

NONLINEAR STATE ESTIMATION
USING
SLIDING OBSERVERS

by

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Submitted to the Department of Mechanical Engineering
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ABSTRACT

A dynamic state estimator, called a sliding observer, is developed for a class of nonlinear/uncertain systems. It is shown that a simple observer, basically a Luenberger observer with a relay or saturation function, can be made robust against neglected nonlinearities, disturbances, and uncertainties. The class of uncertainties that this observer can handle is limited to those bounded by constants, and generated by no more sources than the number of available measurements. Two basic design procedures are suggested. The first one guarantees that the state estimates converge to the actual states asymptotically, but is limited to the case in which it is possible that a particular transfer function matrix inside the estimation error dynamics be made strictly positive real. If this is not possible, a second method is suggested which will provide bounded estimation errors. The designer still has some control over these bounds, but the extent to which the accuracy can be improved depends on certain transmission zeros. This thesis concludes with a discussion of the design method when measurement noise is present. A simple example shows that the sliding observer, with a relay, is naturally robust against changes in the measurement noise intensity.

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To my wife and my son,
Sonia and Erik

To my parents,
Tomoaki and Hisae

To the memory of
Hiroshi Osaka
and
Teruo Hayashida

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Chapter 1

Introduction

The development of state estimation methods has been a very active research field in system engineering since Wiener and Kalman pioneered this field in the late forties [67] and in the early sixties [25]. Many estimation schemes were developed for both linear and nonlinear systems, using both off-line and on-line recursive implementations. This thesis is a contribution to this rich field, using the so-called sliding mode technique.

In this introductory chapter, the state estimation problem, as interpreted in this thesis, is defined and motivated. The scope and the main contributions of this thesis is explicitly enumerated and an outline of the whole thesis concludes this chapter.

1.1 Problem Definition

The general state estimation problem can be defined as follows.

Given the general nonlinear system described in the state-space form:

$$\begin{aligned}\dot{x} &= f(x, \theta, t, u, w) \\ y &= h(x, \theta, t, u, v)\end{aligned}$$

where x is the vector of internal states; y , the vector of measurements; u , the vector of control inputs and w and v , the vectors of process (input) and measurement noises respectively. The system parameters are contained in the vector θ , and the system might have explicit dependence on time t .

Given this general system description, the state estimation problem can be stated as: *Design a dynamic system that estimates the states x given that*

the measurements y – and perhaps u – are available. Sometimes, the system parameters are included as augmented states, and the system identification problem – that is, identification of system parameters θ – can be interpreted as a state estimation problem.

A meaningful state estimator will provide state estimates that will converge, in some sense, to the real states. This convergence is interpreted in different ways in different state estimation methods, but in general it can be interpreted as having the state estimation error, defined as the difference between the estimated and actual states, bounded within desired limits.

A desirable feature is that the state estimator has to be able to provide meaningful estimates even in the presence of any kind of modeling errors, unmeasured inputs u , or disturbances w and v .

The state estimator, as defined here, is often found as a system analysis tool, as a way to infer the system's internal behavior from the usually limited number of measurements. A closely related application is as a system identification tool, when one wants to identify important system parameters given system measurements.

For these applications, one can use off-line techniques [34]; therefore computing power is not, in principle, a serious limitation.

Sometimes one wants information in real time. It can be the case for problems of analysis or identification, but definitely it is essential in automatic control applications.

Most multivariable control systems assume that the whole state is available as system measurements, but clearly such an assumption is frequently violated, since the number of measurements is always limited. In this case, one wants to use some state estimator, and then to replace the actual states with estimates in the controller. Obviously, for control applications, the state estimator has to be able to provide reasonably accurate estimates in a short time, leading to the necessity of recursive, on-line, state estimation schemes.

In this thesis, recursive state estimators are called *observers*.

For general nonlinear systems as defined in this section, usually the observers can only be implemented using microprocessors, which means that, if complex nonlinearities have to be handled and fast response is desired, one has to use powerful microprocessors. Because even powerful microprocessors have their limitations, the available computing power becomes a constraint on those observation schemes which are feasible in practice.

Of course, as the application of automatic control systems becomes more complex, often demanding faster and faster response, the limited power of microprocessors starts demanding simple observer structures in order to be implemented.

From this perspective, the ideal state estimation problem can be interpreted as designing a simple observer that can provide state estimations that converge to the actual states within some desired accuracy, even in the presence of uncertainties.

1.2 Scope and Contribution of this Thesis

Of course, the problem formulated in the previous section portrays the ideal observer. For any practical observer, the class of problems that can be handled has to be constrained. In this thesis, it is assumed that the system can be described as:

$$\begin{aligned}\dot{x} &= Ax + D\eta \\ y &= Cx\end{aligned}$$

that is the vector of measurements y are linear combination of the vector of states x , that are generated by a system that has a “linear part” Ax and a “nonlinear/uncertain part” $D\eta$ which can be function of state variables x , time and external disturbances. The pair (A, C) is assumed to be observable, and the rank of matrix D is assumed to be $d \leq m$, where m is the number of measurements. This class of problems is not as restricted as it looks, as further discussed in Chapter 4. Certainly, the process noise can be included in the uncertainty term, and the effect of noise in the measurement will be discussed in Chapters 3 and 6.

For this system, a simple observer structure is suggested:

$$\dot{\hat{x}} = (A - HC)\hat{x} + Hy + K1_s(y - C\hat{x})$$

It is the Luenberger observer [35,36,37] plus an additional term $K1_s$. The function 1_s is usually defined using the *signum* function or saturation function. This observer will be called **Sliding Mode Observer**, or simply **Sliding Observer**(sometimes abbreviated as SO) in this thesis.

Given these definitions, the main contribution of this thesis can now be listed as follows:

1. A simple observer is suggested.
2. It is shown that this observer can be made robustly stable against the class of nonlinearities and uncertainties as defined above.
3. It is shown that under some conditions the state estimates will converge asymptotically to the actual states. Under less restricted conditions, it will be shown that the state estimation errors will be bounded, and these bounds can be brought into acceptable limits depending on the location of the zeros of a specified transfer function matrix.
4. It is shown that in the presence of measurement noise, the sliding observer using *signum* function in the function 1, is insensitive to changes in the measurement noise intensity.

1.3 Overview

This thesis is organized as follows. Chapter 2 reviews most of the state-of-the-art methods for observer design for nonlinear/uncertain systems. This review will show the design method, without proofs, and the main advantages and drawbacks will be highlighted. The objective of this review is twofold: one objective is to provide the reader with useful results for nonlinear state estimation, and other is to set the stage for the introduction of the sliding observer.

The sliding observer is introduced in Chapter 3. This chapter will show the potential advantages of sliding observers. In this chapter, the class of problems is actually more extensive than the class of problems considered in the rest of this thesis. The observer is also allowed to contain more nonlinear terms in it. The design and analysis of a sliding observer for a second-order system is made using phase-plane analysis and concepts from sliding mode systems. An example illustrates the performance of a sliding observer designed for a nonlinear plant. The conclusions from phase-plane analysis, coupled with the usual properties of sliding mode systems, are extended to higher order systems, on an *ad-hoc* basis. A simple third-order example will show that these extensions can actually fail, and the estimates will diverge. This motivates the need for more detailed studies and improved design techniques.

The next chapter, Chapter 4, presents the design techniques suggested and proved in this thesis. In this chapter all the main results will be presented. The general framework will be set using the *Passivity Theorem*¹. The difficulties related to the straight application of the Passivity Theorem in the observer design will be commented upon, and more useful results will be derived for four particular situations, two of them for single-measurement cases, and two for multiple-measurement cases. The properties of estimation errors will also be discussed.

Chapter 5 shows three “real-life” examples for which the sliding observer is designed. Some digital simulations will show how these observers should perform in actual applications.

The next chapter, Chapter 6, discusses the design changes due to the presence of measurement noise. The main tool used in this chapter is the Random-Input-Describing-Function. It is shown that, for stochastic settings the sliding observer can actually be made equivalent to the optimal filter (Kalman Filter) with the advantage of being insensitive to changes in measurement noise intensity.

Chapter 7 provides a summary of the conclusions and suggestions for future research.

¹All the definitions and results from input-output stability used in this thesis are summarized in appendix A.

Chapter 2

Review of Current Methods for Observer Design for Nonlinear Systems

2.1 Overview

Many authors have worked on the development of state estimators for nonlinear and/or uncertain systems. Some observers were developed for a restricted class of nonlinear systems such as bilinear systems [19,10]. Many results became available for a broader class of problems during recent years.

This chapter presents a brief review of currently available recursive state estimation techniques applicable to a broad class of nonlinear systems. The goal is to provide the reader with the knowledge of what is currently available, in order to put the Sliding Observer in its proper perspective.

Each method will be introduced with a brief description of its theoretical basis, followed by a procedure description. A list of strengths and weakness will conclude the method discussion.

The presentation will start with the Extended Kalman Filter, which pioneered the current approach to state estimation techniques. The Extended Kalman Filter does not have a formal proof of convergence, even though it has been used for quite long time in practice [54]. The only exception is the Constant Gain Extended Kalman Filter, which is known to have guaranteed robustness, and is described in Section 2.3

An alternative way to design state estimators for nonlinear systems is

through the use of Describing Functions. This is the topic of section 2.4 which describes the Statistically Linearized Filter.

A different approach, that applies a nonlinear transformation, known as Global Transformation, converts a set of nonlinear differential equations into a set of linear differential equation, is described in Section 2.5

Because a global transformation is not always possible, Reboulet and Champetier introduced the concept of Pseudo-Linearization, which sometimes allows the transformation of a system description from the original form to a canonical form for which one can design a Luenberger Observer with an additional nonlinear term, which can be known from the measurements. This method is discussed in Section 2.6.

The next method to be discussed is the one that provides the Luenberger Observer with varying gains based on the method of *gain scheduling*; this method is called Extended Linearization method.

In 1973, Thau [58] described the sufficient condition for asymptotic convergence of the state estimates and suggested a particular structure of observer. His approach was further developed by Kou et al. [30], Banks [2] and Tarn and Rasis [57]. The basic observer was an extension of the Luenberger Observer, and is shown in section 2.8.

For time-varying and/or uncertain linear systems adaptive observer is a valid option. This method fits in our discussion if we view nonlinear systems as linear systems with time varying parameters. Section 2.9 discusses an adaptive observer suggested by Gevers and Basin in 1986 with guaranteed convergence for a class of second order system with bounded uncertainties/nonlinearities.

Another method, presented in this review, is the linear filter which applies to a linear system with uncertain parameters; It was presented by Wang et all and uses a set-theoretic point of view associated with Pontryagin's Minimum Principle.

Within the framework of variable-structure systems, Walcott and Żak [63] introduced a variable structure observer which was shown to have perfect modelling error rejection under certain conditions. It is the subject of the last method discussed in this chapter.

A summary will point to the main points of these observer(filter) design techniques and will highlight the gaps that still exist. The claim is that the sliding observer will fill many of these gaps.

2.2 Extended Kalman Filter

Kalman, in 1960 [25], introduced the concept of an *optimal linear filter*; this filter minimizes the mean square estimation error and is now known as the *Kalman Filter*[16]. This filter assumes that the dynamic system whose states we want to estimate can be described as a set of linear differential, or difference, equations. Furthermore a key assumption is that the model reflects the real system exactly.

Because systems are seldom linear, a natural extension called the *Extended Kalman Filter* was developed, and it was the subject of extensive research as documented in the collection of papers edited by Sorenson [54].

The Extended Kalman Filter is derived by minimizing the trace of the covariance matrix of estimation errors. The derivation is shown in full detail in [16], and is not shown here. The final result shown in [16] is reproduced below:

System model:

$$\begin{aligned} \dot{x}(t) &= f(x(t), t) + w(t), \quad w(t) \sim N(0, Q(t)) \\ z(t) &= h(x(t), t) + v(t), \quad v(t) \sim N(0, R(t)) \end{aligned} \quad (2.1)$$

where $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$, $h : \mathbb{R}^n \rightarrow \mathbb{R}^m$; $x \in \mathbb{R}^n$, $w \in \mathbb{R}^n$, $z \in \mathbb{R}^m$, $v \in \mathbb{R}^m$, $m \leq n$, and v and w are assumed to be uncorrelated zero mean Gaussian noise with noise intensities $R(t)$ and $Q(t)$ respectively. The initial condition is assumed to be $x(0) \sim N(\hat{x}_0, P_0)$.

For this system, the filter is implemented as:

$$\begin{aligned} \dot{\hat{x}} &= f(\hat{x}(t), t) + K(t)[z(t) - h(\hat{x}(t), t)] \\ K(t) &= P(t)H^T(\hat{x}(t), t)R^{-1}(t) \end{aligned} \quad (2.2)$$

$$\begin{aligned} \dot{P}(t) &= F(\hat{x}(t), t)P(t) + P(t)F^T(\hat{x}(t), t) + Q(t) + \\ &\quad - P(t)H^T(\hat{x}(t), t)R^{-1}(t)H(\hat{x}(t), t)P(t) \end{aligned} \quad (2.3)$$

$$F(\hat{x}(t), t) = \frac{\partial f(x(t), t)}{\partial x(t)} \quad (2.4)$$

$$H(\hat{x}(t), t) = \frac{\partial h(x(t), t)}{\partial x(t)} \quad (2.5)$$

where equations (2.4) and (2.5) are evaluated at $x(t) = \hat{x}(t)$.

The Extended Kalman Filter is an obvious extension of the Kalman Filter. Because of its similarity to the Kalman Filter, it is widely used [54], but some drawbacks are worth remembering:

- Because P is only an approximation of the true covariance matrix, there is no *a priori* performance or stability guarantee;
- Comparing equations (2.1) and (2.2) (also (2.4) and (2.5)) one can readily see that perfect system knowledge is assumed;
- evaluating equations (2.4) and (2.5) at $x = \hat{x}$ can introduce (even if f is the exact model) arbitrarily large errors in F and H ;
- One conclusion that follows from previous observations is that no robustness against modelling errors can be guaranteed *a priori*;
- The implementation requires a great deal of real time computer power since the filter and covariance equations are coupled.

Some of these drawbacks can be attenuated through additional improvement like the Iterated Extended Kalman Filter or the Second-Order Kalman Filter [16] with the expense of additional computational burden. In spite of the improvements, it is still true that the robustness cannot be guaranteed. The lack of guaranteed robustness and difficulties in implementation motivated the investigation of *Constant Gain Extended Kalman Filter*.

2.3 Constant Gain Extended Kalman Filter

To overcome the substantial real-time computational burden imposed by the Extended Kalman Filter, Safonov and Athans [45,47] suggested the use of *Constant Gain Extended Kalman Filter*. They also showed the conditions under which a class of nonlinear observers is *nondivergent*. The particular case of Constant Gain Extended Kalman Filter was shown to have an intrinsic robustness against modelling errors.

The detailed development is given in [45,47], and will not be shown here. To understand the final results it is necessary to define some terms.

Let \mathcal{F} be an operator, i.e. a mapping of functions into functions, like a *transfer function* that takes the input function and gives the output functions. The derivative of \mathcal{F} , or the *Gateaux derivative* $\nabla\mathcal{F}[z_0]$, is defined via :

$$\nabla\mathcal{F}[z_0]z = \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} (\mathcal{F}(z_0 + \epsilon z) - \mathcal{F}z_0)$$

Note that for memoryless or nondynamical \mathcal{F} (i.e. $(\mathcal{F}z)(t) \equiv f(z(t))$), $\nabla\mathcal{F}[z_0]$ is just the Jacobian $(\partial f/\partial z)(z_0)$.

Using this notation, the estimation problem can be formulated as follows.

Given the system:

$$\begin{aligned} \dot{x} &= A[w]x + B[w]u + \xi \\ y &= C[w]x + \theta \end{aligned} \quad (2.6)$$

where w is a vector of functions including y , u , t as well as all other known or observed functions; ξ and θ are the disturbances.

The observer structure is chosen to be:

$$\begin{aligned} \dot{\hat{x}} &= A[w]\hat{x} + B[w]u - H[w](\hat{y} - y) \\ \hat{y} &= C[w]\hat{x} \end{aligned} \quad (2.7)$$

The dynamics of the estimation error $e = \hat{x} - x$ is found by combining equations (2.6) and (2.7) :

$$\begin{aligned} \dot{e} &= \bar{A}[w, x]e - H[w]r - \xi \\ r &= \bar{C}[w, x]e - \theta \end{aligned} \quad (2.8)$$

where $r = \hat{y} - y$, $\bar{A}[w, x]z = A[w](x + z) - A[w]x$, and $\bar{C}[w, x]z = C[w](x + z) - C[w]x$.

Assuming that equation (2.8) can be linearized one gets:

$$\begin{aligned} \dot{e} &= A[w]e - H[w]r - \xi; \quad e(0) = 0 \\ r &= C[w]e - \theta \end{aligned}$$

At this level of generality, the following theorem ([45,47]) states the conditions under which the estimator will be nondivergent:

Theorem 2.1 *Let the w -dependent matrix $S[w]$ and the constant matrix P be symmetric uniformly positive-definite solutions of the w -dependent Lyapunov equation :*

$$(A[w] - H[w]C[w])P + P(A[w] - H[w]C[w])^T + S[w] = 0$$

If uniformly for all x, w :

$$A[w] - \nabla(A[w])[x] - H[w](C[w]) - \nabla(C[w])[x]P + \frac{1}{2}S[w] > 0$$

then the nonlinear observer (2.7) is nondivergent with finite gain, that is the mapping of the process and measurement noises ξ and θ into estimation error e is such that:

$$\begin{aligned} \eta &= \begin{bmatrix} \xi & \theta \end{bmatrix} \\ e &= x - \hat{x} \in \{e \in \mathcal{L}_e \mid \|e\| \leq k\|\eta\|\} \end{aligned} \tag{2.9}$$

for some $k < \infty$.

Now, if we take the gain $H[w]$ as the Kalman Filter gain, i.e.:

$$H[w] = \Sigma[w]C^T[w]\Theta^{-1}[w]$$

where $\Sigma[w]$ is symmetric, positive definite, and satisfies the filter algebraic Riccati equation:

$$\Sigma[w]A^T[w] + A[w]\Sigma[w] - \Sigma[w]C^T[w]\Theta^{-1}[w]C[w]\Sigma[w] + \Xi[w] = 0$$

where $\Xi[w]$ and $\Theta[w]$ are w -dependent positive definite covariance matrices of the disturbances ξ and θ , respectively.

The *Constant Gain Extended Kalman Filter* is the filter given by equation (2.7) when the gain matrix H becomes a constant, i.e. independent of w . The admissible deviation from the linear model is given by the following theorem [47]:

Theorem 2.2 *If Σ is independent of w and if uniformly for all x and all w*

$$\begin{aligned} & \{A[w] - \nabla(A[w])[x] - H(C[w] - \nabla(C[w])[x])\}\Sigma + \\ & + \frac{1}{2}(\Xi[w] + \Sigma C^T[w]\Theta^{-1}[w]C[w]\Sigma) > 0 \end{aligned}$$

then the Constant Gain Extended Kalman Filter is nondivergent with finite gain.

Through the use of this theorem and by lumping all nonlinearities/uncertainties in the actuators and sensors, Safonov and Athans showed that the Constant Gain Extended Kalman Filter has the typical properties of the LQG-controller/Kalman Filter [45], i.e. infinite gain margin, at least 50 percent gain reduction tolerance, and at least ± 60 deg phase margin in each output of the error dynamics given by equation (2.8).

In spite of their guaranteed robustness, the margins given above can still be quite restrictive, e.g. when one deals with hard nonlinearities.

Also, considering that it is very likely that the matrices $A[w]$, $B[w]$ and $C[w]$ were determined by linearization about operating points, the differentiation of equation of motion involved in the linearization process might also unfavourably affect the guaranteed robustness property.

2.4 Statistically Linearized Filter

Another approach, which uses the *Describing Function*[17], was suggested by Phaneuf[43]. It performed better than the Extended Kalman Filter on re-entry trajectories[5] because it used *Gaussian Statistical Linearization*.

Assume that a system is given as:

$$\begin{aligned} \dot{x}(t) &= f(x(t), t) + w(t); \quad w(t) \sim N(0, Q(t)) \\ z &= h(x) + v; \quad v(t) \sim N(0, R(t)) \end{aligned}$$

with initial conditions given by $x(0) \sim N(\hat{x}_0, P_0)$. Also it is assumed that v and w are uncorrelated.

The describing functions for $f(x)$ and for $h(x)$ are defined as:

$$N_f(P) = E[fx^T]P^{-1}$$

$$\begin{aligned}
N_h(P) &= E[hx^T]P^{-1} \\
P &= \text{covariance matrix of } x \\
&= E[(x - E[x])(x - E[x])^T]
\end{aligned}$$

If the exact probability density function of $x(t)$ is known, then the describing function can be determined exactly. In practice this is impossible, therefore usually the expectations are performed with the stationary gaussian density functions with zero means.

Using these describing functions, the statistically linearized filter becomes [5,16,43]:

$$\begin{aligned}
\dot{\hat{x}} &= f(\hat{x}) + H(z - h(\hat{x})) \\
H &= \tilde{P}N_h^T(\hat{x}, \tilde{P})R^{-1} \\
\dot{\tilde{P}} &= N_f(\hat{x}, \tilde{P})\tilde{P} + \tilde{P}N_f^T(\hat{x}, \tilde{P}) + Q - HRH^T
\end{aligned}$$

This filter suffers the drawback of requiring substantial real-time computing power. An alternative, which requires less computation, was suggested by Beaman [5] :

$$\dot{\hat{x}} = N_f(P)\hat{x} + H[z - N_h(P)\hat{x}] \quad (2.10)$$

$$H = \tilde{P}N_h^T(x, P)R^{-1} \quad (2.11)$$

$$\begin{aligned}
0 &= (N_f(P) - HN_h(P))\tilde{P} + \\
&+ \tilde{P}(N_f(P) - HN_h(P))^T + HRH^T + Q
\end{aligned} \quad (2.12)$$

$$0 = N_f(P)P + PN_f^T(x, P) + Q \quad (2.13)$$

One can notice that equation (2.13) can be solved first; this allows the determination of describing functions N_f and N_h . Then (2.12) can be solved, hence the gain matrix H can be computed from (2.11). The knowledge of describing functions and the matrix gains permit the evaluation of estimates \hat{x} through equation (2.10).

The advantage of the Statistically Linearized Filter is that its performance can be better than the Extended Kalman Filter. Its performance is a function of the matching (or mismatch) between the true statistical properties of x and those assumed for the describing function. One consequence is that performance, and even stability, can not be guaranteed beforehand.

2.5 Global Linearization Methods

For the sake of simplicity, consider the single-output system given by:

$$\begin{aligned}\dot{x} &= f(x) ; & x(t_0) &= x_0 \\ y &= h(x)\end{aligned}\tag{2.14}$$

where y is the single-measurement. Following Bestle and Zeitz [6], who first introduced this concept, *assume* that there exists a nonlinear transformation T given by:

$$x = T(x^*)\tag{2.15}$$

so that in the new coordinates x^* , the system can be described in the observer form :

$$\begin{aligned}\dot{x}^* &= A^*x^* - f^*(x_n^*) \\ y &= x_n^*\end{aligned}\tag{2.16}$$

$$A^* = \begin{bmatrix} 0 & 0 & \dots & 0 & 0 \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & & & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & \dots & 1 & 0 \end{bmatrix}$$

$$f^*(x_n^*) = \begin{bmatrix} f_0^*(x_n^*) \\ f_1^*(x_n^*) \\ f_2^*(x_n^*) \\ \vdots \\ f_{n-1}^*(x_n^*) \end{bmatrix}$$

Because $f^*(x_n^*)$ is only a function of x_n^* (the measurements), the observer can be chosen as:

$$\dot{\hat{x}}^* = A^*\hat{x}^* - f^*(x_n^*) - H(\hat{x}_n^* - y)$$

where H is the gain matrix $H^T = [h_0 \ h_1 \ \dots \ h_{n-1}]$.

Using this observer the error dynamics $\tilde{x}^* = \hat{x}^* - x^*$ is given by:

$$\dot{\tilde{x}}^* = \tilde{A}^*\tilde{x}^*\tag{2.17}$$

$$\bar{A}^* = \begin{bmatrix} 0 & 0 & \dots & 0 & -h_0 \\ 1 & 0 & \dots & 0 & -h_1 \\ 0 & 1 & & & -h_2 \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & \dots & 1 & -h_{n-1} \end{bmatrix}$$

Because the error dynamics given by equation (2.17) are linear, the observer design can be performed using any linear technique (e.g. pole-placement).

The case of multiple outputs is treated in the similar fashion; the difference is in the use of block matrices, each one corresponding to a particular output.

The key step in this method is the transformation (2.15) that takes the original system described in (2.14) into observer canonical form (2.16). Walcott et al [62], using arguments given by Bestle and Zeitz [6] showed that the necessary transformation $x = T(x^*)$, in the single output case, can be determined, under some restrictions, as follows:

$$\begin{bmatrix} L_f^0(dh)(x) \\ L_f^1(dh)(x) \\ \vdots \\ L_f^{n-1}(dh)(x) \end{bmatrix} \frac{\partial T}{\partial x_1^*} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} \quad (2.18)$$

Noting that the matrix on the left hand side is the observability matrix (see Appendix B), one can say that it is nonsingular if the system is locally weakly observable. Therefore $\frac{\partial T}{\partial x_1^*}$ can be determined.

The full Jacobian matrix can then be computed:

$$\frac{\partial T}{\partial x^*} = \left[(ad^0 f, \frac{\partial T}{\partial x_1^*}) \quad (ad^1 f, \frac{\partial T}{\partial x_1^*}) \quad \dots \quad (ad^{n-1} f, \frac{\partial T}{\partial x_1^*}) \right] \quad (2.19)$$

The transformation $T(x^*)$ can be determined by the integration of the Jacobian matrix (2.19).

A similar procedure was shown to exist for multiple-output case [32,62].

The existence of desired transformation (2.15) is associated with the integrability of (2.19). This condition is given by the following theorem presented by Krener and Respondek [32] :

Theorem 2.3 Let $g^1(\xi), \dots, g^p(\xi)$ be the vector fields defined by

$$L_{g^j} L_f(y_i) = \begin{cases} 0 & \text{if } 0 \leq l < k_i - 1 \\ \delta_i^j & \text{if } l = k_i - 1 \end{cases}$$

where δ_i^j is the Kronecker delta symbol and k_i are the observability indices.

The change of coordinates $x = T(x^*)$ exists if and only if

$$[ad^k(-f)g^i, ad^l(-f)g^j] = 0$$

for $i, j = 1, \dots, p; k = 0, \dots, k_i - 1; l = 0, \dots, k_j - 1$

This family of observers was further developed by Keller [27] and Zeitz [68].

The observer design is reduced to a simple linear design, once the transformation is made. Unfortunately this is not always possible, as suggested by the above theorem. Furthermore, successive manipulation of f and h (i.e. taking partial derivatives, Lie derivatives and Lie Brackets) to find the necessary transformations brings up the issue of robustness. In principle this method requires a perfect knowledge of the system being observed.

2.6 Pseudo-Linearization Methods

A different transformation, that takes the original system into an observer canonical form, was proposed by Nicosia et al. [41] and Reboulet and Champetier [44]. The method, called *Pseudo-Linearization*, takes the system given by:

$$\begin{aligned} \dot{x} &= f(x, u) \\ y &= h(x) \end{aligned} \quad (2.20)$$

and transforms it, near operating points, to the form:

$$\begin{aligned} \dot{z} &= A^* z + \gamma(y, u) \\ y &= z_n \end{aligned} \quad (2.21)$$

$$A^* = \begin{bmatrix} 0 & 0 & \dots & 0 & 0 \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & & & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & \dots & 1 & 0 \end{bmatrix}$$

through successive transformations ([41] and [44]) that involves linearizations about the set of operating points and a state variable transformation that brings the system in the form above. When this transformation can not be performed exactly, it is obtained (whenever possible) by neglecting second and higher order terms near the operating points. Actually, comparing this method with the global linearization method one can see that the final observable form is exactly the same, therefore the observer design is carried out in the same manner.

The difference is the procedure used to transform the system into the observable canonical form (2.21). The existence of the desired transformation can be shown in a constructive way for a single-input-single-output (SISO) system.

Assume that the system, given by (2.20), has a set of operating points:

$$S_{x,u} = \{(x_0, u_0) \mid f(x_0, u_0) = 0\}$$

Now, *suppose* there exists a state space transformation :

$$\begin{aligned}\psi_1 &= h(x) \\ \psi_i &= \psi_i(x) ; i = 2, \dots, n\end{aligned}$$

such that $S(\psi, u) = \{(\psi^0, u^0) : \psi_2^0 = \dots = \psi_n^0 = 0; u^0 = u(\psi_1^0)\}$.

In this new coordinate, the system becomes:

$$\begin{aligned}\dot{\psi} &= \tilde{f}(\psi, u) \\ y &= \psi_1\end{aligned}$$

The linearized system about operating point ψ_1 is then:

$$\begin{aligned}\dot{\delta\psi} &= A(\psi_1)\delta\psi + B(\psi_1)\delta u \\ \delta y &= \delta\psi_1\end{aligned}\tag{2.22}$$

Now, *assume* that the linearized system (2.22) is observable. Then one can make an additional transformation $z = z(\psi)$ so that

$$\begin{aligned}\dot{\delta z} &= A^*\delta z + \gamma_1(y^0, u^0)\delta y + \gamma_2(y^0, u^0)\delta u \\ \delta y &= \delta z_n\end{aligned}\tag{2.23}$$

Based on equation (2.23), and neglecting second and higher order terms—provided that they can be neglected— one can describe the original system, near operating points, by:

$$\begin{aligned}\dot{z} &= A^*z + \gamma(y, u) \\ y &= z_n\end{aligned}$$

which is the desired form (2.21).

The advantage of this method is that the observer design becomes very easy, once the observer canonical form is obtained. The transformation is possible for a class of problems for which the global linearization might fail. The above description shows how to construct such a transformation for SISO systems; it does not apply for Multiple-Input-Multiple-Output (MIMO) systems however, and in this case no formal method is known.

The drawbacks of this method are :

- only local properties can be guaranteed;
- perfect knowledge of the system is required, in order to build the transformation. Some uncertainties are allowed if they only affect higher order terms in the observer form (2.21).

2.7 Extended Linearization Methods

The *Extended Linearization Method* is another method which exploits the useful tools available for linear systems. It falls into the category of *gain scheduling* methods, i.e. it gets a desired behavior for a family of linearized systems when in the neighborhood of a set of constant operating points. Specifically for this case, Baumann and Rugh [4] suggest how the eigenvalues of the estimation error dynamics can be made invariant with respect to the operating points parameterized by a set of constant inputs.

Let us assume that the system is given by :

$$\begin{aligned}\dot{x} &= f(x, u) ; x(t=0) = x_0 \\ y &= h(x)\end{aligned}\tag{2.24}$$

and let the observer be:

$$\begin{aligned}\dot{\hat{x}} &= f(\hat{x}, u) + g(y) - g(\hat{y}) \\ y &= h(\hat{x})\end{aligned}\tag{2.25}$$

where the analytical function $g : \mathbb{R}^p \rightarrow \mathbb{R}^n$ is to be determined, with $g(0) = 0$.

The dynamics of estimation error $\tilde{x} = x - \hat{x}$ is :

$$\dot{\tilde{x}} = f(x, u) - f(x - \tilde{x}, u) - g(y) + g(\hat{y}) \quad (2.26)$$

Linearizing about constant operating point, $\tilde{x} = 0$ and $u = \epsilon$, one can write :

$$\dot{\tilde{x}} = [D_1 f(x, \epsilon) - Dg(y) Dh(x)] \tilde{x}$$

where: $D_1 f = \frac{\partial f}{\partial \tilde{x}}$, $Dg = \frac{\partial g}{\partial y}$, $Dh = \frac{\partial h}{\partial \tilde{x}}$.

If the pair $(D_1 f, Dh)$ is observable, then a constant matrix C (parameterized by ϵ) can be determined so that $[D_1 f(x, \epsilon) - C(\epsilon) Dh(x)]$ has arbitrary poles at each operating point parameterized by $u = \epsilon$.

Gain scheduling can be performed if one can find a function g , such that

$$Dg(y_\epsilon) = C(\epsilon)$$

Then, the function $g(\epsilon)$ can be found by integration. The resulting observer (2.25) will have invariant time constants with respect to the operating points.

Bauman and Rugh [4] also showed that a similar procedure can be used for reduced order observers.

The natural question that arises is when does the function g exist? The sufficient condition for the existence of gain function $g(\epsilon)$ is given by the following theorem [4] :

Theorem 2.4 *Suppose that the analytic system :*

$$\begin{aligned} \dot{x} &= f(x, u) ; x(t=0) = x_0 \\ y &= h(x) \end{aligned}$$

is such that $D_1 f(0,0) = [\frac{\partial f}{\partial \tilde{x}}]$ evaluated at $x = 0, u = 0$ is invertible (i.e. the system has no free integrators), $(D_1 f(0,0), Dh(0))$ is an observable pair and $Dy(0) \neq 0$.

Then there exists an analytic function $g(\cdot) : \mathbb{R}^p \rightarrow \mathbb{R}^n$ with $g(0) = 0$ such that the eigenvalues of the linearized equation

$$\dot{\tilde{x}} = [D_1 f(x, \epsilon) - Dg(y) Dh(x)] \tilde{x}$$

are locally invariant with respect to ϵ .

The clear advantage of this method is that it offers an extension of the Luenberger Observer, formalizing the concept of gain scheduling, guaranteeing nominal performance with no reference to robustness, in the neighborhood of constant operating points.

The drawbacks are that only local behavior can be guaranteed, i.e., the eigenvalues are constant in a sufficiently small neighborhood of constant operating points. If disturbances and modelling errors are present, then the performance and stability cannot be guaranteed.

2.8 Thau's Method

The method introduced by Thau, in 1973, is basically a verification method, rather than a design method. In his paper, Thau [58] gave a sufficient condition for estimate convergence. The original work was further extended for deterministic problem by Kou et al. [30] and Banks [2], and for the stochastic case by Tarn and Rasis [57].

The original Thau's method assumes that the system is given by:

$$\begin{aligned}\dot{x} &= f(t, x) \\ y &= h(t, x)\end{aligned}\tag{2.27}$$

The observer is assumed to have a structure given as:

$$\dot{z} = g(t, y, z)$$

where $z = \Phi(\hat{x})$, Φ is invertible.

Thau showed that,

- if g and Φ are such that :

$$\frac{\partial \Phi}{\partial x} f(t, x) = g(t, h(t, x), \Phi(x))$$

- if in the process of linearization of the error dynamics, the second and higher order terms can be neglected by observing that:

$$\frac{\|G_2(t, e)\|}{\|e\|} \rightarrow 0$$

uniformly in t , as e goes to zero; where e and $G_2(t, e)$ are defined as

$$\begin{aligned} e &= z - \phi(x) \\ \dot{e} &= g_1(t, e) - g_2(t) \\ g_1(t, e) - g_2(t) &= G_1(t)e + G_2(t, e)e \end{aligned}$$

- if the linearized system

$$\dot{e} = G_1(t)e$$

is uniformly asymptotically stable

then the local asymptotic stability can be attained through a suitable linearized observer.

A stronger result was derived by Kou et al. [30], for the system given by (2.27). Suppose the observer is

$$\dot{z} = f(z) + g[h(x), h(z)]$$

Then the asymptotic convergence ($\|x(t) - z(t)\| \rightarrow 0$ as $t \rightarrow \infty$) can be determined if a *Lyapunov-like function*[30] can be found.

In particular, if $g[h(x) - h(z)] = H(h(x) - h(z))$, then a more explicit condition can be determined.

First, say that a matrix function $M(x)$ ($x \in \mathbb{R}^n$) is *Uniformly Negative Definite* (U.N.D.) if there exists an $\epsilon > 0$ such that

$$w^T M(x) w \leq -\epsilon \|w\|^2 ; \text{ for all } (x, w) \in \mathbb{R}^n \times \mathbb{R}^n$$

Using this definition the condition for asymptotic convergence is stated by the following theorem:

Theorem 2.5 *For system (2.27), if there exist a constant $n \times m$ constant matrix H and a positive definite, symmetric $n \times n$ matrix Q so that*

$$Q(\nabla f(x) - H\nabla h(x))$$

is U.N.D., for some $\epsilon > 0$, then the dynamic system

$$\dot{z} = f(z) + H(h(x) - h(z))$$

with any initial condition $z(t_0)$ is an asymptotically convergent observer for (2.27) and

$$\|z(t) - x(t)\| \leq \sqrt{\frac{q_2}{q_1}} \|z(t_0) - x(t_0)\| \exp\left[-\frac{\epsilon}{q_2}(t - t_0)\right]$$

for all $t \geq t_0$, where q_1 and q_2 are the smallest and largest eigenvalues of Q respectively.

A better result can be obtained for a particular class of problems, namely for a system that can be described as:

$$\begin{aligned} \dot{x} &= Ax + \phi(x, y, \dot{y}) \\ y &= Cx \\ \phi(x, y, \dot{y}) &= \phi_1(y) + [\nabla\phi_2(y)]\dot{y} + \phi_3(x) \end{aligned} \quad (2.28)$$

with any $\phi_1, \phi_3 \in C^1$ and $\phi_2 \in C^2$ such that $C[\nabla\phi_2(y)]\dot{y} \equiv 0$.

Defining the matrix norm as $\|A\| = [\lambda_{\max}(A^T A)]^{\frac{1}{2}}$, where $\lambda_{\max}(A)$ ($\lambda_{\min}(A)$) denotes the largest(smallest) eigenvalue of constant matrix A . Also define

$$\|\nabla\phi_3\|_{\infty} = \sup_{x \in \mathbb{R}^n} \|\nabla\phi_3(x)\|$$

The existence of a *exponentially stable* observer is guaranteed by the following theorem:

Theorem 2.6 For the nonlinear system (2.28), if

- (A, C) is an observable pair;
- there exists two positive definite, symmetric, $n \times n$ matrices P, Q and a constant matrix H such that

$$Q(A - HC) + (A - HC)^T Q = -P$$

and

$$\frac{1}{2} \frac{\lambda_{\min}(P)}{\lambda_{\max}(Q)} > \|\nabla\phi_3\|_{\infty}$$

then there exists an asymptotically stable observer for (2.28).

These results were extended to a more general system by Banks [2]. One particular case is treated by Thau [58]. Assume the system :

$$\begin{aligned}\dot{\hat{x}} &= Ax + Bu + f(x) \\ y &= Cx\end{aligned}\tag{2.29}$$

and choose the observer as:

$$\dot{\hat{x}} = A_0\hat{x} + Bu + f(\hat{x}) + Hy\tag{2.30}$$

where $A_0 = A - HC$ has eigenvalues in the open left-half plane. Define the error as $\tilde{x} = \hat{x} - x$. The error dynamics are:

$$\dot{\tilde{x}} = A_0\tilde{x} + f(x + \tilde{x}) - f(x)$$

Because A_0 is Hurwitz (i.e. the eigenvalues are in the open left-half plane), given a positive definite matrix P , there is a constant, symmetric, positive definite matrix Q such that

$$A_0^T Q + Q A_0 = -2P$$

Now, assume that $f(\cdot)$ is locally Lipschitz about the origin, i.e.

$$\|f(x_1) - f(x_2)\| \leq L\|x_1 - x_2\|$$

Using a quadratic Lyapunov function $V = \tilde{x}^T Q \tilde{x}$, Thau showed that the estimation error \tilde{x} converges asymptotically to the origin if :

$$\frac{\lambda_{\min}(P)}{\lambda_{\max}(Q)} > L$$

These particular cases show that if the problem at hand can be reduced to one of these cases, then the observer design can be simple; otherwise, Thau's method only allows the verification of stability.

Even this verification is local in general, unless $f(\cdot)$ is globally Lipschitz. Another drawback is that it does not handle modelling errors.

2.9 Adaptive Observers

A robust adaptive observer for a class of nonlinear second-order systems whose coefficients are bounded and have bounded time-variation was presented by Gevers and Bastin in 1986 [18].

It assumes that the system is given by:

$$y + a_1(y, \dot{y}, t)\dot{y} + a_2(y, \dot{y}, t) + f(y) = b(y, \dot{y}, t)u \quad (2.31)$$

where a_1 , a_2 and b are *unknown* functions of time(they may depend on y and \dot{y}); $f(y)$ is a *known* function of y .

Assume that the following conditions hold:

1. The coefficients a_1 , a_2 and b are continuous with respect to y , \dot{y} and t and differentiable with respect to t ; moreover assume that they are bounded as:

$$0 \leq l_1 \leq a_1 \leq l_2, \quad 0 \leq m_1 \leq a_2 \leq m_2, \quad 0 \leq n_1 \leq b \leq n_2$$

for some finite l_2 , m_2 and n_2 that satisfy:

$$l_2 < \frac{m_2 + 2\sqrt{m_1 m_2} + 5m_1}{\sqrt{m_2} - \sqrt{m_1}}, \quad l_1 > \sqrt{m_2} - \sqrt{m_1}$$

2. Given a k , $0 < k < \infty$, assume that:

$$\begin{aligned} |\dot{a}_1| &\leq k, \quad |\dot{a}_2| \leq k \\ |\dot{b}| &\leq k, \quad |a_1| \leq k, \quad \forall t \in [0, \infty) \end{aligned}$$

3. $u(t)$ and $y(t)$ must belong to U_Δ and $|u(t)| \leq M < \infty$, for all $t \in [0, \infty)$. U_Δ is defined as:

- **definition 1:** U_Δ is a set $\{t_i\}$ of points in $[0, \infty)$ for which there exists a Δ such that for any $t_i, t_j \in U_\Delta$ with $t_i \neq t_j$, $|t_i - t_j| \geq \Delta$.
- **definition 2:** A function $u(t)$ belongs to U_Δ if there exist Δ and U_Δ so that
 - (a) $u(t)$ and $\dot{u}(t)$ are continuous on $\{[0, \infty) - U_\Delta\}$;

- (b) there exist constants M_1 and M_2 so that $|u(t)| < M_1$ and $|\dot{u}(t)| < M_2, \forall t \in \{[0, \infty) - U_\Delta\}$;
(c) $\lim_{t \rightarrow t_i^+}$ and $\lim_{t \rightarrow t_i^-}$ are finite for each $t_i \in U_\Delta$.

4. $\exists \delta > 0, \exists t_0 > 0, \exists \alpha > 0$ so that :

$$\int_t^{t+\delta} W(t)W^T(\tau) d\tau \geq \alpha_i I$$

for all $t \geq t_0$. $W^T(\tau)$ is defined as:

$$W^T(\tau) = \frac{1}{(s + \gamma)^3} [u \quad su \quad s^2u \quad s^3u]$$

for some arbitrary $\gamma > 0$.

5. $f(y)$ is a known bounded function of y and $\exists N, 0 < N < \infty$ such that:

$$|f(y)| \leq N, \quad \forall t \in [0, \infty) \text{ and all } u(\cdot)$$

The system (2.31) can be transformed into an *adaptive observer canonical form*[18] using the change of variables:

$$\begin{aligned} x_1 &= y \\ x_2 &= \dot{y} + (a_1 - c_2)y \\ \theta_1 &= c_2 - a_1 \\ \theta_2 &= \dot{a}_1 - a_2 + c_2 a_1 - c_2^2 \end{aligned}$$

for some positive arbitrary constant c_2 .

And the system (2.31) can be rewritten as:

$$\begin{aligned} \dot{x}(t) &= Rx(t) + \Omega(u, y)\Theta(t) + \begin{bmatrix} 0 \\ f(y) \end{bmatrix} \\ y(t) &= x_1 \end{aligned}$$

where $x^T = [x_1 \quad x_2]$, and:

$$R = \begin{bmatrix} 0 & 1 \\ 0 & -c_2 \end{bmatrix}, \quad \Omega(u, y) = \begin{bmatrix} y & 0 & 0 \\ 0 & y & u \end{bmatrix}, \quad \Theta = \begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{bmatrix}, \quad \theta_3 = b \quad (2.32)$$

Given the system in this form, the adaptive observer is given by:

$$\dot{\hat{x}} = R\hat{x}\Omega(u, y)\hat{\Theta} + \begin{bmatrix} 0 \\ f(y) \end{bmatrix} + \begin{bmatrix} c_1\tilde{x}_1 \\ V^T\dot{\hat{\Theta}} \end{bmatrix}$$

where $V^T = [0 \ v_2 \ v_3]$ and,

$$\begin{bmatrix} \dot{v}_2 \\ \dot{v}_3 \end{bmatrix} = \begin{bmatrix} -c_2 & 0 \\ 0 & -c_2 \end{bmatrix} + \begin{bmatrix} y \\ u \end{bmatrix} ; \quad V(0) = 0$$

$$\Phi = \begin{bmatrix} y \\ v_2 \\ v_3 \end{bmatrix}$$

$$\dot{\hat{\Theta}} = \Gamma\Phi\tilde{x}_1 ; \quad \Gamma > 0$$

where $\tilde{x}_1 = x_1 - \hat{x}_1$, and c_1 , c_2 and Γ are the design parameters that determine the dynamics of the observer.

If the problem at hand falls into the category of problems in the form (2.31), and if the assumptions are all satisfied, the global stability can be guaranteed [18].

There are two drawbacks: the first is that the assumptions are difficult to verify, therefore for uncertain systems they have to be assumed *a priori*, and the second is that the scheme requires a substantial amount of real-time computations due to the seventh-order ordinary differential equation and the definition of its parameters that are coupled.

2.10 Set Theoretic Approach

Consider an observable linear time-invariant system:

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned} \tag{2.33}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^p$ (control vector), $y \in \mathbb{R}^m$ (measurements), A , B , and C are constant matrices of appropriate dimensions. C is assumed to be full row rank.

Suppose that A and B contain some uncertainties, so that they can be written as the sum of nominal parts A_0 and B_0 and perturbation terms ΔA and ΔB , which can be nonlinearities, i.e.

$$\begin{aligned} A &= A_0 + \Delta A \\ B &= B_0 + \Delta B \end{aligned}$$

A reduced order observer is considered in this case, and it is assumed to be in the form

$$\begin{aligned} \dot{z} &= Fz + Gy + Hu \\ \hat{x} &= My + Nz \end{aligned}$$

where $z \in \mathbb{R}^{n-m}$ and \hat{x} is the vector of state estimates.

Assuming that C has the structure $C = [I_m \ 0]$, i.e. the measurements are a subset of the states, one can rewrite the system equation (2.33) in terms of submatrices of A_0 and B_0 [37]. In this way, one can write the matrices F , G , H , M and N as:

$$\begin{aligned} F &= A_{22} + LA_{12} \\ G &= LA_{11} + A_{21} - FL \\ H &= LB_1 + B_2 \\ M^T &= [I_{(n-m)} \ L] \\ N^T &= [0 \ I_m] \\ T &= [L \ I_{(n-m)}] \end{aligned}$$

where the matrix L is the observer design parameter.

The estimation error $\tilde{x} = z(t) - Tx(t)$ becomes:

$$\begin{aligned} \dot{\tilde{x}} &= F\tilde{x}(t) + Tv \\ v &= -(\Delta Ax + \Delta Bu) \end{aligned}$$

The “disturbance” $v(t)$ is assumed to be unknown, but bounded, in the bounding ellipsoidal set sense, i.e.:

$$\begin{aligned} v(t) &\in \Omega \\ \Omega &= \{v(t) | v'Q^{-1}v \leq 1\} \end{aligned}$$

where Q is a positive definite matrix defining the bounds. It can be determined in several ways as described in [55,65].

Using this approach, Scheppe [55] shows that the estimation error at any time t is contained within an ellipsoidal set Ω_e given by:

$$\begin{aligned}\tilde{x}(t=0) &\in \{\tilde{x}|\tilde{x}'\Psi^{-1}\tilde{x} \leq 1\} \\ \Omega_e &= \{\tilde{x}|\tilde{x}'\Gamma^{-1}\tilde{x} \leq 1\} \\ \dot{\Gamma}(t) &= F\Gamma(t) + \Gamma(t)F' + \beta(t)\Gamma(t) + \frac{TQT'}{\beta(t)} \\ \Gamma(t=0) &= \Psi\end{aligned}$$

where $\beta(t)$ is a positive real scalar that gives design freedom for the construction of bounding ellipsoids.

For linear time-invariant systems, assuming ergodicity, one is often interested in the steady state. Assuming that $F + (\beta/2)I$ is stable, the Γ_s is the solution of the Lyapunov Equation:

$$(F + \frac{\beta}{2}I)\Gamma_s + \Gamma_s(F + \frac{\beta}{2}I)' + T\tilde{Q}T' = 0 \quad (2.34)$$

where $\tilde{Q} = \frac{1}{\beta}Q$.

The design objective is to minimize the bounding ellipsoidal set, which means that the estimation error will be reduced, that is determine the matrix L so that the following criterion is minimized:

$$J = \text{tr}(\Gamma_s) \quad (2.35)$$

The solution for this problem is given by the following theorem [65]:

Theorem 2.7 *The observer determined uniquely by the matrix L^* minimizes the criterion (2.35) if and only if the following conditions hold for some positive scalar β :*

1. $L^* = -(\Gamma_s^* A'_{12} + \tilde{Q}_{21})\tilde{Q}_{11}^{-1}$, where the symmetric positive definite matrix Γ_s^* satisfies:

$$\begin{aligned}\Gamma_s^*(A_{22} - \tilde{Q}_{21}\tilde{Q}_{11}^{-1}A_{12} + \frac{1}{2}\beta I)' &+ (A_{22} - \tilde{Q}_{21}\tilde{Q}_{11}^{-1}A_{12} + \frac{1}{2}\beta I)\Gamma_s^* \\ &- \Gamma_s^*A'_{12}\tilde{Q}_{11}^{-1}A_{12}\Gamma_s^* + \tilde{Q}_{22} \\ &- \tilde{Q}_{21}\tilde{Q}_{11}^{-1}\tilde{Q}'_{21} = 0\end{aligned}$$

2. $A_{22} + L^* A_{12}$ has eigenvalues in the open left half plane.

The above condition involves a particular choice of β , which as was stated before, determine the estimation error bounding set. Wang et al.[65] suggested two possible ways to choose β :

- As suggested by equation 2.34, the value of β can be used to guarantee that the eigenvalues of $A_{22} + L^* A_{12}$ are well to the left of some prescribed value;
- another approach is to choose β that minimizes the trace of Γ_0^* , i.e. minimizes the “size” of the bounding set. As shown by Wang et al., this procedure requires the use of a numerical iterative optimization method such as a gradient-search method.

This method exploits the convenience of dealing with a linear system, resulting in an elegant method for observer design. The only drawback is that it only uses a linear correction term, which may not be sufficient for many problems [3].

2.11 Walcott and Žak’s Variable Structure Method

The only *Variable Structure Observer* in the literature, besides the one presented in this thesis, is the observer suggested by Walcott and Žak [63,64].

Let the system be given by:

$$\begin{aligned}\dot{x} &= Ax + f(x, t) + B[u(t) + v(t)] \\ y &= Cx\end{aligned}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^p$, $y \in \mathbb{R}^m$, A, B, C are constant matrices of appropriate dimensions, B, C of full rank and $f(x, t)$ is assumed to be continuous. $v(t)$ is the input disturbance.

This observer requires the following assumptions:

1. The pair (A, C) is detectable. This is equivalent to $A_0 = A - HC$ being Hurwitz for an appropriate choice of H ;

2. $\exists Q > 0$, $Q \in \mathbb{R}^{n \times m}$, and vectors $h(t, x)$ and $w(t)$ so that:

$$\begin{aligned} f(x, t) &= P^{-1}C^T h(x, t) \\ Bv(t) &= P^{-1}C^T w(t) \end{aligned}$$

where P is the (positive definite, unique) solution of the Lyapunov Equation:

$$PA_0 + A_0^T P = -Q$$

3. Let $\xi(x, t) = h(x, t) + w(t)$ and assume that $\|\xi\| \leq \rho$, $\rho > 0$ for all x and positive t .

Let the observer be defined as :

$$\begin{aligned} \dot{\hat{x}} &= A_0 \hat{x} + S(\hat{x}, y) + Hy + Bu \\ S(\hat{x}, y) &= \begin{cases} 0 & ; \text{for all } \tilde{x} \in N \\ \frac{-P^{-1}C^T C \tilde{x}}{\|C \tilde{x}\|} \rho & ; \text{otherwise} \end{cases} \end{aligned}$$

where $N = \{\tilde{x} : C \tilde{x} = 0\}$, $\tilde{x} = \hat{x} - x$.

Walcott and Žak [63,64] show that, if the former assumptions are all satisfied, then the estimation error \tilde{x} goes asymptotically to zero as $t \rightarrow \infty$.

Walcott and Žak also presented a version of this observer using a *boundary layer* [49].

Analyzing the assumptions, one can see that assumptions 1 and 3 do not pose severe restrictions, However, assumption 2 imposes a severe restriction on the structure of nonlinearity/modelling error and disturbances. The necessary condition for satisfying this assumption is that $f(x, t)$ and $Bv(t)$ must be in the row space of observation matrix C , i.e., they must “affect” the measured variables *directly*.

The advantage is that if the assumptions are satisfied, then the observer is insensitive to disturbances/nonlinearities/uncertainties.

As will be shown, this Variable Structure observer and the sliding observer presented in this thesis are identical under some conditions.

2.12 Summary

In this chapter, a fairly comprehensive, but by no means exhaustive, review of nonlinear state estimation methods was made. All methods have their

own positive aspects, either as extensions of linear techniques, or as novel nonlinear techniques.

Unfortunately they all have serious drawbacks. Some of them require perfect knowledge of the real system –no modelling error is allowed – and this is seldom true. Some methods will need powerful microprocessors, because of heavy computational loads. Some do not have any stability guarantee, at least in a global sense.

Other methods have known robustness, but they are limited either because the allowable modelling error is too restricted, or because it is simply too difficult to verify the underlying assumptions.

All these points suggest that a simple and robust observer is required for practical application. The following chapters will try to show that the *Sliding Observer* can meet some of these requirements.

Chapter 3

Fundamentals of Sliding Observers

3.1 Overview

This chapter discusses general features offered by sliding mode control, and which motivated the development of the sliding observer.

General basic concepts of the sliding mode control and the behavior of the sliding observer is discussed in the next section. In this section, an intuitive extension to systems in a companion form is shown, together with a brief analysis of the effect of measurement noise. The section concludes with a brief discussion of general multivariable systems pinpointing a difficulty that exists in designing sliding observers.

Section 3.3 describes the design, and performance of a sliding observer for a nonlinear second-order system, based on simulations in a digital computer. As will be clear, it constitutes a successful design.

Unfortunately, the basic design rules that can be derived from the analysis done in this chapter does not guarantee the convergence of state estimations. It is shown, in Section 3.4, that even for a simple third-order system in canonical form the basic rules may indeed fail. This example will motivate the need for more involved study and for reliable design techniques.

This chapter is concluded with a brief section that highlights its main points. The differences between the sliding mode controller and sliding observer – which are *not dual concepts* – is made clear, and the unavoidable

consequences is presented. These consequences motivates the study done in the following chapters.

3.2 General Features of Sliding Observers

3.2.1 The basic concept of Sliding Mode Systems

The basic idea of a sliding mode, linked to the potential advantages of using discontinuous (switching) control laws, is shown in this section. Consider the dynamic system:

$$\mathbf{x}^{(n)}(t) = f(\mathbf{x}; t) + b(\mathbf{x}; t)u(t) + d(t) \quad (3.1)$$

where $u(t)$ is scalar control input, x is the scalar output of interest, and $\mathbf{x} = [x, \dot{x}, \dots, x^{(n-1)}]^T$ is the state. In equation (3.1) the function $f(\mathbf{x}; t)$ (in general nonlinear) is not exactly known, but the *extent of the imprecision* $|\Delta f|$ on $f(\mathbf{x}; t)$ is upper bounded by a known continuous function of x and t ; similarly control gain $b(\mathbf{x}; t)$ is not known exactly, but is of known sign, and is bounded by known, continuous functions of \mathbf{x} and t . Both $f(\mathbf{x}; t)$ and $b(\mathbf{x}; t)$ are assumed to be continuous in \mathbf{x} . The disturbance $d(t)$ is unknown but bounded in absolute value by a known continuous function of time. The control problem is to get the state \mathbf{x} to *track a specific state* $\mathbf{x}_d = [x_d, \dot{x}_d, \dots, x_d^{(n-1)}]^T$ *in the presence of model imprecision* on $f(\mathbf{x}; t)$ and $b(\mathbf{x}; t)$, *and of disturbances* $d(t)$. To guarantee that this is achievable using a finite control u , one has to assume that:

$$\tilde{\mathbf{x}}|_{t=0} = 0 \quad (3.2)$$

where $\tilde{\mathbf{x}} := \mathbf{x} - \mathbf{x}_d = [\tilde{x}, \dot{\tilde{x}}, \dots, \tilde{x}^{(n-1)}]^T$ is the tracking error vector; the relaxation of this assumption is discussed in greater detail later. One must also define a *time-varying sliding surface* $s(t)$ in the state-space \mathfrak{R}^n as $s(\mathbf{x}; t) = 0$ with

$$s(\tilde{\mathbf{x}}; t) := \left(\frac{d}{dt} + \lambda\right)^{n-1} \tilde{x}, \quad \lambda > 0$$

where λ is a positive constant. Given the initial condition (3.2), *the problem of tracking* $\mathbf{x} \equiv \mathbf{x}_d$ is equivalent to that of remaining on the surface $s(t)$ for all $t > 0_-$; indeed, $s \equiv 0$ represents a linear differential equation whose unique solution is $\tilde{x} \equiv 0$ given the initial conditions (3.2). Now a sufficient

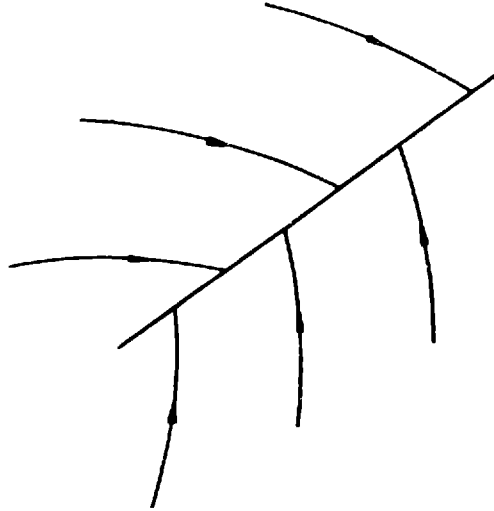


Figure 3.1: The sliding condition

condition for such positive invariance of $s(t)$ is to choose the control law u of (3.1) such that outside of $s(t)$

$$\frac{1}{2} \frac{d}{dt} s^2(\mathbf{x}; t) \leq -\eta |s| \quad (3.4)$$

where η is a positive constant. Inequality (3.4) constrains trajectories to point towards the surface $s(t)$ (Figure 3.1), and is referred to as the *sliding condition*.

The idea behind equations (3.3) and (3.4) is to choose a well-behaved function of the tracking error, s , according to (3.3), and then select the feedback control law u (3.1) so that s^2 remains a Lyapunov function of the closed-loop system *despite the presence of model imprecision and of disturbances*. Furthermore, satisfying (3.3) guarantees that if condition (3.2) is not exactly verified, i.e. if $\mathbf{x}|_{t=0}$ is actually off $\mathbf{x}_d|_{t=0}$, the surface $s(t)$ will nonetheless be reached in a finite time inferior to $|s(\mathbf{x}(0); 0)|/\eta$, while definition (3.3) then guarantees that $\bar{\mathbf{x}} \rightarrow 0$ as $t \rightarrow \infty$. Control laws that satisfy (3.4), however, have to be *discontinuous* across the sliding surface, thus leading in practice to control chattering.

The obvious problem in similarly exploiting sliding behavior in the design of *observers*, rather than controllers, is precisely that the full state is not available for measurement, and thus that a sliding surface definition analog to (3.3) is not adequate. Some intuition can be developed for

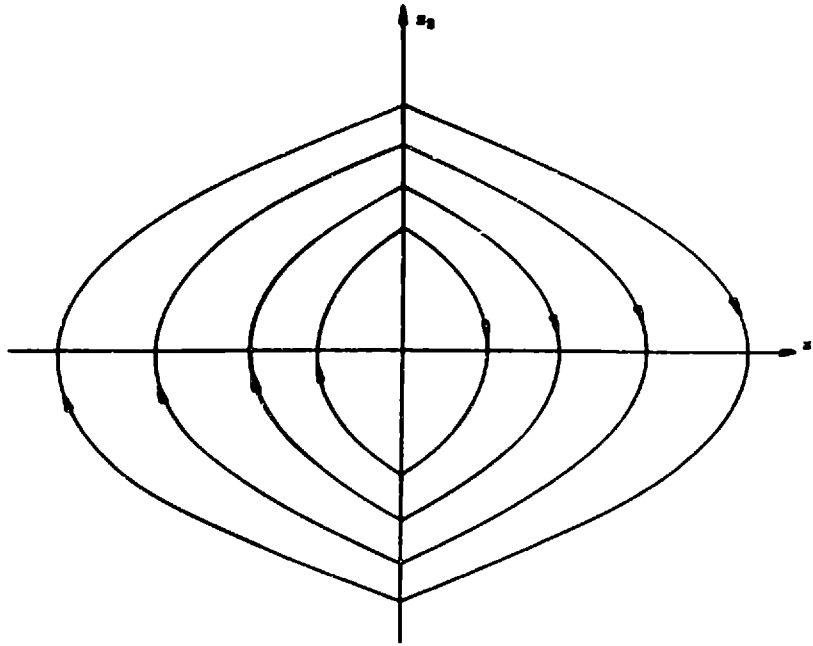


Figure 3.2: Second-order system with single input switching

addressing this difficulty by considering simple second-order dynamics.

3.2.2 Shearing Effect and Sliding Patches

Consider, now, the generation of sliding behavior in a second-order system through input switching according to the value of a *single* component of the state, rather than a linear combination of both components, as in (3.3). The system

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= -k_2 \operatorname{sgn}(x_1)\end{aligned}$$

where k_2 is a positive constant and “sgn” is the sign function, clearly exhibits no sliding behavior (Figure 3.2). On the other hand, consider the system

$$\begin{aligned}\dot{x}_1 &= x_2 - k_1 \operatorname{sgn}(x_1) \\ \dot{x}_2 &= -k_2 \operatorname{sgn}(x_1)\end{aligned}$$

where k_1 and k_2 are positive constants. The corresponding phase-plane trajectories are illustrated in Figure 3.3, which can be constructed from Figure 3.2 by shifting the trajectories on the right half-plane upwards, by

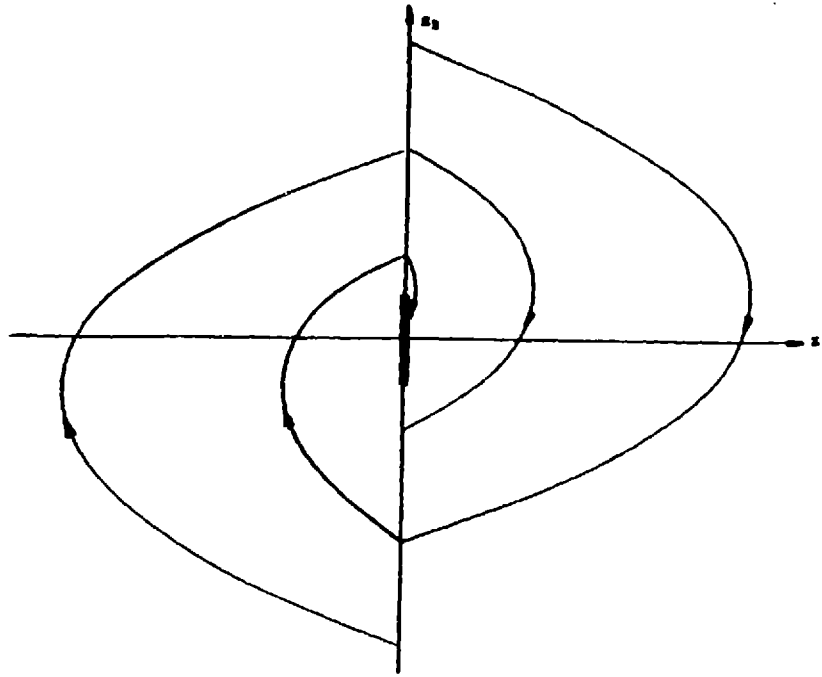


Figure 3.3: Shearing effect

the quantity k_1 , and by similarly shifting the left half-plane trajectories by $-k_1$. This shearing effect generates sliding behavior in the region

$$|x_2| \leq k_1, x_1 = 0 \quad (3.5)$$

which is referred to as the *sliding patch*. The detailed analysis follows. The condition

$$\frac{d}{dt}(x_1)^2 < 0$$

is satisfied if condition (3.5) holds, which defines the sliding patch. The dynamics on the sliding patch itself can be derived from Filippov's solution concept [13], which formalizes engineering intuition: the dynamics on the patch can only be a convex combination (i.e., an *average* of the dynamics on each side of the discontinuity surface

$$\begin{aligned} \dot{x}_1 &= \gamma(x_2 + k_1) + (1 - \gamma)(x_2 - k_1) \\ \dot{x}_2 &= \gamma k_2 + (1 - \gamma)(-k_2) \end{aligned}$$

The value of γ , and therefore the resulting dynamics, are then formally determined by the invariance of the patch itself:

$$\dot{x}_1 = 0 \rightarrow \dot{x}_2 = -\frac{k_2}{k_1} x_2$$

Thus, x_2 exponentially decreases to 0 after reaching the sliding patch, with a time-constant k_1/k_2 . Furthermore, one can easily show that all trajectories starting on the x_2 axis reach the patch in a time lesser than $|x_2(t=0)|/(k_1k_2)$. Actually, sliding can be guaranteed from time $t=0$ by making k_1 and k_2 time-varying, with

$$\begin{aligned}\frac{k_2}{k_1} &\geq a \\ k_1 &> |x_2(t=0)|e^{-at}\end{aligned}$$

where $a = a(t)$ is any positive function of time.

3.2.3 System Damping

Consider now the system

$$\begin{aligned}\dot{x}_1 &= -h_1x_1 + x_2 - k_1 \operatorname{sgn}(x_1) \\ \dot{x}_2 &= -h_2x_1 - k_2 \operatorname{sgn}(x_1)\end{aligned}$$

Repeating the previous analysis, the sliding condition is verified in the extended region

$$\begin{aligned}x_2 &\leq k_1 + h_1x_1 && \text{if } x_1 > 0 \\ x_2 &\geq -k_1 + h_1x_1 && \text{if } x_1 < 0\end{aligned}$$

as illustrated in Figure 3.4. Thus, the addition of the damping term in h_1 increases the region of direct attraction. Furthermore, the value of h_2 only affects the capture phase but not the dynamics on the patch itself, which remains unchanged:

$$\dot{x}_2 = -\frac{k_2}{k_1}x_2$$

3.2.4 Systems in Canonical Form with Single Measurement

Systems with a Single Measurement

Now consider the system

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= f\end{aligned}$$

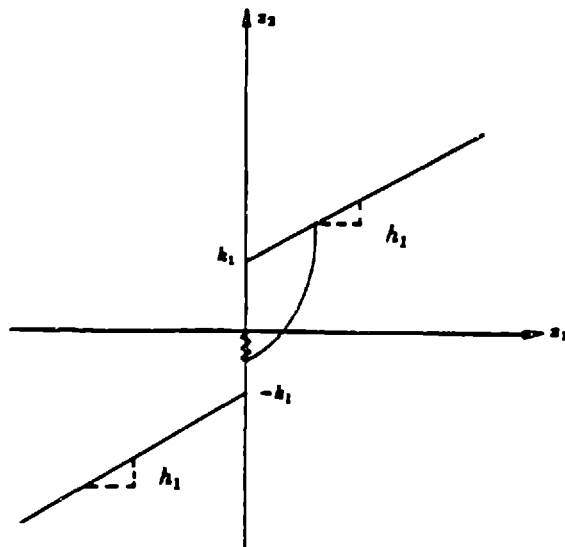


Figure 3.4: Effect of system damping

where f is a nonlinear, uncertain function of the state $\mathbf{x} = [x_1, x_2 = \dot{x}_1]^T$. Exploit the preceding development to design an observer for this system, based on the measurement of x_1 alone. From the previous discussion, one can use an observer structure of the form

$$\begin{aligned}\dot{\hat{x}}_1 &= h_1 \tilde{x}_1 + \hat{x}_2 + k_1 \operatorname{sgn}(\tilde{x}_1) \\ \dot{\hat{x}}_2 &= h_2 \tilde{x}_1 + k_2 \operatorname{sgn}(\tilde{x}_1) + \hat{f}\end{aligned}$$

where $\tilde{x}_1 = x_1 - \hat{x}_1$, \hat{f} is the estimated value of f , and the constants h_i are chosen as in a Luenberger observer (which would correspond to $k_1 = 0, k_2 = 0$) so as to place the poles of the linearized system at desired locations $-\zeta_j$. The resulting error dynamics can be written:

$$\begin{aligned}\dot{\tilde{x}}_1 &= -h_1 \tilde{x}_1 + \tilde{x}_2 - k_1 \operatorname{sgn}(\tilde{x}_1) \\ \dot{\tilde{x}}_2 &= -h_2 \tilde{x}_1 - k_2 \operatorname{sgn}(\tilde{x}_1) + \Delta f\end{aligned}$$

The value of $\Delta f = f - \hat{f}$ depends both on the modelling effort and on the computational complexity allowable in the observer itself. Assume that dynamic uncertainty Δf is explicitly bounded. Known nonlinear terms may also, for simplicity, be treated as bounded error (using known bounds on the

actual system state) and included in Δf . The effect of Δf is compensated by exploiting this knowledge of its generally time-varying bound, as will be later illustrated.

The methodology could, in principle, be directly extended to n^{th} -order systems in companion form:

$$\dot{x}_1^{(n)} = f$$

where x_1 is the single measurement available. The observer structure is then of the form

$$\begin{aligned}\dot{\hat{x}}_1 &= h_1 \hat{x}_1 + \hat{x}_2 + k_1 \text{sgn}(\tilde{x}_1) \\ \dot{\hat{x}}_2 &= h_2 \hat{x}_1 + \hat{x}_3 + k_2 \text{sgn}(\tilde{x}_1) \\ &\vdots \quad \vdots \\ \dot{\hat{x}}_n &= h_n \hat{x}_1 + \hat{f} + k_n \text{sgn}(\tilde{x}_1)\end{aligned}$$

The $n - 1$ poles associated with the implicit dynamics on the patch are defined by

$$\det(pI_{n-1} - \begin{bmatrix} -k_2/k_1 & 1 & 0 & \cdots & 0 \\ -k_3/k_1 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \cdots & 1 \\ -k_n/k_1 & 0 & 0 & \cdots & 0 \end{bmatrix}) = 0 \quad (3.6)$$

Where the I_{n-1} is the identity matrix of order $n - 1$. Thus, the poles on the patch can be placed arbitrarily by proper selection of the ratios k_i/k_1 , $[i = 2, \dots, n]$. A possible choice is to define k_1 as the desired precision in \tilde{x}_2 . Let

$$k_n \geq |\Delta f|$$

and let the ratio k_n/k_1 be constant. Finally define the remaining gains k_i , $[i = 2, \dots, n - 1]$ so that the implicit dynamics associated with the patch is critically damped, i.e., have all poles real and equal to a positive constant λ .

3.2.5 Effects of Measurement Noise

Consider again a second order system with a single measurement, now corrupted by noise $v = v(t)$

$$\dot{\tilde{x}}_1 = -h_1(\tilde{x}_1 + v) + \tilde{x}_2 - k_1 \text{sgn}(\tilde{x}_1 + v)$$

$$\dot{\tilde{x}}_2 = -h_2(\tilde{x}_1 + v) - k_2 \operatorname{sgn}(\tilde{x}_1 + v) + \Delta f$$

Although the presence of the terms in $\operatorname{sgn}(\tilde{x}_1 + v)$ makes an exact stochastic analysis fairly involved, useful insight can be obtained by using appropriate simplifying approximations.

Assume, first, that v is a deterministic C^1 signal of bounded spectrum:

$$0 \leq \omega < \omega_- \text{ or } \omega > \omega_+ \rightarrow F_v(\omega) = 0$$

where F_v is the Fourier transform of v . Sliding behavior, if any, can then only occur on the surface

$$\tilde{x}_1 + v = 0$$

Repeating the analysis previously made, the sliding region is then defined by

$$|\tilde{x}_2 + \dot{v}| \leq k_1$$

and the equivalent dynamics are given by

$$\begin{aligned} \tilde{x}_1 &= -v \\ \dot{\tilde{x}}_2 + \frac{k_2}{k_1} \tilde{x}_2 &= -\frac{k_2}{k_1} \dot{v} + \Delta f \end{aligned}$$

Two limiting cases deserve particular attention:

1. $\omega_+ \ll \frac{k_2}{k_1}$. Then, if $\Delta f = 0$

$$\tilde{x}_2 \approx -\dot{v}, \quad |\dot{v}| \ll \frac{k_2}{k_1} |v|_{max}$$

In particular, the estimate of x_2 is exact if the measurement error in x_1 is constant.

2. $\omega_- \gg \frac{k_2}{k_1}$. Then, if $\Delta f = 0$

$$\tilde{x}_2 = 0$$

The above discussion implies that, as could be expected, the system cannot remain in a pure sliding mode in the presence of arbitrary measurement noise. Further analysis can be made using *Random Input Describing Function*, as shown in Chapter 6.

3.2.6 General Multivariable systems

Consider the n^{th} order nonlinear system:

$$\dot{\mathbf{x}} = f(\mathbf{x}, t) , \mathbf{x} \in \mathbb{R}^n \quad (3.7)$$

and, for convenience, consider a vector of measurements that are linearly related to the state vector:

$$\mathbf{y} = C\mathbf{x} , \mathbf{y} \in \mathbb{R}^m \quad (3.8)$$

Define an observer with the following structure:

$$\dot{\hat{\mathbf{x}}} = \hat{f}(\mathbf{x}, t) + H\tilde{\mathbf{y}} - K\mathbf{1}_s,$$

where $\tilde{\mathbf{x}} \in \mathbb{R}^n$, \hat{f} is our *model* of f , H and K are $n \times m$ gain matrices to be specified, and $\mathbf{1}_s$ is the $m \times 1$ vector

$$\mathbf{1}_s = \left[\text{sgn}(\tilde{y}_1) \quad \text{sgn}(\tilde{y}_2) \quad \cdots \quad \text{sgn}(\tilde{y}_m) \right]^T$$

where

$$\tilde{y}_i = y_i - c_i \hat{\mathbf{x}}$$

and c_i is the i -th row of the $m \times n$ C matrix. Also define the error vectors:

$$\begin{aligned} \mathbf{s} &= \tilde{\mathbf{y}} = C(\mathbf{x} - \hat{\mathbf{x}}) \\ \tilde{\mathbf{x}} &= \mathbf{x} - \hat{\mathbf{x}} \end{aligned} \quad (3.9)$$

Using equations (3.7) and (3.8) one can write:

$$\dot{\tilde{\mathbf{x}}} = \Delta f - H\tilde{\mathbf{y}} - K\mathbf{1}_s, \quad (3.10)$$

where

$$\Delta f = f(\mathbf{x}, t) - \hat{f}(\hat{\mathbf{x}}, t)$$

For convenience the equation (3.10) can be rewritten as,

$$\dot{\tilde{\mathbf{x}}} = \tilde{f} , \tilde{f} = \Delta f - H\tilde{\mathbf{y}} - K\mathbf{1}_s,$$

One would say that the straight extension of the discussion regarding the sliding observer for a second-order system could lead to the conclusion that the m dimensional surface, $\mathbf{s} = 0$, would be *attractive* if,

$$s_i \dot{s}_i < 0 , i = 1, \dots, m$$

Unfortunately, it is not enough as shown in Section 3.4. This condition constraints some of the gains, and the remaining ones are free to be chosen according to some other constraint, e.g. equivalent dynamics. What happens is that the final choice of the gains are not necessarily the ones that will guarantee stability, even inside the sliding patch.

The methods discussed in this section – not valid in general – are now applied to a second-order system.

3.3 A Second-Order System

Consider a second order nonlinear system, consisting of a mass connected to a nonlinear spring in the presence of dry friction and stiction, in companion form :

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\kappa x_1^3 - f(x_2) + u \\ y &= x_1 + v\end{aligned}\tag{3.11}$$

where v is the measurement noise, κ is a constant nonlinear spring coefficient, and $f(x_2)$ represents dry friction with stiction. For this system, the sliding mode observer can be written :

$$\begin{aligned}\dot{\hat{x}}_1 &= h_1 \tilde{y} + \hat{x}_2 + k_1 \operatorname{sgn}(\tilde{y}) \\ \dot{\hat{x}}_2 &= h_2 \tilde{y} - \hat{\kappa} \hat{x}_1^3 - \hat{f}(\hat{x}_2) + u + k_2 \operatorname{sgn}(\tilde{y}) \\ \tilde{y} &= -y \hat{x}_1\end{aligned}$$

The numerical values used in the simulations are :

- $\kappa = 1.0$;
- $F_s = 1.0 =$ static friction ;
- $F_d = 0.75 =$ dynamic friction ;

while the estimated values used in the observer are :

- $\hat{\kappa} = 0.75$;
- $\hat{F}_s = 1.25 =$ static friction ;

- $\hat{F}_d = 1.00 = \text{dynamic friction ;}$

Dry friction with stiction represents a multivalued function at the origin, and therefore, no linearization technique can be applied. Assume a white measurement noise with standard deviation .1. Modelling the effect of parametric uncertainty as a white process noise acting on state x_2 with standard deviation 1, approximating (incorrectly) dry friction by viscous friction, and linearizing, the Kalman Filter gains h_1 and h_2 can be obtained :

$$h_1 = 3.8$$

$$h_2 = 7.2$$

The switching gain k_1 is chosen as a bound on the steady state estimation error on x_2 , and k_2 is chosen to be larger than modelling errors.

$$k_1 = 0.1$$

$$k_2 = 2.0$$

The true system is excited by a sinusoidal input :

$$u = \sin(t)$$

and the initial conditions are: $x_1(0) = 0.0$ and $x_2(0) = 0.5$; with the estimated initial conditions: $\hat{x}_1(0) = 0.0$ and $\hat{x}_2(0) = 0.2$.

The simulation results, in the absence of measurement noise, are shown in Figure 3.6.

The results with gaussian measurement noise of standard deviation .1, are shown in Figure 3.7. In this simulation, the sign function was replaced by the relay with dead-zone (Figure 3.5), with dead-zone width δ equal to the standard deviation of measurement noise 0.1. The reason for this choice is that for small estimation error the linear correction term must provide adequate correction because the effect of the uncertainty term will be small. When the estimation errors become larger, the switching term becomes active, compensating for the modelling errors. Another reason for replacing the signum function with relay-with- dead-zone is to avoid the high-gain effect of pure relay at the origin ¹. These points will be discussed more extensively in Chapter 6.

¹One can run a simulation with pure relay and verify that actually the sliding observer will perform poorly.

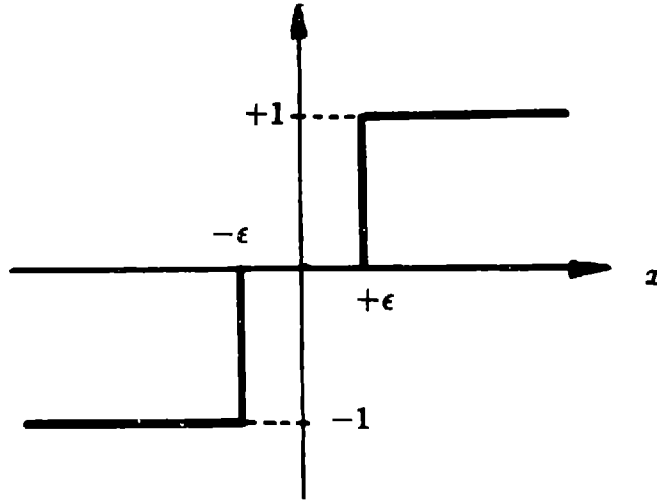


Figure 3.5: Relay with dead-zone

These simulations show that, in spite of the parameter mismatch, the sliding mode observer provides adequate performance.

The reader may be curious about how a simple “Kalman filter” would perform in this problem. This would correspond with using the linearized model alone and with letting k_1 and k_2 equal zero in the sliding mode observer. The corresponding comparative results are shown in Figure 3.8. As one can see, the sliding observer yields a considerable increase in performance, with a minor increase in complexity.

This example shows the actual potential of sliding observer. The following example shows that the simple rules shown here are actually not sufficient to guarantee stable observers.

3.4 Motivation for Further Study: a Third-Order Example

Assume that a sliding observer was designed for a third-order plant with a measurement of x_1 only. The error dynamics would result as:

$$\begin{aligned}\dot{\tilde{x}}_1 &= -h_1\tilde{x}_1 + \tilde{x}_2 - k_1l, \\ \dot{\tilde{x}}_2 &= -h_2\tilde{x}_1 + \tilde{x}_3 - k_2l, \\ \dot{\tilde{x}}_3 &= -h_3\tilde{x}_1 + w - k_3l,\end{aligned}$$

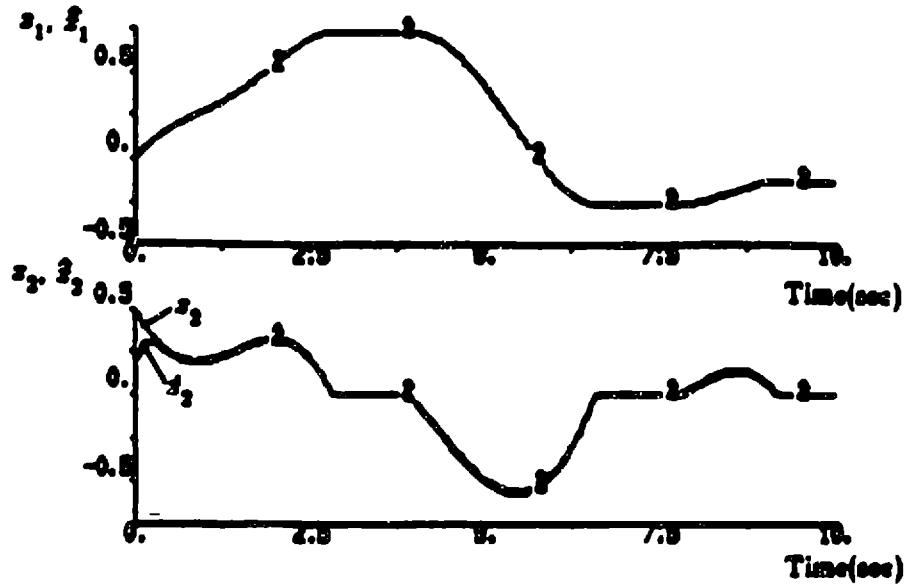


Figure 3.6: Nonlinear deterministic case: true and estimated states. One can see that the estimation is essentially perfect

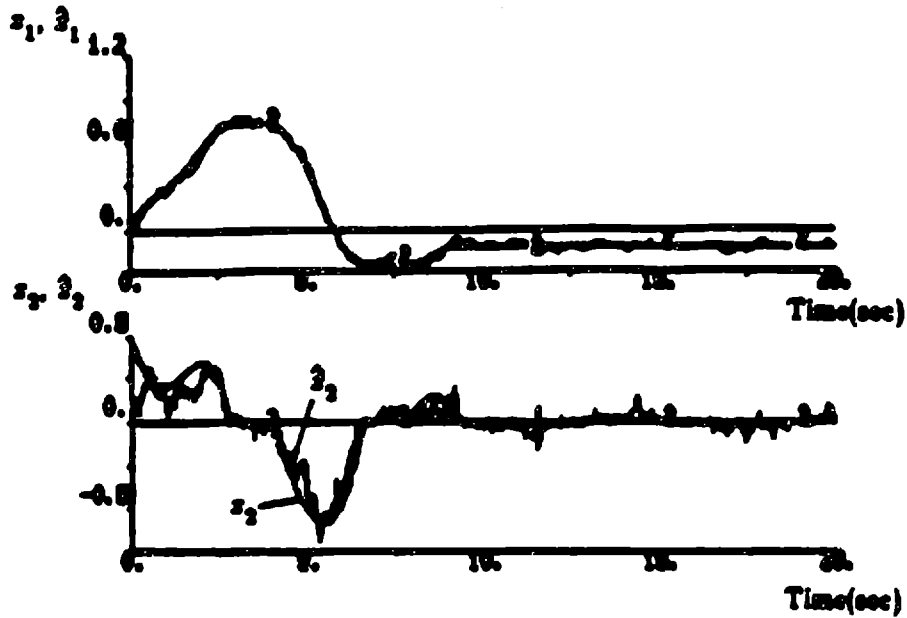
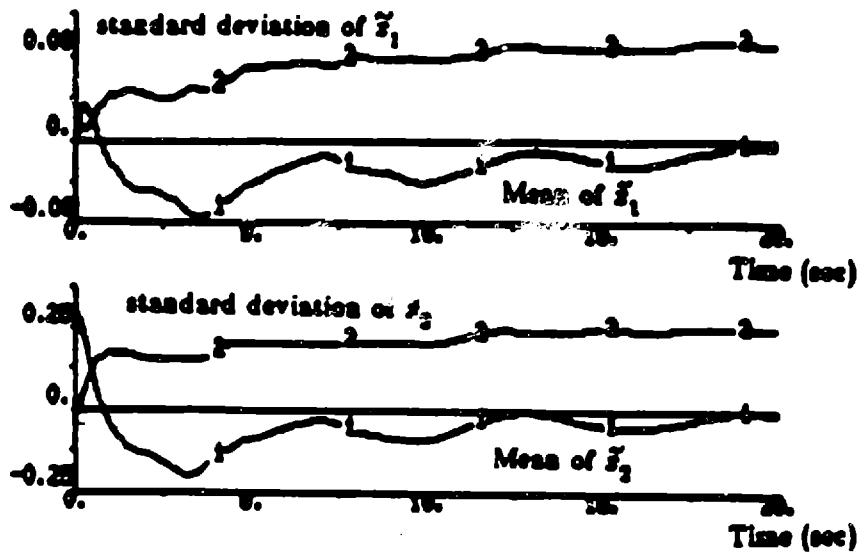
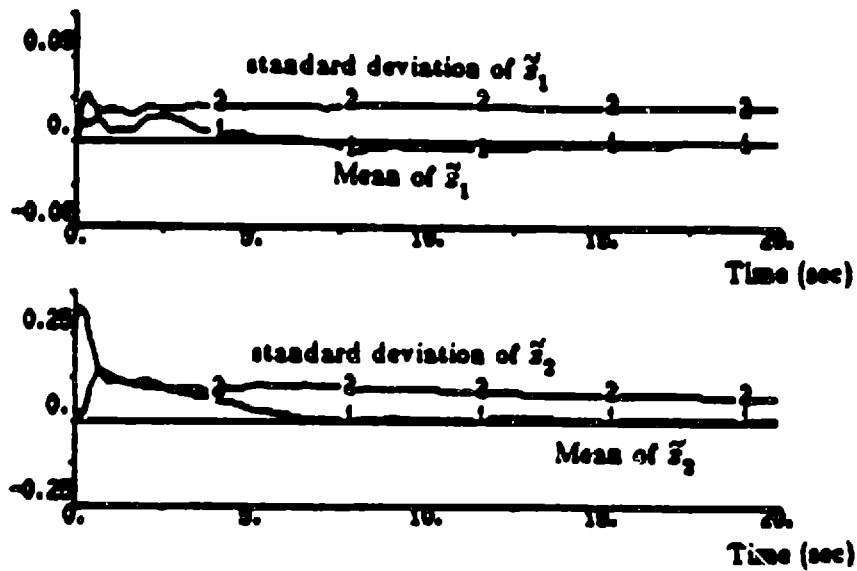


Figure 3.7: Nonlinear stochastic case: true and estimated states



a) Kalman filter



b) sliding observer

Figure 3.8: Nonlinear stochastic case: statistics of estimation errors. a) Kalman Filter; b) Sliding Observer

Clearly, the measurement is done on the first state only. Under this condition, the sliding surface is chosen as $s = \tilde{x}_1 = \hat{x}_1 - x_1$. Therefore the sliding condition

$$\tilde{x}_1 \dot{\tilde{x}}_1 < 0$$

is satisfied if $|\tilde{x}_2| < k_1$.

Because x_1 is measured, it is reasonable to assume that \hat{x}_1 can be taken arbitrarily close to x_1 , i.e. one can assume that the trajectories start exactly on the sliding patch.

If this is true, the “linear gains” h_1 , h_2 and h_3 have no effect, therefore they can be set to zero.

The behavior on the sliding patch – if sliding actually occurs – can be found by applying the Utkin’s equivalent dynamics [59], or the Filippov’s arguments [13]. The results is:

$$\begin{aligned}\dot{\tilde{x}}_{2a} &= \tilde{x}_{3a} - \frac{k_2}{k_1} \tilde{x}_{2a} \\ \dot{\tilde{x}}_{3a} &= w - \frac{k_3}{k_1} \tilde{x}_{2a}\end{aligned}$$

Suppose that the modelling errors, input disturbances and neglected dynamics are acting as

$$w = 1.4 \operatorname{sgn}(\tilde{x}_3)$$

Since $|w| \leq 1.4$, k_3 is chosen to bound its effect, e.g. let $k_3 = 1.5$.

As was shown, k_1 is linked to the size of the sliding patch, or equivalently associated with the desired accuracy in the estimation. Say that one wants the estimation error to be less than 0.1, therefore take $k_1 = 0.1$.

It is reasonable, as was shown before, that the behavior on the sliding patch is made critically damped. This defines the gain k_2 . The final set of gains is:

$$\begin{aligned}k_1 &= 0.1 \\ k_2 &= 0.774 \\ k_3 &= 1.5\end{aligned}$$

In this case, the “sliding condition”, $\tilde{x}_1 \dot{\tilde{x}}_1 < 0$, would simply imply that the sliding patch would be defined by $|\tilde{x}_2| \leq k_1$. With this condition in mind, the initial estimation errors are chosen as $\tilde{x}_1 = 0$, $\tilde{x}_2 = 0.05$, and

$\bar{x}_3 = 1.0$. Obviously the initial estimation error is well inside the “sliding patch”. The result of a simulation, using SIMNON, is shown in Figure 3.9.

Even though the system started inside the sliding patch, satisfying the sliding condition, the estimations are diverging. The instability of error dynamics, in this example, is unexpected from the analysis made in this chapter.

In this example, the flaw in the design process is in assuming that the sliding condition is sufficient to guarantee the stability of error dynamics, which is not true, as shown. In this example the input disturbance was chosen so that, even though it is been bounded by k_3 , it is able to drive the trajectory to outside the sliding patch. Clearly, the sliding condition alone does not give a satisfactory guidance over the choice of the gains. In particular, it gives no explicit clue on how to choose the gains h_i , and the choice of the gains k_i are not necessarily the ones that will guarantee global stability.

Actually such difficulties could be expected. As shown by White [66] variable control systems with switching based on some, but not all, states have difficulties in terms of reachability of the sliding surface. This means that, in some situations, the set of possible trajectories might not include trajectories on the sliding surface.

3.5 General remarks

Clearly, the analysis made in this chapter shows that sliding observer theory requires further studies. It was shown that for a simple second-order system the sliding condition can be derived from the phase-plane analysis, and that the observer can be made very robust against modelling errors, neglected dynamics, input disturbances and uncertainties. It was also shown that for this case the accuracy can also be determined by choosing the size of the sliding patch.

Some of the analyses done for this particular class of second-order systems were extended to higher-order systems. Unfortunately, the design rules drawn from the second-order system are not sufficient to guarantee stable observers for high-order, that is more than second-order, systems.

Actually, the conditions for the stable sliding observer are more involved

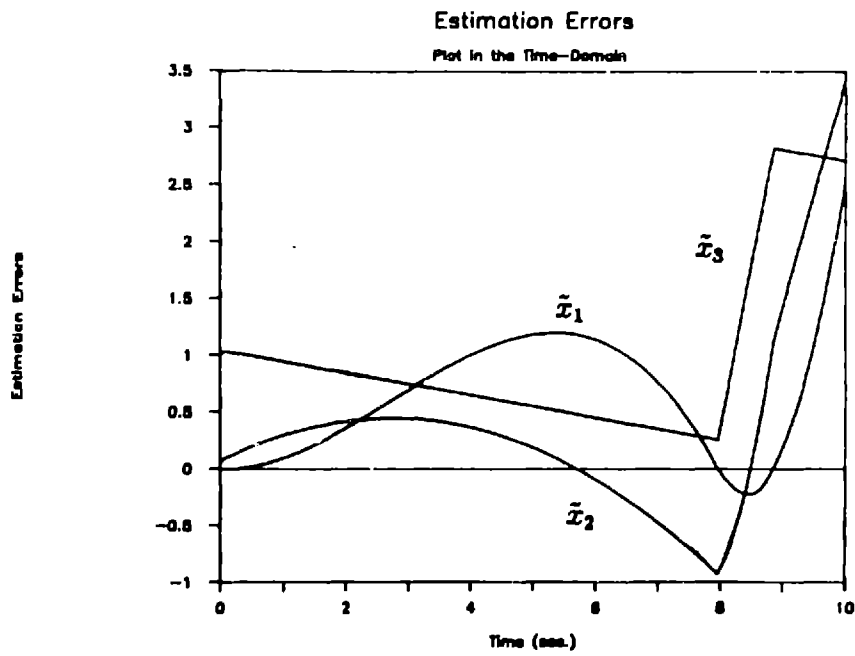


Figure 3.9: Simulation of error dynamics of a third-order system

than the ones shown in this chapter. These are given in Chapter 4. As will become clear, the combined choice of linear gains and switching gains guarantees a stable and robust sliding observer. The drawback that appears is that, in many instances, the sliding patch will not exist any more, and actual sliding does not occur.

Chapter 4

Design of Sliding Observers for Stability Robustness

4.1 Overview

This chapter describes how a control engineer can design a sliding observer which is naturally robust against certain types of bounded modelling error and/or disturbance inputs. The descriptions are followed by mathematical justifications of the given design methods.

The mathematical tools used in this chapter, namely the techniques for input-output stability, are often found in standard textbooks on nonlinear system analysis (e.g. [11] and [61]). The main definitions and results, as used in this thesis, are summarized in Appendix A.

The sections in this chapter are organized as follows: Section 4.2 defines the working environment, that is, the plant and the model that will be used for the sliding observer design, and the observer structure that will be used throughout this thesis. Section 4.3 presents the main conditions for robust observer design. Because the main conditions are often difficult to verify, Section 4.4 shows and proves the sliding observer design method that will guarantee stability robustness for a single measurement case. Section 4.5 extends the design methodology to the multiple measurement case using *passivity* and multivariable circle criterion arguments. The chapter is concluded by making some observations, for example on how the performance issues can be handled in the described design procedures.

4.2 General Structure and Problem Formulation

The plant is assumed to be described by a system of first-order differential equations in the form:

$$\begin{aligned}\dot{x} &= Ax + w \\ y &= Cx\end{aligned}\tag{4.1}$$

where: $x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$, $A \in \mathbb{R}^{n \times n}$, $C \in \mathbb{R}^{n \times m}$ and w is a n -dimensional column vector where all nonlinearities and uncertainties were lumped.

The control inputs, if any, can be added in the system description as known inputs (if they are actually measured) without changing the basic problem, or they can be regarded as uncertainties and therefore included in the vector w .

The system (4.1) is assumed to be observable and the nonlinear term plus the external disturbance w to be bounded, more specifically:

- The pair (A, C) must be observable;
- the entries w_i in the vector w are bounded by constants γ_i

Notice that this description can be obtained from a more general description by isolating the (perfectly known) linear part so that the above conditions are met. Another way to obtain this description is through a linearization by keeping the linear terms from the Taylor expansion of a nonlinear model. In this latter case, w would contain the input disturbances and the high-order terms from the Taylor expansion.

Regardless of the method used to reach to this model, the bounds γ_i can be obtained in several ways, in general problem dependent, but two basic cases may be considered:

- Sometimes the bounds exist naturally, e.g. when one deals with dry friction, the friction force is bounded and the bound can be measured experimentally.
- Often the functions w_i 's are unbounded function of x , e.g. when they result from the Taylor Series expansion. In this case it is necessary to assume that the plant is stable in order to have bounded states.

Assuming the given system description, a sliding mode observer that will provide estimates of states x , given the measurements y , has the structure given by equation (4.2):

$$\dot{\hat{x}} = A\hat{x} + H(y - C\hat{x}) + K1_s(\tilde{y}) \quad (4.2)$$

where:

$$\begin{aligned} \tilde{y} &= y - C\hat{x} \\ H^T &= [h_1^T, h_2^T, \dots, h_n^T], H \in \mathbb{R}^{n \times m} \\ K^T &= [k_1^T, k_2^T, \dots, k_n^T], K \in \mathbb{R}^{n \times m} \end{aligned} \quad (4.3)$$

This sliding observer is basically the conventional Luenberger Observer with the additional “switching” term $K1_s(1_s(\tilde{y}))$, with $\tilde{y} = C(x - \hat{x})$ is defined in terms of *signum* function or saturation function) that will be used to guarantee robustness against modelling errors/uncertainties w .

With this state estimator, the error dynamics can be described as:

$$\begin{aligned} \dot{\tilde{x}} &= x - \hat{x} \\ \tilde{y} &= y - C\hat{x} = C\tilde{x} \\ \dot{\tilde{x}} &= (A - HC)\tilde{x} - K1_s + w \end{aligned} \quad (4.4)$$

The purpose of a state estimator is to provide the state estimate \hat{x} which is an approximation of the actual state x as fast, and as accurate, as possible. The estimation error \tilde{x} should (ideally) go to zero as fast as possible.

With this picture in mind, the problem to be solved can be stated as: *Determine the gain matrices H and K to make the estimation errors \tilde{x} smaller than some specification within a specified time, in the presence of bounded modelling errors/uncertainties.*

4.3 The Main Result for Robust Sliding Observer Design

4.3.1 Observer Design Using the Passivity Theorem

This section describes the main condition that guarantee the L_p -stability of the estimation error.

In order to maintain generality, at this point, consider the following conditions:

- Define the terms $\mathbf{1}_s$ such that $\tilde{y}^T \mathbf{1}_s > 0$;
- let

$$w = D\eta$$

where the rank of D is $d \leq m$, whose elements are all non-negative, and $|\eta| \leq 1$;

- Choose the matrix K as

$$K = D\rho$$

where ρ is a diagonal matrix $\text{diag}(\rho_1, \rho_2, \dots, \rho_m)$;

- write $\mathbf{1}_s$ as:

$$\mathbf{1}_s = \mathbf{1}_s(\tilde{y}) = \alpha(\tilde{y})\tilde{y} = \alpha(\tilde{y})C\tilde{x}$$

With these conditions one can combine the disturbance and “switching” term as:

$$-K\mathbf{1}_s + w = -K(\mathbf{1}_s - \rho^{-1}\eta) = -K(\alpha(\tilde{y})\tilde{y} - \rho^{-1}\eta)$$

resulting in the estimation error dynamics described by:

$$\dot{\tilde{x}} = (A - HC)\tilde{x} - K(\alpha(\tilde{y})\tilde{y} - \rho^{-1}\eta) \quad (4.5)$$

This equation is described in the block diagram of Figure 4.1. Comparing the Figure 4.1 with the ones shown in the Appendix A used to define L_p - stability, one can readily see that the equation (4.5) is now given in the exact form required by the **Passivity Theorem**[11](see Appendix A).

As should be clear by comparing the equation (4.5 and the figure, the operators H_1 and H_2 , and the input u_2 are defined as follows:

$$\begin{aligned} H_1 e_1 &:= C \int_0^t \Phi(\tau) K(\tau) e_1(\tau) d\tau \\ H_2 e_2 = H_2 \tilde{y} &:= \alpha(\tilde{y})\tilde{y} - \rho^{-1}\eta \\ u_2 &:= C\Phi(t)\tilde{x}(0) \end{aligned}$$

where $\Phi(t) = e^{(A-HC)t}$ is the transition matrix [9,24].

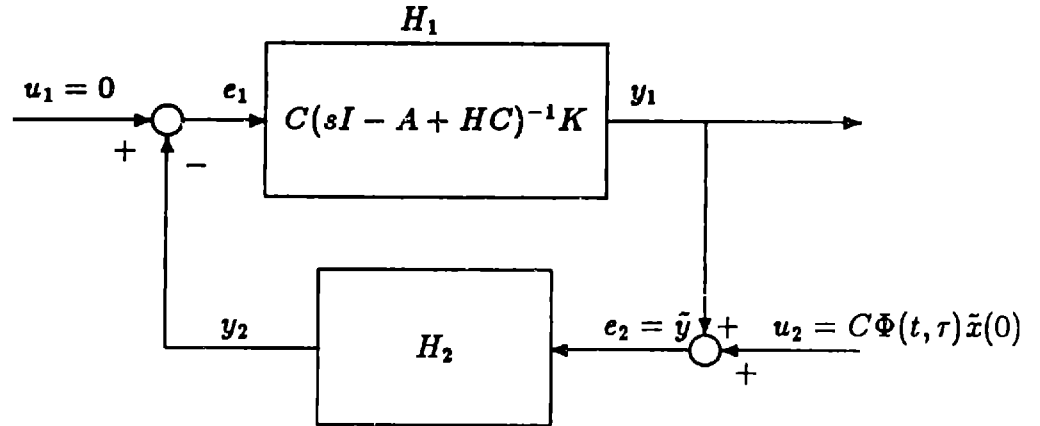


Figure 4.1: Estimation error dynamics

In this scheme, e_1 and e_2 are defined as:

$$e_1 = u_1 - H_2 e_2$$

$$e_2 = u_2 + H_1 e_1$$

The main estimation convergence (stability) result can be stated as a theorem:

Theorem 4.1 *If :*

- $A - HC$ is made stable;
- The gain matrices H and K , and the function Φ , are chosen such that there are constants α_i and β_i , $i = 1, 2, 3$, so that:

$$\begin{aligned} \|H_1 z\|_T &\leq \alpha_1 \|z\|_T + \beta_1 \\ \langle z | H_1 z \rangle_T &\geq \alpha_2 \|z\|_T^2 + \beta_2 \end{aligned} \quad (4.6)$$

$$\langle H_2 z | z \rangle_T \geq \alpha_3 \|H_2 z\|_T^2 + \beta_3 \quad (4.7)$$

$$\forall z \in \mathcal{X}_e, \forall T \in [0, \infty);$$

- $\alpha_2 + \alpha_3 > 0$;

Then: $e_1, e_2 = \tilde{y}, H_1 e_1, H_2 e_2 \in \mathcal{H}$ (definition in Appendix A) , and $\tilde{y} = C\tilde{x} \rightarrow 0$ as $t \rightarrow \infty$.

This theorem guarantees L_2 -stability, but it does not state any conclusion regarding the asymptotic stability of all state estimation errors. However, the state estimation error is actually bounded because they can be interpreted as the outputs of a stable linear system with bounded inputs as seen in equation (4.4).

The first result in Theorem 4.1, namely that $e_1, e_2 = \tilde{y}, H_1 e_1, H_2 e_2 \in \mathcal{H}$ for properly chosen gain matrices H and K and function 1_s , is simply a restatement of the *Passivity Theorem* (Appendix A, and Desoer and Vidyasagar [11]), therefore it will not be proved here. The second result, namely that $\tilde{y} = C\tilde{x} \rightarrow 0$ as $t \rightarrow \infty$ requires more explanation.

Clearly, from the passivity theorem

$$\int_0^{\infty} \tilde{y}^T \tilde{y} dt < \infty$$

It is also true that $\tilde{y} = C\tilde{x}$ and its derivative are bounded because \tilde{x} is bounded as mentioned before, therefore the only way the integral can be finite is that \tilde{y} has to go to zero as time goes to infinity[53].

The main result stated in this section gives a fairly powerful result for the problem at hand. However, the conditions that are easy to state are also difficult to verify. Therefore one will quickly realize that it is very difficult to use it as a design procedure.

4.3.2 Linear Strictly Positive Real Subsystem

The passivity theorem can be applied more explicitly when the function 1_s is based on *signum* function. It will be shown that it will require that the operator H_1 must be strictly positive real.

Let the function 1_s be

$$1_s = \begin{bmatrix} \text{sign}(\tilde{y}_1) \\ \text{sign}(\tilde{y}_2) \\ \vdots \\ \text{sign}(\tilde{y}_m) \end{bmatrix}$$

where the *signum* function is

$$\text{sign}(\tilde{y}_i) = \frac{\tilde{y}_i}{|\tilde{y}_i|}$$

with this choice of function 1_s , the operator H_2 can be seen to be:

$$H_2 \tilde{y} = \begin{bmatrix} \left(\frac{\tilde{y}_1}{|\tilde{y}_1|} - \frac{\eta_1}{\rho_1} \right) \\ \left(\frac{\tilde{y}_2}{|\tilde{y}_2|} - \frac{\eta_2}{\rho_2} \right) \\ \vdots \\ \left(\frac{\tilde{y}_m}{|\tilde{y}_m|} - \frac{\eta_m}{\rho_m} \right) \end{bmatrix}$$

therefore, the left hand side of the condition (4.7) becomes:

$$\int_0^T (H_2 \tilde{y})^T \tilde{y} dt = \int_0^T \sum_{i=1}^m \left(|\tilde{y}_i| - \frac{\tilde{y}_i \eta_i}{\rho_i} \right) dt \quad (4.8)$$

for $\rho_i \geq 1$. It has to be compared with

$$\int_0^T \sum_{i=1}^m \left(1 + \frac{\eta_i^2}{\rho_i^2} - \frac{2\tilde{y}_i \eta_i}{|\tilde{y}_i| \rho_i} \right) dt$$

Clearly, due to the uncertainty terms η_i , nothing can be said except that the left hand side of the condition (4.7), given by equation (4.8) is always positive because the integrand is always positive (recall that $|\eta_i| \leq 1$ and $\rho_i \geq 1$). That is

$$\langle H_2 \tilde{y} | \tilde{y} \rangle_T \geq 0$$

so $\alpha_3 = 0$. Therefore, the passivity theorem requires that

$$\langle \tilde{y} | H_1 \tilde{y} \rangle_T \geq \alpha_2 \|\tilde{y}\|_T^2 + \beta_2$$

with $\alpha_2 > 0$. It means that the operator H_1 has to be strictly passive.

Because the operator H_1 is causal, the concept of strict passivity and the notion of strict positivity are identical [11], therefore the use of *signum* function is requiring a quite restrictive condition on the operator H_1 . As will be shown, if the observer can be designed satisfying these conditions then, the state estimation errors go asymptotically to zero. Actually, under these restrictive conditions, the sliding observer presented in this thesis, with the matrix ρ taken as identity matrix, is identical to the variable structure observer shown by Walcott and Žak in [63,64] and discussed in Chapter 2. These identical results were derived using different routes: Walcott and Žak derived them introducing the necessary assumptions without any clear motivation, apparently as a convenient algebraic hypothesis that makes a

quadratic form to be a Lyapunov function, whereas in this thesis the similar conditions are derived using the passivity theorem and the *signum* function.

The explicit conditions that will guarantee that H_1 is strictly passive (positive) will be discussed in the following sections as *Case 1* for the single measurement case, and as *Case 3* for the multiple measurement case.

The usual situation is when H_1 is not strictly passive, that is when $\alpha_2 \leq 0$. In this case, the passivity theorem requires that

$$\langle H_2 z | z \rangle_T \geq \alpha_3 \|H_2 z\|_T^2 + \beta_3$$

with $\alpha_3 > -\alpha_2$. It is a very difficult condition to verify directly.

An alternative way to look for design criteria when H_1 cannot be made positive real, is to use the definition of strictly positive real systems.

Consider the situation when only one measurement is available. If the gain K is constant, and $A - HC$ has eigenvalues in the open left half plane then the transfer function

$$H_1(s) = C(sI - A + HC)^{-1}K$$

is strictly positive real if

$$\operatorname{Re}[H_1(j\omega)] > \mu \quad \forall \omega \in \mathfrak{R}$$

for some fixed $\mu > 0$. From this definition one can see that the transfer function $H_1(s)$ is not strictly positive real if $H_1(j\omega)$ is in the closed left half plane for some values of ω .

From this observation, one can use concepts from absolute stability to conclude that additional constraint on the sector that contains the operator H_2 must be introduced. This means that the *signum* function in the definition of the function 1, must be replaced with another function that can be placed inside a sector that does not cover the whole first and third quadrant. An obvious candidate is the saturation function.

The single measurement case, where $H_1(s)$ is not positive real will be discussed as *Case 2*, using the circle criterion from absolute stability theory [21,40]. The extension to the multiple-measurement case will be discussed as *Case 4* using the multivariable circle criterion [46,48].

4.4 Sliding Observer for Systems with Single Measurement

4.4.1 General Consideration

The basic design method will first be shown for the single measurement case. The reasons for explicitly studying this case are:

- The underlying theory can be easily translated as a design tool;
- It permits the designer to use classical concepts like transfer functions and the circle criterion for absolute stability;
- It is important in its own right, as far as the practical application goes (e.g. the two link planar manipulator dynamics with position measurements can be taken as two independent single measurement systems, with the coupling terms in the equations of motions taken as bounded modelling errors).

Two cases, that have different constraints, will be considered. The first case requires that the operator H_1 must be strictly positive real. The second case is the case when the strict positivity of H_1 cannot be imposed.

The methodology is first shown for the first case, which is less likely to occur in practice, but it shows the application of the circle criterion in the Sliding Observer Design Technique. Then the second case will be considered. It will be shown that this case introduces some additional constraints that are reflected as changes in the basic design method.

4.4.2 Case 1

Development of the Design Method

The first case to be considered is the case when the linear subsystem of estimation error dynamics can be made strictly positive real.

In this case, it is assumed that only one measurement from the plant is available, and that all disturbances/modelling errors are generated by the same scalar function η^1 :

$$w = D\eta, \quad |\eta| \leq 1$$

¹This case is included in the class of systems considered by Walcott and Žak [63,64]

$$D^T = [d_1 \ d_2 \ \dots \ d_n]$$

where $d_i > 0$ ($i = 1, 2, \dots, n$) are real valued constants ($\in \mathfrak{R}$), and η is a real valued function that can depend on time, states, and external disturbances.

For this case, the function 1_s can be defined as the usual *signum* function:

$$1_s = \text{sign}(\tilde{y}) = \frac{\tilde{y}}{|\tilde{y}|} \quad (4.9)$$

Define the constant gain matrix K as

$$K = D\rho$$

As was shown before the estimation error dynamics can be described as

$$\begin{aligned} \dot{\tilde{y}} &= C\tilde{x} \\ \dot{\tilde{x}} &= (A - HC)\tilde{x} - K\phi\tilde{y} \end{aligned} \quad (4.10)$$

where the operator ϕ is:

$$\phi\tilde{y} = \frac{\tilde{y}}{|\tilde{y}|} - \frac{\eta}{\rho}$$

It is easy to see that the operator ϕ is sector constrained due to the presence of \tilde{y} in the denominator:

$$\underline{G}\tilde{y} \leq \phi(t, w, \tilde{x})\tilde{y} < \overline{G}\tilde{y}$$

where $\underline{G} = 0$ and $\overline{G} = \infty$.

This representation of error dynamics (see Figure 4.2) is exactly in the form required by the passivity theorem and also by the absolute stability theory, where the operators H_1 and H_2 are defined as:

$$\begin{aligned} H_1 e_1 &= \int_0^t C\Phi(\tau)K e_1 d\tau \\ H_2 e_2 &= \phi e_2 \\ \Phi(\tau) &= e^{(A-HC)\tau} \end{aligned}$$

where the solutions e_1 and e_2 ($= \tilde{y}$) are assumed to exist, either in the usual sense, or in the sense of Filippov [13], in \mathcal{X}_e .

For this case, it was already seen that the operator H_1 has to be strictly passive, which is equivalent to strict positivity. Because H_1 is a SISO

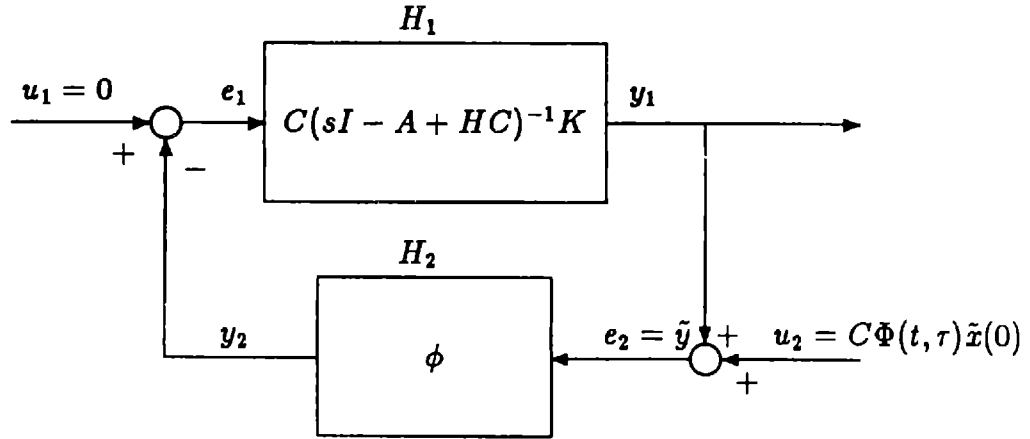


Figure 4.2: Estimation error dynamics

system, the strict positivity means that the real part of $H_1(s) = C(sI - A + HC)^{-1}K$ must be positive, that is the Nyquist plot must always be to the right of the imaginary axis.

A control engineer familiar with absolute stability would readily verify that the conditions just derived using the *Passivity Theorem* are exactly the same conditions that would result if one applies the circle criterion to the system given by the equation (4.10) and Figure 4.2.

The conclusion is that, if the gain matrix H and the gain ρ can be chosen so that the

$$\operatorname{Re}[H_1(j\omega)] = \operatorname{Re}[C(j\omega I - A + HC)^{-1}K] > 0$$

for all values of $\omega \in \mathfrak{R}$, then the use of $1_s = \operatorname{sign}(\tilde{y})$ will guarantee that $\tilde{y} \rightarrow 0$ as $t \rightarrow \infty$. Actually the whole state estimation error \tilde{x} will go to zero. This result will be shown within more general framework in the *Case 3* using Lyapunov theory.

The design procedure, for this single measurement case is summarized as follows:

Design Procedure for Single Measurement Case - Case 1

The design of a sliding observer for *this single measurement case* can be easily carried out by applying the following procedure:

1. Verify that the problem really falls in this category, i.e. the vector that lumps all uncertainties/nonlinearities can be written as

$$w^T = \eta[d_1 \ d_2 \ \dots \ d_n]$$

where the scalar function $|\eta| \leq 1.0$;

2. Let $1_s = \text{sign}(\bar{y})$ as defined in (4.9);
3. Let the constant gain matrix $K = \rho D$, with $\rho \geq 1$;
4. Choose the gain vector H and the scalar parameter ρ , so that *Nyquist Plot* of the transfer function $H_1(s) = C(sI - A + HC)^{-1}K$ is completely contained in the open right half plane, that is $\text{Re}[H_1(j\omega)] > 0$ for all $\omega \in \mathfrak{R}$.
5. If this last condition cannot be satisfied, then one can iterate either by changing the matrix D (if possible) or by adding convenient blocks, in the error dynamics, that change the blocks H_1 and H_2 according to the multiplier theory ([11], section 6.9). If these changes does not help then one can use the *Case 2*

A First-Order System

As was mentioned previously, the design method described in this section is more likely to find application in first-order systems, even though a few high-order systems for which this case apply can be found.

This section shows such a situation. Consider the first-order system:

$$\begin{aligned}\dot{x} &= -ax + f(x, t) + w' \\ y &= x\end{aligned}$$

where $x \in \mathfrak{R}$, $y \in \mathfrak{R}$, $w \in \mathfrak{R}$, and the input disturbance and the nonlinearity are assumed to be bounded as

$$|w| = |f(x, t) + w'| \leq \gamma$$

where the bound γ is assumed to be constant.

The sliding observer for this system is:

$$\dot{\hat{x}} = -a\hat{x} + h(y - \hat{x}) + k \text{sign}(y - \hat{x})$$

The error dynamics is given by:

$$\dot{\tilde{x}} = -(a + h)\tilde{x} - k \operatorname{sign}(\tilde{x}) + w$$

where the gain k is taken to be greater or equal to the bound γ .

It can be rewritten as:

$$\dot{\tilde{x}} = -(a + h)\tilde{x} - k\phi\tilde{x}$$

where the operator ϕ is contained in the sector that covers the whole first and third quadrant. In this case the circle criterion will require that the Nyquist plot of

$$H_1(s) = \frac{k}{s + a + h}$$

must lie in the open right half plane. Of course one will design a stable observer, therefore $a + h > 0$ and $k > 0$. It means that the condition given by the circle criterion is trivially verified because the real part of $H_1(j\omega)$ is always positive, for all ω . It was known beforehand, because a stable proper first-order system is always strictly positive real.

It is also clear that if the original plant is unstable, i.e. $a < 0$ then the gain h has to be $h > -a$ in order to guarantee stable sliding observer.

4.4.3 Case 2

The second case consider the case when the system has only one measurement and only one source of disturbance η and when the transfer function $H_1(s) = C(sI - A + HC)^{-1}K$ cannot be made strictly positive real.

The design procedure still relies on the absolute stability theory (easier than applying the Passivity Theory directly) in order to guarantee automatic robustness against bounded uncertainties/nonlinearities, but additional constraints are introduced due to the loss of strict positive realness. For this reason the proof of the design procedure is slightly more involved, therefore for the sake of clarity the design procedure is first presented as an algorithm, and then the design method is derived formally showing that it actually guarantees stability robustness.

In this case the design process has two parts. The first part guarantees that the output estimation error $\tilde{y} = y - C\hat{x}$ remains inside the boundary layer once it gets into it and the second part guarantees that the boundary layer is attractive (see Figure 4.3).

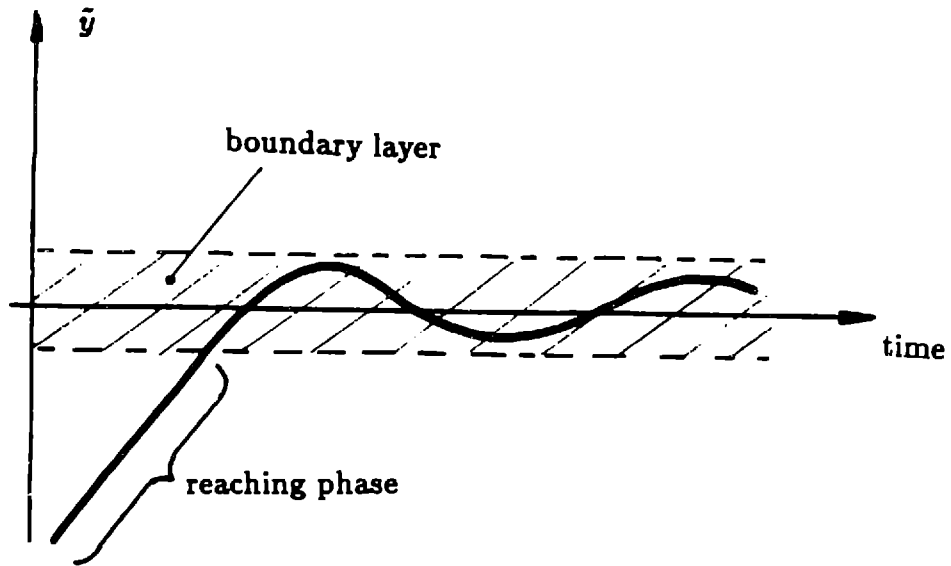


Figure 4.3: The boundary-layer

In this case the design procedure can be organized as follows:

Design Procedure for Single Measurement Case - Case 2

1. Let the 1_s to be the saturation function defined as:

$$1_s(\tilde{y}) = \text{sat}(\tilde{y}/\epsilon) = \begin{cases} \tilde{y}/|\tilde{y}| & \text{if } |\tilde{y}| \geq \epsilon \\ \tilde{y}/\epsilon & \text{if } |\tilde{y}| < \epsilon \end{cases} \quad (4.11)$$

where $\tilde{y} = y - C\hat{x}$.

2. Let $K = D\rho$, where $\rho \geq 1$, and K is a constant gain matrix;
3. Determine the width of the boundary layer, called ϵ , which coincides with saturation limit ϵ in the $\text{sat}(\cdot)$ - function defined by (4.11);
4. Design for boundedness inside the boundary layer. It can be done in two ways:

- Choose the gain matrices H and ρ such that $A - (H + K\Delta^{-1})C$ has distinct eigenvalues and such that for all $\omega \in \mathfrak{R}$:

$$\max[\sigma_{\max}(C(j\omega I - A + (H + K\Delta^{-1})C)^{-1})D] +$$

$$+\sigma_{max}(C)\text{cond}(V)\max(|\tilde{x}_0|) \leq \epsilon$$

where $\sigma_{max}(M)$ is the maximum singular value of the complex matrix A , the norm $|z| = (z^*z)^{\frac{1}{2}}$ (z^* denotes the complex conjugate), and the $\text{cond}(V)$ is the condition number of the matrix of eigenvectors V of $A - (H + K\Delta^{-1})C$. $\max(|\tilde{x}_0|)$ is an upper bound of initial state estimation error \tilde{x}_0 . This approach suffers the drawback that it depends on the scaling factors of eigenvectors.

- Choose the gain matrices H and ρ such that for all $t \in \mathbb{R}_+$ and for all $\omega \in \mathbb{R}$:

$$\begin{aligned} \max[\sigma_{max}(C(j\omega I - A + (H + K\Delta^{-1})C)^{-1})D] + \\ + \sigma_{max}(Ce^{(A-(H+K\Delta^{-1})C)t})\max(|\tilde{x}_0|) \leq \epsilon \end{aligned}$$

In practice this test has to be verified in the time interval of interest, typically during the initial transient.

Both methods are very conservative, as will be shown in the examples.

These conditions are expressed in terms of the bound on the estimation error $\tilde{x}(t_0)$ for the states at the instant t_0 when the output estimation error \tilde{y} entered the boundary-layer, and the maximum magnitudes of uncertainties/nonlinearities d_i of vector D .

The above conditions basically state that the “loop” gain has to be high enough to guarantee that the output estimation error \tilde{y} will not leave the boundary layer once it gets into it. Or more realistically, it can be assumed that, because y has been measured, $\tilde{y} = y - \hat{y}$ can be taken as zero at initial time, therefore the gains have to be high enough so that \tilde{y} will not leave the boundary layer. Certainly these comments will be limited when the transfer function $G(s)$ defined by

$$G(s) = C(sI - A + (H + \frac{1}{\epsilon}K)C)^{-1}D$$

has some finite zeros. In this case, it might be difficult, or even impossible to obtain the desired decrease in the maximum singular values;

5. Check whether the state estimation error bounds are within desirable limits:

$$|\tilde{x}| \leq \max_{-\infty < \omega < +\infty} [\sigma_{\max}((j\omega I - A + (H + \frac{1}{\epsilon}K)C)^{-1})D]$$

6. Check for absolute stability. The error dynamics can be written as:

$$\begin{aligned} \dot{\tilde{y}} &= C\tilde{x} \\ \dot{\tilde{x}} &= (A - HC)\tilde{x} - K\phi\tilde{y} \end{aligned} \quad (4.12)$$

therefore, it is necessary to verify that (4.12) is stable for ϕ such that ϕ is constrained in the sector:

$$0 < \phi < \frac{1}{\epsilon} \left(1 + \frac{1}{\rho}\right) = \bar{G}$$

For instance, if the circle criterion is used, then it is enough to show that the Nyquist Plot of the transfer function

$$H_1(s) = C(sI - A + HC)^{-1}K$$

is to the right of vertical line that intersects the real axis at $-\frac{1}{\bar{G}}$ (see Figure 4.4).

7. Obviously if one concludes that the error dynamics is absolutely stable, then the design is, in principle, complete. If this is not the case one iterates through previous steps.

As this design procedure shows, even though additional constraints make the design procedure more complicated than in the first case, it is still feasible and fairly straightforward as will be shown in the next example.

The proof of this method is presented after the example.

A Second-Order System: Mass with Dry Friction and Nonlinear Spring

The method shown here is applied to the example used in Chapter 3. The plant was described in the equation (3.11), in Chapter 3.

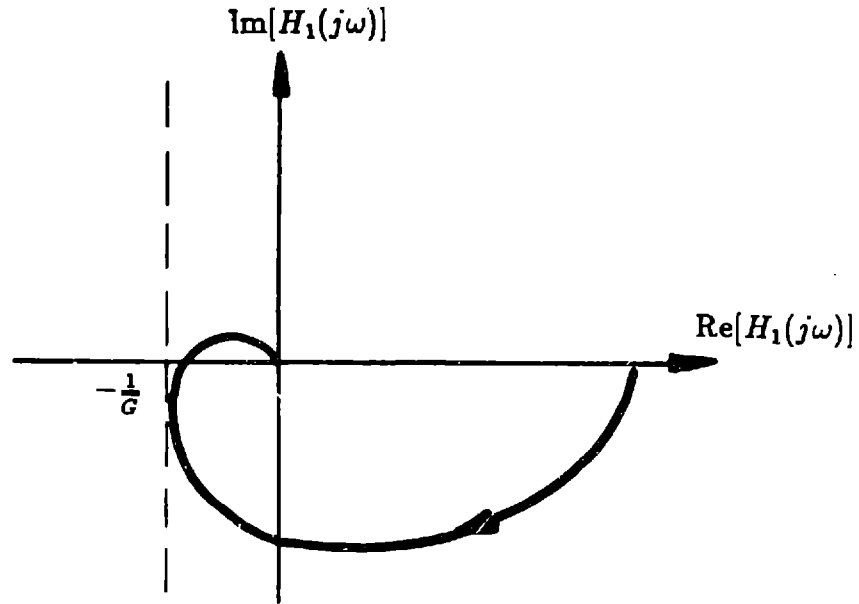


Figure 4.4: Circle criterion for the Case 2

Using the model described as

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= w\end{aligned}$$

where w contains all nonlinear terms as well as the input which is assumed to be unknown. Then the sliding observer becomes:

$$\begin{aligned}\dot{\hat{x}}_1 &= \hat{x}_2 - h_1 \tilde{x}_1 \\ \dot{\hat{x}}_2 &= -h_2 \tilde{x}_1 - k_2 \tilde{x}_2\end{aligned}$$

where k_1 was set to zero because there is no uncertainty, or modelling error in \hat{x}_1 .

As shown before, the modelling errors, nonlinearities and uncertainties, all lumped into a single term, can be bounded by $\gamma = 2$, therefore $k_2 = 2$ is a reasonable starting value.

Now, assume that one wants to have the estimation of x_1 to be accurate within 2.5%, so take $\epsilon = 0.025$.

Having these values, one can first design the observer so that output estimation errors \tilde{y} that come into the boundary-layer never leave it. So

a reasonable strategy is to make the system critically damped inside the boundary layer, that is, if ω_n is the the desired undamped natural frequency, the gains h_1 and h_2 are chosen as:

$$\begin{aligned} h_1 &= 2\omega_n \\ h_2 &= \omega_n^2 - \frac{k_2}{\epsilon} \end{aligned}$$

In order to choose the gains using the singular values, it is necessary to estimate the bounds of estimation errors, when the output estimation error \tilde{y} gets into the boundary layer, or more realistically the maximum estimation error that one can expect for \tilde{x}_2 , since \tilde{x}_1 can be assumed to be zero.

A safe guess for this problem would be that

$$|\tilde{x}_2| \leq 0.5$$

Using standard CACSD software, and after two iterations, the gains

$$\begin{aligned} h_1 &= 44 \\ h_2 &= 400 \end{aligned}$$

resulted in the plot of singular values shown in Figure 4.5, from which one can conclude that

$$|\tilde{x}_1| \leq 0.0042 + 0.5 \times 1 = 0.50042 > .025$$

The effect of initial condition was bounded using the time domain approach because the eigenvalues were chosen to be coincident, which means that the approach using the condition number of the matrix of eigenvectors does not work. Even though the boundedness condition inside the boundary layer is violated, the simulation shows that the output estimation error \tilde{y} remains inside the boundary layer. This fact shows how conservative this criterion may be.

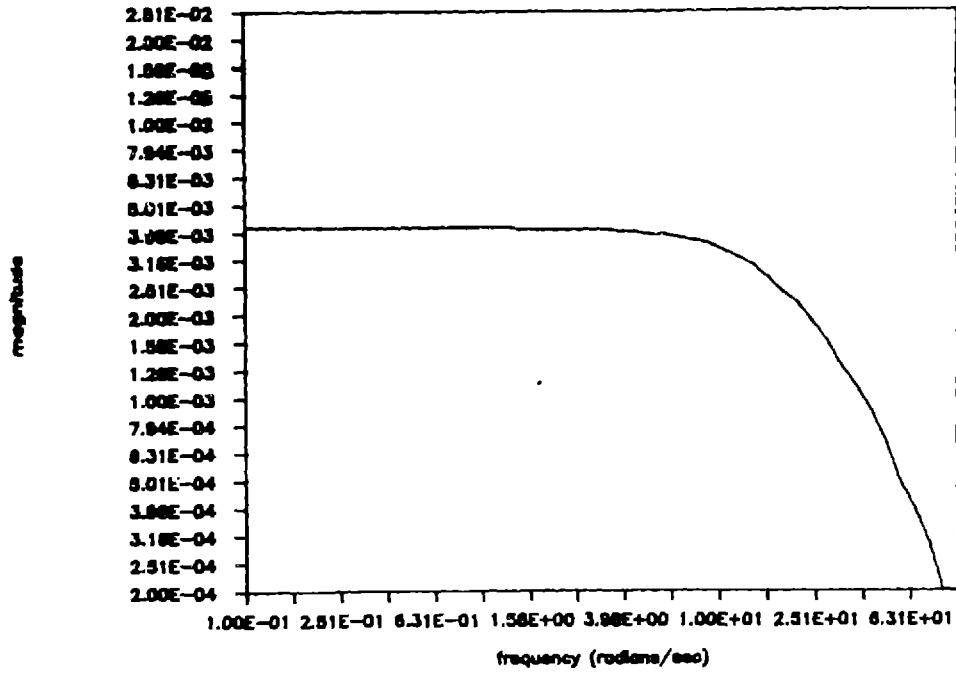
The last step is to check whether the circle criterion is satisfied.

For this particular example, the nonlinearity ϕ is constrained to be inside the sector

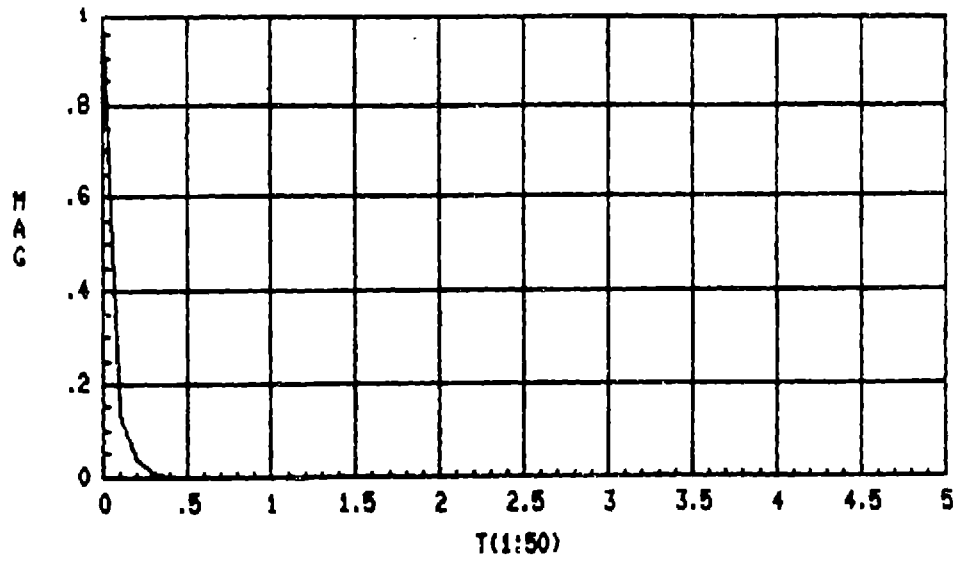
$$0 < \phi < (1 + 1)/.025 = 80 = \bar{G}$$

The Nyquist Plot of

$$H_1(S) = \frac{2}{s^2 + 44s + 480}$$



$$a) G_1(j\omega) = C(j\omega - A + (H + \frac{1}{2}K)C)^{-1}D$$



$$b) \sigma_{max}(C e^{(A - (H + \frac{1}{2}K)C)t})$$

Figure 4.5: Singular value plots for the second-order example

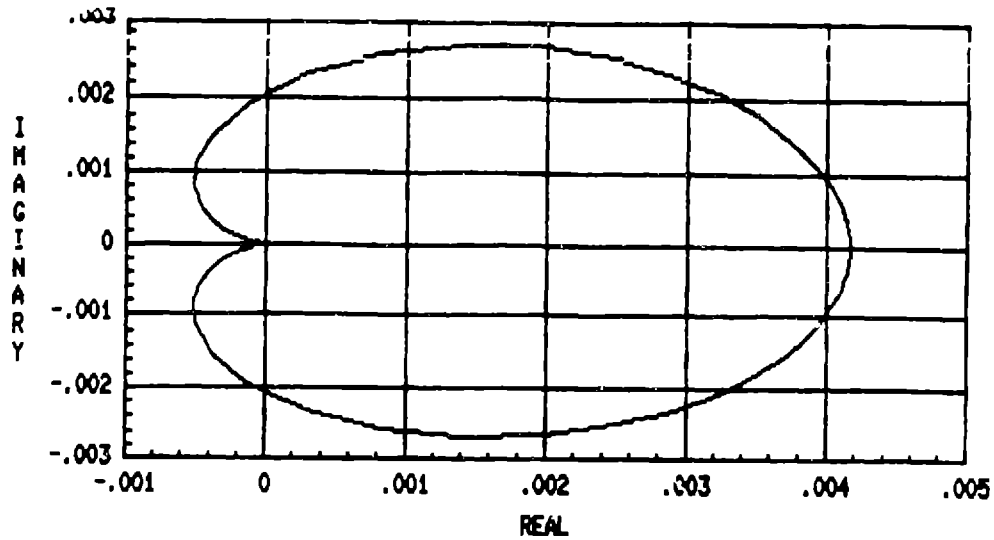


Figure 4.6: Nyquist plot for the second-order system

is shown in Figure 4.6, and clearly the sector condition is satisfied. The system composed with the plant and the sliding observer was simulated using SIMNON and the results are shown in Figure 4.7. The phase-plane plot shows that, in fact, the output estimation error $\tilde{y} = \tilde{x}_1$ remains inside the boundary layer, and the estimation errors are in general better than the results obtained from the Kalman Filter shown in Chapter 3.

The Proof of the Design Method

The proof of this method is done in a constructive way so that the reader can get further insight into the method just presented.

The estimation error dynamics was given in (4.4) and it is repeated here for convenience:

$$\begin{aligned}\dot{\tilde{x}} &= x - \hat{x} \\ \dot{\tilde{x}} &= (A - HC)\tilde{x} - K_1 \epsilon + D\eta\end{aligned}$$

where $|\eta| \leq 1$

Let K be defined as:

$$K^T = [k_1 \ k_2 \ \dots \ k_n] = D^T \rho$$

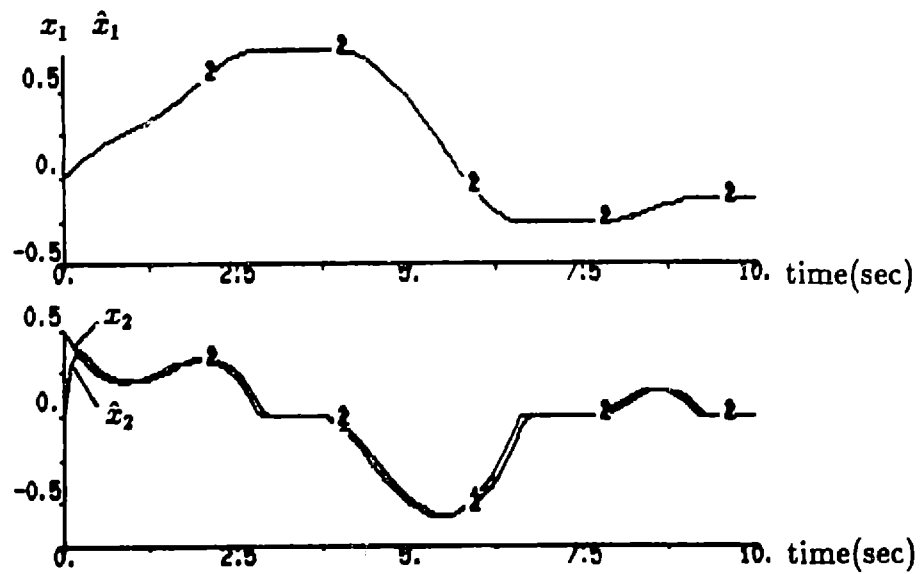
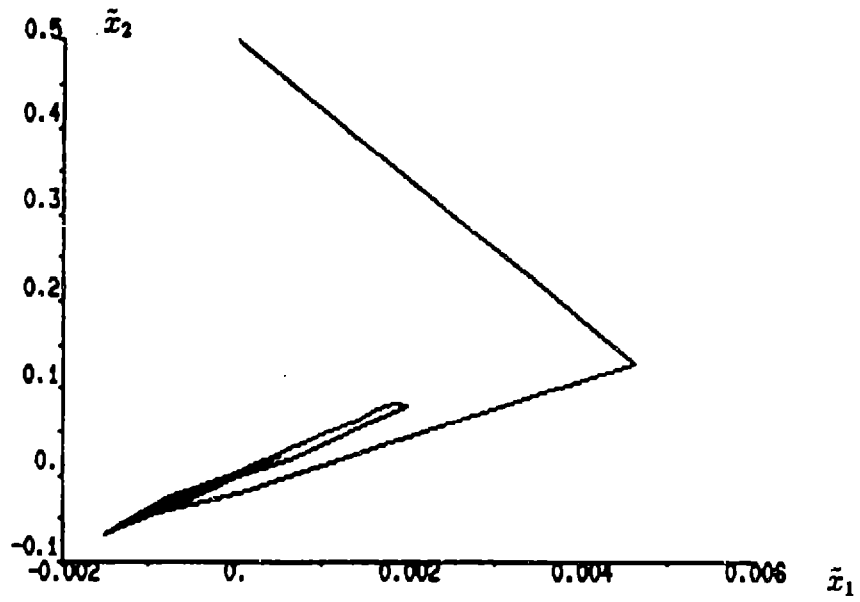


Figure 4.7: Simulation of second-order system: a) Phase-plane plot; b) Time domain plot

where $\rho \geq 1$.

The combination of these two terms is:

$$-K1_s + D\eta = -K(1_s - \rho^{-1}\eta)$$

which can be written, using the operator ϕ as:

$$\begin{aligned} -K1_s + D\eta &= -K\phi\tilde{y} \\ \phi\tilde{y} &= 1_s(\tilde{y}) - \frac{\eta}{\rho} \end{aligned} \quad (4.13)$$

The error dynamics can now be written:

$$\begin{aligned} \dot{\tilde{y}} &= C\tilde{x} \\ \dot{\tilde{x}} &= (A - HC)\tilde{x} - K\phi\tilde{y} \end{aligned}$$

In this case, it has been assumed that the transfer function

$$H_1(s) = C(sI - A + HC)^{-1}K$$

cannot be made strictly positive real, that is part of the Nyquist Plot will lie in the left half plane, therefore using the concepts from absolute stability, in particular using the circle criterion, one can see that the sector in which the function $\phi(t, \tilde{x}, w)$ has to be contained has an upper limit, say \overline{G} , as described in Figure 4.4

Recognizing that there will be an upper limit for the sector, the first step is to try to make the observer stable in the presence of this sector bounded nonlinearity.

The obvious sector bounded nonlinearity that also satisfies the conditions imposed on function 1_s is the saturation function as defined in (4.11).

In order to proceed, it is necessary to make an additional assumption, that will later be dropped, and the consequences will be analyzed.

Assume that the following inequalities hold:

$$\begin{aligned} |\eta| &\leq 1 \\ \lim_{\tilde{y} \rightarrow 0} \frac{|\eta|}{\phi|1_s|} &\leq 1 \end{aligned}$$

Actually it is very easy to see that it seldom holds (e.g. think about the dry friction as a disturbance and 1_s based on position measurement of

a mechanical system) and this is the reason why it will be dropped shortly, but it is invoked here so that the basic setup can be derived.

With this assumption and using the saturation function with saturation level ϵ (see Figure 4.8), the function ϕ will now be constrained to be inside the sector defined by:

$$0 < \phi \leq \frac{1}{\epsilon} \left(1 + \frac{1}{\rho}\right) = \bar{G}$$

Now, assuming that ϕ stays inside the above sector the problem becomes a simple extension of Case 1, namely the stability robustness of the sliding observer can be guaranteed by choosing the matrix H and the gain ρ so that the Nyquist Plot remains to the right of the vertical line that intersects the real axis at $-\frac{1}{\bar{G}} = -\epsilon \frac{\rho}{\rho+1}$. It results from the application of the circle criterion.

The additional constraint is introduced when the assumption regarding the limit :

$$\lim_{\tilde{y} \rightarrow 0} \frac{|\eta|}{\rho|1_s|} \leq 1$$

is dropped. By dropping this assumption, the proof has to be done in two parts, the first one considering the situation when the output estimation errors \tilde{y} are outside the boundary layer (called the “reaching phase”), when the previous analysis still hold , and the second part when the output estimation error \tilde{y} is inside the boundary layer, that is when the saturation function is actually acting as a linear gain, and when all the absolute stability analysis is not valid anymore.

The first part is actually showing that regardless of uncertainties/non-linearities, the boundary layer is “attractive”, and its proof is identical to Case 1, and to the case just analyzed with the strong assumption regarding the limits.

To complete the proof, it is necessary to show that the behavior inside the boundary layer is such that a output estimation error \tilde{y} that goes inside the boundary layer never leaves it (the variation of this same theme was commented upon when the procedure was described).

The proof starts by recognizing that the equations of error dynamics inside the boundary layer are linear. The error dynamics is described by:

$$\dot{\tilde{x}} = (A - (H + K\Delta^{-1})C)\tilde{x} + w$$

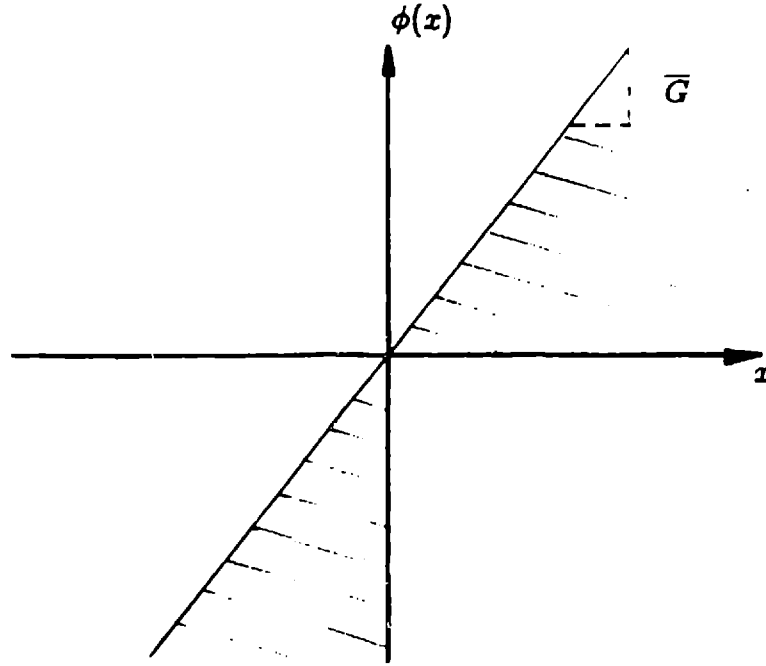


Figure 4.8: Sector constraint for the function ϕ

$$\begin{aligned}\bar{y} &= C\bar{x} \\ \bar{x} &= \bar{x}(0) \text{ at } t = 0\end{aligned}$$

The output estimation error \bar{y} can be written as

$$\bar{y}(t) = \bar{y}^\eta(t) + \bar{y}^0(t)$$

The first term is due to the uncertainties and the second term is due to the initial conditions. The first term can be written in the frequency domain as:

$$\bar{y}^\eta(j\omega) = C(j\omega I - A + (H + K\Delta^{-1})C)^{-1}D\eta(j\omega)$$

so, using the inverse Fourier Transform $\bar{y}(t)$ and $\bar{y}(j\omega)$ are related as

$$\bar{y}^\eta(t) = \int_{-\infty}^{+\infty} \bar{y}^\eta(j\omega) e^{j\omega t} d\omega$$

by using the bounds of frequency response, one can bound the time response as

$$|\bar{y}^\eta(t)| \leq \max_{\omega} [\sigma_{\max}[C(j\omega I - A + (H + K\Delta^{-1})C)^{-1}D]] \int_{-\infty}^{+\infty} |\eta(j\omega)| d\omega$$

Clearly the last term, with the integral, is the function $\eta(t)$, whose norm is bounded by one, by definition. It means that

$$|\bar{y}^\eta(t)| \leq \max_{\omega} [\sigma_{\max}[C(j\omega I - A + (H + K\Delta^{-1})C)^{-1}D]]$$

The term due to the initial condition can be bounded in two ways. The first possibility is to use spectral factorization, i.e. by taking the matrix Λ of eigenvalues of $(A - (H + K\Delta^{-1})C)$, and the matrix V of eigenvectors, one has

$$A - (H + K\Delta^{-1})C = V\Lambda V^{-1}$$

so,

$$\tilde{y}^0(t) = C e^{(A - (H + K\Delta^{-1})C)t} = CV e^{(\Lambda)t} V^{-1} \tilde{x}_0$$

therefore the solution $\tilde{y}^0(t)$ can be bounded provided that some upper bound $\max |\tilde{x}_0|$ of initial state estimation error is available.

$$|\tilde{y}^0(t)| \leq \sigma_{\max}(C) \sigma_{\max}(V) \max_i [\sigma_{\max}(e^{\Lambda t})] \sigma_{\max}(V^{-1}) \max |\tilde{x}_0|$$

The exponential term is bounded by one, because it is a diagonal matrix whose entries are all bounded by one since the underlining system is made stable. Using the definition of condition number of a matrix, and properties of singular values, one obtains:

$$|\tilde{y}^0(t)| \leq \sigma_{\max}(C) \text{cond}(V) \max |\tilde{x}_0|$$

An alternative way to bound $\tilde{y}^0(t)$ is by taking the bound of matrix exponential:

$$|\tilde{y}^0(t)| \leq \max_i \sigma_{\max} [C e^{(A - (H + K\Delta^{-1})C)t}] \max |\tilde{x}_0|$$

Using either bound of $\tilde{y}^0(t)$, and combining with the bound of \tilde{y}^n , one can impose that

$$|\tilde{y}| \leq |\tilde{y}^n| + |\tilde{y}^0| \leq \min_i(\epsilon_i)$$

This condition guarantees that the output estimation error \tilde{y} will remain inside the boundary layer, and completes the proof.

As was described in the design procedure for this case, one can estimate the bounds of state estimation errors. It can be estimated easily in the frequency domain by recognizing that the behavior inside the boundary-layer, where the output estimation error \tilde{y} should be in the steady state, is essentially linear. Using the singular values, one can easily see that the state estimation error is bounded by:

$$|\tilde{x}| \leq \max_{-\infty < \omega < +\infty} [\sigma_{\max}((j\omega I - A + (H + \frac{1}{\epsilon}K)C)^{-1})D]$$

4.4.4 Some Remarks Regarding the Design Method for the Single Measurement Problem

Some comments, related to the generality and conservativeness of the design methods just presented, can be made:

- **Generality:** The two classes of single measurement problems presented in this section are not the most general of their kind. The first one is too restricted, and (except for first order plants) it is very unlikely to find application. However, the second case is general enough to handle systems given in the following canonical form:

$$\begin{aligned}
 x^T &= [x_1 \ x_2 \ \dots \ x_n] \in \mathbb{R}^n \\
 \dot{x} &= Ax + w \\
 y &= Cx \\
 A &= \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ a_{n1} & a_{n2} & a_{n3} & \dots & a_{nn} \end{bmatrix} \\
 w^T &= [0 \ 0 \ 0 \ \dots \ 0 \ w_n] \\
 C &= [1 \ 0 \ 0 \ \dots \ 0 \ 0]
 \end{aligned}$$

which is fairly usual, for example in mechanical systems where position is measured and disturbances and nonlinearities in the equation of motion appear in the acceleration.

- **Conservativeness:** As is well known, the methods from input-output stability tend to be very conservative (which is good when the desired goal is stability robustness), and can require large observer gains in order to have enough attenuation inside the boundary layer. As is well known, large gains are not desirable when measurement noise is present. An heuristic argument, combined with an additional restriction on the nonlinearity/uncertainty term w can be used to lower the gains in a suitable way. The drawback is that it will take several switchings before the output estimation error \tilde{y} gets into the

boundary layer(that is the boundary layer will be crossed a finite number of times).

Assume that the combined uncertainty/nonlinearity vector w is such that it does not contain any impulse (Dirac-delta) function ;it does not have to be continuous.

Under these conditions, it is possible to guarantee (due to the integration effect) that the output estimation error \tilde{y} will not have any discontinuity (i.e. $x(t)$ will be of class C^0), in particular when it crosses the boundary layer (if this happens), there will not be any "jumps".

If this is true, one can design an observer which is guaranteed to be robust outside the boundary layer (i.e. the boundary layer is "attractive"), and by making the dynamics inside the boundary layer critically damped, or overdamped, one can argue that the error dynamics cannot be going unstable because the output estimation errors \tilde{y} will be going "inwards" in the output estimation error space (in terms of error dynamics). Based on this fact, the designer can set simple (heuristic) rules. One possible rule would be, " choose the gains to achieve critically damped error dynamics inside the boundary layer, so that if one starts with a estimation error $\tilde{y} = \epsilon$, and all its derivatives equal to zero, the output estimation error \tilde{y} will not go outside the boundary layer".

This rule applied to systems in the canonical form described in this section would give:

Choose the gains H and ρ so that:

$$\max_{\omega} |C(j\omega I - A + HC + \frac{1}{\epsilon}KC)^{-1}D| \leq \epsilon$$

where the vector D was defined previously.

- **Comparison with Luenberger Observer :** One can argue that in the *case 2* a Luenberger Observer with a linear gain given by:

$$H^* = H + \frac{1}{\epsilon}K$$

where the gains H , K and ϵ are the values to be used in the Sliding Observer for a given problem, will essentially give the same steady

state estimation error, and it must also be robust against bounded nonlinearity/uncertainty term $w = D\eta$. It is actually true, however their behavior will be different if some measurement noise is present (what is always true). In this case, the fact the function 1, acts as pure relay outside the boundary layer, is reflected as a virtual gain reduction outside the boundary layer, which actually affect favorably in the presence of measurement noise, if compared with the *high gain* Luenberger Observer. This aspect of sliding observer is discussed in the corresponding chapter in this thesis.

The extension of the methods presented in these two sections to the multiple-measurement case can be done using the *Positive Real Lemma* and *Multivariable circle criterion* for a class of multivariable systems. It is shown in the following section.

4.5 Sliding Observer for Multiple Measurements

4.5.1 General Consideration

This section extends the sliding observer design method presented in the previous section to the multiple measurement case. The plant is assumed to be in the form defined previously:

$$\dot{x} = Ax + D\eta \quad (4.14)$$

$$y = Cx \quad (4.15)$$

where: $x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$, $A \in \mathbb{R}^{n \times n}$, $C \in \mathbb{R}^{m \times n}$ and $D\eta$ is the column vector that incorporates all the effects of input disturbances, uncertainties and nonlinearities in the system description. The matrix $D \in \mathbb{R}^{n \times m}$ is a matrix that carries the information about the magnitudes, with $\text{rank } d \leq m$ and $\eta \in \mathbb{R}^m$ is the source of disturbances/nonlinearities (it is the obvious extension of the function η defined in the single measurement case), with the condition $|\eta_i| \leq 1$.

The sliding observer has the structure shown earlier in the equation (4.2) :

$$\dot{\hat{x}} = A\hat{x} + H(y - C\hat{x}) + K1,$$

where:

$$\begin{aligned} H^T &= [h_1^T, h_2^T, \dots, h_n^T], H \in \mathbb{R}^{n \times m} \\ K^T &= [k_1^T, k_2^T, \dots, k_n^T], K \in \mathbb{R}^{n \times m} \end{aligned}$$

With this state estimator, the error dynamics can be described as:

$$\begin{aligned} \dot{\tilde{x}} &= x - \hat{x} \\ \dot{\tilde{x}} &= (A - HC)\tilde{x} - K\mathbf{1}_n + D\eta \end{aligned}$$

There are, as was true for the single-measurement case, essentially two cases to be considered for this class of problem. These cases will be shown next.

4.5.2 Case 3

The design procedure described in this section works for a particular class of problems for which the transfer function matrix

$$H_1(s) = C(sI - A + HC)^{-1}D$$

can be made strictly positive real.

Design Procedure

1. Write the vector w as :

$$w = D\eta = \begin{bmatrix} d_{11} & \dots & d_{1m} \\ \vdots & \ddots & \vdots \\ d_{n1} & \dots & d_{nm} \end{bmatrix} \begin{bmatrix} \eta_1 \\ \vdots \\ \eta_m \end{bmatrix}$$

where $D \in \mathbb{R}^{n \times m}$ is a constant matrix with rank $d \leq m$, and $d_{ij} \geq 0$.

2. Assign K as $K = D\rho$, where ρ is a diagonal matrix $\text{diag}(\rho_1, \dots, \rho_m)$.

Example : These two first steps can be illustrated by this simple example. Assume that the uncertainty vector is:

$$w^T = [0 \ 0 \ 0 \ w_1 \ w_2]$$

where $|w_1| \leq \gamma_1$ and $|w_2| \leq \gamma_2$.

Assume that the first three states is been measured, that is

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

In this system, the matrices D and η would be constructed as:

$$D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ \gamma_1 & 0 & 0 \\ 0 & \gamma_2 & 0 \end{bmatrix}$$

$$\eta = \begin{bmatrix} \eta_1 \\ \eta_2 \\ 0 \end{bmatrix}$$

For this case, the gain matrix K would be

$$K = D\rho = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ \gamma_1 & 0 & 0 \\ 0 & \gamma_2 & 0 \end{bmatrix} \begin{bmatrix} \rho_1 & 0 & 0 \\ 0 & \rho_2 & 0 \\ 0 & 0 & \rho_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ \gamma_1\rho_1 & 0 & 0 \\ 0 & \gamma_2\rho_2 & 0 \end{bmatrix}$$

where $\rho_i \geq 1$.

3. Check if the gain K can be written as:

$$K^T P = C$$

or, equivalently:

$$\begin{aligned} D^T P^* &= C \\ P^* &= \rho P \end{aligned} \tag{4.16}$$

for some symmetric positive definite matrix² P^* . If it is true, then one can proceed. Otherwise, one has to try to change the definition of matrix D , so to change the matrix K . If it does not work, then this method does not apply, and the case 4 must be tried.

4. Let the function 1_s be

$$1_s = \begin{bmatrix} \text{sign}(\tilde{y}_1) \\ \text{sign}(\tilde{y}_2) \\ \vdots \\ \text{sign}(\tilde{y}_m) \end{bmatrix}$$

5. Choose the gain matrices H and ρ so that the following conditions are satisfied:

$$\begin{aligned} P(A - HC) + (A - HC)^T P &= -Q \\ K^T P &= C \\ K &= D\rho \end{aligned} \quad (4.17)$$

for symmetric positive definite matrices P and Q . This condition is usually hard to verify and hard to satisfy, and to the best of author's knowledge there is no numerical method that solves these equations. In this step a symbolic manipulator program, like MACSYMA or MUSIMP, might be useful.

6. If these conditions are met, then the design is complete. If they are not met one can either iterate changing the matrix D (if possible) or using the multiplier theory (see Desoer and Vidyasagar [11] section 6.9) to change the blocks H_1 and H_2 by adding convenient operators in the error dynamics. If these changes don't help then one can try the method described in case 4.

The conditions imposed by this method can be too restrictive (in analogy to case 1, for the single measurement case) for a particular problem at hand. An example of this case is illustrated next.

²It requires that the matrix D^T has to be in the range space of C .

Example: Assume that C and D was found to be:

$$D = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ d_1 & 0 \\ 0 & d_2 \end{bmatrix}$$

and

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

In this case the condition (4.18) implies that the following condition must be satisfied

$$D^T P^* = C$$

It is easy to verify that there is no symmetric positive definite matrix P^* that satisfies this condition.

The purpose of this example is to show that, quite often it might be difficult (even impossible) to design the observer using the positive real lemma, as described here.

This design procedure will actually guarantee that $\tilde{y} \rightarrow 0$, and $\tilde{x} \rightarrow 0$ as $t \rightarrow \infty$.

The proof of this method is given in the next subsection.

Proof of the Design Procedure

The proof of the design method, just presented, relies on the results from the passivity theorem that was discussed before, and the condition on the operator H_1 to be strictly passive translated into a more useful form using the Kalman-Yakubovich Lemma (Positive real lemma).

As was extensively discussed in subsection 4.3.2, the use of pure relay in the function 1, as adopted in this case, requires that the linear forward operator H_1 must be strictly positive real. Because this operator is causal, it was shown that it implied that the operator H_1 has to be strictly positive real.

This condition can be translated in a more useful form, using the Kalman-Yakubovich Lemma (the Positive Real Lemma)[33,40].

Lemma 4.1 H_1 is strictly positive real if and only if the gain matrices H and K are chosen so that there exist a symmetric positive definite matrices P and Q such that the following equalities hold:

$$\begin{aligned} P(A - HC) + (A - HC)^T P &= -Q \\ K^T P &= C \end{aligned} \quad (4.18)$$

which is exactly the last condition shown in the design procedure. As was commented upon, before, the condition imposed by the Positive Real Lemma is often hard to verify.

One can see that the conditions given by the Lemma 4.1, with ρ taken as an identity matrix (recall that $K = D\rho$), are exactly the hypothesis assumed by Walcott and Žak [63,64], discussed in Chapter 2. Walcott and Žak, apparently, stated those assumptions as a convenient way to guarantee the estimation convergence using a quadratic Lyapunov function. In [64] a brief interpretation, is given in terms of positive real systems. The derivation shown in this thesis, using the passivity theorem, provides additional insight on the assumptions made by Walcott and Žak.

The last result in this section that has to be proved is the result regarding the fact that $\tilde{x} \rightarrow 0$ as $t \rightarrow \infty$.

This fact can be shown using the Lyapunov Stability theory. This result was derived independently by this author and by Walcott and Žak [63,64].

The estimation error dynamics is given by,

$$\begin{aligned} \dot{\tilde{x}} &= (A - HC)\tilde{x} - K\tilde{y} + D\eta \\ \tilde{y} &= C\tilde{x} \end{aligned}$$

And the Positive Real Lemma requires that,

$$\begin{aligned} (A - HC)^T P + P(A - HC) &= -Q \\ K^T P &= C \end{aligned}$$

for symmetric positive definite matrices P and Q . Recall that the gain was defined as:

$$K = D\rho \implies D = K\rho^{-1}$$

Let the candidate Lyapunov Function be the quadratic form:

$$V = \tilde{x}^T P \tilde{x}$$

where P is restricted to be symmetric positive definite matrix. Its total derivative is

$$\dot{V} = \dot{\tilde{x}}^T P \tilde{x} + \tilde{x}^T P \dot{\tilde{x}}$$

which, with the substitution of the equation of estimation error dynamics, and the definition of gain K becomes

$$\begin{aligned} \dot{V} &= \tilde{x}^T [(A - HC)^T P + P(A - HC)] \tilde{x} - 2\tilde{x}^T P K (1_s - \rho^{-1} \eta) \implies \\ \implies \dot{V} &= -\tilde{x}^T Q \tilde{x} - 2\tilde{x}^T P P^{-1} C^T (1_s - \rho^{-1} \eta) \implies \\ \implies \dot{V} &= -\tilde{x}^T Q \tilde{x} - 2(C\tilde{x})^T (1_s - \rho^{-1} \eta) \end{aligned}$$

Because $|\eta_i| \leq 1$ and $\rho_i \geq 1$, one can verify that

$$\tilde{y}^T (1_s - \rho^{-1} \eta) = \sum_{i=1}^m \tilde{y}_i \left(\frac{\tilde{y}_i}{|\tilde{y}_i|} - \frac{\eta_i}{\rho_i} \right) \geq 0$$

therefore,

$$\dot{V} < 0$$

which means that $\tilde{x} \rightarrow 0$ as $t \rightarrow \infty$. In this case, Ogata [42] shows that

$$\dot{V} \leq -\lambda V$$

where λ is the minimum eigenvalue of $P^{-1}Q$.

A last comment regards the case when the bounds of nonlinearities/uncertainties are functions of time. In this case, by assigning the gain matrix K as $K = D\rho$, would give a time-varying gain matrix $K(t)$. In this case, an alternative lemma can be used to guarantee that the operator H_1 is still strictly positive real[33]:

Lemma 4.2 *The system described by*

$$\begin{aligned} \dot{\tilde{x}} &= (A - HC)\tilde{x} + K(t)u \\ \tilde{y} &= C\tilde{x} \end{aligned}$$

is strictly positive real if there exist a symmetric positive definite matrix $P(t)$, a symmetric matrix $Q(t)$, and a matrix $S(t)$ so that

$$\begin{aligned} \dot{P}(t) + (A - HC)^T P(t) + P(t)(A - HC) &= -Q(t) \\ K^T(t)P(t) + S^T(t) &= c \\ \begin{bmatrix} Q(t) & S(t) \\ S^T(t) & 0 \end{bmatrix} &> 0 \end{aligned}$$

for all positive t .

4.5.3 Case 4

This case is the extension of the method described in “case 2”, for the single-measurement case, for the multiple-measurement case.

One can verify that the case 3 was restricted by the Positive Real Lemma, which resulted from the strict passivity condition imposed by the use of pure switching (*signum* function) for the function 1_s .

It suggests that, as for the case 2, taking the saturation function for 1_s , a less restrictive condition would be found. It is actually the case, and the design procedure uses the extension of circle criterion to the multivariable case.

The design procedure is first presented, and proof of this method is presented at the end of this subsection.

Design Procedure

The design procedure for this case is very similar to the case 2:

1. Let the 1_s to be the vector function whose components are saturation function defined in the usual way:

$$1_s = \begin{bmatrix} \text{sat}(\tilde{y}_1/\epsilon_1) \\ \text{sat}(\tilde{y}_2/\epsilon_2) \\ \vdots \\ \text{sat}(\tilde{y}_m/\epsilon_m) \end{bmatrix}$$

and

$$\text{sat}(\tilde{y}_i/\epsilon_i) = \begin{cases} \tilde{y}/|\tilde{y}| & \text{if } |\tilde{y}_i| \geq \epsilon_i \\ \tilde{y}_i/\epsilon_i & \text{if } |\tilde{y}_i| < \epsilon_i \end{cases} \quad (4.19)$$

2. Let $w = D\eta$
3. Set the gain matrix K as:

$$\begin{aligned} K &= D\rho \\ \rho &= \text{diag}[\rho_1, \rho_2, \dots, \rho_m] \end{aligned}$$

4. Determine the width of boundary layer for \tilde{y}_i , called ϵ_i , and which coincides with saturation limit ϵ_i in the *sat*-function defined by (4.19); define the matrix

$$\Delta = \text{diag}(\epsilon_1, \epsilon_2, \dots, \epsilon_m)$$

5. Design for boundedness inside the boundary layer. It can be done in two ways:

- Choose the gain matrices H and ρ such that $A - (H + K\Delta^{-1})C$ has distinct eigenvalues and such that for all $\omega \in \mathfrak{R}$:

$$\max[\sigma_{\max}(C(j\omega I - A + (H + K\Delta^{-1})C)^{-1})D] + \sigma_{\max}(C)\text{cond}(V)\max(|\tilde{x}_0|) \leq \min_i(\epsilon_i)$$

where $\sigma_{\max}(M)$ is the maximum singular value of the complex matrix A , the norm $|z| = (z^*z)^{\frac{1}{2}}$ (z^* denotes the complex conjugate), and the $\text{cond}(V)$ is the condition number of the matrix of eigenvectors V of $A - (H + K\Delta^{-1})C$. $\max(|\tilde{x}_0|)$ is an upper bound of initial state estimation error \tilde{x}_0 . This approach suffers the drawback that it depends on the scaling factors of eigenvectors.

- Choose the gain matrices H and ρ such that for all $t \in \mathfrak{R}_+$ and for all $\omega \in \mathfrak{R}$:

$$\max[\sigma_{\max}(C(j\omega I - A + (H + K\Delta^{-1})C)^{-1})D] + \sigma_{\max}(Ce^{(A - (H + K\Delta^{-1})C)t})\max(|\tilde{x}_0|) \leq \min_i(\epsilon_i)$$

In practice this test has to be verified in the time interval of interest, typically during the initial transient.

Both methods are very conservative, as will be shown in the examples.

6. Check whether the state estimation error bounds are within reasonable limits:

$$|\tilde{x}| \leq \max_{-\infty < \omega < +\infty} [\sigma_{\max}((j\omega I - A + (H + K\Delta^{-1})C)^{-1})D]$$

7. Check for stability outside the boundary layer. The matrices H and K must be so that:

- $(A - HC - KMC)$ has eigenvalues in the open left half plane; the matrix M is a diagonal matrix whose entries are $m_{ii} = \frac{1}{2\epsilon_i}(1 + \frac{1}{\rho_i})$;

- The following condition is verified for all values of $\omega \in \mathfrak{R}$ at which G^{-1} exists:

$$\sigma_{\min}([M + G^{-1}(j\omega)]M^{-1}) \geq 1$$

where $H_1(s) = C(sI - A + HC)^{-1}K$.

8. If this last condition is satisfied, then the design is complete. Otherwise, one has to iterate.

Proof of the Design Procedure

The proof of this design procedure has to be done in two parts: the first part corresponds to the situation when the output estimation error \tilde{y} is outside the boundary layer (i.e. during the reaching phase), and the second part is the situation when the output estimation error \tilde{y} gets inside the boundary-layer.

The second situation, that is inside the boundary layer, is exactly equal to the situation discussed in Case 2, therefore the proof will be omitted.

The new situation is when the output estimation error \tilde{y} is outside the boundary layer. To show that the design procedure presented for this case is correct, the multivariable circle theorem [45,46,48] is used³.

It is necessary to recognize that the estimation error dynamics can be described as:

$$\begin{aligned}\dot{\tilde{x}} &= (A - HC)\tilde{x} - K\phi\tilde{y} \\ \tilde{y} &= C\tilde{x}\end{aligned}$$

as shown in Figure 4.2. In this case ϕ is:

$$\phi\tilde{y} = \mathbf{1}_s(\tilde{y}) - \rho^{-1}\eta$$

therefore, outside the boundary layer, each component ϕ_i of the operator ϕ is contained inside the sector:

$$0 \leq \phi_i\tilde{y}_i \leq \left[\frac{1}{\epsilon_i}\left(1 + \frac{1}{\rho_i}\right)\right]\tilde{y}_i \quad (4.20)$$

The last step in the proof is to show that the multivariable circle theorem (see Appendix A) is satisfied:

³See Appendix A

- The functions ϕ 's are in the cone (M, R) (using the nomenclature defined by Safonov [45]), that is it is inside the "cone with center M and radius R . Because each component of the operator ϕ is inside the sector defined by (4.20), the center M and the radius R are defined by:

$$M = R = \text{diag}(r_1, r_2, \dots, r_m)$$

$$r_i = \frac{1}{2\epsilon_i} \left(1 + \frac{1}{\rho_i}\right)$$

- By constraining the eigenvalues of $(A - HC - KMC)$ to the open left half plane, one can conclude that the estimation error dynamics with $\phi = M$ is asymptotically stable, therefore $\mathcal{L}_{2\epsilon}$ -stable.
- The last condition that has to be verified is the condition that involves the minimum singular value :

$$\sigma_{\min}([R + G^{-1}]M^{-1}) \geq 1$$

for all ω at which G^{-1} exists. In this case $R = M$, and the last condition is verified.

Therefore, if all the steps in the design procedure are satisfied, then the multivariable circle theorem is verified, and the \mathcal{L}_2 -stability, can be concluded [45,46,48]. It shows that the boundary layer is attractive, and it completes the proof.

4.6 General Remarks

In this chapter, the sliding observer design procedures for stability robustness were presented and proved.

The procedures were presented as algorithms, so that the control engineer could try to apply these methods without deep understanding of the proofs of each of the methods.

Nowhere in these procedure descriptions was the issue of convergence rate mentioned. Even though the design methods do not guarantee any performance measure, it is possible to guide the design in such a way to improve the speed of response.

The easiest rule is to place the poles of error dynamics (either of the linear part $A - HC$ at the "reaching phase" - i.e. outside the boundary layer, or the whole $A - (H + \frac{1}{\epsilon}K)C$ inside the boundary layer) at suitable positions, so that the conditions for stability robustness are not violated and yet guarantee quick response.

In Case 4, the performance, in terms of the speed of response can be more explicitly reinforced if $j\omega - \alpha$ is substituted for $j\omega$ in the stability tests: it will guarantee that the error dynamics will have an exponential rate of decay $e^{-\alpha t}$.

As was mentioned several times in this chapter, the design procedure does not cover the whole range of possible state estimation problems. If the problem at hand can be described in the form given by equations (4.14) and (4.15), but if the simplified design procedures described in this chapter do not apply, then the design can still be carried out by applying Theorem 4.1.

Chapter 5

Design Examples

In this chapter, three examples are developed to illustrate the use of sliding observers for state estimation of real systems.

In all examples, the presented results were obtained from simulations in the digital computer, due to the unavailability of actual hardware.

In the first example, the sliding observer is used to estimate the angular velocity (“yaw rate”) and the angular acceleration of a super-tanker. The second example shows the application of a sliding observer to estimate the states of a one link manipulator with flexible joint. The third example shows the application of a sliding observer for shaft torque estimation in a automatic transmission for cars.

5.1 Super-Tanker Lateral Dynamics

In this example, the sliding observer is used to estimate the yaw rate and its derivatives, from the measurement of the “heading”, i.e. the yaw angle of a Super-Tanker.

The plant is described as an unstable third-order system, whose equation of motion was identified by Frimm [14]. It is described as:

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_3 \\ \dot{x}_3 &= -\frac{k}{T_1 T_2} H(x_2) - \left(\frac{1}{T_1} + \frac{1}{T_2}\right) x_3 + \frac{k}{T_1 T_2} (\delta + T_3 \dot{\delta})\end{aligned}$$

where:

- $x_1 = \psi$ is the yaw angle (degree);
- $x_2 = \dot{\psi}$ is the yaw rate (degree/sec);
- x_3 is the derivative of yaw rate;
- δ is the rudder angle (degree);

This is an observable system, given the measurement of x_1 .

The function $H(x_2) = H(\dot{\psi})$, and the constants were identified, from the actual ship as:

$$\begin{aligned}
 H(\dot{\psi}) &= 1.8419 - 21.294\dot{\psi} - 8.0534\dot{\psi}^2 + 96.5283\dot{\psi}^3 - 24.9247\dot{\psi}^5 \\
 T_1 &= -60.26 \\
 T_2 &= 7.77 \\
 T_3 &= 17.5 \\
 \dot{\kappa} &= -.04696
 \end{aligned}$$

Obviously, if the measurement of x_1 is truly noiseless, one can simply take the derivatives of x_1 in order to get the estimates of x_2 and x_3 . In practice all sensors have some noise therefore an alternative method must be used. Due to the presence of nonlinearities, the obvious candidate for alternative estimator is the sliding observer. The goal of this problem is to design a sliding observer that provides the estimates for the second and third states, given the measurement of yaw angle. A linear model, given by Arie[1] will be used and the rudder inputs will be lumped with the nonlinearity and regarded as an uncertainty.

The model used for the observer design is then:

$$\begin{aligned}
 \dot{x}_1 &= x_2 \\
 \dot{x}_2 &= x_3 \\
 \dot{x}_3 &= -\frac{1}{T_1 T_2} x_2 - \left(\frac{1}{T_1} + \frac{1}{T_2}\right) x_3 + w \\
 |w| &\leq \gamma
 \end{aligned}$$

where the uncertainty bound γ will be determined based on the knowledge of the actual system as will be shown shortly.

The sliding observer, after setting the gain matrix as $K = \rho D$, becomes:

$$\begin{aligned}\dot{\hat{x}}_1 &= \hat{x}_2 + h_1 \tilde{x}_1 \\ \dot{\hat{x}}_2 &= \hat{x}_3 + h_2 \tilde{x}_1 \\ \dot{\hat{x}}_3 &= -\frac{1}{T_1 T_2} \hat{x}_2 - \left(\frac{1}{T_1} + \frac{1}{T_2}\right) \hat{x}_3 + h_3 \tilde{x}_1 + k_3 \text{sat}(\tilde{x}_1/\epsilon) \\ \tilde{x}_1 &= x_1 - \hat{x}_1\end{aligned}$$

where the h_i 's are the "linear gains" and k_3 is the switching gain to be determined.

Following the design procedure for the single-measurement problem – Case 2 – one has to determine the bound for the modelling error.

In this case, the modelling errors are mathematically unbounded, therefore it is necessary to use the physical knowledge to determine the bounds for modelling errors. The experimental data was obtained by feeding back the position measurement through a relay, inducing a limit cycle.

The experimental data shows that

$$\begin{aligned}|x_1| &< 1(\text{degree}) \\ |x_2| &< .2(\text{degree/sec}) \\ |\delta| &\leq 10(\text{degree})\end{aligned}$$

Because the relay output looks like a square wave, the derivative of the input $\dot{\delta}$ is theoretically infinite, therefore unbounded. Since this singularity only occurs in a very few instants – actually $\dot{\delta} = 0$ during most of the time – it was neglected in the bounding process. The effect of this simplification is that, if this singularity ever becomes important, then the trajectory would leave the boundary layer momentarily. Soon after this singularity disappears, the stability is again guaranteed and the trajectory would return to the boundary layer.

Using the physical bounds given above, and using the triangle and Schwarz inequalities, the bound on the uncertainty term is found to be:

$$|w| = \left| \frac{1}{T_1 T_2} (x_2 - kH(x_2)) + \frac{k}{T_1 T_2} (\delta + T_3 \dot{\delta}) \right| < .023$$

Taking $d = .023$, and $k = .1$ (that is $\rho = 4.3$), one can now proceed through the following steps.

Say that, for control applications, one allows the estimation error for the estimation of x_1 of 0.1 degree, that is one can set $\epsilon = .1$.

With this information, the behavior inside the boundary layer is determined assuming that the initial estimation error would roughly be bounded by

$$\begin{aligned}\tilde{x}_1 &= 0 \\ |\tilde{x}_2|_{max} &\leq 1.5 \\ |\tilde{x}_3|_{max} &\leq 0.5\end{aligned}$$

which gives $|\tilde{x}| \leq 1.58$.

Using standard CACSD ¹ software, the singular value analysis given in Chapter 4 is easily carried out. By constraining the behavior inside the boundary layer to be critically damped, the gains were chosen as:

$$\begin{aligned}h_1 &= 14.89 \\ h_2 &= 73.3 \\ h_3 &= 115.8\end{aligned}$$

One can see that the boundedness condition is violated (Figures 5.1 and 5.2) due to the effect of initial conditions, which is not surprising considering its conservativeness. The simulations shows that the output estimation error \tilde{y} remains inside the boundary layer, once it gets into it, even though this condition is violated.

The final step in the design process requires that the absolute stability must be checked. This is done by drawing the Nyquist plot of the transfer function $G(s) = C(sI - A + HC)^{-1}K$ (the necessary matrices are evident from the model description).

From the given information, it is desired that the Nyquist plot stay to the right of the vertical line that intersects the real axis at -0.081 .

The Nyquist plot is shown in Figure 5.3, and the absolute stability is guaranteed, by large margin, using the circle criterion.

Simulating the plant and the observer, using the program SIMNON, the time history for the real and estimated states were obtained. They are

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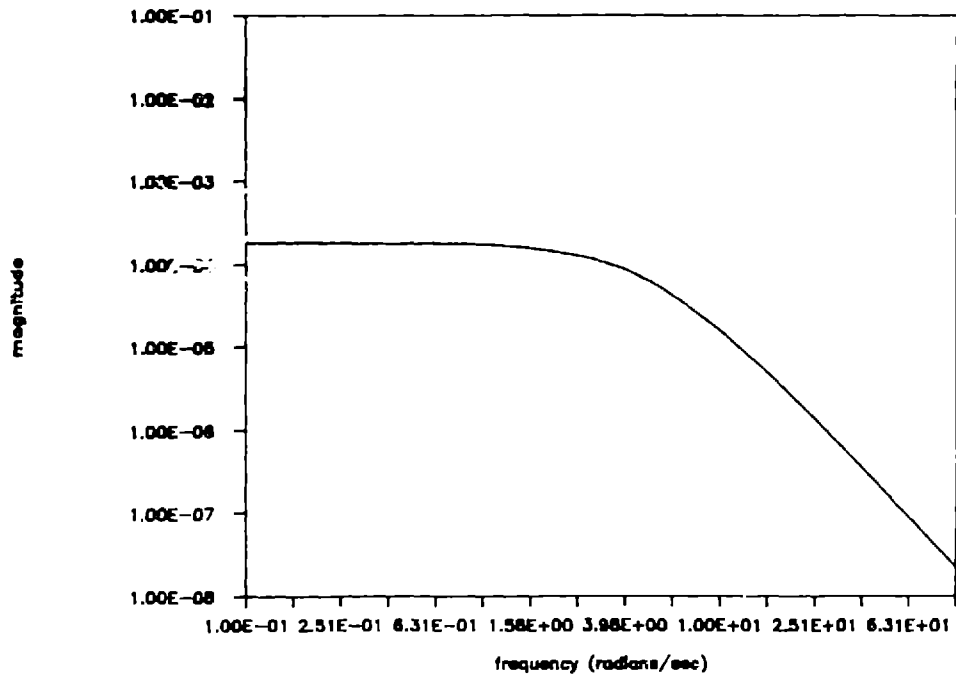


Figure 5.1: Singular values of $G(s) = C(sI - A + (H + \frac{1}{2}K)C)^{-1}D$

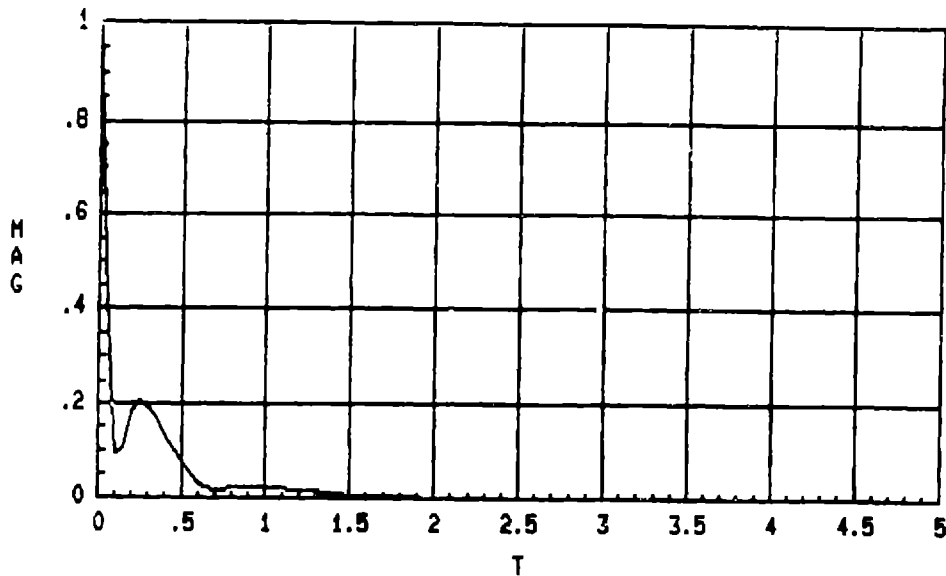


Figure 5.2: Singular values of $C e^{A - (H + \frac{1}{2}K)C} t$

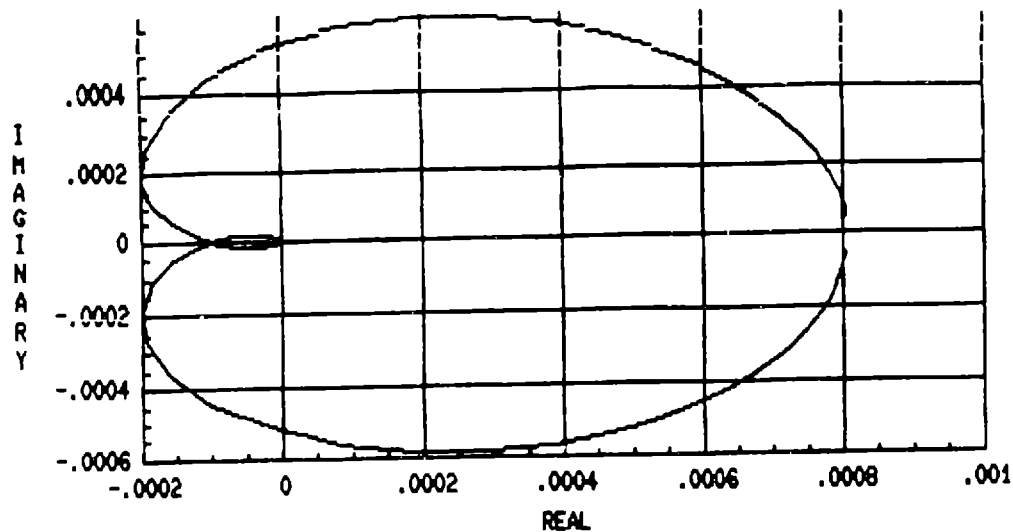


Figure 5.3: Nyquist plot for the observer design for the super-tanker

shown in the Figures 5.4 and 5.5 for the time plot and the trajectories in the state-space of error dynamics.

Clearly, we obtained convergence in less than 2 sec, and the trajectory actually goes into the boundary layer, and never leaves it.

One might wonder how the heuristic “rule” mentioned in the design procedure would perform in this case. Following that rule, the absolute stability is guaranteed with $h_1 = 5.89$, $h_2 = 11.34$, $h_3 = 4.74$, $k = 0.1$, $\epsilon = 0.05$, as long as the trajectory is outside the boundary layer. Running the simulations with these numbers, one can verify (the simulation results are not shown here, in order to avoid proliferation of Figures) that the behavior is very close to the first one: The convergence is obtained in about 4 sec. and the largest overshoot of \tilde{x}_1 is less than 0.1 degree. The largest estimation error for x_2 and x_3 are twice the ones obtained before, but they are still small.

This remark gives some sense of how conservative this design process is.

Another question that the reader might ask is what is the effect of the switching gain, since the linear gains are proportionally much larger. The answer is that, for this particular problem, the performance is not really affected if the switching gain is set to zero as seen in Figure 5.6. However, by

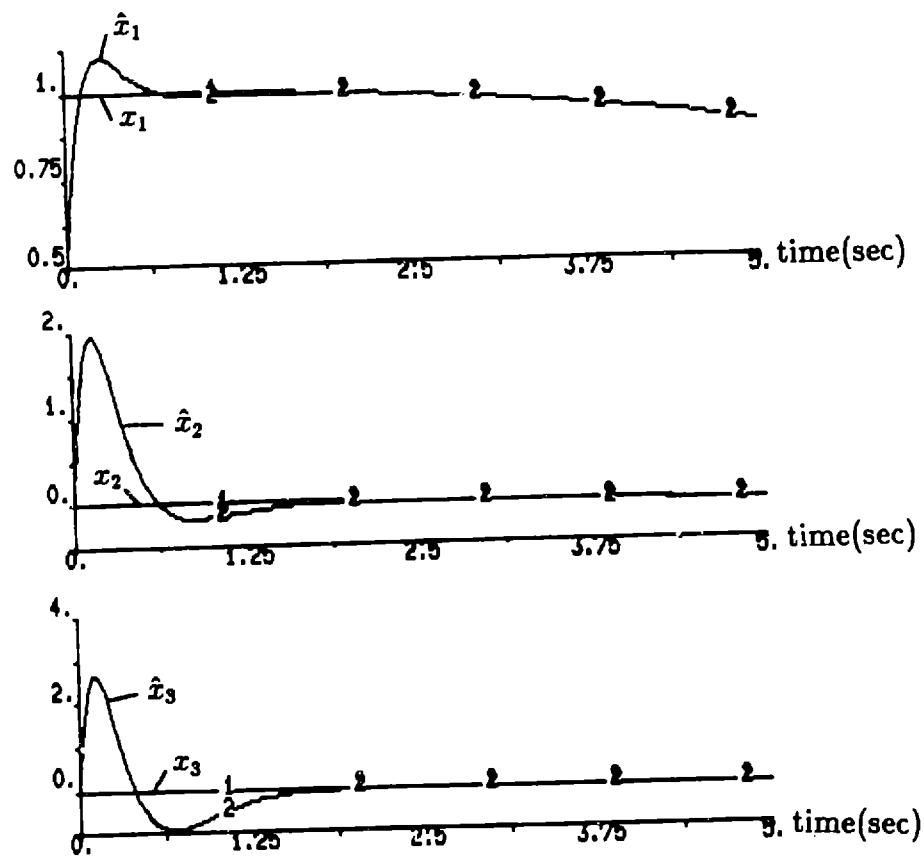


Figure 5.4: Time plot of actual and estimated states super-tanker states

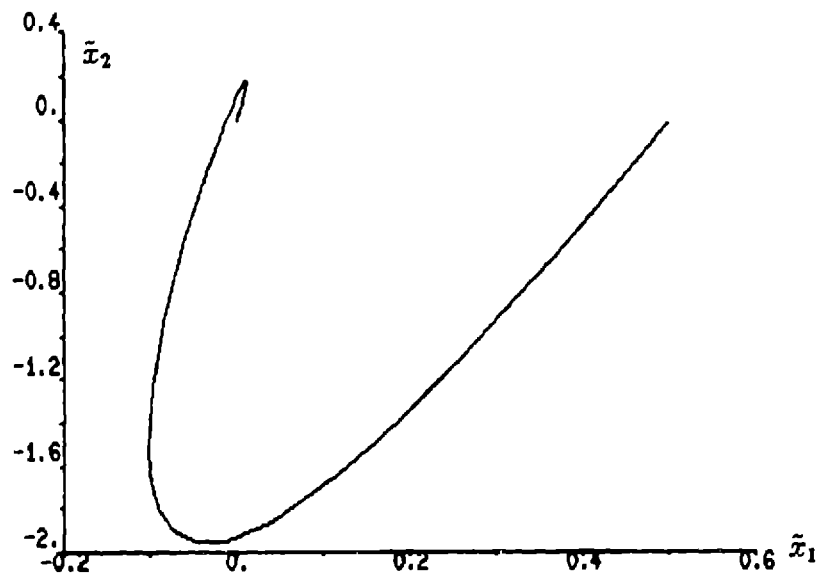
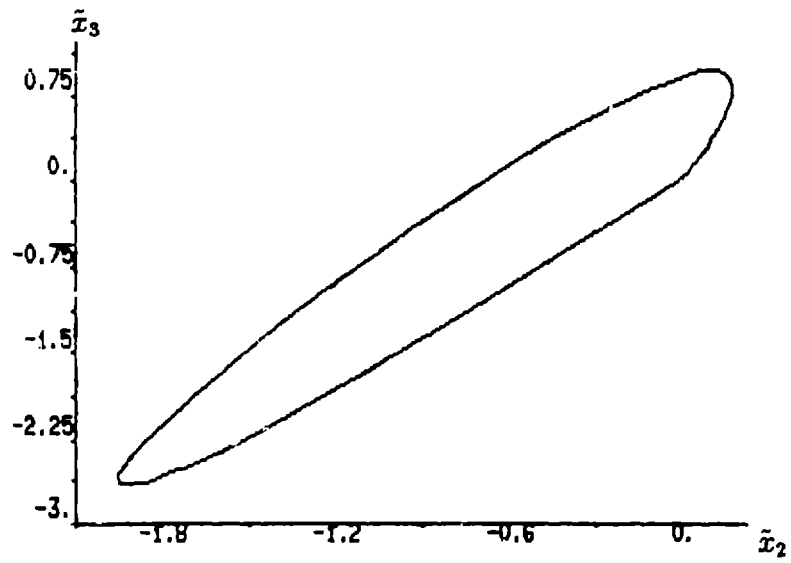


Figure 5.5: State-space plots of the super-tanker estimation errors

doing it the robustness cannot be guaranteed because the only knowledge of the uncertain term is its bound.

5.2 One Link Manipulator with Flexible Joint

An interesting application of the sliding observer can be found in the estimation of angular velocities, given the measurement of positions in a one-link, elastic-joint, manipulator.

5.2.1 The Elastic Joint Manipulator

The system, illustrated in Figure 5.7, was described by Spong [56]. The equations of motion for this system in state space form are:

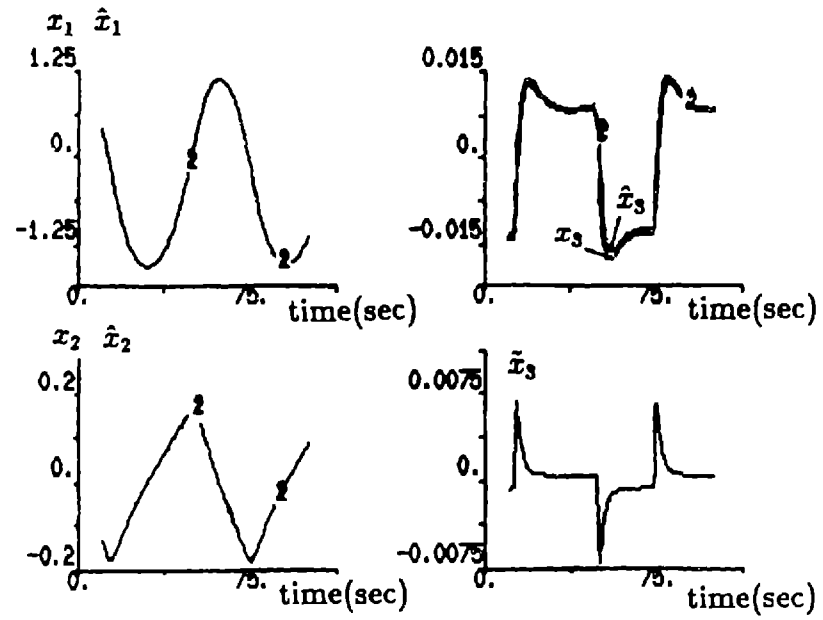
$$\begin{aligned}
 \dot{x}_1 &= x_2 \\
 \dot{x}_2 &= -\frac{mgl}{I} \sin(x_1) - \frac{k}{I}(x_1 - x_3) \\
 \dot{x}_3 &= x_4 \\
 \dot{x}_4 &= \frac{k}{J}(x_1 - x_3) + \frac{1}{J}u
 \end{aligned} \tag{5.1}$$

where $x_1 = \theta_1$, $x_2 = \dot{\theta}_1$, etc... ; with the parameters given in the International System of Units (meter-kilogram-second) are:

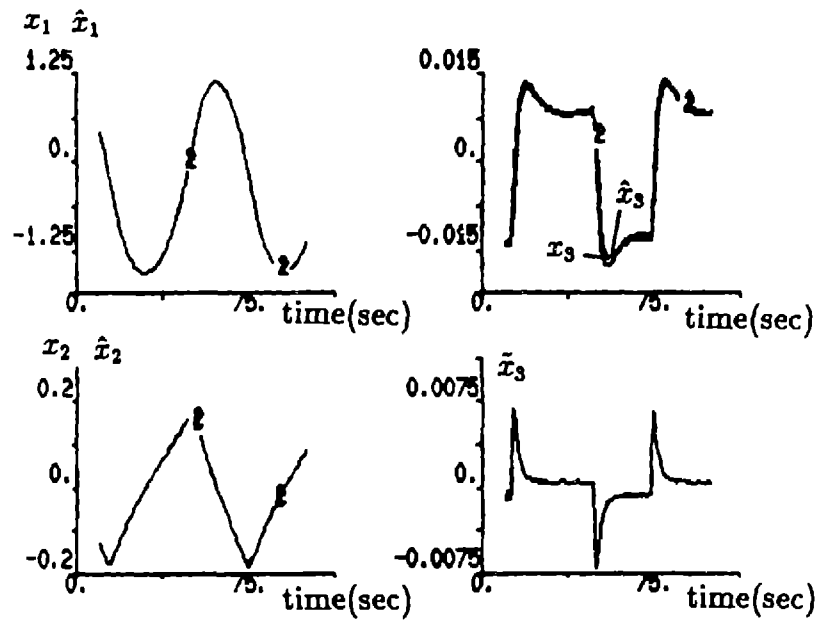
- mass = $m = 1$
- stiffness $k = 100$
- length ($2l$) $l = 1$
- gravity $g = 9.8$
- inertias $I = J = 1$

The manipulator is controlled by a compensator designed using Global Linearization [22], shown in [56]. The controller is given by:

$$u = \frac{IJ}{k}(v - F(x_1, x_2, x_3, x_4))$$



a) linear filter



b) sliding observer

Figure 5.6: Comparison of estimation errors from the linear filter and the sliding observer

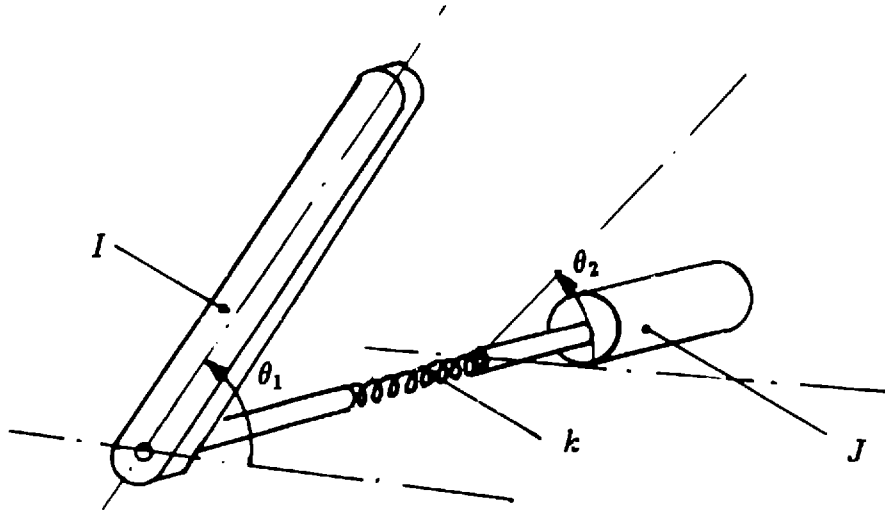


Figure 5.7: Single link with elastic joint

$$\begin{aligned}
 v &= \dot{y}_4^d + \sum_{i=1}^4 a_{i-1}(y_i^d - y_i) \\
 y_1^d &= \sin(\omega t) \\
 \dot{y}_i^d &= y_{i+1}^d, \quad \forall i = 1, 2, 3 \\
 y_1 &= x_1 \\
 y_2 &= x_2 \\
 y_3 &= -\frac{mgl}{I} \sin x_1 - \frac{k}{I}(x_1 - x_3) \\
 y_4 &= -\frac{mgl}{I} \cos x_1 x_2 - \frac{k}{I}(x_2 - x_4) \\
 F(.) &= \frac{mgl}{I} \sin x_1 x_2^2 + \left(\frac{mgl}{I} \cos x_1 + \frac{k}{I}\right) \left(\frac{mgl}{I} \sin x_1 + \frac{k}{I}(x_1 - x_3)\right) + \\
 &\quad + \frac{k^2}{IJ}(x_1 - x_3) \tag{5.2}
 \end{aligned}$$

where the controller parameters a_i were chosen so that the poles of the globally linearized system are at $\lambda = -10$. It gives:

- $a_0 = 10^4$
- $a_1 = 4000$
- $a_2 = 600$

- $a_3 = 40$

Given this system description, the goal is to design a sliding observer, given the measurement of the positions $x_1 = \theta_1$ and $x_2 = \theta_2$.

5.2.2 Model for Sliding Observer Design

A slightly simplified model is used for Sliding Observer Design. Neglecting the nonlinear term due to the gravity term, and assuming that the input is not measured, the simplified equations of motion are:

$$\begin{aligned}
 \dot{x}_1 &= x_2 \\
 \dot{x}_2 &= -\frac{k}{I}(x_1 - x_3) + w_1 \\
 \dot{x}_3 &= x_4 \\
 \dot{x}_4 &= \frac{k}{J}(x_1 - x_3) + w_2
 \end{aligned} \tag{5.3}$$

Comparing equations (5.1) and (5.3), one can identify the *disturbance* terms w_1 and w_2 explicitly and find bounds for them.

Because the sine function is bounded, the term w_1 is bounded as:

$$|w_1| \leq 9.8 < 10.0$$

Bounding the term $w_2 = u/J$ is difficult. In this case, the bound were obtained by looking at the input signal in some simulations. By taking the reference input as $y_1^d = \sin(3t)$, one can verify that :

$$|w_2| \leq 10.0$$

Actually, depending on the initial conditions, this bound is violated during the initial transient. The effect of this violation is that the estimation may diverge in the beginning, but once the tracking is obtained, the estimation will also converge properly. When the observer is used inside the control loop, a more conservative bound must be used in order to account for the transient large inputs.

5.2.3 Sliding Observer Design

The model (5.3) can be described in the matrix form as:

$$\begin{aligned}\dot{x} &= Ax + D\eta \\ y &= Cx\end{aligned}$$

where:

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -100 & 0 & 100 & 0 \\ 0 & 0 & 0 & 1 \\ 100 & 0 & -100 & 0 \end{bmatrix}$$

$$D = \begin{bmatrix} 0 & 0 \\ 10 & 0 \\ 0 & 0 \\ 0 & 10 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$|\eta_1| \leq 1.0$$

$$|\eta_2| \leq 1.0$$

One can readily see that this system cannot be designed using the method discussed in *case 3* because the condition

$$D^T P = C$$

cannot be satisfied by any symmetric positive definite matrix P .

The alternative is to use the method discussed as *Case 4*.

Let

$$K = D\rho = \begin{bmatrix} 0 & 0 \\ 10\rho_1 & 0 \\ 0 & 0 \\ 0 & 10\rho_2 \end{bmatrix}$$

Also, let $\rho_1 = 1$ and $\rho_2 = 1$.

And choose the boundary layer width to be

$$\delta_1 = \delta_2 = 0.01$$

that is,

$$\Delta^{-1} = \begin{bmatrix} 100 & 0 \\ 0 & 100 \end{bmatrix}$$

The next step is to design the observer, so that the trajectories inside the boundary layer remain inside the boundary layer. The gain matrix H was selected using the constant gain Kalman filtering theory, with artificial process and measurement noise intensities. This is an iterative process, and the choice of process noise intensity

$$Q = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

and the measurement noise intensity

$$R = 10^{-7} \times \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

gave the gain matrix H as:

$$H = \begin{bmatrix} 67.0 & 1.0 \\ 2249 & 67.0 \\ 1.0 & 67.0 \\ 67.0 & 2249 \end{bmatrix}$$

With this choice of gain matrices the condition on the maximum singular value is satisfied. In fact, the plot of singular values of

$$G_1(j\omega) = C(j\omega I - A + (H + K\Delta^{-1})C)^{-1}D$$

is as illustrated in Figure 5.8, and the maximum singular values of

$$C e^{(A - (H + K\Delta^{-1})C)t}$$

is illustrated in Figure 5.9. Clearly, the condition

$$|\tilde{y}| = |C(x - \hat{x})| \leq 0.01 = \delta_1 = \delta_2$$

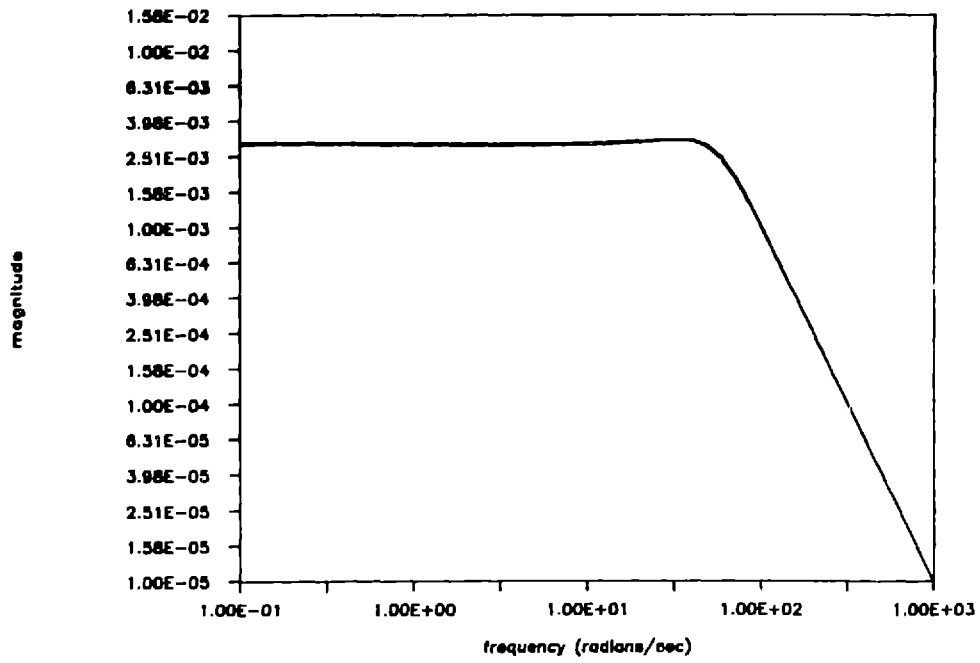


Figure 5.8: Singular values of $G_1(j\omega)$

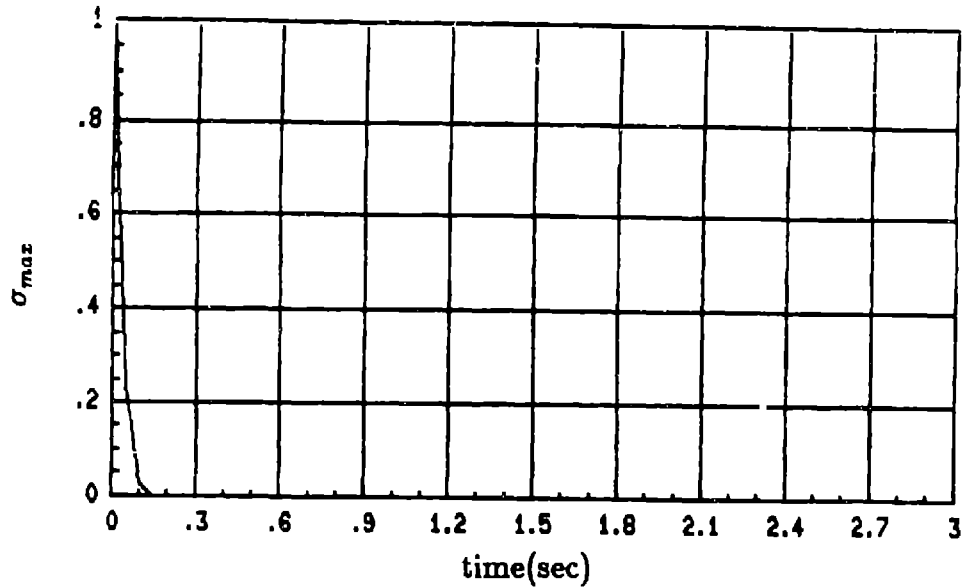


Figure 5.9: Singular value $\sigma_{max}(C e^{(A-(H+K\Delta^{-1})C)t})$

is satisfied, provided that $|\tilde{x}_0| \leq 0.096$. It is quite conservative, and the simulation shows that much larger initial estimation error will not pose any problem.

The next step is to check robustness using the multivariable circle criterion. The cone (M, R) is defined by the saturation function plus the uncertainty term. It is found to be:

$$m_{ii} = r_{ii} = \frac{1}{2\delta_i} \left(1 + \frac{1}{\rho_i}\right) = 100$$

therefore,

$$M = R = \begin{bmatrix} 100 & 0 \\ 0 & 100 \end{bmatrix}$$

Because the eigenvalues of

$$A - (H + KM)C$$

are in the open left half plane, the only condition that must be checked is the one that says that the minimum singular value of

$$V(j\omega) = [M + G_3^{-1}(j\omega)]R^{-1}$$

with

$$G_3(j\omega) = C(j\omega I - A + HC)^{-1}K$$

must be always larger or equal to one. This condition is easily satisfied, as shown in Figure 5.10

One can also estimate the bounds of the estimation errors at the steady state. The estimation error can be bounded by the largest maximum singular value, over all frequencies, of

$$G_4 = (j\omega I - A + (H + K\Delta^{-1})C)D$$

The frequency domain plot is shown in Figure 5.11. One can see that the maximum singular value is always smaller than 0.25, therefore

$$|\tilde{x}| \leq 0.25$$

One can see that the observer design was carried out easily for this problem. Some digital simulation shows how this observer works for some initial conditions.

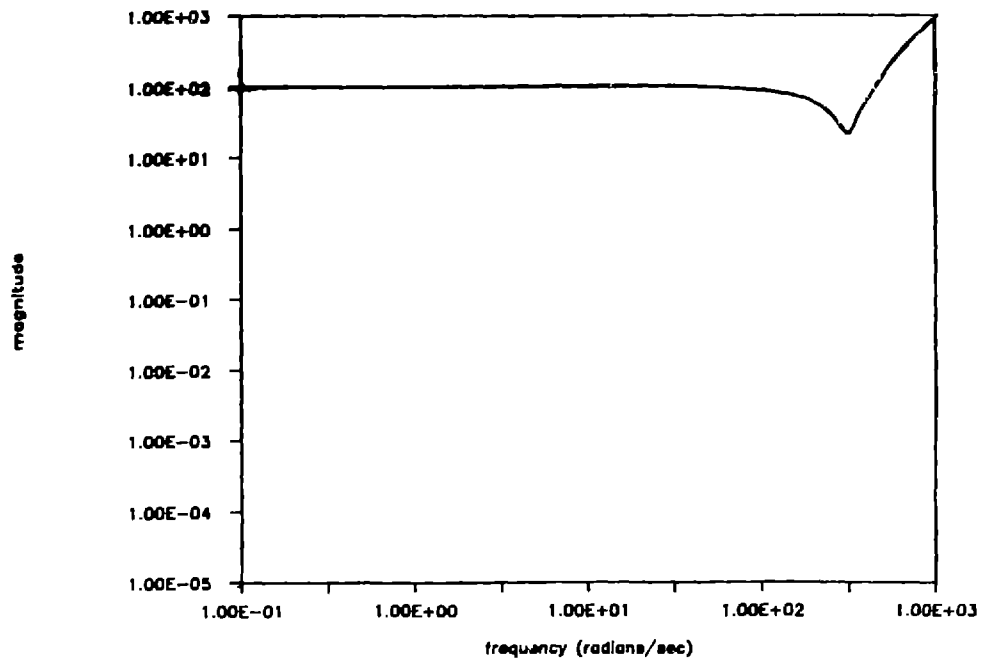


Figure 5.10: Singular values of $G_3(j\omega)$

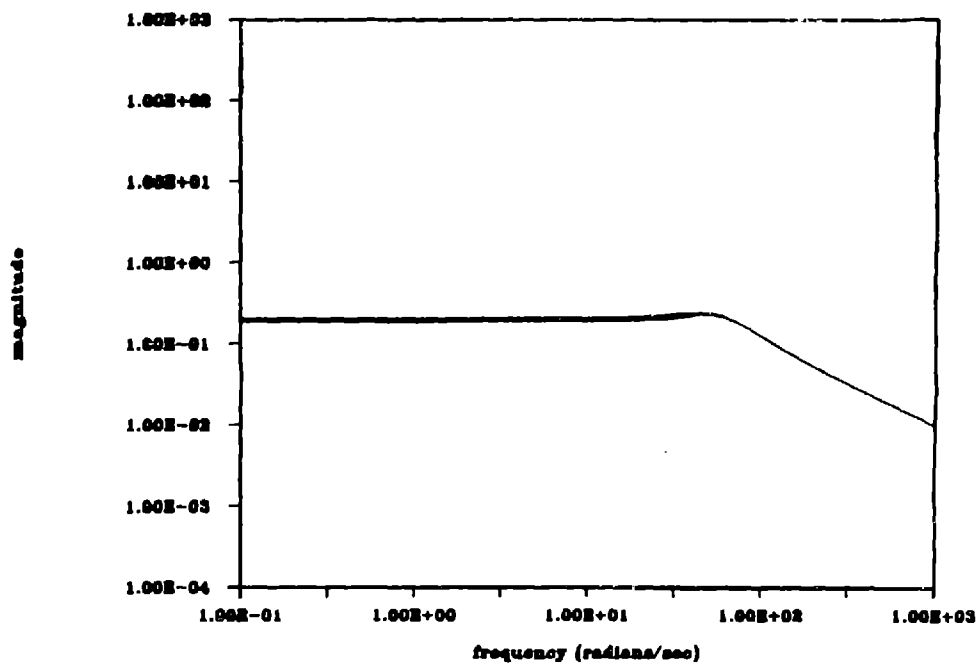


Figure 5.11: Singular values of $G_4(j\omega)$

<i>states</i>	<i>initial conditions</i>	
	actual	estimated
x_1	0.0	0.0
x_2	3.0	2.5
x_3	0.0	0.0
x_4	3.0	2.5

Table 5.1: Flexible joint manipulator – first initial conditions

5.2.4 Simulations

The simulations were run using SIMNON, for basically two different initial conditions.

The first set of initial conditions are shown in Table 5.1. Clearly the estimation error is starting inside the boundary layer. It is a natural assumption since the state x_1 and x_3 are both been measured. The simulation for this set of initial conditions are shown in Figure 5.12. The corresponding estimation errors are shown in Figure 5.13. It is clear that the sliding observer is working properly, and the estimated bounds of the estimation errors actually agree with the estimation errors shown in Figure 5.13.

A more challenging situation is when the estimation error starts outside the boundary layer. In this case one can expect that the trajectory will come inside the boundary layer after a finite number of switchings.

Consider the initial conditions shown in Table 5.2. The simulation considering these initial conditions gave the estimates shown in Figure 5.14 and the estimation error shown in Figure 5.15.

Obviously, the trajectories come into the boundary layer after two switchings, and the resulting estimation error in steady state remains within the estimated bounds.

5.3 Shaft Torque Estimation in Automatic Transmission

In this section, a sliding observer for shaft torque estimation in automotive automatic transmission is presented. The basic model was previously published by Masmoudi and Hedrick in [38].

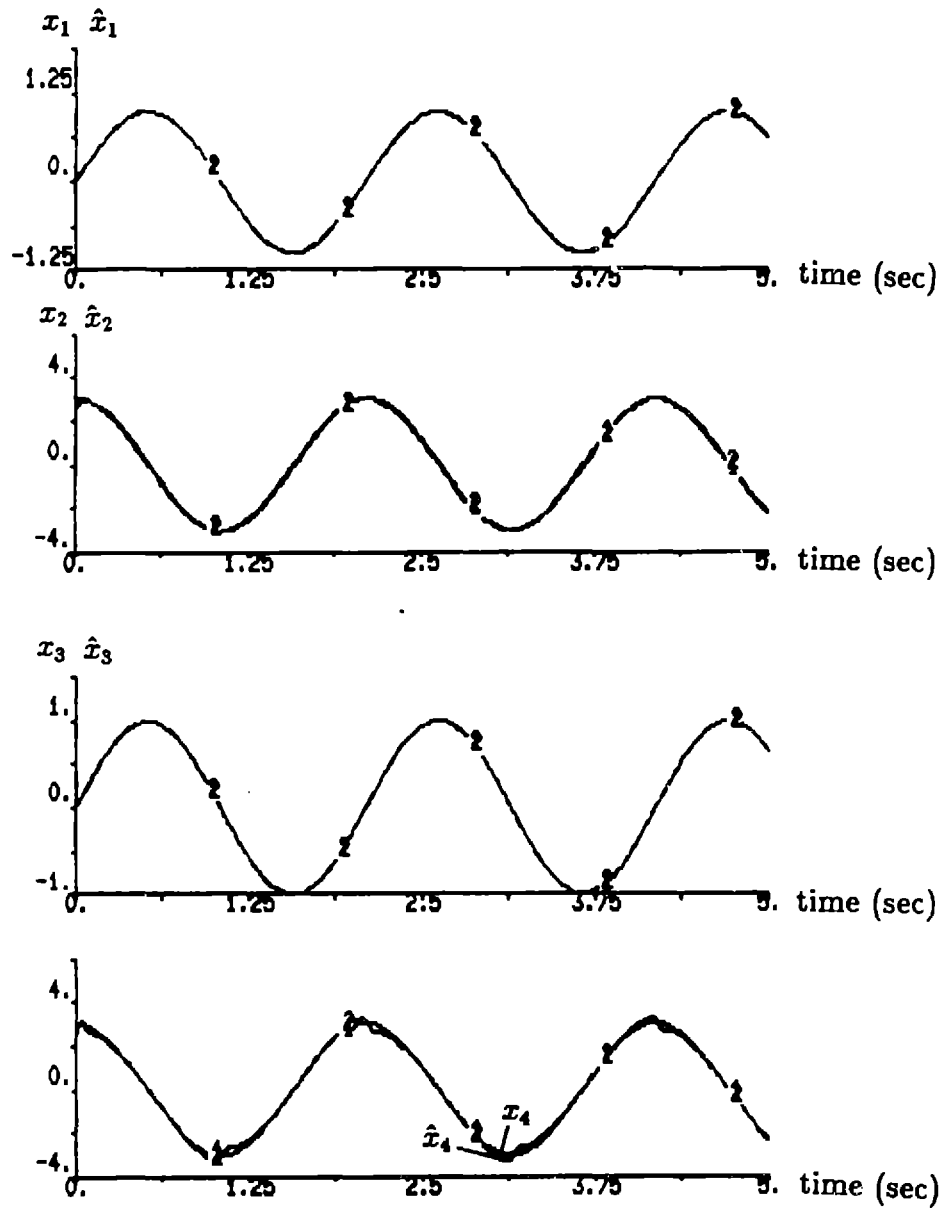


Figure 5.12: Actual and estimated states – first initial conditions

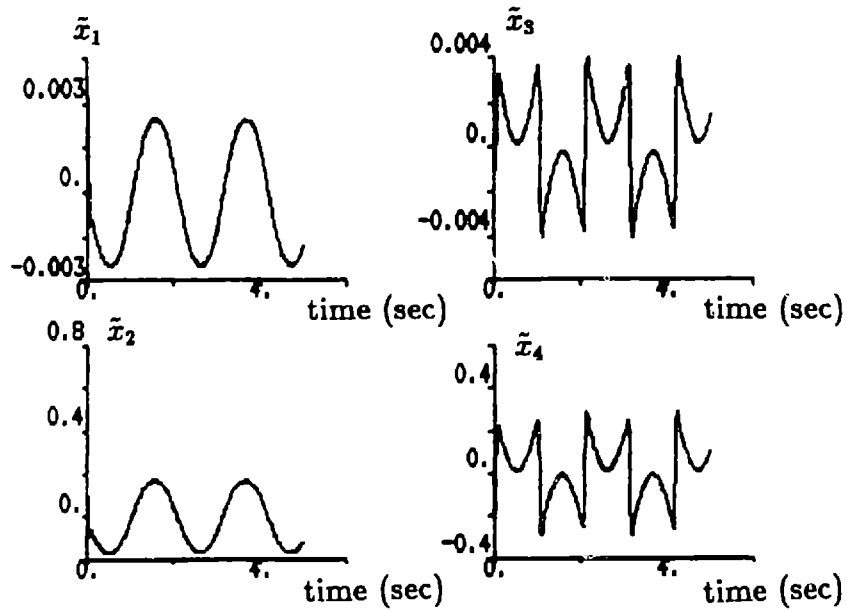


Figure 5.13: Estimation errors – first set of initial conditions

<i>states</i>	<i>initial conditions</i>	
	actual	estimated
x_1	3.0	0.0
x_2	3.0	2.5
x_3	3.0	0.0
x_4	3.0	2.5

Table 5.2: Flexible joint manipulator – second initial conditions

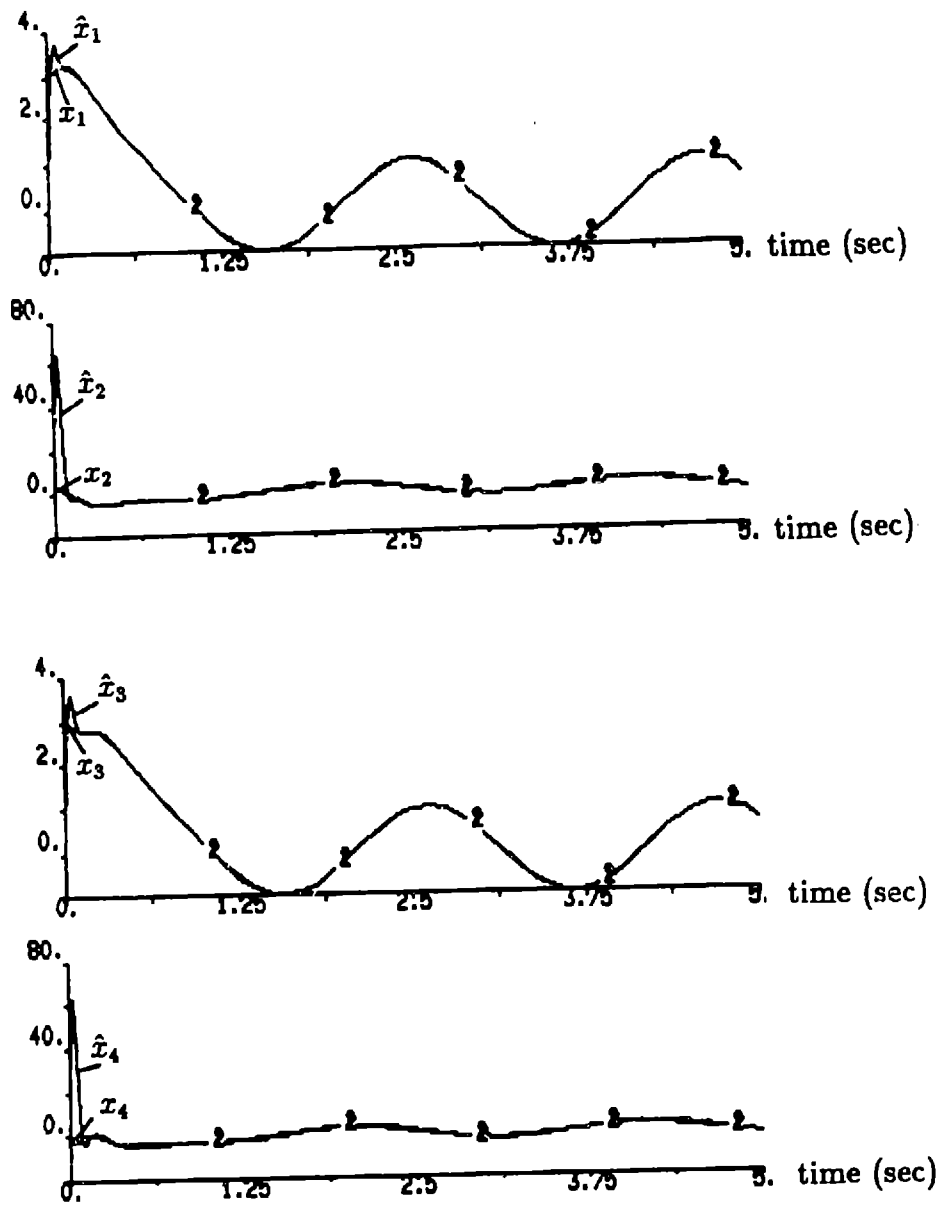


Figure 5.14: Actual and estimated states – second initial conditions

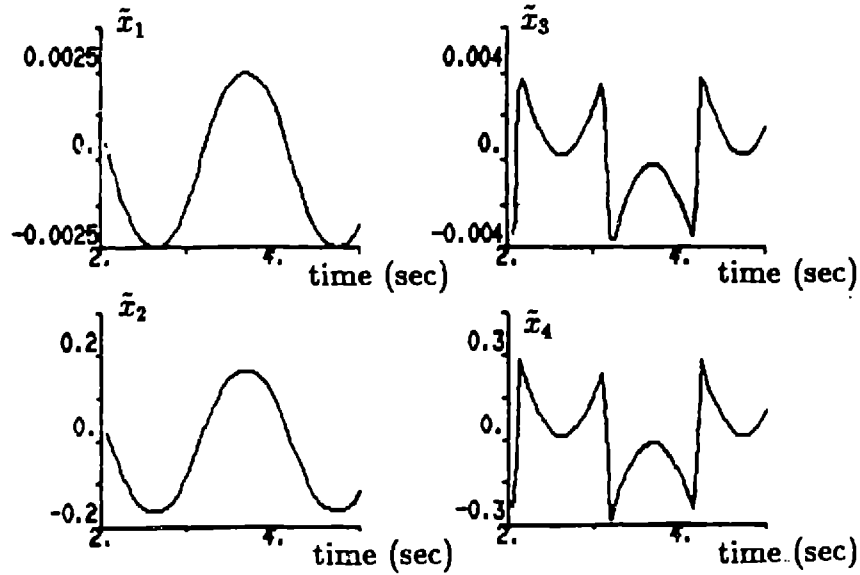


Figure 5.15: Estimation errors – second initial conditions

The basic setup, describing the model of the system as well as the model for the sliding observer design, is first shown. The bounds on the neglected nonlinearities are then derived. In the sequel, it will be first shown that the case 3 –that is, the method that requires strictly positive real *linear part*–cannot be applied, thus the more involved method described in case 4 is applied to the sliding observer design.

5.3.1 The Automatic Transmission Model

The problem considered here is the shaft torque estimation during the torque phase of the first to second gear upshift. The model that describes the system is the one shown in [38]:

$$\begin{aligned} \dot{x}_1 &= -56.93R_D x_3 + 1.6629u_1 - 0.7705A_c R_c u_2 \mu_{friction} \\ \dot{x}_2 &= \frac{100}{I_v} x_3 - T_{load} \\ \dot{x}_3 &= 76.25R_D x_1 - 76.25x_2 \end{aligned} \quad (5.4)$$

$$\mu_{friction} = \text{sign}(x_1)[0.0631 + 0.0504 \exp(-.0446|x_1|)] \quad (5.5)$$

$$T_{load} = 0.000116x_2^2 + \frac{50.85}{I_v}$$

with the outputs:

$$y_1 = x_1$$

$$y_2 = x_2$$

where:

- x_1 is the reaction carrier speed, x_2 is the wheel speed and x_3 is the shaft torque
- $u_1 = 210$ is the engine torque
- $u_2 = 2.0 \times 10^6 t$ is the clutch pressure (t is the time)

All units are given in the International System of Units (meter-kilogram-seconds).

The parameters are

- $R_D = 0.3521$
- $I_v = 169.8218$
- $A_c R_c = 0.0045$

Given this system description, the goal is to estimate the shaft torque, state x_3 , using the measurements y_1 and y_2 . The observer design uses a more simplified model as shown next.

5.3.2 The Model for Sliding Observer Design

The Sliding Observer is designed assuming that the nonlinearities and uncertainties are bounded disturbances. In this case, they are results of neglected nonlinearities and unmeasured inputs:

$$\begin{aligned} \dot{x}_1 &= -ax_3 + w_1 \\ \dot{x}_2 &= bx_3 + w_2 \\ \dot{x}_3 &= cx_1 - dx_2 \end{aligned} \tag{5.6}$$

with the parameters

- $a = 56.93R_D = 20.05$
- $b = \frac{100}{I_v} = 0.59$
- $c = 76.25R_D = 26.85$
- $d = 76.25$

and the uncertainty terms w_1 and w_2 are assumed to be bounded by constants γ_1 and γ_2 . The actual value of these constants can be evaluated using the model given by (5.4):

$$w_1 = 1.6629u_1 - 0.7705A_cR_cu_2\mu_{friction}$$

knowing that $|x_1| < 150$, as shown by [38] and knowing that the shift duration is less than 1.0 seconds, one can find the value of γ_1 considering the worst case and using the triangle and Schwarz inequality. It results as

$$|w_1| \leq 349.0 = \gamma_1$$

Analogously, with the knowledge that $|x_2| \leq 60.0$, and using the explicit formula for w_2 :

$$w_2 = -1.16 \times 10^{-4}x_2^2 - \frac{50.85}{I_v}$$

one can find the bound γ_2 to be:

$$|w_2| \leq 0.72 = \gamma_2$$

The model can be written in matrix form as:

$$\begin{aligned} \dot{x} &= Ax + D\eta \\ y &= Cx \\ x &= \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \\ A &= \begin{bmatrix} 0 & 0 & -a \\ 0 & 0 & b \\ c & -d & 0 \end{bmatrix} \end{aligned} \tag{5.7}$$

$$D = \begin{bmatrix} 349 & 0 \\ 0 & 0.72 \\ 0 & 0 \end{bmatrix} \quad (5.8)$$

$$y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \quad (5.9)$$

$$|\eta_1| \leq 1.0$$

$$|\eta_2| \leq 1.0$$

It is worth noting that the choice of matrix D is, to some extent, arbitrary in the sense that the first and second row could be exchanged with suitable exchange in the roles of η_1 and η_2 . The adopted form of matrix D is important, because it will reflect on the positive realness of the *linear part* as will be shown in the next subsection.

5.3.3 Design using *Case 3*

The observer structure is the usual one:

$$\dot{\hat{x}} = A\hat{x} + H(y - C\hat{x}) + K1_s$$

where 1_s is chosen to be:

$$1_s = \begin{bmatrix} \text{sign}(x_1 - \hat{x}_1) \\ \text{sign}(x_2 - \hat{x}_2) \end{bmatrix}$$

The *switching* gain matrix K is chosen as:

$$K = D\rho$$

where ρ is the diagonal matrix whose entries are ρ_1 and ρ_2 , which are greater than or equal to one. The gain matrix K becomes:

$$K = \begin{bmatrix} 349\rho_1 & 0 \\ 0 & 0.72\rho_2 \\ 0 & 0 \end{bmatrix}$$

The next step in the design procedure is to check whether the system given by $C(sI - A + HC)^{-1}K$ is strictly positive real (S.P.R.). One of the conditions is:

$$K^T P = C$$

which gives

$$\begin{aligned} p_{11} &= \frac{1}{349\rho_1} \\ p_{22} &= \frac{1}{0.72\rho_2} \\ p_{33} &> 0 \text{ (arbitrary)} \\ p_{12} &= p_{13} = p_{23} = 0 \end{aligned}$$

The other condition requires that the following equality holds for symmetric positive matrix Q :

$$\begin{aligned} (A - HC)^T P + P(A - HC) &= -Q \\ (A - HC) &= \begin{bmatrix} -h_{11} & -h_{12} & -a \\ -h_{21} & -h_{22} & b \\ c - h_{31} & -d - h_{32} & 0 \end{bmatrix} \end{aligned}$$

Clearly, one would choose a gain matrix H using some technique like pole-placement or Kalman Filtering, and check whether the matrix Q is positive definite². First it is necessary to check if one can hope to find such gain matrix.

The matrix Q can be found to be:

$$-Q = \begin{bmatrix} -2h_{11}p_{11} & -h_{21}p_{22} - h_{12}p_{11} & p_{33}(c - h_{31}) - ap_{11} \\ -h_{21}p_{22} - h_{12}p_{11} & -2h_{22}p_{22} & bp_{22} - p_{33}(d + h_{32}) \\ p_{33}(c - h_{31}) - ap_{11} & bp_{22} - p_{33}(d + h_{32}) & 0 \end{bmatrix}$$

so Q is the symmetric matrix:

$$Q = \begin{bmatrix} q_{11} & q_{12} & q_{13} \\ q_{12} & q_{22} & q_{23} \\ q_{13} & q_{23} & 0 \end{bmatrix}$$

²Depending on the number of state and the number of measurement, it is sometimes possible to first choose the matrices P and Q , and then compute the gain matrix H so that the Lyapunov Function $V = x^T P x$ decreases exponentially with the bound $\dot{V} \leq -\lambda V$, where λ is the minimum eigenvalue of $P^{-1}Q$.

It will be positive definite if and only if

$$\begin{aligned} q_{11} &> 0 \\ q_{11}q_{22} - q_{12}^2 &> 0 \\ 2q_{12}q_{13}q_{23} - q_{11}q_{23}^2 - q_{22}q_{13}^2 &> 0 \end{aligned}$$

Taking q_{11} and q_{22} as positive numbers that satisfy the first and second inequalities and noting that the effect of q_{13} and q_{23} are similar, one can try to find the interval for q_{23} , on the real axis, in which the third inequality is satisfied. It would give

$$q_{23_l} < q_{23} < q_{23_u}$$

where q_{23_l} and q_{23_u} are the solution of

$$q_{11}q_{23}^2 - (2q_{12}q_{13})q_{23} + q_{22}q_{13}^2 = 0$$

Solving this equation, one finds that

$$q_{23} = \frac{q_{12}q_{13}}{q_{11}} \pm \frac{q_{13}}{q_{11}} \sqrt{q_{12}^2 - q_{11}q_{22}}$$

Because q_{11} , q_{22} and q_{12} must satisfy the second inequality, the solutions are actually a complex pair, therefore the desired interval on the real axis does not exist. The conclusion is that, due to the $q_{33} = 0$, the matrix Q cannot be positive definite, and so there is no gain matrix H that can make $C(sI - A + HC)^{-1}K$ strictly positive real. This conclusion could be expected due to the existent of zero in the diagonal position.

It means that the method described as *case 3* in the chapter 4 cannot be applied in this problem, therefore the method described as *case 4* must be used.

5.3.4 Design using *Case 4*

Because the design procedure just described cannot be applied to this problem, the method described as *Case 4*, is used.

The estimation error dynamics inside the boundary layer is given by:

$$\begin{aligned} \dot{\tilde{x}} &= (A - (H + K\Delta^{-1})C)\tilde{x} + D\eta \\ \tilde{y} &= C\tilde{x} \end{aligned} \tag{5.10}$$

$$\begin{aligned}\tilde{x} &= x - \hat{x} \\ \tilde{y} &= y - C\tilde{x} \\ \Delta &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\end{aligned}$$

where the matrices A, D and C are given by equations (5.7), (5.8) and (5.9) respectively.

In this case, one can verify that the system described by the equations (5.10) has a transmission zero at the origin. One can expect that this zero limits strongly the design freedom; actually, using the Kalman Filtering Theory to choose the linear gains one can verify that it is very easy to limit the trajectory to be inside the boundary layer with widths $\epsilon_1 = 1$ and $\epsilon_2 = 1$. Also it is very easy to verify the stability outside the boundary layer, but the bounds of estimation error of shaft torque is as big as the shaft torque itself. This result is not acceptable, so some different route has to be used.

In this particular system, one can see that the bounds of disturbance inputs are:

$$\gamma_2 = 0.72 \ll \text{shaft torque } (\approx 20)$$

therefore, one might want to neglect the effect of disturbance input w_2 .

On the other hand, the shaft stiffness, modelled as a linear spring, may have considerable uncertainty in its spring constant. Assuming that one has 30% uncertainty in the spring constant, one has the disturbance term in the third equation of (5.6), given by:

$$w_3 = 0.3(cx_1 - dx_2)$$

that is the model used for the observer design becomes:

$$\begin{aligned}\dot{x}_1 &= -ax_3 + w_1 \\ \dot{x}_2 &= bx_3 \\ \dot{x}_3 &= cx_1 - dx_2 + w_3\end{aligned}\tag{5.11}$$

with the measurements of speeds x_1 and x_2 .

The bound of w_3 can be found using known bounds, from simulations, of x_1 and x_2 . A tighter bound can be easily found from simulations, computing the term w_3 explicitly. It was found to be:

$$|w_3| \leq \gamma_3 = 15$$

One can verify that the estimation error dynamics, inside the boundary layer, resulting from this system has no finite transmission zeros, in this case the design process must be easier.

The first step in the design process is to design the observer, inside the boundary layer.

Using the Kalman filtering theory to choose the gains, with fictitious noise intensities

$$\begin{aligned} Q &= D_2 D_2^T \\ R &= 0.001 I \\ D_2 &= \begin{bmatrix} 349 & 0 \\ 0 & 0 \\ 0 & 15 \end{bmatrix} \end{aligned}$$

with I as the identity matrix, and taking the gains ρ_1 and ρ_2 to be one, the gain matrix H was found to be:

$$H = \begin{bmatrix} 110020 & 0 \\ 0 & 74 \\ -81.4 & 4652 \end{bmatrix}$$

which placed the poles at:

$$\begin{aligned} \lambda_1 &= -1.1036 \times 10^5 \\ \lambda_{2,3} &= -37 \pm 37.8j \end{aligned}$$

One can see that the gain matrix H has one very high gain. Noting that it is related to the very fast pole, one can just reduce this gain. By choosing the gain matrix H to be:

$$H = \begin{bmatrix} 1100 & 0 \\ 0 & 74 \\ -81.4 & 4652 \end{bmatrix}$$

the poles becomes:

$$\begin{aligned} \lambda_1 &= -1.45 \times 10^3 \\ \lambda_{2,3} &= -37.8 \pm 37.8j \end{aligned}$$

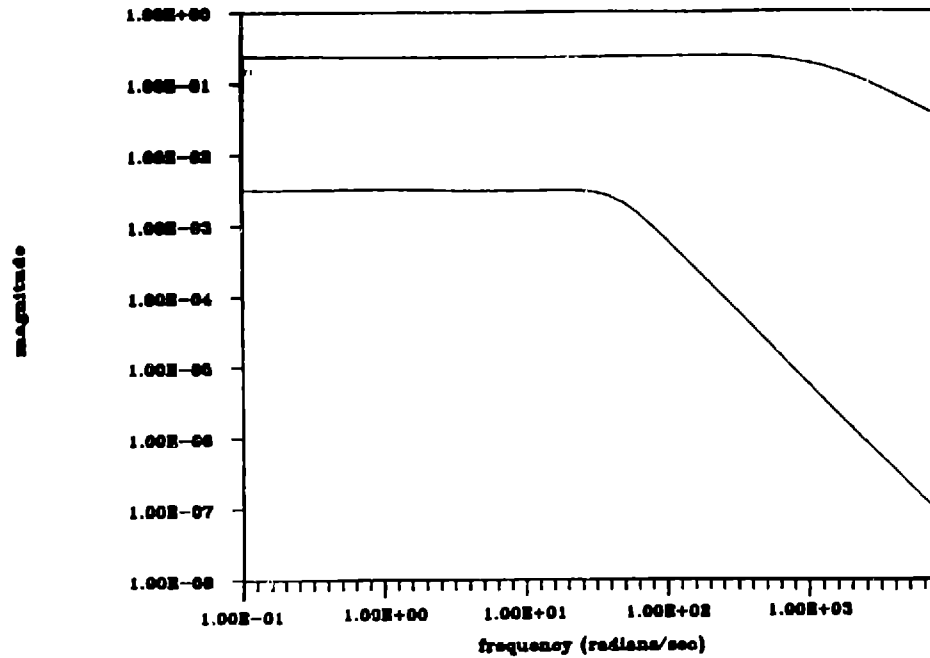


Figure 5.16: Singular values of $G(s) = C(sI - A + (H + K)C)^{-1}D_2$

which seems to be a satisfactory choice of poles. The singular values that determine the boundedness inside the boundary layer are shown in figures 5.16 and 5.17. It can be seen that if the initial estimation error are so that:

$$\begin{aligned}\tilde{x}_1 &= 0 \\ \tilde{x}_2 &= 0 \\ |\tilde{x}_3| &\leq 1\end{aligned}$$

then the trajectory will not leave the boundary-layer.

The next step is to check for stability outside the boundary layer. In this case, because the saturation levels were taken as unity, the matrices M and R used in the multivariable circle criterion are also identity matrices. The application of the multivariable circle criterion is shown in Figure 5.18. Clearly, the minimum singular values are larger than one, therefore the \mathcal{L}_2 -stability can be concluded.

The overall bound of estimation error can be found using the singular value plot shown in Figure 5.19. The estimation error, mainly of \tilde{x}_3 is less than 1. This observer was simulated, using the original nonlinear description given by equation (5.4) as the plant. The result of this simulation

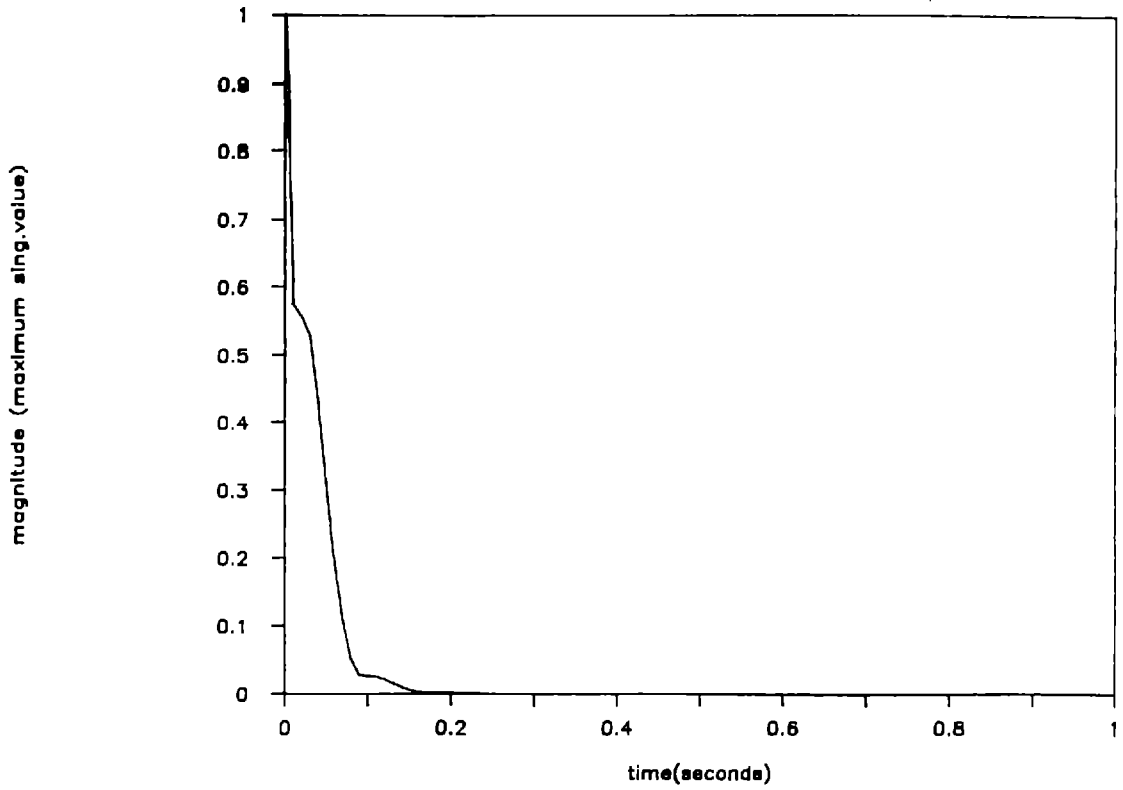


Figure 5.17: Singular value $\sigma_{max}(C e^{(A-(H+K\Delta^{-1})C)t})$

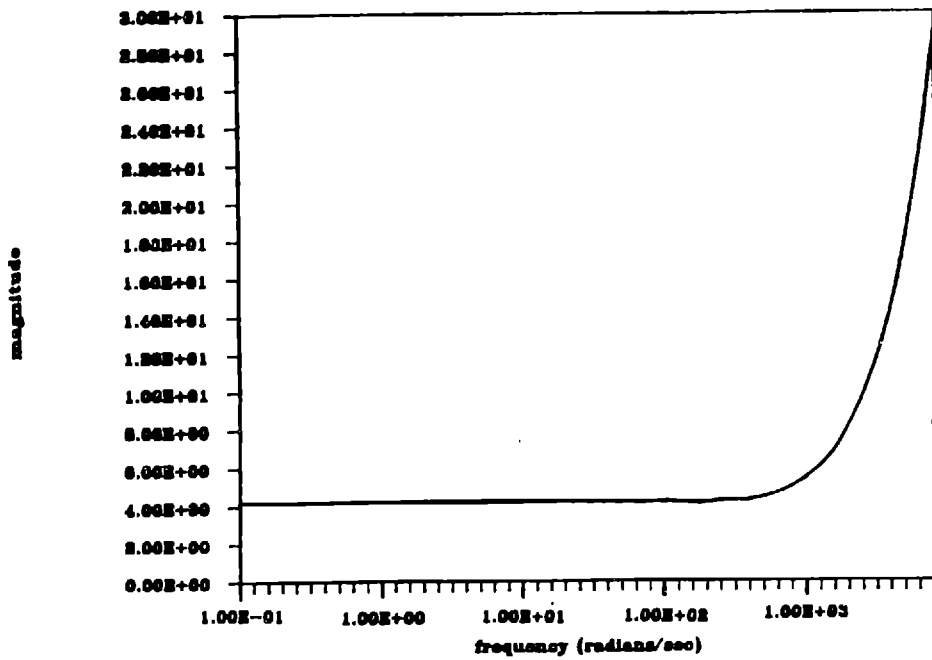


Figure 5.18: Multivariable circle criterion – shaft torque estimation

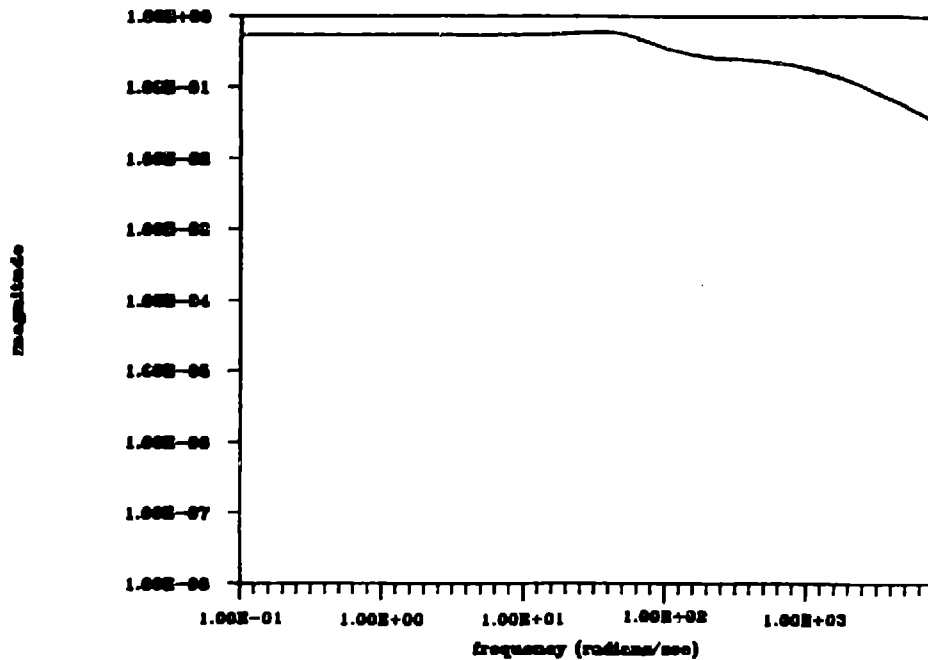


Figure 5.19: Maximum singular value of $(j\omega I - A + (H + K)C)^{-1}D_2$

is shown in Figure 5.20. For all practical purpose, the convergence in less than 0.2 seconds was achieved, and satisfactory estimation accuracy was obtained, even with initial estimation error larger than one.

5.4 Summary

In this chapter three examples were used to illustrate the application of sliding observers.

The first example used a model of lateral dynamics of a super-tanker, and a sliding observer was designed in order to estimate the yaw rate and its derivative given the measurement of the yaw angle only. The design was shown to be successful, and in principle no limitation was found.

A model of a one-link manipulator with a flexible-joint was the subject of the second example. A sliding observer was designed to estimate the angular velocities, given the angular position measurements. Again, the design process was fairly easy, and the design was again successful.

The third example was used to illustrate the eventual limitations that the observer designer might find. This example uses a nonlinear model of

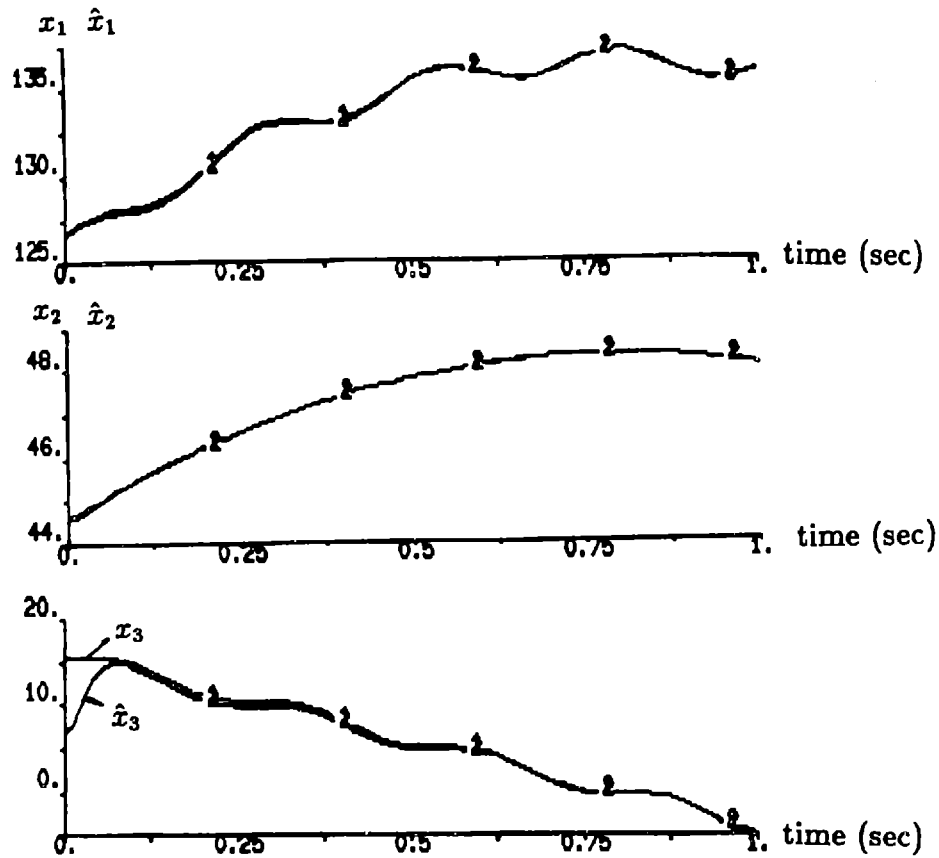


Figure 5.20: Actual versus estimated states – shaft torque estimation problem

an automotive automatic transmission, and the goal is to estimate the shaft torque, given the two speed measurements. Two difficulties were found in the design process: the first limitation was found when the design method described as Case 3 was used and verified that this method could not be used; the second limitation appeared when the design was attempted using the method described as the Case 4. The problem was the existence of zero at the origin, in the estimation error dynamics inside the boundary layer. This zero limited the accuracy of shaft torque estimation, and a different model was used. Assuming that the shaft spring constant had some uncertainty, a new model for observer design was chosen, and then the observer design became successful.

These examples also demonstrated that the condition for boundedness inside the boundary layer determined by the maximum singular value of a function with a matrix exponential is very conservative.

In these examples, the measurement noise was always assumed to be absent – even though the measurement noise is the only reason why one would not use pure differentiator in the first two examples –, however they are known to be always present. In this case, some of the high gains might

not be acceptable.

The effect of measurement noise, and an alternative design procedure is shown in next chapter.

Chapter 6

Design for Noisy Measurements

6.1 Overview

Because there is no such thing as noiseless sensors, measurement noise is always present and a useful state estimator must reject the sensor noises as efficiently as possible.

Drakunov [12], in 1983, suggested an adaptive filter which is actually the sliding observer presented in this thesis without the linear term “ $HC\tilde{x}$ ”. Using the averaging theory, this paper showed that this kind of filter can be robust against changes in the intensity of measurement noise, resulting in a suboptimal filter as the measurement noise becomes white, for a specific choice of gains[12]. Drakunov’s results actually inspired this chapter, and in particular the first-order example presented in this chapter used to show, qualitatively, the intrinsic robustness properties of sliding observer. The results presented in this chapter confirms the results in [12].

Drakunov[12], however, did not include the linear gain “ $HC\tilde{x}$ ” which, as shown in the previous chapters, is important to guarantee stability robustness of a sliding observer. This chapter includes this linear term in the observer. Also the analysis done for a first-order system complements the results in [12], because the effect of changes in the process noise intensity, as well as the effect of parameter mismatch, are included.

In the case of the sliding observer, assuming that the sensor noises have known statistical properties – Gaussian noise preferably – and if all

uncertainties in the input can be modeled as white noises then one can use *Random Input Describing Function* (RIDF) in the design process. In this case, some optimality can be claimed to some extent, however robustness has to be checked separately.

In this chapter, the plant is assumed to be described as in Chapter 4, with the addition of measurement noise:

$$\begin{aligned}\dot{x} &= Ax + w \\ y &= Cx + v\end{aligned}$$

where v is the measurement noise.

For this system, using the usual sliding observer structure, the estimation error dynamics can be described as:

$$\begin{aligned}\dot{\tilde{x}} &= (A - HC)\tilde{x} - K1_s + w + Hv \\ 1_s^T &= \left[\text{sgn}(\tilde{y}_1) \quad \text{sgn}(\tilde{y}_2) \quad \cdots \quad \text{sgn}(\tilde{y}) \right] \\ \tilde{y}_i &= y_i - C_i\hat{x} = C_i\tilde{x} + v \\ \tilde{x}(t=0) &= \tilde{x}_0\end{aligned}\tag{6.1}$$

Given this estimation error dynamics, the sliding observer design can be carried out as follows.

6.2 Random Input Describing Function Approach

Recall that the estimation error dynamics is described by equation (6.1).

Using the RIDF, the “switching” function 1_s can be approximated by:

$$1_s \approx N_1\tilde{y} = N_1(C\tilde{x} + v)$$

where N_1 is the $m \times m$ matrix of RIDF of 1_s , which is function of statistical properties of \tilde{y} . In practical terms, because these properties are not known, an approximation is made assuming that \tilde{y} is a zero mean Gaussian Process and therefore determined by covariance matrix of \tilde{y} .

By doing this approximation, the gains H and K can be combined in a single term H^* :

$$H^* = H + KN_1$$

where, obviously, H^* is a function of the covariance matrix of \tilde{y} .

From this point, one can perform the initial observer design following different routes depending on the problem at hand, which will determine the lumped gain matrix H^* . For example, two possibilities are:

- if the uncertainty term w only contains known nonlinear terms then the nonlinear terms in w can be approximated using RIDF and the optimal observer design can be done using the technique shown by Beaman [5] and reviewed in Chapter 2.
- if the uncertainty term w is actually a process noise that can be assumed to be a white noise and if the measurement noise can also be assumed to be white, then the design can be made using the conventional Constant Gain Kalman Filter Theory.

Whichever method one decide to use, the gain matrix H^* can be, in the optimization process, regarded as a matrix of constant gains. It can be done because the gain matrix H and K can always be adjusted conveniently so that the optimal gain H^* can be obtained.

When the gain matrix H^* is determined, one also has the covariance matrix of estimation errors \tilde{x} , usually denoted as P . With the knowledge of covariance matrix of measurement noise – assumed to be white – denoted as R , and the matrix P , one can easily verify that the covariance matrix of \tilde{y} is given by:

$$E[\tilde{y}(t)\tilde{y}^T(t)] = CPC^T + R$$

where $E[\cdot]$ denotes ensemble average.

It is clear that once the covariance matrix $E[\tilde{y}\tilde{y}^T]$ is determined the matrix of RIDF's N_1 for a particular choice of function 1_1 can be evaluated. Because H^* is now fixed, implying that the combined effect of gains H and K is to give an optimal filter – provided that the approximation made to get the RIDF holds true – one can now proceed to choose each of the gains H and K , so that the robustness can be guaranteed.

In the process of robustness verification, as described in Chapter 4, one has to assume noiseless measurements because the measurement noise cannot be included explicitly in these methods.

One important choice that is left to the designer is the choice of the function 1_1 . It must be chosen so that the robustness can be guaranteed, but also the particular choice of function 1_1 will have a great impact on the

behavior of sliding observer in the presence of modelling errors or deviations in the process noise intensity. More precisely, a common situation is when the uncertainty term w , due to the unmodelled dynamics or neglected nonlinearities, has its frequency content in the low frequency band compared with the frequency content of the measurement noise. In this case, because the role of function 1_s is to make corrections based on the information from the uncertainty terms, one useful procedure is to low pass the error signal \tilde{y} , cutting-off the effect of measurement noise, before entering the function 1_s .

Another situation occurs when, not only the uncertainty term has low frequency content, but also the matrix A in the plant model is a good representation of the system near working operating point. It often occurs when the linear part in the plant model is obtained by linearization using Taylor expansion. If the white noise statistics are also close to the real ones, obviously the Kalman Filter is expected to be the best filter to use when the system is operating near the operating point. In this case, one would take the gain H to be the Kalman Filter gain and one would choose the function 1_s as the relay with dead-zone, with the dead-zone width of order of standard deviation of the measurement noise. With this choice, when the estimation error is small, one can expect that the Kalman Filter (which is the case because the gain K will be virtually zero due to the dead-zone) will be working properly. When the estimation error grows, which means that the effect of uncertainty term became important, then the error signal \tilde{y} will go beyond the dead-zone width and the switching term will become active. The benefit of this procedure was already seen in the example shown in Chapter 3.

As will be shown, the use of pure relays has some advantages, namely robustness against deviations in the measurement noise intensity, but one should always remember that looking strictly at the case of small signals entering the signum function (relay), the relay can be working as a high gain element due to the infinite slope at the origin which can be disastrous in some applications.

It was a general outline of the design method using RIDF. In the following subsection, the design process is detailed using a third possible route, namely that the uncertainty term w is actually a white noise and that the measurement noise is correlated. In general it would require that the gain matrix H^* , has to be determined using some iterative optimization method.

In the case of a simple first-order system considered in the example, the optimal gain can be found in the closed form, and some numerical parametric analysis will show how this method works.

6.2.1 The Design Process

Let the plant be described in the usual way:

$$\begin{aligned}\dot{x} &= Ax + w \\ y &= Cx + v\end{aligned}$$

Assume that the uncertainty term w can be modeled as a white noise process:

$$E[w(t)w(t + \tau)^T] = Q\delta(\tau)$$

Suppose that the measurement noise is colored, i.e. it is correlated. It is assumed that the measurement noise can be modeled through the use of a “shaping filter”:

$$\dot{v} = Ev + Fv_1$$

where the matrices E and F are chosen properly and v_1 is a white noise process:

$$E[v_1(t)v_1(t + \tau)^T] = R\delta(\tau)$$

One can define an augmented state \mathbf{x} as $\mathbf{x}^T := [x, v]$, in this way, the plant can be modeled as:

$$\dot{\mathbf{x}} = \begin{bmatrix} \dot{x} \\ \dot{v} \end{bmatrix} = \begin{bmatrix} A & 0 \\ 0 & E \end{bmatrix} \begin{bmatrix} x \\ v \end{bmatrix} + \begin{bmatrix} I & 0 \\ 0 & F \end{bmatrix} \begin{bmatrix} w \\ v_1 \end{bmatrix}$$

where I is the identity matrix with proper dimension.

Using RIDF, the sliding observer can be written as:

$$\dot{\hat{x}} = A\hat{x} + H^*(y - C\hat{x}) \quad (6.2)$$

therefore, the estimation error dynamics can be described, using augmented estimation error states:

$$\dot{\tilde{\mathbf{x}}} = \begin{bmatrix} \dot{\tilde{x}} \\ \dot{\tilde{v}} \end{bmatrix} = \begin{bmatrix} A - H^*C & -H^* \\ 0 & E \end{bmatrix} \begin{bmatrix} \tilde{x} \\ \tilde{v} \end{bmatrix} + \begin{bmatrix} I & 0 \\ 0 & F \end{bmatrix} \begin{bmatrix} w \\ v_1 \end{bmatrix} = A\tilde{\mathbf{x}} + \mathcal{G}\mathcal{W}$$

For this description, the covariance matrix of estimation error can be partitioned as:

$$P = \begin{bmatrix} \mathbf{E}[\tilde{x}\tilde{x}^T] & \mathbf{E}[\tilde{x}v^T] \\ \mathbf{E}[v\tilde{x}^T] & \mathbf{E}[vv^T] \end{bmatrix}$$

which can be propagated as:

$$\dot{P} = AP + PA^T + G\Phi G^T$$

where $\Phi = \text{diag}(Q, R)$.

Assuming that the original process is ergodic, one is often interested in designing the state estimator with constant gains. It is done by solving the covariance propagation equation for the steady-state, that is by setting $\dot{P} = 0$ the resulting equation is the well-known Lyapunov Equation, for a fixed matrix H^* .

In order to have the optimal gain for the filter given by equation (6.2), in the least-square sense, it is desirable to find the gain matrix H^* that minimizes the cost function:

$$J = \text{trace}(\mathbf{E}[\tilde{x}\tilde{x}^T]) = \text{trace}(MPM^T)$$

where M is the $n \times (n + m)$ matrix where the first $n \times n$ submatrix is the identity matrix, and the rest is zero.

What is desired here is an optimal filter with fixed structure, and to the best of this author's knowledge, the closed form solution for this minimization problem is unknown, however the optimal gain can still be found by solving this minimization problem using some iterative optimization method, like the gradient-search method.

Once the optimal gain H^* is found, one can now go back and try to choose the original gains H and K so that the resulting filter is robust for the given uncertainties/modelling errors. If the uncertainty term is truly a white noise, the robustness is actually guaranteed from this design process. Moreover, if w is really white, and if the noise statistics are the real ones, then the Kalman Filter is the optimal one, so there is no need to use the switching term in the observer. Of course it is not always true, and actually, in many situations the use of switching function has some advantages as will be shown in the example.

It is worth noting that the gains H that would result if one assumes, right from the beginning, that the measurement noise is white is often

smaller than the gain H^* that results from the method described in this section. This suggests that the gain H can be taken as the plain Kalman Filter gain, and the difference $H^* - H$ can be assigned to the switching term KN_1 .

This method will be illustrated in a simple first-order system, for which the optimization procedure suggested in this section can be solved in closed form.

6.2.2 Example

In this example consider a simple first-order system, with correlated measurement noise that can be modeled as a first-order Gauss-Markov Process:

$$\begin{aligned}\dot{x} &= -ax + w \\ y &= x + v \\ \tau\dot{v} &= -v + v_1\end{aligned}$$

where w and v_1 are assumed to be zero mean White Gaussian Noise with noise intensities q^2 and r^2 respectively.

Assume that one wants to get the estimate of the state x , given the measurement of y , and the statistical properties of process and measurement noise:

$$\begin{aligned}q^2 &= 1.0 \\ r^2 &= 0.01\end{aligned}$$

and the numerical values of the parameters:

$$\begin{aligned}a &= 1 \\ \tau &= .05\end{aligned}$$

The estimate is to be obtained using a suitable filter. The linear filter — which is the Kalman Filter if the gain is the optimal one in the least-square sense — looks like:

$$\dot{\hat{x}} = -a\hat{x} + h(y - \hat{x})$$

whereas, the sliding observer is

$$\dot{\hat{x}} = -a\hat{x} + k\text{sign}(y - \hat{x})$$

Notice that, because it is a first-order system, there is no need for the linear correction term, in order to guarantee stability robustness ¹.

Because the bandwidth of the measurement noise is much larger than the plant bandwidth, the first filter that one might want to design is the classical Kalman Filter. Assuming that the measurement noise v is white, with noise intensity $r^2 = 0.01$, the Constant Kalman Filter gain becomes²:

$$h = 9.05$$

The Kalman Filter with this gain will be called, in this example, as the *Kalman Filter 1* or simply *KF1*.

Because the measurement noise is actually correlated, one can hope to design a better filter by exploiting the available measurement noise model. The improved linear filter will use the optimal gain H^* described in the last section.

The first step in the design process would be to get the statistical steady state variance of estimation error $\tilde{x} = x - \hat{x}$. It can be solved explicitly, resulting:

$$p = E[\tilde{x}^2] = \frac{q^2[(a+h)\tau+1] + h^2r^2}{2(a+h)[(a+h)\tau+1]}$$

where h is the filter gain. The design goal is to find the gain h that will result in the minimum value of p .

In this case, this optimization process can be solved explicitly. By taking the first partial derivative $\partial p/\partial h$ and setting it equal to zero, one can find the optimal gain h^* :

$$h^* = \frac{(1+a\tau)[\tau q^2 - a^2r^2 \pm \sqrt{a^2r^4 + q^2r^2}]}{r^2(2a\tau+1) - q^2\tau^2}$$

from which the positive solution is taken. Numerically it will give:

$$h^* = 17.356$$

The linear filter with this optimal gain will be called the *Kalman Filter 2* or simply *KF2* in the analysis that will be done shortly.

¹See example for Case 1, in Chapter 4

²See [16]; the gains can easily be obtained using standard CACSD software

One can now proceed to compute the gain for the sliding observer, because the optimal gain h^* is known. For this case, the covariance of estimation error is

$$p = 0.069$$

therefore the covariance $E[\tilde{y}^2]$, $\tilde{y} = y - \hat{x}$, can be found as:

$$\sigma_{\tilde{y}} = E[\tilde{y}^2] = 0.282$$

therefore, the gain k in the sliding observer becomes:

$$k = h^* \sigma_{\tilde{y}} \sqrt{\pi/2} = 6.13$$

Using these gains for the Kalman Filter (KF1 and KF2) and the sliding observer (SO) a series of simulations were done, in order to study the sensitivity of Sliding Observer with respect to changes in the intensities of process noise and measurement noise. Both Kalman Filters , KF1 and KF2, were also simulated under the same conditions for comparative purposes.

At the same time, the agreement between the simulations and the prediction of steady state covariance of estimation error for the sliding observer, using Random Input Describing Function, was verified. The steady state covariance, for this case, is given by:

$$\sigma_{\tilde{y}}^2 = \frac{1}{2(a + \sqrt{2/\pi} \frac{k}{\sigma_{\tilde{y}}})} \left[\frac{(a^2 - 1/\tau^2)\tau^2/\tau}{a + 1/\tau + \sqrt{2/\pi} \frac{k}{\sigma_{\tilde{y}}}} + q^2 + \frac{r^2}{\tau^2} \right]$$

$$p_{so} = E[\tilde{x}^2] = \sigma_{\tilde{y}}^2 + \left(1 - 2 \frac{a + 1/\tau}{a + 1/\tau + \sqrt{2/\pi} \frac{k}{\sigma_{\tilde{y}}}} \right) \frac{r^2}{2\tau}$$

In the following analyses the following notation will be used: $v2so$ for the covariance of estimation error from the sliding observer, $v2kf1$ for the covariance of estimation error from the Kalman Filter 1 (with the gain $h = 9.05$), and $v2kf2$ for the covariance of estimation error from the Kalman Filter 2 (with the gain $h = 17.356$).

Effect of Measurement Noise

Obviously, when the noise intensities are the ones considered in the design process the Kalman Filter KF1 is the optimal one. In this analysis, the

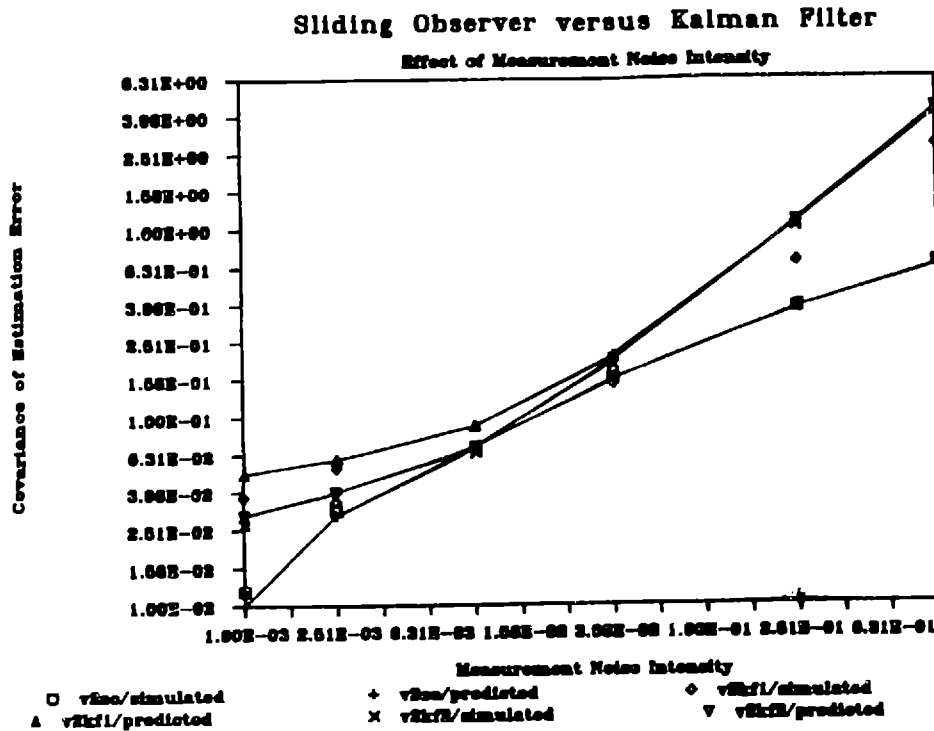


Figure 6.1: Effect of measurement noise

effect of deviation of measurement noise intensity from the nominal value, is investigated.

Both the predictions and the results from the averaging the time averages over 10 simulations are shown in Figure 6.1. Clearly, at the nominal condition the KF1 is the optimal one, seen both from the predicted values and from the values from simulations. When the measurement noise intensity deviates from the nominal value, the inherent robustness of Sliding Observer with respect of changes in the measurement noise becomes evident: for all values of measurement noise intensities off from the nominal value the sliding observer is the best one.

This result is due to the fact that the sign function has an inherent *adaptive*-type of behavior with respect to changes in the measurement noise³. This fact can be clearly observed if one recalls the definition of

³This result was first observed by Drakunov [12]

RIDF for the sign function:

$$\text{sign}(\hat{y}) = \sqrt{2/\pi} \frac{1}{\sigma_{\hat{y}}}$$

because $\sigma_{\hat{y}}$ is directly affected by changes in the measurement noise, one can see that increase in the measurement noise reflects as a virtual decrease in the filter gain. The opposite is also true, that is a decrease in the measurement noise intensity is reflected as virtual increase in the filter gain. This behavior is exactly the behavior desired from an adaptive filter in order to maintain a good performance for a wide range of changes in the measurement noise intensity.

Effect of Process Noise

The effect of changes in the process noise intensity was also verified. The predicted results and the results from the simulations – as averages of time averages of 10 simulations – are shown in Figure 6.2. In this case, for large deviations of process noise intensity the Kalman Filters were the clear winners. For large increase in the process noise intensity, the KF2 was the best one, while the sliding observer was the worst filter; it is probably due to the fact that for large process noise intensity a larger filter gain is required so that suitable corrections is provided to the filter. Apparently the sliding observer was unable to provide enough compensation for the large noise intensity.

For very small process noise intensity, the KF1 was clearly the best one. It is exactly the symmetric situation from the one just discussed, For small process noise intensity, with fixed measurement noise intensity, one wants to decrease the filter gain in order to have suitable filter. The KF1 has smaller gain than the other filters used in this comparison. The sliding observer, for this situation worked as well as the KF2, even though the prediction and the simulated values deviated from each other.

For deviations close to the nominal value, one can see that the KF2 was the best one, followed closely by the sliding observer.

Effect of Parametric Mismatch

Here the effect of mismatch between the parameter a used in the filter and the real one used in the plant is investigated. Because the RIDF cannot be

Sliding Observer versus Kalman Filter

Effect of Process Noise Intensity

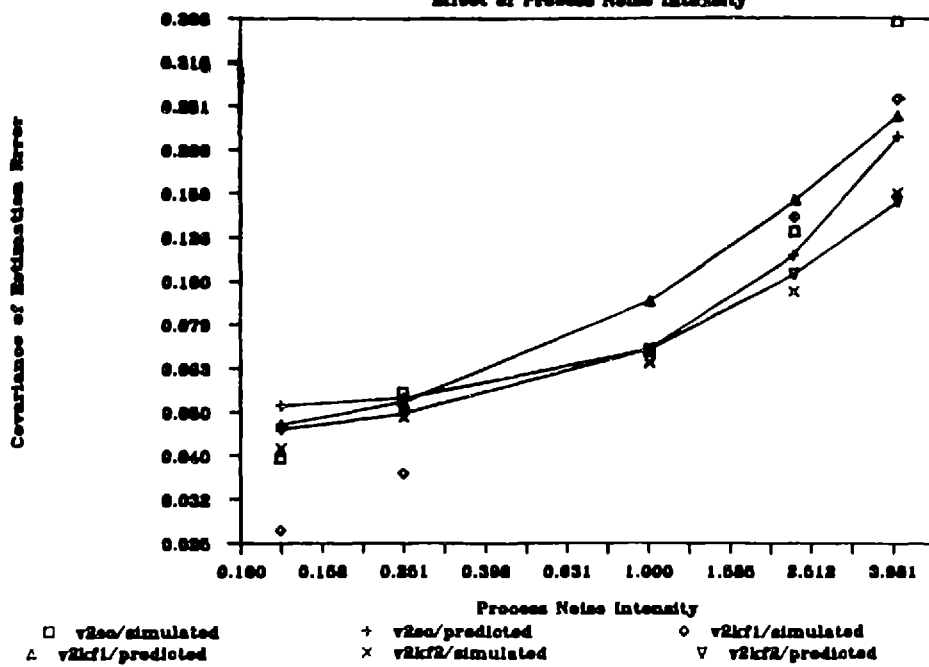


Figure 6.2: Effect of process noise intensity

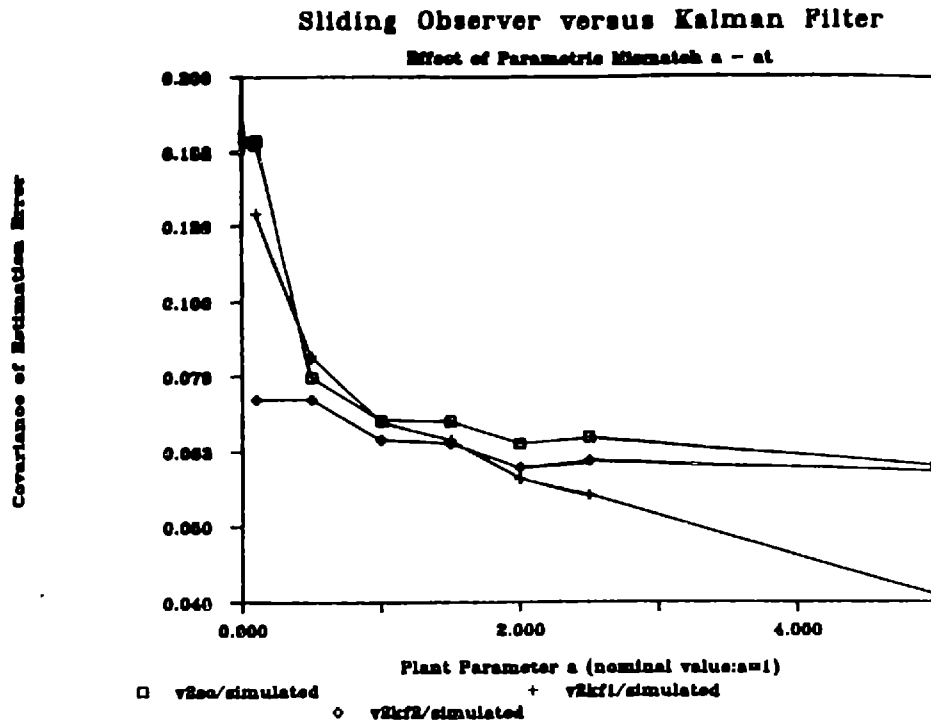


Figure 6.3: Effect of parameter mismatch

used to predict such effect in the sliding observer, the parametric analysis was done based only on the simulations (as always ensemble average of 10 time averages).

The results are summarized in Figure 6.3: For this type of situation, the SO and KF2 worked very similarly for values close to and larger than the nominal value. For values of actual a much smaller than the nominal value, the Kf2 is much better than the SO. For values of a much larger than the nominal one, the KF1 is clearly the best one.

6.3 Summary

In this chapter, the effect of measurement noise was considered as changes in the design procedure. Basically, when the measurement noise is present then the design should be primarily done using the Random Input De-

scribing Function –RIDF– and the robustness must be checked using the methods discussed in Chapter 4.

It was shown that several routes exist, once one quasi-linearizes the error dynamics, or the sliding observer, using RIDF. One of them that can claim some optimality was discussed with some detail, and a simple example was used to show the potential advantages and drawbacks of using sliding observers for systems with noisy measurements.

The example showed that the sliding observer has inherent robustness against deviations in the measurement noise intensity. For this first-order system, the behavior of the sliding observer and the Kalman Filter were comparable, in terms of deviations in the process noise and parameter mismatch. This is due to the fact that the uncertainty term w and the measurement noises have energies in the same range of frequencies. When the uncertainty term has energy at low frequencies compared to the frequency content of the measurement noise – as is often the case – the use of switching function with dead-zone plus the use of linear gain, or the use of low-passed error signal in the switching function can improve the performance of sliding observer significantly as was shown in Chapter 3.

Chapter 7

Conclusion and Suggestions for Further Research

7.1 Thesis Summary

This thesis showed that a simple observer, basically a Luenberger Observer with an additional term defined in terms of either *signum* or saturation function, can be made robust against bounded neglected nonlinearities and/or uncertainties.

In Chapter 2, an extensive review of the current methods for state estimation applicable to nonlinear uncertain systems was made.

In Chapter 3, the basic properties of the sliding observer were presented, showing that it can be very robust against the class of uncertainties that was considered in this thesis. This chapter motivated the need for more careful investigation using a simple third-order example.

Chapter 4 is the main chapter in this thesis. It showed that, in a fairly general framework the sliding observer can be said to be stable, i.e. nondivergent, if the conditions stated by the passivity theorem are satisfied. It was shown that this general framework was difficult to apply as a design tool, so more useful design procedures were sought. Four basic cases were considered, and the design of a robust sliding observers became feasible. These design procedures are summarized in the next section.

Three examples presented in Chapter 5 illustrated the application of sliding observers. The first example considered the estimation of yaw rate, and its time derivative, given the measurement of yaw angle, for a super-

tanker. It was shown that the sliding observer worked nicely in spite of neglected nonlinearities and control inputs. The second example considered a one-link, elastic joint, manipulator. It has two angular position measurements, and the goal was to estimate angular velocities in the presence of neglected nonlinear gravity effect and the control torque input. In this multiple-measurement case, the observer also worked properly. The third example, an automotive automatic transmission, assumes that two angular speeds are measured, and the goal is to estimate the shaft torque, in the presence of neglected nonlinearities and inputs. This is a more involved example, and it was used to show the limitations of this method. It was shown that the method described as *Case 3*, which was a candidate for this case ¹, could not be applied to this case. The method described as *Case 4* was then considered.

Chapter 6 discussed the change in the design procedure when measurement noise is present. The Random Input Describing Function is used to quasi-linearize the error dynamics, and some procedures to choose the gains are suggested. The case of colored measurement noise and white gaussian process noise is considered with some detail, and a first-order system illustrates the suggested design procedure and its performance is analyzed.

7.2 Conclusions

The main results presented in this thesis are summarized in this section. These results are given in terms of estimate convergence properties and in terms of the recommended design procedure.

The system is assumed to be described by

$$\begin{aligned}\dot{x} &= Ax + D\eta \\ y &= Cx\end{aligned}$$

with some conditions on the matrices described in Chapter 4.

The strongest result, which is also the one with the most restricted applicability, uses the concept of strictly positive real system. In this case it was shown that the state estimation error $\tilde{x} = x - \hat{x}$ goes to zero as time goes to infinity. In this case, the sliding observer:

$$\dot{\hat{x}} = (A - HC)\hat{x} + Hy + K1,$$

¹which is actually a rare situation.

with

$$1_s = \begin{bmatrix} \text{sign}(\tilde{y}_1) \\ \text{sign}(\tilde{y}_2) \\ \vdots \\ \text{sign}(\tilde{y}_m) \end{bmatrix}$$

and the gain K defined as

$$K = D\rho$$

where ρ is a m -dimensional diagonal matrix with elements $\rho_i \geq 1$, is shown to be asymptotically convergent if the transfer function matrix

$$G(s) = C(sI - A + HC)^{-1}K$$

can be made strictly positive real.

If only one measurement is available, $G(s)$ is a transfer function, therefore the strict positivity means that the real part of the stable transfer function $G(j\omega)$ has to be in the open complex right half plane for all values of real ω . The design procedure can use the transfer function itself, or the Nyquist plot in order to choose the gain matrix H and the gain ρ that guarantees that this condition is satisfied.

For multiple measurement system, the strict positivity can be translated to a useful form using the positive real lemma. The designer has to choose the gain matrices H and ρ so that the following conditions are simultaneously satisfied:

$$\begin{aligned} (A - HC)^T P + P(A - HC) &= -Q \\ K^T P &= C \end{aligned}$$

for symmetric positive definite matrices P and Q , and $A - HC$ has poles in the open left half plane.

One can see that this last condition is quite restrictive. If the system is so that these conditions cannot be satisfied, one can iterate changing the matrix D , or using convenient operators that change the blocks H_1 and H_2 according to the multiplier theory ([11], section 6.9). If these approaches does not work, one still has the option to design the observer using an alternative method described as *Case 2* and *Case 4*.

In this case, one should choose the function 1_s as:

$$1_s = \begin{bmatrix} \text{sat}(\tilde{y}_1/\epsilon_1) \\ \text{sat}(\tilde{y}_2/\epsilon_2) \\ \vdots \\ \text{sat}(\tilde{y}_m/\epsilon_m) \end{bmatrix}$$

and

$$\text{sat}(\tilde{y}_i/\epsilon_i) = \begin{cases} \tilde{y}_i/|\tilde{y}_i| & \text{if } |\tilde{y}_i| \geq \epsilon_i \\ \tilde{y}_i/\epsilon_i & \text{if } |\tilde{y}_i| < \epsilon_i \end{cases}$$

The first step in the design procedure is to guarantee that the trajectories inside the boundary layer never leave it. It is guaranteed by choosing the gain matrices H and ρ so that the following frequency domain inequalities are satisfied:

$$\max_{-\infty < \omega < +\infty} [\sigma_{\max}(C(j\omega I - A + (H + K\Delta^{-1})C)^{-1})D] + \sigma_{\max}(C)\text{cond}(V)\max(|\tilde{x}_0|) \leq \min_i(\epsilon_i)$$

where $\sigma_{\max}(A)$ is the maximum singular value of the complex matrix A , and the norm $|z| = (z^*z)^{\frac{1}{2}}$ (z^* denotes the complex conjugate). The matrix Δ is defined as:

$$\Delta = \text{diag}(\epsilon_1, \epsilon_2, \dots, \epsilon_m)$$

One should also check whether the state estimation error bounds are within reasonable limits:

$$|\tilde{x}| \leq \max_{-\infty < \omega < +\infty} [\sigma_{\max}((j\omega I - A + (H + K\Delta^{-1})C)^{-1})D] \quad (7.1)$$

The next step is to check for stability outside the boundary layer.

For systems with single measurement, the stability can be verified using the circle criterion. This criterion will state that the matrices H and K must be so that the Nyquist plot of $G(s) = C(sI - A + HC)^{-1}K$ has to be to the right of vertical line that intersects the real axis at $-1/\bar{G}$, where \bar{G} is defined by:

$$\bar{G} = \frac{1}{\epsilon} \left(1 + \frac{1}{\rho}\right)$$

For systems with more than one measurement, the matrices H and K must be so that:

- $(A - HC - KMC)$ has eigenvalues in the open left half plane; the matrix M is a diagonal matrix whose entries are $m_{ii} = \frac{1}{2\epsilon_i}(1 + \frac{1}{\rho_i})$;
- The following condition is verified for all values of $\omega \in \mathfrak{R}$ at which G^{-1} exists:

$$\sigma_{\min}([M + G^{-1}(j\omega)]R^{-1}) \geq 1$$

where $G(s) = C(sI - A + HC)^{-1}K$.

In this case, one can verify that the placement of eigenvalues of $(A - HC - K\Delta^{-1}C)$ will be reflected in the convergence time.

Clearly, if the system has finite transmission zeros, in the error dynamics inside the boundary layer, there will be some limits in the achievable performance, that will allow the designer to meet or not meet the above conditions.

For this case, for which the strict positivity cannot be reinforced, the state estimation error will be bounded and the bound will be given by the inequality (7.1).

These results were derived assuming that no measurement noise was present. In this case, the achievable estimation error from the sliding observer using saturation function is comparable to the estimation error resulting from a Luenberger observer with the gain matrix $H^* = H + K\Delta^{-1}$. It results from the fact that, inside the boundary layer both observers are essentially the same. The difference emerges when the measurement noise is present.

When the measurement noise is present, the analysis/synthesis problem becomes fairly involved due to the terms like $\text{sign}(\tilde{y} + v)$, where v is the measurement noise. An approximation was shown in Chapter 6 and, depending on the particular problem that is being considered one of the suggested optimal design – in the least square sense – can be used. The first-order example shown in Chapter 6 illustrates the robustness against changes in the measurement noise intensity when the sliding observer uses *signum* function.

Surely, in principle there is no reason why the optimal gains obtained through quasi-linearization and some optimization criterion, for a particular problem, would also satisfy the robustness criteria shown in Chapter 4. If they are conflicting criteria one has to use engineering judgment to trade-off between sub-optimal filter and robust sliding observer.

7.3 Suggestions for Further Research

7.3.1 Use of Nonlinear Model in the Observer

Intuitively, when the estimation errors are small, the use of an available nonlinear model should improve the accuracy of estimation. The passivity theorem provides a general framework that can include additional nonlinearities in the observer. Further research is necessary to translate the general statement provided by passivity theorem into a useful observer design technique.

The use of a nonlinear model in the observer should be investigated carefully, because two drawbacks might exist:

- If the state estimation error is large, as in the beginning of estimation process, the use of nonlinear model might have a negative effect and a linear model might be preferable.
- The introduction of nonlinear model will increase the computational load, that will go against initial goal of having a simple observer. It has to be traded against the improvement in the observer performance.

7.3.2 Use of Available Degrees of Freedom

When the gain K was set

$$K = D\rho$$

several elements in the gain matrix K were set to zero, and they were not used. This eliminates the possibility of sliding behavior in many instances. Keeping all the elements of the gain matrix K , and defining the operators H_1 and H_2 as:

$$\begin{aligned} H_1 e_1 &= C \int_0^t e^{(A-HC)\tau} (K(\tau)1_s(\tau) - w(\tau)) d\tau \\ H_2 e_2 &= 1_s(e_2) \end{aligned}$$

one can, in principle, use the passivity theorem to design the observer, so that if $\tilde{y} \rightarrow 0$, sliding behavior is obtained. The degrees of freedom available in the gain matrix K can be used to assign desired behavior on the sliding surface, as shown in Chapter 3.

7.3.3 Rigorous Analysis of Random Measurement Noise Effect

As commented upon before, the coupling between estimation error signal and measurement noise inside the nonlinear element 1, makes rigorous analysis very difficult. As suggested by Drakunov [12], the use of a weak convergence method for stochastic processes might lead to further insight into the problem and useful design techniques might emerge.

7.3.4 The Use of Sliding Observer Inside the Control Loop

The use of sliding observer inside the control loop must be analyzed in detail. Intuitively, if the sliding observer provides estimates quickly, it should not affect the stability of the closed loop system. However, because the separation theorem does not hold for nonlinear systems, it is an issue that has to be analyzed in great detail.

In the case of unstable plants, for which the assumption on bounded uncertainties may not hold, the designer might design the observer assuming that the states remain within some desired bounds. A separate design would provide the necessary controller that would stabilize the plant and bring the states into desired bounds. If the coupled sliding observer - controller does not destabilize the closed loop system, the system would work properly.

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Appendix A

Mathematical Concepts

This appendix summarizes the main mathematical definitions and results used in this thesis. These mathematical tools are mainly related to stability of dynamical systems, namely input-output stability. Only the essential information is given and the readers are referred to Desoer and Vidyasagar [11], Vidyasagar [61], and Safonov [45,46,48] for rigorous treatment and more details.

A.1 Normed Inner Product Space

Consider some preliminary definitions: Let \mathcal{E} be a vector space (in this thesis $\mathcal{E} = \mathbb{R}^n$), and let \mathcal{V} be a normed linear vector space; a norm is defined based on scalar product as:

$$\|x(t)\| = \langle x(t)|x(t) \rangle^{1/2}$$

where the scalar product is defined as:

$$\langle x(t)|y(t) \rangle = \int_0^{\infty} x(t)^T y(t) dt$$

typically $x \in \mathbb{R}^n$.

Also define \mathcal{F} as the class of function that maps the non-negative real numbers into \mathcal{V} .

A inner product function space \mathcal{X} – also denoted as \mathcal{L}_2 – is defined as:

$$\mathcal{X} = \{x(t) \in \mathcal{F} \mid \|x(t)\|^2 = \langle x(t)|x(t) \rangle < \infty\}$$

Notice that this class of functions requires that the integral of the square of the function, from zero to infinity, must be finite. Intuitively one might expect that such functions have to be bounded and they must go to zero, at least asymptotically. This is actually not true, as shown in Chen [9] and Slotine [53]: the function can actually be unbounded, i.e. it might have some “spikes” that grow unbounded but so that their occurrence gets rarer and rarer as time goes to infinity. As shown by Slotine in [53], if the function is differentiable and if it has bounded derivatives then the function that has bounded integral of its square actually goes to zero asymptotically.

Another important concept is the concept of truncation of a function. The function $x_T(t)$ is called the truncation of $x(t)$ to the interval $[0, T]$ and is defined as

$$x_T(t) = \begin{cases} x(t) & 0 \leq t \leq T \\ 0 & t > T \end{cases}$$

for all $T \in [0, \infty)$. The function $x(t)$ is assumed to be measurable [11,61] (“almost” all functions considered in the engineering practice are measurable).

The extended inner product function space \mathcal{X}_e – also indicated as \mathcal{L}_{2e} – is defined as:

$$\mathcal{X}_e = \{x(t) \in \mathcal{F} | \forall T \in \mathbb{R}_+, \|x_T\|^2 = \langle x_T | x_T \rangle < \infty\}$$

Note the equivalence:

$$\|x_T\|^2 = \|x\|_T^2 = \langle x | x \rangle_T = \langle x_T | x_T \rangle$$

An example of function that belongs to \mathcal{X}_e and does not belong to \mathcal{X} is a constant function, or bias. A function that does not belong to neither \mathcal{X} nor \mathcal{X}_e is the tangent function.

A fact that is worth noting, because it is of special interest for sliding mode systems, is the fact that a chattering function as defined using limiting process as shown by Filippov [13] and Utkin [59] actually belongs to \mathcal{X} . This fact can be shown using the Dominated Convergence Theorem [11,28].

Indeed, consider a oscillating function $f_n(t)$ with amplitude ϕ_n , perhaps generated by a system with a relay with delay or dead-zone, as shown in figure A.1. Assume that the sequence of amplitude ϕ_n is such that

$$\phi_n \rightarrow 0 \text{ as } n \rightarrow \infty \text{ monotonically}$$

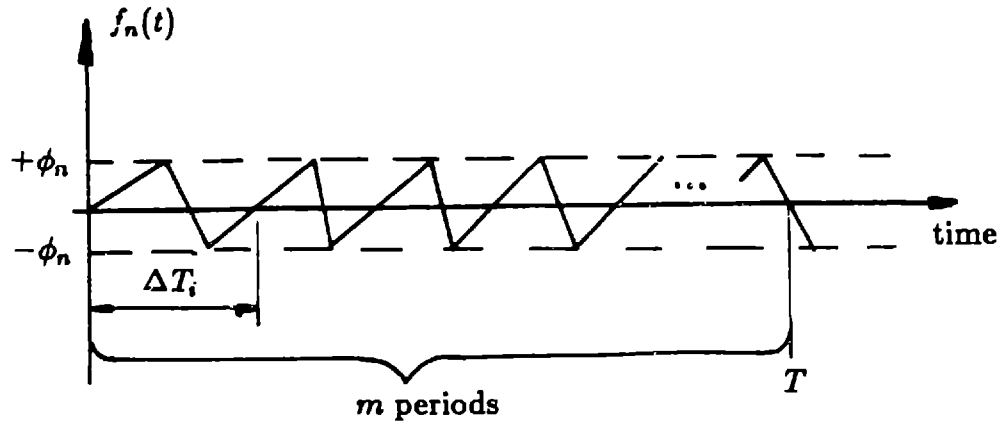


Figure A.1: A chattering function

Clearly, by definition $f_n(t) \rightarrow f = 0$, and f , as defined by Utkin [59] will be oscillating at infinite frequency.

The classical theory of integration [28] states that:

$$\lim_{n \rightarrow \infty} \int_D f_n dt = \int_D (\lim_{n \rightarrow \infty} f_n) dt$$

where D denotes the domain of integration.

In order to show that $f \in \mathcal{H}$, it is necessary to show that

$$\int_0^\infty f^2(t) dt < \infty$$

Consider the truncated version of this statement:

$$\int_0^\infty f_T^2(t) dt = \int_0^T f^2(t) dt = \sum_{i=1}^m \frac{1}{2} \phi_n^2 \Delta T_i = \frac{\phi_n^2}{2} \sum_{i=1}^m \Delta T_i = \frac{\phi_n^2}{2} T$$

So,

$$\int_0^T f(t)^2 dt = \lim_{n \rightarrow \infty} \int_0^T f_n^2 dt = 0$$

It shows that $f(t) \in \mathcal{H}_e$. Since this result does not depend on any particular interval T , then

$$\int_0^\infty f(t)^2 dt = 0 < \infty$$

With the definition of \mathcal{L}_2 spaces, one can define \mathcal{L}_2 -stability as follows. Let $H : \mathcal{L}_{2e} \rightarrow \mathcal{L}_{2e}$. The mapping H (or the system represented by mapping

H) is said to be \mathcal{L}_2 -stable if $Hx \in \mathcal{L}_2$ whenever $x \in \mathcal{L}_2$, and if there exist finite constants k and b such that

$$\|Hx\|_2 \leq k\|x\|_2 + b \quad \forall x \in \mathcal{L}_2$$

A.2 Positive Real System

The concept of a positive real system, and the concept of a strictly positive real system as used in this thesis, is well discussed in [33,40]. The reader is referred to these books for further details.

A rational function $h(s)$ of the complex variable $s = \sigma + j\omega$ is said to be strictly positive real if

1. $h(s)$ is real for real s .
2. $h(s)$ has no poles in the closed right half plane $\text{Re}[s] \geq 0$.
3. $\text{Re}[h(j\omega)] > 0$, for all $\omega \in \mathfrak{R}$.

Moreover, the linear time-invariant system

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned}$$

is strictly positive real and the transfer function matrix

$$H(s) = C(sI - A)^{-1}B$$

is a strictly positive real transfer function matrix if and only if there exist a symmetric positive definite matrices P and Q so that

$$\begin{aligned} PA + A^T P &= -Q \\ B^T P &= C \end{aligned} \tag{A.1}$$

This result is also known as the Kalman-Yakubovich Lemma, or the Positive Real Lemma.

A.3 Passivity

The concept of passivity is explained in [11], and the reader is referred to this book for all the details.

First consider the definitions.

$H : \mathcal{X}_e \rightarrow \mathcal{X}_e$ is passive if and only if there exists some constant β so that:

$$\langle Hx|x \rangle_T \geq \beta$$

for all $x \in \mathcal{X}_e$ and $T \in \mathfrak{R}_+$.

$H : \mathcal{X}_e \rightarrow \mathcal{X}_e$ is strictly passive if and only if there exists $\delta > 0$ and exists β so that:

$$\langle Hx|x \rangle_T \geq \delta \|x_T\|^2 + \beta$$

for all $x \in \mathcal{X}_e$ and $T \in \mathfrak{R}_+$.

Basically this definition is stating that H is more dissipative than a resistor of δ ohms.

The powerful result used in this thesis, called the Passivity Theorem, is stated as:

Theorem A.1 *Consider a feedback system as shown in Figure A.2, and described by:*

$$\begin{aligned} e_1 &= u_1 - H_2 e_2 \\ e_2 &= u_2 + H_1 e_1 \end{aligned}$$

where H_1 and H_2 map \mathcal{X}_e into \mathcal{X}_e . Assume that for any u_1 and u_2 in \mathcal{X} , there are solutions e_1 and e_2 in \mathcal{X}_e . Suppose that there are constants $\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2$ and β_3 such that:

$$\begin{aligned} \|H_1 z\|_T &\leq \alpha_1 \|z\|_T + \beta_1 \\ \langle z|H_1 z \rangle_T &\geq \alpha_2 \|z\|_T^2 + \beta_2 \\ \langle H_2 z|z \rangle_T &\geq \alpha_3 \|H_2 z\|_T^2 + \beta_3 \end{aligned}$$

$\forall z \in \mathcal{X}_e, \forall T \in [0, \infty)$; Under these conditions, if $\alpha_2 + \alpha_3 > 0$, then: $e_1, e_2, H_1 e_1, H_2 e_2 \in \mathcal{X}$.

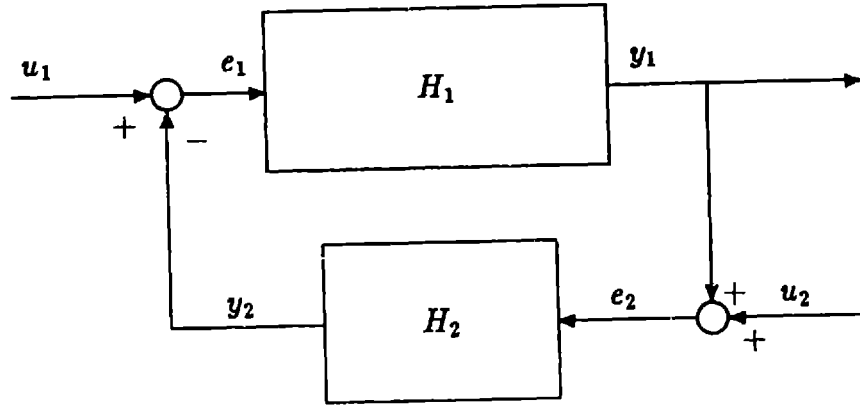


Figure A.2: "closed loop" system considered in the passivity theorem

A.4 Multivariable Circle Criterion

The multivariable, or multiloop, circle criterion is a very powerful result and is shown in complete form in [45,46,47]. The simplified form is presented here.

Consider the feedback system shown in Figure A.3 and described by:

$$\begin{aligned} y &= H(.)x \quad H_i = h_i(x_i, t) \text{ or } H_i = H_i(s) \\ x &= -G(s)(y + v) + u \end{aligned} \quad (\text{A.2})$$

H_i is said to be strictly inside the $\mathcal{L}_{2\epsilon}$ -cone (C_i, R_i, S_i) if:

$$\|S_i(h_i(\cdot) - C_i x_i(t))\|_T^2 \leq \|R_i x_i(t)\|_T^2 - \epsilon \|x_i(t)\|^2$$

or

$$\|S_i(j\omega)(H_i(j\omega) - C_i(j\omega)X_i(j\omega))\|_T^2 \leq \|R_i(j\omega)X_i(j\omega)\|_T^2 - \epsilon \|X_i(j\omega)\|^2$$

for some matrices C_i , R_i and S_i . C_i is called the center of the cone, and (R_i, S_i) its radius.

The multivariable circle criterion is stated as a theorem:

Theorem A.2 Suppose the feedback system shown in Figure A.3 is \mathcal{L}_2 -stable for the case when $h_i = c_i x_i$ or $H_i = C_i(s)$. Then a sufficient condition for the feedback system to be \mathcal{L}_2 -stable for every $H(\cdot)$ satisfying the conicity condition is that, for all real ω :

$$\sigma_{\min}(S(j\omega)(C(j\omega) + G^{-1}(j\omega))R^{-1}(j\omega)) \geq 1$$

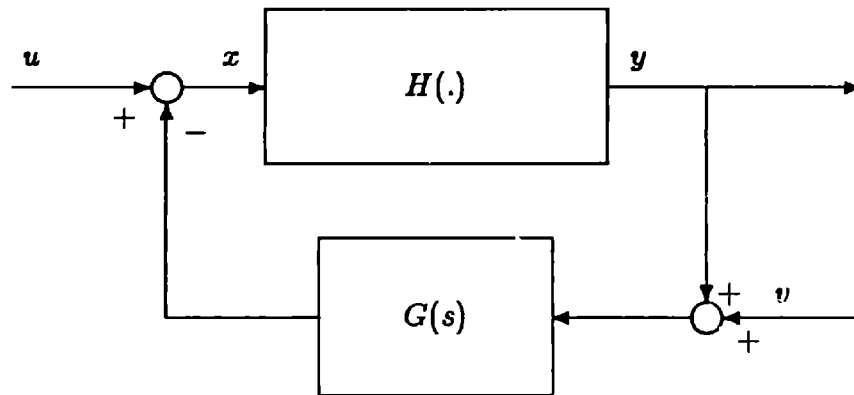


Figure A.3: "Closed loop" system considered in the Multivariable Circle Criterion

for all ω at which $G^{-1}(j\omega)$ exists.

Appendix B

Observability

In this chapter the concept of observability of dynamical systems is reviewed, and the algebraic conditions for observability are given. Roughly speaking, observability studies the possibility of estimating the states of a dynamical system given measurements of its outputs.

B.1 Observability of Linear Time Invariant Systems

The concept of observability of linear time invariant systems was developed by Kalman [26], and it is a very well known result.

The linear time invariant system

$$\begin{aligned} \dot{x} &= Ax \quad x \in \mathbb{R}^n \\ y &= Cx \quad y \in \mathbb{R}^m \end{aligned} \tag{B.1}$$

is observable if and only if the observability matrix

$$O = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

has rank n [9,24]. This result can be derived by looking at the measurement y and its derivatives as an hypothetical way to estimate the states. Differentiation is not feasible in practice, but it gives a limit on the observability

inherent in the system (B.1):

$$\begin{aligned} y &= Cx \\ \dot{y} &= C\dot{x} = CAx \\ &\vdots \\ y^{(n)} &= CA^{n-1}x \end{aligned}$$

or in the matrix form,

$$\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} x = \begin{bmatrix} y \\ \dot{y} \\ \vdots \\ y^{(n)} \end{bmatrix}$$

Therefore, given y and its derivatives the vector of states can be obtained at any instant provided that the observability matrix O has full rank. Clearly, due to the Cayley-Hamilton theorem, there is no need to take more derivatives since they will be linear combination of terms already obtained.

For linear time invariant systems, observability is an easy condition to verify. For nonlinear systems, the situation is quite different.

B.2 Observability of Nonlinear Systems

When the dynamical system being considered is nonlinear, the concept of observability becomes quite involved. Several researchers had investigated this issue in the past [7,15,20,29,31,32]. As shown by Herman and Krener [20], even though global observability is desirable, only a local and weak version yields useful results.

Consider a nonlinear dynamical system, with trajectories on a manifold $M \in \mathfrak{R}^n$, given as

$$\begin{aligned} \dot{x} &= f(x) \quad x \in \mathfrak{R}^n \\ y &= h(x) \quad y \in \mathfrak{R}^m \end{aligned} \tag{B.2}$$

Let $x^0(t)$ and $x_1(t)$ be the trajectories starting from different initial conditions x_0 and x_1 . Let the outputs $y(t)$ resulting from these trajectories be $y^0(t) = h(x^0)$ and $y^1(t) = h(x^1(t))$ respectively. The system given by

(B.2) is said to be locally weakly observable if for each $x_0 \in M$ there exists an open neighborhood U in which a different initial condition x_1 can be taken, so that the trajectories $x^0(t)$ and $x^1(t)$ contained in every open neighborhood V of x_0 – contained in U – result in distinct outputs $y^0(t)$ and $y_1(t)$. Intuitively, the system (B.2) is locally weakly observable if one can instantaneously distinguish each point from its neighbors based on the available measurements.

Several algebraic conditions for locally weakly observable systems are found in the literature.

A condition similar to the one derived for linear time invariant systems by taking the measurements and its derivatives can be found for nonlinear systems. As suggested by Krener and Respondek [32] and by Krener and Isidori [31], by taking the measurements y and its derivatives,

$$\begin{aligned} y &= h(x) \\ \dot{y} &= \frac{\partial h(x)}{\partial x} f(x) \\ &\vdots = \vdots \end{aligned}$$

Defining the Lie derivatives recursively, as:

$$\begin{aligned} L_f^0(h_i)(x) &= h_i(x) \\ L_f^j(h_i)(x) &= \frac{\partial}{\partial x} (L_f^{j-1}(h_i)(x)) f(x) \end{aligned}$$

one can rewrite the measurements and its derivatives as:

$$\begin{aligned} y_i &= L_f^0(h_i)(x) \\ \dot{y}_i &= L_f^1(h_i)(x) \\ &\vdots = \vdots \end{aligned} \tag{B.3}$$

$$y_i^{(n)} = L_f^{(n-1)}(h_i)(x) \tag{B.4}$$

for all $i = 1, \dots, m$. The system will be locally weakly observable if for all $x \in M$, the nonlinear algebraic equation (B.3) has unique solution. This condition is given as an algebraic condition by Krener and Isidori [31]. The system (B.2) is locally weakly observable if

$$L_f^j(dh)(x) \quad j = 0, \dots, n-1 \tag{B.5}$$

are linearly independent for $x \in U$, U contained in M . Note the definitions:

$$\begin{aligned} dh(x) &= \left(\frac{\partial h}{\partial x_1}(x), \dots, \frac{\partial h}{\partial x_n}(x) \right) \\ L_f(dh)(x) &= d(L_f(h))(x) \end{aligned}$$

The condition (B.5) is called the observability rank condition. Detailed discussion of this point of view is given in [20,32,31].