, one needs only to verify the inequality (E) for radial perturbations $\chi_{\{r \leqq f(\theta)\}} = \chi_f$, $\int_D \chi_f \, dx\, dy = \int_D \omega_0$, where $f(\theta)$ is C^1-close to γ.

5. A Second Order Taylor Expansion of $E(\chi_g)$

Set $\mathscr{R} = \{f: S^1 \to \mathbb{R} \mid f \in C^1,\ 0 < a < f(\theta) < R\}$, where $S^1 = IR \bmod 2\pi$, and a is a small fixed number, $0 < a < \gamma$. To each $f \in \mathscr{R}$, denote by χ_f the characteristic function of the set $\{r \leqq f(\theta)\}$. Thus, χ_f is a vortex patch of unit strength, with boundary given by a radial function $r = f(\theta)$. For convenience, in what follows we shall use the complex notation: $z = x + iy = re^{i\theta}$.

Denote by K the Green's function for the Dirichlet problem (cf. Morrey, 1966) $\begin{cases} -\Delta\psi = \omega \text{ in } D \\ \psi = 0 \text{ on } \partial D \end{cases}$ on the disk of radius R. Thus, $\psi(\xi) = \int_D K(z,\xi)\omega(z)\,dx\,dy$ with $K(z,\xi) = 1/2\pi[\ln|\xi - z'|/|\xi - z| + \ln|z|/R]$, $z,\ \xi \in \mathbb{C}$, $|z|,\ |\xi| \leqq R$, where $z' = (R^2/|z|^2)z$, the reflection point of z with respect to the circle ∂D of radius R. It is not hard to see that $1 \leqq (|\xi - z'|/|\xi - z|)\cdot(|z|/R) \leqq (|\xi| + |z'|)|z|/|\xi - z|R \leqq 2R/|\xi - z|$. Let $|\theta - \theta'| < \pi/2$, $r' \geqq a$; thus,

$$
\begin{aligned}
|K(re^{i\theta}, r'e^{i\theta'})| &\leqq \frac{1}{2\pi}\ln\frac{2R}{|re^{i\theta} - r'e^{i\theta'}|} \\
&\leqq \frac{1}{2\pi}\ln\frac{2R}{|r'\sin(\theta - \theta')|} \\
&\leqq \frac{1}{2\pi}\ln\frac{1}{|\sin(\theta - \theta')|} + M_1 = \bar{K}
\end{aligned}
\tag{1}
$$

for some constant $M_1 > 0$ (indeed, $M_1 = (1/2\pi)\ln(2R/a)$).

For a continuous function $h: S^1 \to \mathbb{R}$, set $|h|_1 = \int_0^{2\pi} |h(\theta)|\,d\theta$, $|h|_2 = \left(\int_0^{2\pi} |h(\theta)|^2\,d\theta\right)^{1/2}$, and $|h|_\infty = \max\{|h(\theta)| \mid \theta \in S^1\}$. For $h \in C^1$, $|h|_\infty \leqq 1$, and $|h'|_\infty \leqq 1$, we have the Sobolev type inequality

$$|h|_2 \geqq \tfrac{2}{3}|h|_\infty^3. \tag{2}$$

Therefore, $|h|_\infty$ is small if $|h|_2$ is small (and $|h|_\infty$, $|h'|_\infty \leqq 1$).

Now, let g, $g + h \in \mathscr{R}$, and write

$$
\begin{aligned}
E(\chi_{g+h}) - E(\chi_g) &= \tfrac{1}{2}\langle \chi_{g+h}, G\chi_{g+h}\rangle - \tfrac{1}{2}\langle \chi_g, G\chi_g\rangle \\
&= \langle \chi_{g+h} - \chi_g, G\chi_g\rangle + \tfrac{1}{2}\langle \chi_{g+h} - \chi_g, G\chi_{g+h} - G\chi_g\rangle,
\end{aligned}
$$

$$\langle \chi_{g+h} - \chi_g, G\chi_g\rangle = \int_0^{2\pi}\int_{g(\theta)}^{g(\theta)+h(\theta)} G(\chi_g)(re^{i\theta})r\,dr\,d\theta. \tag{3}$$

Fix θ, and expand in r at $g(\theta)e^{i\theta}$, noting that $G(\chi_g)\in C^1$,

$$G(\chi_g)(re^{i\theta}) = G(\chi_g)(g(\theta)e^{i\theta}) + \frac{\partial G(\chi_q)}{\partial r}(g(\theta)e^{i\theta})(r - g(\theta)) + o(|r - g(\theta)|),$$

$$rG(\chi_g)(re^{i\theta}) = rG(\chi_g)(g(\theta)e^{i\theta}) + r\frac{\partial G}{\partial r}(g(\theta)e^{i\theta})(r - g(\theta)) + o(|h(\theta)|)$$

$$= rG(\chi_g) + g\frac{\partial G}{\partial r}(r - g) + o(|h(\theta)|).$$

Hence at $g(\theta)e^{i\theta}$ we have

$$\langle \chi_{g+h} - \chi_g, G\chi_g \rangle = \int_0^{2\pi} G(\chi_g)gh\,d\theta + \frac{1}{2}\int_0^{2\pi}\left[G(\chi_g)h^2 + \left(g\frac{\partial G}{\partial r}\right)h^2\right]d\theta + o(|h|_2^2). \quad (4)$$

Lemma 3. *For g, $g + h\in\mathscr{R}$, $|h|_\infty$, $|h'|_\infty \leqq 1$, we have*

$$[G(\chi_{g+h}) - G(\chi_g)](r'e^{i\theta'}) = \int_0^{2\pi} K(g(\theta)e^{i\theta}, r'e^{i\theta'})g(\theta)h(\theta)\,d\theta + o(|h|_2),$$

uniformly in $r'e^{i\theta'}$.

Proof. For each $\varepsilon > 0$, choose $\delta > 0$ small ($\delta < \pi/2$), so that $\int_{\theta'-\delta}^{\theta'+\delta} \bar{K}^2 d\theta < \varepsilon^2$, where $\bar{K} = (1/2\pi)\ln(1/|\sin(\theta - \theta')|) + M_1$ is defined by Eq. (1). Since $[G(\chi_{g+h}) - G(\chi_g)](r'e^{i\theta'}) = \int_0^{2\pi}\int_{g(\theta)}^{g(\theta)+h(\theta)} K(re^{i\theta}, r'e^{i\theta'})r\,dr\,d\theta$, we get

$$\left|[G(\chi_{g+h}) - G(\chi_g)](r'e^{i\theta'}) - \int_0^{2\pi} K(g(\theta)e^{i\theta}, r'e^{i\theta'})g(\theta)h(\theta)\,d\theta\right|$$

$$\leqq \left|\int_{|\theta-\theta'|\geqq\delta}\int_{g(\theta)}^{g(\theta)+h(\theta)} K(re^{i\theta}, r'e^{i\theta'})r\,dr\,d\theta - \int_{|\theta-\theta'|\geqq\delta} K(g(\theta)e^{i\theta}, r'e^{i\theta'})g(\theta)h(\theta)\,d\theta\right|$$

$$+ \left|\int_{\theta'-\delta}^{\theta'+\delta}\int_{g(\theta)}^{g(\theta)+h(\theta)} K(re^{i\theta}, r'e^{i\theta'})r\,dr\,d\theta\right| + \left|\int_{\theta'-\delta}^{\theta'+\delta} K(g(\theta)e^{i\theta}, r'e^{i\theta'})g(\theta)h(\theta)\,d\theta\right|$$

by the mean-value theorem,

$$\leqq \int_{|\theta-\theta'|\geqq\delta} |K(\bar{g}(\theta)e^{i\theta}, r'e^{i\theta'})\bar{g}(\theta) - K(g(\theta)e^{i\theta}, r'e^{i\theta'})g(\theta)||h(\theta)|\,d\theta$$

$$+ \int_{\theta'-\delta}^{\theta'+\delta} |K(g^*(\theta)e^{i\theta}, r'e^{i\theta'})g^*(\theta)h(\theta)|\,d\theta$$

$$+ \int_{\theta'-\delta}^{\theta'+\delta} |K(g(\theta)e^{i\theta}, r'e^{i\theta'})g(\theta)h(\theta)|\,d\theta.$$

Since $|h|_{L^\infty}$ is small, $\bar{g}$ and g^* are uniformly close to g, and so by the Schwarz inequality, Eqs. (1), (2) and our choice of δ, the above is

$$\leqq \varepsilon|h|_2 + \varepsilon(|g|_\infty + 1)|h|_2 + \varepsilon|g|_\infty|h|_2$$
$$= \varepsilon C_1|h|_2 \text{ for } |h|_2 \text{ sufficiently small.} \quad \blacksquare$$

Lemma 3 implies that

$$\begin{aligned}\langle \chi_{g+h} - \chi_g, G\chi_{g+h} - G\chi_g\rangle &= \int_0^{2\pi} d\theta' \int_{g(\theta')}^{g(\theta')+h(\theta')} r'\,dr' \cdot\left\{\int_0^{2\pi} K(g(\theta)e^{i\theta}, r'e^{i\theta'})g(\theta)h(\theta)\,d\theta + o(|h|_2)\right\}\\ &= \int_0^{2\pi} d\theta' \int_g^{g+h} r'\,dr'\left\{\int_0^{2\pi} Kgh\,d\theta\right\} + o(|h|_2^2) \end{aligned} \tag{5}$$

Lemma 4. $\displaystyle\int_0^{2\pi} d\theta' \int_{g(\theta')}^{g(\theta)'+h(\theta')} r'\,dr'\left\{\int_0^{2\pi} K(g(\theta)e^{i\theta}, r'e^{i\theta'})g(\theta)h(\theta)\,d\theta\right\}$

$$= \int_0^{2\pi}\int_0^{2\pi} K(g(\theta)e^{i\theta}, g(\theta')e^{i\theta'})g(\theta)g(\theta')h(\theta)h(\theta')\,d\theta\,d\theta' + o(|h|_2^2).$$

Proof. To each $\varepsilon > 0$, there exists $\delta > 0$ ($\delta < \pi/2$) such that $\displaystyle\int_{\theta'-\delta}^{\theta'+\delta} \bar{K}^2\,d\theta < \varepsilon^2$.

$$\begin{aligned}&\left|\int_0^{2\pi} d\theta' \int_{g(\theta')}^{g(\theta')+h(\theta')} r'\,dr'\left\{\int_0^{2\pi} K(g(\theta)e^{i\theta}, r'e^{i\theta'})g(\theta)h\theta)\,d\theta\right\}\right.\\
&\qquad\left. - \int_0^{2\pi}\int_0^{2\pi} K(g(\theta)e^{i\theta}, g(\theta')e^{i\theta'})g(\theta)g(\theta')h(\theta)h(\theta')\,d\theta\,d\theta'\right|\\
&\leqq \left|\int_0^{2\pi} d\theta' \int_{g(\theta')}^{g(\theta')+h(\theta')} r'\,dr'\left\{\int_{|\theta-\theta'|\geqq\delta} K(g(\theta)e^{i\theta}, r'e^{i\theta'})g(\theta)h(\theta)\,d\theta\right\}\right.\\
&\qquad\left. - \int_0^{2\pi} d\theta' \int_{|\theta-\theta'|>\delta} K(g(\theta)e^{i\theta}, g(\theta')e^{i\theta'})g(\theta)g(\theta')h(\theta)h(\theta')\,d\theta\right|\\
&\qquad + \left|\int_0^{2\pi} d\theta' \int_{g(\theta')}^{g(\theta')+h(\theta')} r'\,dr'\left\{\int_{|\theta-\theta'|\leqq\delta} K(g(\theta)e^{i\theta}, r'e^{i\theta'})g(\theta)h(\theta)\,d\theta\right\}\right|\\
&\qquad + \left|\int_0^{2\pi} d\theta' \int_{|\theta-\theta'|\leqq\delta} K(g(\theta)e^{i\theta}, g(\theta')e^{i\theta'})g(\theta)g(\theta')h(\theta)h(\theta')\,d\theta\right|.\end{aligned}$$

Applying the mean-value theorem, the Schwarz inequality, and Eqs. (1) and (2), we get

$$\leqq \varepsilon|h|_2^2 + \varepsilon|g|_\infty(|g|_\infty + 1)\sqrt{2\pi}\,|h|_2^2 + \varepsilon|g|^2\sqrt{2\pi}\,|h|_2^2$$
$$= \varepsilon C_2|h|_2^2 \text{ for } |h|_2 \text{ sufficiently small.} \quad \blacksquare$$

Now, we can write down the second order Taylor expansion of $E(\chi_g)$ together with an estimate on its remainder term.

Proposition 2. *For $g, g+h \in \mathscr{R}$ $|h|_\infty$, $|h'|_\infty \leqq 1$, we have*

$$\begin{aligned} E(\chi_{g+h}) - E(\chi_g) &= \int_0^{2\pi} G(\chi_g)(g(\theta)e^{i\theta})g(\theta)h(\theta)\,d\theta \\ &\quad + \frac{1}{2}\int_0^{2\pi} [G(\chi_g)(g(\theta)e^{i\theta})(h(\theta))^2 \\ &\quad + g(\theta)\frac{\partial G}{\partial r}(g(\theta)e^{i\theta})(h(\theta))^2]\,d\theta \\ &\quad + \frac{1}{2}\int_0^{2\pi}\int_0^{2\pi} K(g(\theta)e^{i\theta}, g(\theta')e^{i\theta'})g(\theta)g(\theta')h(\theta)h(\theta')\,d\theta\,d\theta' \\ &\quad + o(|h|_2^2). \end{aligned}$$

Proof. This follows from Eqs. (3), (4), (5) and Lemmas 3 and 4. ∎

Notice the expansion is good for $g \in \mathscr{R}$, and uniform in C^1-norm on g. Therefore,

Corollary 1. *$E(\chi_g)$ is of class C^2 in g with respect to the C^1-norm.*

6. The Spectral Analysis of the Second Variation and its Consequences

Let us consider the Taylor expansion of $E(\chi_g)$ at $g=\gamma$ as given by Proposition 2. We also restrict ourselves to area-preserving deformations, i.e. $\int \chi_{\gamma+h} = \int \chi_\gamma$. Thus, $1/2 \int_0^{2\pi} (\gamma+h)^2\,d\theta = 1/2\int \gamma^2\,d\theta$, and

$$\int_0^{2\pi} \gamma h\,d\theta + \frac{1}{2}\int_0^{2\pi} h^2\,d\theta = 0. \tag{6}$$

Since $G = (1/4)\gamma^2 \ln(R^2/\gamma^2) = \bar{c}$ and $\partial G/\partial r = -(1/2)\gamma$ on $|z| = \gamma$, we get

$$\begin{aligned} E(\chi_{\gamma+h}) - E(\chi_\gamma) &= \bar{c}\int_0^{2\pi} (\gamma h + \tfrac{1}{2}h^2)\,d\theta + \frac{1}{2}\int_0^{2\pi} \gamma(-\tfrac{1}{2}\gamma)h^2\,d\theta \\ &\quad + \tfrac{1}{2}\langle h(\theta), \gamma^2 \int_0^{2\pi} K(\gamma e^{i\theta}, \gamma e^{i}\theta')h(\theta')d\theta'\rangle + o(|h|_2^2) \\ &= \tfrac{1}{2}\langle h, Lh\rangle + o(|h|_2^2), \end{aligned} \tag{7}$$

where $Lh = -(\gamma^2/2)\,h + \mathscr{H}h$, and

$$\mathscr{H}h(\theta) = \gamma^2 \int_0^{2\pi} K(re^{i\theta}, r'e^{i\theta'})h(\theta')d\theta.$$

Next, we need to compute the eigenvalues of $\mathscr{H}$ as a linear transformation on $L^2[0,2\pi]$. Clearly, $\mathscr{H}$ is a bounded self-adjoint linear transformation on $L^2[0,2\pi]$. Let $\theta' = \theta + \bar{\theta}$, and $h(\theta') = e^{in\theta'}$. Since K is circularly invariant,

$$\mathscr{H}(e^{in\theta}) = \gamma^2 \int_0^{2\pi} K(\gamma e^{i\theta}, \gamma e^{i\theta} e^{i\bar{\theta}}) e^{in\theta} e^{in\bar{\theta}}\, d\bar{\theta} = \gamma^2 \int_0^{2\pi} K(\gamma, \gamma e^{i\bar{\theta}}) e^{in\theta} e^{in\bar{\theta}}\, d\bar{\theta}$$

Hence $e^{in\theta}$ are eigenfunctions of $\mathscr{H}$ with real eigenvalues,

$$a_n = \gamma^2 \int_0^{2\pi} K(\gamma, \gamma e^{i\bar{\theta}}) e^{in\bar{\theta}} d\bar{\theta} = \gamma^2 \int_0^{2\pi} \frac{1}{2\pi}\left[\ln \frac{|(\gamma/R)^2 e^{i\theta} - 1|}{|e^{i\theta} - 1|} + \ln\left(\frac{R}{\gamma}\right)\right] \cos n\theta\, d\theta.$$

Since $\ln(1 - (\gamma/R)^2 e^{i\theta}) = -\sum_{n>0} (\gamma/R)^{2n} e^{in\theta}/n$ $(\ln(1-z) = -z - z^2/2 - z^3/3 \cdots)$, we get

$\int_0^{2\pi} \ln|1 - (\gamma/R)^2\, e^{i\theta}| \cos n\theta\, d\theta = -(\gamma/R)^{2n}\pi/n$, for $0 < \gamma/R < 1$, $n > 0$. Thus,

$$\begin{cases} a_n = \frac{1}{2}\gamma^2 \dfrac{1 - (\gamma/R)^{2|n|}}{|n|} \leqq \frac{1}{2}\gamma^2\left(1 - \left(\dfrac{\gamma}{R}\right)^2\right) \dfrac{(1 + (\gamma/R)^2 + \cdots + (\gamma/R)^{2|n|-2})}{|n|} \\ \qquad \leqq \frac{1}{2}\gamma^2\left(1 - \left(\dfrac{\gamma}{R}\right)^2\right) \quad \text{for } n \neq 0. \\ a_0 = \gamma^2 \ln\left(\dfrac{R}{\gamma}\right). \end{cases}$$

The linear transformations L has eigenfunctions $e^{in\theta}$ with eigenvalues $-\gamma^2/2 + a_n$.

Let $h = h_t + h_n$, $h_n = 1/2\pi \int_0^{2\pi} h\ d\theta$, and $\int_0^{2\pi} h_t\, d\theta = 0$, be an L^2-orthogonal eigenspace decomposition of L on $L^2[0, 2\pi]$. The linear transformation L has eigenvalues $-\gamma^2/2 + a_n \leqq -(\gamma^2/2)(\gamma^2/R^2)$, when restricting to the eigenspace $\{h_t\}$. Therefore,

$$\langle h_t, Lh_t \rangle \leqq -\frac{\gamma^4}{2R^2} \langle h_t, h_t \rangle. \tag{8}$$

Proposition 3. *Given* $\varepsilon > 0$, *there exists* $\eta > 0$ (*small*) *such that* $\langle h, Lh \rangle \leqq -(\gamma^4/2R^2 - \varepsilon)\, |h|_2^2$, *if* $h \in L^2\,[0, 2\pi]$, $|h_n|_2/|h|_2 < \eta$.

Proof. $\langle h, Lh \rangle = \langle h_t + h_n, L(h_t + h_n) \rangle = \langle h_t, Lh_t \rangle + \langle h_n, Lh_n \rangle$

$$\leqq -\frac{\gamma^4}{2R^2} \langle h_t, h_t \rangle + |L|_2 \eta^2 |h|_2^2, \text{(by Eq. (8), and the Schwarz inequality)}$$

$$= -\frac{\gamma^4}{2R^2} \langle h - h_n, h - h_n \rangle + |L|_2 \eta^2 |h|_2^2$$

$$\leqq -\frac{\gamma^4}{2R^2} |h|_2^2 + \frac{\gamma^4}{2R^2} 2\eta |h|_2^2 + \frac{\gamma^4}{2R^2} \eta^2 |h|_2^2 + |L|_2 \eta^2 |h|_2^2.$$

Here, $|L|_2$ denotes the operator norm of the bounded linear operator L,

$$|L|_2 \leqq \tfrac{1}{2}\gamma^2 + \max\left(\frac{1}{2}\gamma^2, \gamma^2 \ln \frac{R}{\gamma}\right).$$

The Proposition follows by taking η sufficiently small. ■.

Finally, with all the preparations made in Sect. 4 through 6, we prove the validity of our inequality (E) as stated in Sect. 2.

For $\gamma + h \in \mathscr{R}$ (i.e. $h \in C^1$, $a < \gamma + h < R$), with $|h|_\infty \leqq 1$, $|h'|_\infty \leqq 1$,

$$\begin{aligned} |\chi_{\gamma+h}|_{L^1} &= \frac{1}{2}\int_0^{2\pi} |(\gamma+h)^2 - \gamma^2|\, d\theta \\ &\leqq \int_0^{2\pi} |\gamma h|\, d\theta + \frac{1}{2}\int_0^{2\pi} h^2\, d\theta \\ &\leqq \gamma\sqrt{2\pi}\,|h|_2 + \tfrac{1}{2}|h|_2^2, \text{ by the Schwarz inequality,} \\ &\leqq \sqrt{2\pi}(\gamma + \tfrac{1}{2})|h|_2, \quad \text{for } |h|_2 \leqq \sqrt{2\pi}. \end{aligned} \tag{9}$$

Suppose also that $\int_D \chi_{\gamma+h} = \int_D \chi_\gamma$ (i.e. area-preserving deformation). Then Eq. (6), namely, $\int_0^{2\pi} \gamma h\, d\theta + \frac{1}{2}\int_0^{2\pi} h^2\, d\theta = 0$ gives $|h_n|_2/|h|_2 = O(|h|_2)$. By Proposition 3, and Eq. (7) and (9), there exists a positive constant c_2 such that

$$E(\chi_{\gamma+h}) - E(\chi_\gamma) \leqq -c_2|\chi_{\gamma+h} - \chi_\gamma|_{L^1}^2, \tag{10}$$

provided $|h|_2$ is sufficiently small.

For $|\omega_1 - \omega_0|_{L^1}$ small, the boundary $r = f(\theta)$ of $\bar{\omega}_1$ is C^1-close to γ (for the definition of $\bar{\omega}_1$, see Sect. 4). Hence, one can take $h = f - \gamma$ and assume $|h|_\infty$, $|h'|_\infty$ small. Therefore, $|h|_2$ must be small, and Eq. (10) becomes:

$$E(\bar{\omega}_1) - E(\omega_0) \leqq -c_2|\bar{\omega}_1 - \omega_0|_{L^1}^2$$

for ω_1 sufficiently L^1-close to ω_0. Applying Proposition 1, we obtain the desired inequality (E).

Consequently our stability theorem follows as was shown in Sect. 2.

7. Stability for Vortex Distributions

So far we have proved that the circular vortex patch is nonlinearly stable within the class of vortex patches. A somewhat stronger stability result is the following.

Stability Theorem. *Let ω_0 be circular vortex patch of radius γ in a disk of radius R. For any $\eta > 0$, there is a $\delta > 0$ such that if ω is a vorticity distribution in L^∞ satisfying*

$$\text{(i)} \quad 0 \leqq \omega(,y) \leqq \lambda, |\lambda - 1| < \frac{2\delta}{|\omega_0|_{L'}},$$

and $\quad$ (ii) $\quad |\omega - \omega_0|_{L^1} < \delta$,

then $$\|\Phi_t(\omega) - \omega_0\|_{L^1} < \eta$$

for all $t \geqq 0$.

As in the case of patches, the proof is based on inequality (E) for more general $\hat{\omega}$. More precisely, if ω_c is of strength λ, λ close to 1, (E) is valid if $\hat{\omega}$ is a vorticity distributions satisfying

$$0 \leqq \hat{\omega}(x) \leqq \lambda, \quad \text{and} \quad \int_D \hat{\omega} = \int_D \omega_C.$$

The proof that (E) implies the stability theorem proceeds essentially the same as before.

In the earlier arguments in Sect. 2, we first establish $|\omega^* - \omega_0|_{L^1} \leqq 5\delta$, a fact which follows by assuming ω is a vortex patch. Here, we simply take $|\lambda - 1||\omega_0| < 2\delta$ as hypothesis. The rest of the proof is formally the same.

It suffices to justify the inequality (E) for more general $\hat{\omega}$, in the special case where $\omega_c = \omega_0$, and $\hat{\omega} = \omega_1$ with ω_1 a vortex distribution obeying $0 \leqq \omega_1(x) \leqq 1$. The definitions of ζ_0, ζ_1, ζ_2 and the statement and proof of Lemma 1 is unchanged.

Proof of Lemma 2. Let $U = \{(x, y, v) \in D \times \mathbb{R} \mid \bar{\omega}_1(x, y) = 1 \text{ and } \omega_1(x, y) \leqq v \leqq 1\}$ and $V = \{(x, y, v) \in D \times \mathbb{R} \mid \bar{\omega}_1(x, y) = 0 \text{ and } 0 \leqq v \leqq \omega_1(\lambda, y)\}$. Since $\int_D \bar{\omega}_1 = \int_D \omega_1$, we have $\operatorname{vol} V = \operatorname{vol} U$ and

$$\begin{aligned} \langle \omega_1 - \bar{\omega}_1, G\omega_1 \rangle &= \int_D \left[\int_{\bar{\omega}_1(x,y)}^{\omega_1(x,y)} \zeta(x, y) dv \right] dxdy \\ &= \int_U \zeta(x, y) dxdydv - \int_V \zeta(x, y) dxdydv. \end{aligned}$$

For $|\omega_1 - \omega_0|_{L^1}$ sufficiently small, choose $\zeta_2 < \zeta_0 < \zeta_1$ so that

$$\text{area}\,\{\xi_0 \geqq \xi \geqq \xi_2\} = \operatorname{vol} U \quad \text{and} \quad \text{area}\,\{\xi_1 \geqq \xi \geqq \xi_0\} = \operatorname{vol} V.$$

Thus the hypotheses of Lemma 1 hold. From $\partial\zeta/\partial r < 0$ one sees that

$$\int_U \zeta dxdydv \leqq \int_0^1 \int_{\zeta_0 \geqq \zeta \geqq \zeta_2} \zeta dxdydv = \int_{\zeta_0 \geqq \zeta \geqq \zeta_2} \zeta dxdy$$

and

$$\int_V \zeta dxdydv \leqq \int_0^1 \int_{\zeta_1 \geqq \zeta \geqq \zeta_0} \zeta dxdydv = \int_{\zeta_1 \geqq \zeta \geqq \zeta_0} \zeta dxdy.$$

Thus,

$$\langle \omega_1 - \bar{\omega}_1, G\omega_1 \rangle \leqq \int_{\zeta_0 \geqq \zeta \geqq \zeta_2} \zeta dxdy - \int_{\zeta_1 \geqq \zeta \geqq \zeta_0} \zeta dxdy,$$

and so the lemma follows from Lemma 1 as before. ■

One might not have guessed that inequality (E) would be valid for all vorticity distributions that are nonnegative and L^1 close to ω_0 in view of the fact that formally, this class includes many coadjoint orbits. However, the coadjoint orbit of vortex patches is very singular and E in fact has a nondegenerate L^1 local maximum on a whole convex set of coadjoint orbits containing ω_0. This agrees with Turkington [1983].

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