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Nonlinear Stability of Overcompressive Shock Waves in a Rotationally Invariant System of Viscous Conservation Laws

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Abstract. This paper proves that certain non-classical shock waves in a rotationally invariant system of viscous conservation laws possess nonlinear large-time stability against sufficiently small perturbations. The result applies to small intermediate magnetohydrodynamic shocks in the presence of dissipation.

1. Introduction

In this paper, we consider the parabolic system

$$u_t + (|u|^2 u)_x = \mu u_{xx} \,, \tag{1.1}$$

where $x \in \mathbb{R}$, $t \in \mathbb{R}$, $u(x, t) \in \mathbb{R}^n$ $(n \geqq 2)$, $\mu > 0$, and study the question of large time stability of some of its shock wave solutions

$$u_*(x, t) = \phi_*((x - st)\mu), \quad \phi_*(\pm\infty) = u^\pm, \quad u^- \neq u^+ \,. \tag{1.2}$$

System (1.1) and these shock waves have physically relevant interpretations as we will detail soon below. The system is rotationally invariant and thus its inviscid part

$$u_t + (|u|^2 u)_x = 0 \tag{1.3}$$

is *non-strictly* hyperbolic: The characteristic speeds

$$\lambda_1(u) = |u|^2, \quad \lambda_2(u) = 3|u|^2 \tag{1.4}$$

touch at the umbilic point 0; the corresponding eigenspaces

$$R_i(u) = \ker((|u|^2 - \lambda_i(u))I + 2uu^t), \quad i = 1, 2 \,,$$

rotate as

$$R_1(u) = \{u\}^\perp, \quad R_2(u) = \mathbb{R}u, \quad u \neq 0 \,, \tag{1.5}$$

with the base point u and couple at 0 to

$$R_1(0) = R_2(0) = \mathbb{R}^n . \tag{1.6}$$

Correspondingly, system (1.3)/(1.1) is endowed with a variety of shock waves of different types which do not occur for strictly hyperbolic systems and small amplitude (nor, typically, for large amplitude: e.g., they do not exist in gas dynamics under standard assumptions). Of these non-classical shock waves, we are interested here in *overcompressive* shocks, which are characterized by

$$\lambda_1(u^-) > s > \lambda_2(u^+) . \tag{1.7}$$

The goal of this paper is to prove that overcompressive shocks for (1.1) can be stable against small perturbations. i.e., given the profile ϕ_* of an (appropriate) overcompressive shock wave and a function $u_0 \colon \mathbb{R} \to \mathbb{R}^n$ (of appropriate type) such that

$$\overline{u_0} \equiv u_0 - \phi_*(\cdot/\mu)$$

is small (in an appropriate sense), then the solution u of (1.1) with initial data u_0 converges time-asymptotically to another profile ϕ:

$$\lim_{t\to\infty} \sup_{x\in\mathbb{R}} |u(x, t) - \phi(x - st/\mu)| = 0 . \tag{1.8}$$

The situation is thus similar to that of small shocks associated with genuinely nonlinear modes [14]. There are, however, two important differences between the stability of overcompressive shocks and that of classical Laxian shocks. To describe the first of them, recall that from any viscous shock wave (1.2), any phase shift $x \mapsto x + \delta$, $\delta \in \mathbb{R}$, trivially produces another (just shifted) shock wave with the same end states. For classical Laxian shock waves, the profile ϕ is vice versa uniquely determined modulo such phase shifts. By contrast, in the case of overcompressive shocks, whole families of orbitally different profiles exist. Correspondingly, the asymptotic profile ϕ in (1.8) generically differs from the profile ϕ_* of the unperturbed solution not only by a phase shift, but by a true change in shape. This additional freedom compensates for the fact that the ordering of shock and characteristic speeds associated with an overcompressive shock allows for a smaller number of diffusion waves [14] than in the classical case. Like the phase shift in the classical case, the phase *and* shape of the asymptotic overcompressive shock wave can be determined directly from specific components of the integral of the perturbation $\overline{u_0}$. The second important difference between overcompressive and classical shocks consists in the fact that while it is generally believed that the stability of Laxian shocks is uniform in the viscosity μ, this is certainly not so for the overcompressive shocks that we consider. This non-uniformity is obvious from the fact that their inviscid versions

$$u(x, t) = \begin{cases} u^-, & x < st \\ u^+, & x > st \end{cases}$$

are completely unstable against generic perturbations [4, 8]. Indeed, with decreasing μ, ever smaller perturbations, e.g., of the form

$$\overline{u_0} \equiv g(\cdot/\mu), \quad g \text{ appropriate},$$

give rise to wave patterns which differ markedly from any stationary shock wave with whatever profile ϕ [5, 8, 10, 15].

Relative to a given $u^- \in \mathbb{R}^n \setminus \{0\}$, let $\pi : \mathbb{R}^n \to \mathbb{R}^n$ denote the orthogonal projection on the line $L = \mathbb{R}u^-$ and $\pi^\perp = \mathrm{id}_{\mathbb{R}^n} - \pi$ that on the complement $L^\perp$. We summarize the above discussion by stating our main result in precise terms:

Theorem 1. *Consider a viscous shock wave solution*

$$u_*(x, t) = \phi_*((x - st)/\mu), \quad \phi_*(\pm\infty) = u^\pm$$

of (1.1), *with* $u^- \neq 0, u^+ = \alpha u^-$, $-\frac{1}{8} < \alpha < 0$ (*— this implies that* (1.7) *holds —*), *and a perturbation* $\overline{u_0} \in \mathscr{L}^1(\mathbb{R}, \mathbb{R}^n)$. *If the quantities*

$$m_* = \int_{-\infty}^{\infty} \pi^\perp(u_*(x, 0))\, dx, \quad \overline{m} = \int_{-\infty}^{\infty} \pi^\perp(\overline{u_0}(x))\, dx$$

are sufficiently small, then a unique profile ϕ *with* $\phi(\pm\infty) = \phi_*(\pm\infty) = u^\pm$ *is determined through the relation*

$$\int_{-\infty}^{\infty} \phi(x/\mu) - \phi_*(x/\mu)\, dx = \int_{-\infty}^{\infty} \overline{u_0}(x)\, dx\,.$$

If $\overline{u_0}$ *is small in the sense that the function* U_0 *given by*

$$U_0(x) = \int_{-\infty}^{x} (\phi_*(\tilde{x}/\mu) + \overline{u_0}(\tilde{x})) - \phi(\tilde{x}/\mu))\, d\tilde{x}$$

satisfies

$$\| U_0 \|_{H^{2,2}(\mathbb{R})} \ll 1\,,$$

(*— how small* $\| U_0 \|_{H^{2,2}(\mathbb{R})}$ *has to be depends on* α, $|u^-|$ *and* μ(!); *but not on* m_*, $\overline{m}$ *as long as these latter are small —*), *then the solution* u *of* (1.1) *with data*

$$u(x, 0) = \phi_*(x/\mu) + \overline{u_0}(x), \quad x \in \mathbb{R}\,,$$

converges in the sense of (1.8) *to the viscous shock wave with profile* ϕ.

This will be proved in Sect. 3, after preliminary observations on the viscous profiles in Sect. 2. For a previous stability result on overcompressive shock waves in a different model we refer the reader to [17].

In the rest of this introduction, we want to comment in some detail on the motivation for studying system (1.1), (1.3) and its overcompressive shock waves.

In physical systems of conservation laws, rotational invariance typically arises due to natural isotropy. While in the classical and most prominent example for conservation law theory, i.e., gas dynamics, such isotropy is superposed with Galilean invariance in a geometrically non-generic way, other systems such as those describing magnetohydrodynamic or elastic plane waves display rotational symmetry in its generic form, which induces a mode coupling such as described in (1.4), (1.5), (1.6). Their modeling character has motivated previous mathematical study of (1.3) and similar systems; cf. [13, 16, 5, 9, 15]. It turned out that the specific model (1.1)/(1.3) can be formally derived from larger systems via asymptotic expansion techniques (see [2]) and that this had even been done as early as in 1974, when Cohen and Kulsrud obtained the inviscid version (1.3) as an approximation of one-species magnetofluid-dynamics that they used to study certain features of the solar wind [3]. The overcompressive shock waves of the viscous version (1.1) which we study in this paper are good descriptions of certain non-classical shock waves that arise in physical systems. Especially, in magnetohydrodynamics they represent what is traditionally called intermediate shocks. In the inviscid framework, intermediate magnetohydrodynamic shocks are extremely unstable waves, a property for which, according to [18], they have been excluded from certain considerations on physical situations in which they might indeed be relevant building blocks. In recent years, this possibility has repeatedly been pointed out by Brio and Wu, see [1, 18, 19] and references therein. (Cp. also [6] and references there.) It seems that our result yields the first *proof* that intermediate magnetohydrodynamic shock waves can be stable in the presence of dissipation.

2. Profiles of Overcompressive Shocks

The profile ϕ of any viscous shock wave is a heteroclinic trajectory of the o.d.e. system

$$\phi' = (|\phi|^2 - s)\phi - b\,, \tag{2.1}$$

in which the speed s and the relative flux b are given by the Rankine–Hugoniot conditions

$$(|u^+|^2 - s)u^+ = (|u^-|^2 - s)u^- = b \tag{2.2}$$

with

$$u^\pm = \phi(\pm\infty)\,. \tag{2.3}$$

The phase portrait of (2.1) is easy to analyze; see [8]. Here, we focus attention on the case of overcompressive shocks. We start with

Lemma 1. (i) $u^-, u^+ \in \mathbb{R}^n$ *satisfy the inequality* $\lambda_1(u^-) > s > \lambda_2(u^+)$ ((1.7) *with* s *from* (2.2)) *if and only if*

$$u^- \neq 0 \quad and \quad u^+ = \alpha u^-, \quad \alpha \in \left(-\frac{1}{2}, 0\right). \tag{2.4}$$

(ii) *In this case, there is an* $(n-1)$*-parameter family of viscous profiles* (2.1)–(2.3).

Proof. (2.2) is satisfied for some b, if either $|u^-|^2 = |u^+|^2 = s$, or $u^- = 0$, or $u^- \neq 0$ and $u^+ = \alpha u^-$ for some $\alpha \in \mathbb{R}$. In the first two of these three cases, inequality (1.7) is obviously wrong. In the third case, however, it is equivalent to $1 > 1 + \alpha + \alpha^2 > 3\alpha^2$, and thus to $\alpha \in (-\frac{1}{2}, 0)$. This proves (i). To deduce (ii), assume that (2.4) holds. By (1.7), u^- is an unstable node, u^+ is a stable node of (2.1). As is easy to check from (2.2), system (2.1) has precisely one more rest point besides u^- and u^+, namely

$$(u^+)^* = \alpha^* u^- \quad \text{with} \quad \alpha^* = -1 - \alpha \in \left(-1, -\frac{1}{2}\right).$$

This third rest point lies on the invariant line through u^- and u^+. Since it lies outside the segment between u^- and u^+, this segment is contained in the unstable manifold of u^- as well as in the stable manifold of u^+. As these manifolds are in fact open sets in $\mathbb{R}^n$, the proof of Lemma 1 is complete. □

Remarks. (i) The closure $\mathscr{M}^*$ of the intersection of the unstable manifold of u^- and the stable manifold of $(u^+)^*$ is homeomorphic to S^{n-1}. It induces a family of non-classical viscous shock waves of a different kind (cf. [13, 7]), which we do not consider in this paper. (ii) The set $\mathscr{M}$ of points which belong to the image of heteroclinic trajectories ϕ of (2.1) from u^- to u^+, i.e., in other words, of states that can appear in viscous shock waves u satisfying (1.2), (2.1)–(2.4), is bounded. (Its boundary consists in the union of $\mathscr{M}^*$ and the closed straight line segment between u^+ and $(u^+)^*$.) Since furthermore all three rest points u^-, u^+, $(u^+)^*$ lie on a common line, there exists (cf. ([15]) a constant $M^* > 0$ such that each of these shock waves satisfies

$$\left| \int_{-\infty}^{\infty} \pi^\perp(u(x,t))\,dx \right| = \mu \left| \int_{-\infty}^{\infty} \pi^\perp(\phi(y))\,dy \right| < \mu M^* .$$

This shows that stability in the sense of Theorem 1 requires smallness of the perturbation in the sense that $|m_* + \overline{m}| < \mu M^*$.

Among the overcompressive shock waves characterized by Lemma 1, we consider especially those which assume values near the line L through u^- and u^+. We collect some useful properties of them in

Lemma 2. *For any* $\alpha \in (-\frac{1}{2}, 0)$ *and* $u^+ = \alpha u^- \neq 0$, *there are constants* M_1, M_2, $\delta_1 > 0$ *such that the following holds.*

(i) *For any* $m \in L^\perp$ *with* $|m| \leqq M_1$, *there is precisely one function* ϕ *that satisfies* ((2.1)–(2.3)),

$$\int_{-\infty}^{\infty} \pi^\perp(\phi(x))\,dx = m \quad \textit{and} \quad d \equiv \phi(0) - \frac{1}{2}u^- \in L^\perp, \quad |d| \leqq \delta_1 .$$

(ii) *The mapping* $m \mapsto d$ *is a diffeomorphism from* $\{m \in L^\perp\colon |m| \leqq M_1\}$ *to* $\{d \in L^\perp\colon |d| \leqq \delta_1\}$.

(iii) *These profiles are uniformly bounded*:

$$|\phi(x)| \leqq 1 \quad \textit{for all} \quad x \in \mathbb{R} .$$

(iv) *Near* u^+, *their direction is uniformly "almost longitudinal": To any* $\kappa > 0$, *there exists a value* $\xi^+ > \alpha$ *such that*

$$\pi(\phi(x)) < \xi^+ \Rightarrow \left| \frac{\phi'(x)}{|\phi'(x)|} - (-l) \right| < \kappa$$

holds with $l = u^-/|u^-|$ *for all* $x \in \mathbb{R}$, *if* $|m| \leqq M_2$.

Proof. Fix $\alpha \in (-\frac{1}{2}, 0)$. For $d \in L^\perp$, consider the trajectory ϕ^d of (2.1) satisfying

$$\phi^d(0) = \frac{1}{2}l + d . \tag{2.5}$$

For sufficiently small $|d|$, $\phi^d(\pm\infty) = u^\pm$ and the transverse mass

$$m^d = \int_{-\infty}^{\infty} \pi^\perp(\phi^d(x))\,dx$$

is finite. Obviously,

$$\phi^d(\mathbb{R}) \subset L + [0, \infty)d\,, \tag{2.6}$$

and thus

$$m^d = \hat{m}(\rho)\,l^\perp \quad \text{if } d = \rho l^\perp \text{ with } \rho \in \mathbb{R}, l^\perp \in L^\perp, |l^\perp| = 1$$

with some smooth function $\hat{m}$. Equations (2.5) and (2.6) imply that $\hat{m}'(0) > 0$ and thus

$$\det\left(\frac{\partial m^d}{\partial d}\right)\bigg|_{d=0} = \det(\hat{m}'(0)\,\mathrm{id}_{L^\perp}) \neq 0\,;$$

so the implicit function theorem yields assertions (i) and (ii). To prove part (iii), we consider $d \neq 0$, let $l^\perp = d/|d|$ and represent ϕ^d in polar coordinates as

$$\phi^d(x) = r(x)\,(l\cos\theta(x) + l^\perp \sin\theta(x)) \tag{2.7}$$

with $r\colon \mathbb{R} \to (0, \infty)$, $\theta\colon \mathbb{R} \to (0, \pi)$. Equation (2.1) becomes

$$r' = (r^2 - s)r - |b|\cos\theta\,,$$

$$r\theta' = |b|\sin\theta\,.$$

r cannot have a maximum at any finite point $x \in \mathbb{R}$, since $r'(x) = 0$ implies $r''(x) = (|b|\sin\theta(x))^2/r(x) > 0$. Since $r(-\infty) = 1$, $r(\infty) = |\alpha|$, (iii) follows. To see (iv), note that due to

$$\lambda_1(u^+) - s < \lambda_2(u^+) - s < 0$$

and

$$R_1(u^+) = L^\perp, \quad R_2(u^+) = L\,,$$

integral curves of (2.1) which are uniformly close enough to L are asymptotically tangent to L when they approach u^+. □

We introduce the notation

$$a(u) \equiv (|u|^2 - s)u - b$$

and

$$A(u) \equiv |u|^2 I + 2uu^t\,.$$

Our proof of Theorem 1 is based on the following technical

Lemma 3. *Consider fixed* $\alpha \in (-\frac{1}{8}, 0)$, u^- *with* $|u^-| = 1$, *and* $u^+ = \alpha u^-$. *If* $d \in \{u^-\}^\perp$ *is sufficiently small, then there exist a constant* $k > 0$ *and a uniformly* *bounded smooth function* $\hat{w}: \mathbb{R} \to \mathbb{R}$ *such that the inequality*

$$-\frac{1}{2}((a\cdot\nabla)A)(u) + \hat{w}(\pi(u))|a(u)|(A(u) - sI) - \frac{3}{2}|\hat{w}(\pi(u))\,a(u)|^2 I \geqq k|a(u)|I \tag{2.8}$$

holds for all points u that lie on the orbit O^d *of* (2.1) *which contains the point* $\frac{1}{2}u^- + d$.

Proof. We assume first that $d = 0$ and show that in this case (2.8) holds with

$$\hat{w}(\xi) = \begin{cases} -3, & \xi < -\dfrac{\alpha}{4} \\ 0, & \xi \geqq -\dfrac{\alpha}{4}. \end{cases} \tag{2.9}$$

Since $d = 0$, we must check (2.8) for all points $u = \xi l$, $\alpha < \xi < 1$, $l \equiv u^-$. We have

$$a(u) = -|a(\xi l)|l\,,$$

$$-\frac{1}{2}((a\cdot\nabla)A)(u) = |a(\xi l)|\xi(I + 2ll^\top)\,,$$

and

$$A(u) - sI = \xi^2(I + 2ll^\top) - (1 + \alpha + \alpha^2)I\,.$$

The left-hand side of (2.8) is thus equal to $|a(\xi l)|B(\xi)$ with

$$\begin{aligned} B(\xi) \equiv \xi(I + 2ll^\top) &+ \hat{w}(\xi)\xi^2(I + 2ll^\top) \\ &- \hat{w}(\xi)(1 + \alpha + \alpha^2)I - \frac{3}{2}\hat{w}^2(\xi)|a(\xi l)|I\,, \end{aligned}$$

and we are left with showing that

$$B(\xi) > kI \tag{2.10}$$

with some constant $c > 0$. From (2.9) we see that

$$B(\xi) = \xi(I + 2ll^\top) > -\frac{\alpha}{4}I \quad \text{for} \quad \xi \geqq -\frac{\alpha}{4}. \tag{2.11}$$

For $\xi \in \left(\alpha, -\dfrac{\alpha}{4}\right)$, we write

$$B(\xi) = (\xi - 3\xi^2)(I + 2ll^\top) + 3(1 + \alpha + \alpha^2)I - \frac{27}{2}|a(\xi l)|I\,,$$

check that

$$|a(\xi l)| = |(\xi^2 - (1 + \alpha + \alpha^2))\xi + (\alpha + \alpha^2)| < -\frac{5}{4}\alpha\,,$$

and deduce that for these ξ

$$B(\xi) \geqq (3(\alpha - 3\alpha^2) + 3\left((1 + \alpha + \alpha^2) + \frac{135}{8}\alpha\right)I \geqq \frac{3}{64}I\,.$$

This and (2.11) prove that (2.10), and thus (2.8) (for $d = 0$), hold with $k = -\frac{\alpha}{4}$. To remove the assumption $d = 0$, we now observe that by regular perturbation and upon lowering k if necessary, (2.8) continues to hold for all points u which satisfy

$$\xi^+ \leqq \pi(u) \leqq \xi^- \tag{2.12}$$

with $\alpha < \xi^+ < \xi^- < 1$ and lie on the orbits O^d as long as $|d|$ is smaller than some $\delta(\xi^+, \xi^-) > 0$. Using assertion (iv) of Lemma 2, we can say the same even if $\xi^+ = \alpha$. When $\xi^- \nearrow 1$, however, the direction of a, which is discontinuous near u^- even for small d, plays a sensitive role in (2.8). To deal with this problem, we use (1.7) and part (iii) of Lemma 2 and observe that there are a number $c_\alpha > 0$ and, for every $\varepsilon > 0$, values $\xi_\varepsilon \in (0, 1)$ and $\delta_\varepsilon > 0$ such that

$$A(u) - sI > c_\alpha I$$

and

$$|a(u)| < \varepsilon$$

hold for all

$$u \in O^d \quad \text{with} \quad \xi_\varepsilon \leqq \pi(u) < 1, \quad \text{if} \quad |d| < \delta_\varepsilon\,.$$

Letting

$$\hat{w}_\varepsilon(\xi) = \begin{cases} -3, & \xi < -\frac{\alpha}{4} \\ 0, & -\frac{\alpha}{4} \leqq \xi \leqq \xi_\varepsilon \\ 4c_\alpha^{-1}, & \xi \geqq \xi_\varepsilon\,, \end{cases} \tag{2.13}$$

we see that with $\hat{w}_\varepsilon$ instead of $\hat{w}$, (2.8) holds for all points $u \in O^d$ as soon as $\varepsilon > 0$ is small enough and $|d| < \min\{\delta(\alpha, \xi_\varepsilon), \delta_\varepsilon\}$. We pick such an ε and redefine $\hat{w}$ as a smoothed variant of $\hat{w}_\varepsilon$. □

3. Stability Proof

We turn to proving Theorem 1. Consider thus $u^- \neq 0$, $\alpha \in (-\frac{1}{8}, 0)$, $u^+ = \alpha u^-$. Assume that $|u^-| = \mu = 1$. By the scaling properties of (1.1), this means no loss of generality. Consider a profile ϕ_*, i.e., a solution of (2.1)–(2.3), with small transverse mass $m_* = \int_{-\infty}^{\infty} \pi^\perp(\phi_*(x))dx$, and a perturbation $\overline{u_0} \in \mathscr{L}^1(\mathbb{R}, \mathbb{R}^n)$ with small transverse mass $\overline{m} = \int_{-\infty}^{\infty} \pi^\perp(\overline{u_0}(x))\,dx$. Using part (i) of Lemma 2 and phase shifting if necessary, we find precisely one (other) profile ϕ such that

$$\int_{-\infty}^{\infty} \phi(x) - \phi_*(x)\,dx = \int_{-\infty}^{\infty} \overline{u_0}(x)\,dx\,.$$

It remains to show that if the function U_0, given by

$$U_0(x) = \int_{-\infty}^{x} \phi_*(\tilde{x}) + \overline{u_0}(\tilde{x}) - \phi(\tilde{x})\,d\tilde{x}\,,$$

is sufficiently small in $H^{2,2}(\mathbb{R})$, then the solution u of (1.1) with initial data $\phi_* + \overline{u_0}$ satisfies (1.8). In order to do this, we work with the so-called integrated equation, i.e., the system

$$U_t(x, t) + (A(\phi(x)) - sI)\,U_x(x, t) + Q(\phi(x), U_x(x, t)) = U_{xx}(x, t) \tag{3.1}$$

which governs the integrated perturbation

$$U(x, t) = \int_{-\infty}^{x} u(\tilde{x} + st) - \phi(\tilde{x})\,d\tilde{x}\,.$$

Equation (3.1) is obtained by subtracting the equation for u_* (u_* is given by (1.2)) from that for u, integrating the result with respect to x, and Taylor expanding the flux difference $|u|^2 u - |u_*|^2 u_*$. The latter gives rise to the quadratic remainder Q, which is easily checked to obey the estimate

$$|Q(\phi, z)| \leqq 3(|\phi| + |z|)|z|^2\,. \tag{3.2}$$

We treat (3.1) by an energy method. In describing this treatment now, we only sketch the basic steps, since they are known from the pertinent literature [11, 14]. The decisive specific estimate, which uses new technical ingredients to deal with the mode coupling (1.4)–(1.6), is formulated as Theorem 2 and proved in the second half of this section.

For given data and any $\beta > 0$, let $T_\beta \geqq 0$ be the largest time such that the solution exists until T_β and satisfies

$$\sup_{\mathbb{R}\times[0,\,T_\beta]} \{|U|, |U_x|\} \leqq \beta\,. \tag{3.3}$$

A standard short time estimate shows that there is a $\beta_1 > 0$ with the property that for all $\beta \in (0, \beta_1)$ there exists $\gamma_\beta > 0$ such that

$$\|U(\cdot, 0)\|_{H^{2,2}(\mathbb{R})} \leqq \gamma_\beta \quad \Rightarrow \quad T_\beta > 0\,.$$

Moreover, for such β, γ_β and $U(\cdot, 0)$, well-known considerations on (3.1) yield the energy estimate

$$\|U_x(\cdot, T)\|^2_{H^{1,2}(\mathbb{R})} \leqq c\left(\|U_x(\cdot, 0)\|^2_{H^{1,2}(\mathbb{R})} + \int_0^T \int_{-\infty}^{\infty} |U_x|^2\,dx\,dt\right), \tag{3.4}$$

which holds for all $T \in [0, T_{\beta_1}]$ with some c that does not depend on T. Suppose now one has also

Theorem 2. *There is a $\beta_2 \in (0, \beta_1)$ such that for all $\beta \in (0, \beta_2)$, the solution U of* (3.1) *satisfies*

$$\int_{-\infty}^{\infty} |U(x, T)|^2\,dx + \int_0^T \int_{-\infty}^{\infty} |U_x|^2\,dx\,dt \leqq c \int_{-\infty}^{\infty} |U(x, 0)|^2\,dx \tag{3.5}$$

for all $T \in [0, T_{\beta_2}]$ with some $c > 0$ which does not depend on T.

Then one can combine (3.4) and (3.5) to find $\beta, \delta > 0$ such that

$$\|U(\cdot, 0)\|_{H^{2,2}(\mathbb{R})} \leqq \delta \quad \Rightarrow \quad \|U(\cdot, T)\|_{H^{2,2}(\mathbb{R})} \leqq \gamma_\beta \quad \text{for all } T \in [0, T_\beta]\,.$$

This, however, means that for such data $T_\beta = \infty$. Then, (3.5) implies

$$\int_0^\infty \int_{-\infty}^\infty |U_x|^2 \, dx \, dt < \infty . \tag{3.6}$$

Through an integration step of the form

$$|U_x(x, t)|^2 \leqq c(\tau_1, \tau_2) \int_{t-\tau_2}^{t-\tau_1} \int_{-\infty}^\infty |U_x|^2 \, dx \, dt, \quad t > \tau_2 > \tau_1 > 0 ,$$

(3.6) implies the desired decay result

$$\limsup_{t \to \infty} \sup_{x \in \mathbb{R}} |U_x(x, t)| = 0 .$$

Establishing Theorem 1 is thus reduced to the

Proof of Theorem 2. The main idea is to circumvent technical difficulties associated with the mode coupling (1.4)–(1.6) by avoiding diagonalization of the symmetric system (3.1). Correspondingly, we replace the traditionally used monotonicity $(\lambda \circ \phi)' < 0$ of the principal speed by the matrix inequality $(A \circ \phi)' < 0$. This latter holds along most of the profile orbit $\phi(\mathbb{R})$; the fact that it does not hold everywhere is dealt with by considerations whose results have been expressed in Lemmas 2 and 3 above. To simplify the proof, we have chosen to use a weight function $w\colon \mathbb{R} \to (0, \infty)$ (see [12]) and define $V = V(x, t)$ through

$$U(x, t) = w(x) V(x, t) . \tag{3.7}$$

Substituting (3.7) into (3.1) and dividing by w yields

$$V_t + \frac{1}{w}(A(\phi) - sI)(wV)_x + \frac{1}{w} Q(\phi, (wV)_x) = V_{xx} + 2\frac{w'}{w} V_x + \frac{w''}{w} V .$$

We multiply by V^t and integrate with respect to x and t to obtain

$$\begin{aligned} &\int_{-\infty}^\infty |V(x, T)|^2 \, dx + \int_0^T \int_{-\infty}^\infty \frac{1}{w} V^t (A(\phi) - sI)(wV)_x \, dx \, dt \\ &\quad + \int_0^T \int_{-\infty}^\infty \frac{1}{w} V^t Q(\phi, (wV)_x) \, dx \, dt \\ &= \int_{-\infty}^\infty |V(x, 0)|^2 \, dx - \int_0^T \int_{-\infty}^\infty |V_x|^2 \, dx \, dt \\ &\quad \int_0^T \int_{-\infty}^\infty V^t \left(2\frac{w'}{w} V_x + \frac{w''}{w} V \right) dx \, dt . \end{aligned}$$

Integrating by parts and using the symmetry (!) of A, we write the second term on the left-hand side as

$$\int_0^T \int_{-\infty}^\infty V^t \left(-\frac{1}{2}(A \circ \phi)' + \frac{w'}{w}(A(\phi) - sI) \right) V \, dx \, dt .$$

Another integration by parts shows that the last term on the right-hand side is equal to

$$\int_0^T \int_{-\infty}^{\infty} \left(\frac{w'}{w}\right)^2 |V|^2 \, dx \, dt \, .$$

Due to (3.2), (3.3), the third term on the left-hand side is

$$\leqq 3 \int_0^T \int_{-\infty}^{\infty} \frac{1}{w} |V| \, (|\phi| + |U_x|) \, |(wV)_x|^2 \, dx \, dt$$

$$\leqq 6 \int_0^T \int_{-\infty}^{\infty} |wV| \, (|\phi| + |U_x|) \left(|V_x|^2 + \left(\frac{w'}{w}\right)^2 |V|^2 \right) dx \, dt$$

$$\leqq 6\beta \, (1 + \beta) \left(\int_0^T \int_{-\infty}^{\infty} |V_x|^2 \, dx \, dt + \int_0^T \int_{-\infty}^{\infty} \left(\frac{w'}{w}\right)^2 |V|^2 \, dx \, dt \right) .$$

Combining these facts and making β small enough, we find

$$\int_{-\infty}^{\infty} |V(x, T)|^2 \, dx \int_0^T \int_{-\infty}^{\infty} V^t \left(-\frac{1}{2} (A \circ \phi)' + \frac{w'}{w} (A(\phi) - sI) \right) V \, dx \, dt$$

$$\leqq \int_{-\infty}^{\infty} |V(x, 0)|^2 \, dx - \frac{1}{2} \int_0^T \int_{-\infty}^{\infty} |V_x|^2 \, dx \, dt + \frac{3}{2} \int_0^T \int_{-\infty}^{\infty} \left(\frac{w'}{w}\right)^2 |V|^2 \, dx \, dt \, . \qquad (3.8)$$

By statement (ii) of Lemma 2 and (sufficient) smallness of m_*, $\overline{m}$, the image $\phi(\mathbb{R})$ of the asymptotic profile ϕ is the orbit O^d of (2.1) through the point $\frac{1}{2} u^- + d$ with sufficiently small d so that Lemma 3 applies. With the function $\hat{w}$ whose existence Lemma 3 establishes, we now choose the weight w through

$$\frac{w'(x)}{w(x)} = \hat{w}(\pi(\phi(x))) \, |\phi'(x)|, \quad x \in \mathbb{R}, \quad \text{and} \quad w(-\infty) = 1 \, .$$

Since $|\phi'(x)|$ decays exponentially for $|x| \to \infty$ and $\hat{w}$ is uniformly bounded, w is well-defined and satisfies

$$c_1^{-1} < |w(x)| < c_1, \quad |w'(x)| < c_2 \qquad (3.9)$$

as well as

$$|w'(x)| < c_3 |\phi'(x)| \qquad (3.10)$$

for all $x \in \mathbb{R}$, with appropriate $c_1, c_2, c_3 > 0$. Remembering that $\phi' = a \circ \phi$, we use Lemma 3 to deduce

$$\int_0^T \int_{-\infty}^{\infty} V^t \left(-\frac{1}{2} (A \circ \phi)' + \frac{w'}{w} (A(\phi) - sI) \right) V \, dx \, dt - \frac{3}{2} \int_0^T \int_{-\infty}^{\infty} \left(\frac{w'}{w}\right)^2 |V|^2 \, dx \, dt$$

$$\geqq k \int_0^T \int_{-\infty}^{\infty} |\phi'| \, |V|^2 \, dx \, dt \, .$$

Combining this with (3.9) and (3.10), we conclude that

$$\int_{-\infty}^{\infty} |V(x,T)|^2\,dx + c_3^{-1}\,k \int_0^T \int_{-\infty}^{\infty} |w'|\,|V|^2\,dx\,dt + \frac{1}{2}\int_0^T \int_{-\infty}^{\infty} |V_x|^2\,dx\,dt$$

$$\leqq c \int_{-\infty}^{\infty} |V(x,0)|^2\,dx\,.$$

By means of (3.9), this implies

$$\int_{-\infty}^{\infty} |wV(x,T)|^2\,dx + \int_0^T \int_{-\infty}^{\infty} |(wV)_x|^2\,dx\,dt \leqq c \int_{-\infty}^{\infty} |wV(x,0)|^2\,dx\,.$$

for some $c > 0$; this is (3.5). □

References

1. Brio, M.: Propagation of weakly nonlinear magnetoacoustic waves. Wave Motion **9**, 455–458 (1987)
2. Brio, M., Hunter, J.: Rotationally invariant hyperbolic waves. Commun. Pure Appl. Math. **43**, 1037–1053 (1990)
3. Cohen, R.H., Kulsrud, R.M.: Nonlinear evolution of parallel-propagating hydromagnetic waves. Phys. Fluids **17**, 2215–2225 (1974)
4. Freistühler, H.: Rotational degeneracy of hyperbolic systems of conservation laws. Arch. Rational Mech. Anal. **113**, 39–64 (1991)
5. Freistühler, H.: Instability of vanishing approximation to hyperbolic systems of conservation laws with rotational invariance. J. Differ. Eqs. **87**, 205–226 (1990)
6. Freistühler, H.: Some remarks on the structure of intermediate magnetohydrodynamic shocks. J. Geophys. Res. **95**, 3825–3827 (1991)
7. Freistühler, H.: Linear degeneracy and shock waves. Math. Z. **207**, 583–596 (1991)
8. Freistühler, H.: Dynamical stability and vanishing viscosity: A case study of a non-strictly hyperbolic system of conservation laws. Commun. Pure Appl. Math. **45**, 561–582 (1992)
9. Freistühler, H.: On the Cauchy problem for a class of hyperbolic systems of conservation laws. J. Differ. Eqs., to appear
10. Freistühler, H., Pitman, E.B.: A numerical study of a rotationally degenerate hyperbolic system. Part I: The Riemann problem. J. Comput. Phys. **100**, 306–321 (1992)
11. Goodman, J.: Nonlinear asymptotic stability of viscous shock profiles for conservation laws. Arch. Rational Mech. Anal. **95**, 325–344 (1986)
12. Goodman, J.: Remarks on the stability of viscous shock waves. In *Viscous profiles and numerical methods for shock waves*, ed.: Shearer, M., Philadelphia: SIAM, 1991, pp. 66–72
13. Keyfitz, B., Kranzer, H.: A system of non-strictly hyperbolic conservation laws arising in elasticity theory. Arch. Rational Mech. Anal. **72**, 219–241 (1980)
14. Liu, T.-P.: Nonlinear stability of shock waves for viscous conservation laws. Am. Math. Soc. Mem. **328**, Providence: AMS 1985
15. Liu, T.-P.: On the viscosity criterion for hyperbolic conservation laws. *Viscous profiles and numerical methods for shock waves*, ed.: Shearer, M., Philadelphia: SIAM, 1991, pp. 105–114
16. Liu, T.-P., Wang, C.-H.: On a non-strictly hyperbolic system of conservation laws. J. Differ. Eq. **57**, 1–14 (1985)
17. Liu, T.-P., Xin, Z.: Stability of viscous shock waves associated with a system of non-strictly hyperbolic conservation laws. Commun. Pure Appl. Math. **45**, 361–388 (1992)
18. Wu, C.C.: On MHD intermediate shocks. Geophys. Res. Lett. **14**, 668–671 (1987)
19. Wu, C.C.: Formation, structure, and stability of MHD intermediate shocks. J. Geophys. Res. **95**, 8149–8175 (1990)

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