Nonlinear Stefan problem with convective boundary condition in Storm's materials*

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Abstract

We consider a nonlinear one-dimensional Stefan problem for a semi-infinite material $x > 0$, with phase change temperature T_f. We assume that the heat capacity and the thermal conductivity satisfy a Storm's condition and we assume a convective boundary condition at the fixed face $x = 0$. An unique explicit solution of similarity type is obtained. Moreover, asymptotic behavior of the solution when $h \to +\infty$ is studied.

Key Words: Stefan problem, free boundary problem, phase-change process, similarity solution

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1 Introduction

As in [4, 8, 10] we consider the following one phase nonlinear unidimensional Stefan problem for a semi-infinite material $x > 0$, with phase change temperature T_f

$$s(T)\frac{\partial T}{\partial t} = \frac{\partial}{\partial x}\left[k(T)\frac{\partial T}{\partial x}\right] \quad , \quad 0 < x < X(t) \ \ , \ \ t > 0 \ \ , \tag{1}$$

$$k(T(0,t))\frac{\partial T}{\partial x}(0,t) = \frac{h}{\sqrt{t}}[T(0,t) - T_m] \ , \ h > 0 \ , \ \ t > 0 \ \ , \tag{2}$$

$$T(X(t),t) = T_f \ , \tag{3}$$

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$$k(T_f)\frac{\partial T}{\partial x}(X(t),t) = \alpha \overset{\bullet}{X}(t) \ , \quad t > 0 \ , \tag{4}$$

$$X(0) = 0 \tag{5}$$

where the positive constant α is ρL, L is the latent heat of fusion of the medium, ρ is the density (assumed constant), T_m is the temperature of the medium $T_m < T(0,t) < T_f$ and h_0 is the positive heat transfer coefficient.

We assume that the metal exhibits nonlinear thermal characteristics such that the heat capacity $c_p(T) > 0$ and the thermal conductivity $k(T) > 0$ satisfy a Storm's condition [1, 2, 5, 6, 7, 9]

$$\frac{\frac{d}{dT}\left(\sqrt{\frac{s(T)}{k(T)}}\right)}{s(T)} = \lambda = const. > 0 \ , \tag{6}$$

where $s(T) = \rho c_p(T)$.

Condition (6) was originally obtained by [9] in an investigation of heat conduction in simple monoatomic metals. There, the validity of the approximation (6) was examined for aluminium, silver, sodium, cadium, zinc, copper and lead.

In [7] the free boundary problem $(1)-(6)$ (fusion case) for the particular case $k(T) = \rho c/\left(a+bT\right)^2$ and $s(T) = \rho c =$ constant was studied. The explicit solution of this problem was obtained through the unique solution of an integral equation with time as a parameter. A similar case with the constant temperature at the fixed face $x = 0$ was also studied.

In [2] two nonlinear Stefan problems analogous to $(1)-(5)$ with phase change temperature T_f and the Storm's condition (6) are considered. In one case a heat flux boundary condition of the type $q(t) = \frac{q_0}{\sqrt{t}}$ and in the other case a temperature boundary condition $T = T_s < T_f$ at the fixed face $x = 0$ are assumed. Solutions of similarity type are obtained in both cases and the equivalence of the two problems is demonstrated.

The goal of this paper is to determine the temperature $T = T(x,t)$ and the position of the phase change boundary at time t, $X = X(t)$, which satisfy the problem $(1)-(6)$. In the section 2 we show how to find a unique solution of the similarity type for this problem. In Section 3 we study the asymptotic behavior when $h \to +\infty$. We prove that the solutions $T = T_h(x,t), X = X_h(t)$ of $(1)-(5)$ converges to the solution $T = T_\infty(x,t), X = X_\infty(t)$ of an analogous Stefan problem with temperature condition $T(0,t) = T_m$ when $h \to +\infty$.

2 Existence and uniqueness of the solution to the Stefan problem with convective boundary condition on the fixed face

We consider the problem $(1)-(6)$ and we propose a similarity type solution given by [2, 3, 4]

$$T(x,t) = \Phi(\xi) \ , \ \xi = \frac{x}{X(t)} \tag{7}$$

where

$$X(t) = \sqrt{2\gamma t} \ , \ t > 0 \tag{8}$$

is the free boundary and γ is assumed a positive constant to be determined.

Then we have that the problem $(1) - (5)$ is equivalent to

$$k(\Phi)\Phi''(\xi) + k'(\Phi)\Phi'^2(\xi) + \gamma s(\Phi)\Phi'(\xi)\xi = 0 \quad , \quad 0 < \xi < 1 \ , \tag{9}$$

$$k(\Phi(0))\Phi'(0) = h\sqrt{2\gamma}[\Phi(0) - T_m] \ , \tag{10}$$

$$\phi(1) = T_f \ , \tag{11}$$

$$k(\Phi(1))\Phi'(1) = \alpha\gamma \ . \tag{12}$$

If we define

$$y(\xi) = \sqrt{\frac{k}{s}\left(\Phi(\xi)\right)} \ , \tag{13}$$

then a parametrization of the Storm condition (6) is

$$s(\Phi) = -\frac{1}{\lambda y^2}\frac{dy}{d\Phi} \quad , \quad k(\Phi) = -\frac{1}{\lambda}\frac{dy}{d\Phi} \tag{14}$$

and then we have that the following problem is equivalent to $(9) - (12)$

$$\frac{d^2y}{d\xi^2} + \frac{\gamma\xi}{y^2}\frac{dy}{d\xi} = 0 \quad , \quad 0 < \xi < 1 \ , \tag{15}$$

$$y'(0) = -\lambda h\sqrt{2\gamma}\left[P(y^2(0)) - T_m\right] \ , \tag{16}$$

$$y'(1) = -\alpha\lambda\gamma \ , \tag{17}$$

$$y(1) = y_1 = \sqrt{\frac{k(T_f)}{s(T_f)}} \ . \tag{18}$$

where P is the inverse function of the decreasing function $\frac{k}{s}$.

Lemma 1 *A parametric solution to the problem* $(15) - (18)$ *is given by*

$$\xi = \varphi_1(u) = \frac{F_{u_0}(u)}{F_{u_0}(u_1)}, \tag{19}$$

$$y = \varphi_2(u) = \frac{\sqrt{\gamma}\sqrt{\frac{\pi}{2}}\left[erf\left(\frac{u}{\sqrt{2}}\right) - g\left(\frac{u_0}{\sqrt{2}}, \frac{1}{\sqrt{\pi}}\right)\right]}{F_{u_0}(u_1)}, \tag{20}$$

for

$$u_0 \leq u \leq u_1$$

where the function $F_{u_0} = F_{u_0}(u)$ *was defined in [2] as follow*

$$F_{u_0}(u) = \exp(-\tfrac{u^2}{2}) + u\left(\int_{u_0}^{u} \exp(-\tfrac{z^2}{2})dz - \frac{\exp(-\frac{u_0^2}{2})}{u_0}\right) = \sqrt{\tfrac{\pi}{2}}u\left[g\left(\tfrac{u}{\sqrt{2}}, \tfrac{1}{\sqrt{\pi}}\right) - g\left(\tfrac{u_0}{\sqrt{2}}, \tfrac{1}{\sqrt{\pi}}\right)\right],\ u \geq u_0$$

where u_0, u_1 *are the parameter values which verify that* $\xi = \varphi_1(u_0) = 0$ *and* $\xi = \varphi_1(u_1) = 1$,

$$g(x,p) = erf(x) + p\frac{\exp(-x^2)}{x},\ p > 0,\ x > 0 \tag{21}$$

and

$$erf(x) = \frac{2}{\sqrt{\pi}}\int_0^x \exp(-z^2)dz,\ x > 0.$$

The unknowns γ, u_0 *and* u_1 *must verify the following system of equations*

$$u_0 = \sqrt{2}\lambda h\left[P\left(\frac{\gamma exp(-u_0^2)}{[u_0 F_{u_0}(u_1)]^2}\right) - T_m\right], \tag{22}$$

$$\sqrt{\gamma} = \frac{\exp(-\frac{u_1^2}{2})}{\sqrt{\frac{\pi}{2}}\alpha\lambda\left[g\left(\frac{u_0}{\sqrt{2}}, \frac{1}{\sqrt{\pi}}\right) - erf\left(\frac{u_1}{\sqrt{2}}\right)\right]} \tag{23}$$

$$y_1 = \frac{-\exp(-\frac{u_1^2}{2})}{\alpha\lambda F_{u_0}(u_1)} \tag{24}$$

Proof. A parametric solution of (15) was deduced in [4] and it is given by

$$\xi = \varphi_1(u) = C_2\left(\exp(-\frac{u^2}{2}) + u\left(\int_0^u \exp(-\frac{x^2}{2})dx + C_1\right)\right) \tag{25}$$

$$y = \varphi_2(u) = \sqrt{\gamma}C_2\left(\int_0^u \exp(-\frac{x^2}{2})dx + C_1\right),\ u > 0 \tag{26}$$

where C_1 and C_2 are integration constants to be determined.

We choose u_0 and u_1 be such that $\varphi_1(u_0) = 0$ and $\varphi_1(u_1) = 1$, we obtain that

$$C_1 = -\frac{\exp(-\frac{u_0^2}{2})}{u_0} - \int_0^{u_0} \exp(-\frac{x^2}{2})dx, \tag{27}$$

$$C_2 = \left\{\exp(-\frac{u_1^2}{2}) + u_1\left(-\frac{\exp(-\frac{u_0^2}{2})}{u_0} + \int_{u_0}^{u_1} \exp(-\frac{x^2}{2})dx\right)\right\}^{-1}. \tag{28}$$

Then, we have

$$\xi = \varphi_1(u) = \frac{\exp(-\frac{u^2}{2}) + u\left(\int\limits_{u_0}^{u} \exp(-\frac{x^2}{2})dx - \frac{\exp(-\frac{u_0^2}{2})}{u_0}\right)}{\exp(-\frac{u_1^2}{2}) + u_1\left(-\frac{\exp(-\frac{u_0^2}{2})}{u_0} + \int\limits_{u_0}^{u_1} \exp(-\frac{x^2}{2})dx\right)},\ u_0 \le u \le u_1 \tag{29}$$

and

$$y = \varphi_2(u) = \frac{\sqrt{\gamma}\left\{-\frac{\exp(-\frac{u_0^2}{2})}{u_0} + \int\limits_{u_0}^{u} \exp(-\frac{x^2}{2})dx\right\}}{\exp(-\frac{u_1^2}{2}) + u_1\left(-\frac{\exp(-\frac{u_0^2}{2})}{u_0} + \int\limits_{u_0}^{u_1} \exp(-\frac{x^2}{2})dx\right)},\ u_0 \le u \le u_1 \tag{30}$$

that is (15) – (18).

Next we prove that the unknowns u_0, u_1 and γ must satisfy (22) – (24). From (29) and (30) we have

$$y'(\xi) = \frac{\varphi_2'(u)}{\varphi_1'(u)} = \frac{\sqrt{\gamma}\exp(-\frac{u^2}{2})}{\int\limits_{u_0}^{u} \exp(-\frac{x^2}{2})dx - \frac{\exp(-\frac{u_0^2}{2})}{u_0}} \tag{31}$$

then

$$y'(0) = -\sqrt{\gamma}u_0 \tag{32}$$

and taking into account that

$$y(0) = \frac{-\sqrt{\gamma}exp(-\frac{u_0^2}{2})}{u_0 F_{u_0}(u_1)}$$

and from (16) we have (22).

Analogously we have

$$y'(1) = \frac{\varphi_2'(u_1)}{\varphi_1'(u_1)} = \frac{\sqrt{\gamma}\exp(-\frac{u_1^2}{2})}{\int\limits_{u_0}^{u_1} \exp(-\frac{x^2}{2})dx - \frac{\exp(-\frac{u_0^2}{2})}{u_0}} \tag{33}$$

and by (17) we have

$$\frac{\sqrt{\gamma}\exp(-\frac{u_1^2}{2})}{\int\limits_{u_0}^{u_1} \exp(-\frac{x^2}{2})dx - \frac{\exp(-\frac{u_0^2}{2})}{u_0}} = -\alpha\lambda\gamma \tag{34}$$

that is (23).
Last, we have

$$y(1)=\varphi_2(u_1)=\frac{\sqrt{\gamma}\left\{-\frac{\exp(-\frac{u_0^2}{2})}{u_0}+\int\limits_{u_0}^{u_1}\exp(-\frac{x^2}{2})dx\right\}}{\exp(-\frac{u_1^2}{2})+u_1\left(-\frac{\exp(-\frac{u_0^2}{2})}{u_0}+\int\limits_{u_0}^{u_1}\exp(-\frac{x^2}{2})dx\right)} \tag{35}$$

and taking into account (18) and (23) we obtain (24). ■

Next we want to find u_0, u_1 and γ the solutions to the equations $(22)-(24)$. We can rewrite the system $(22)-(24)$ as follow

$$P^{-1}\left(\frac{u_0}{\sqrt{2}h\lambda}+T_m\right)=\frac{\gamma exp(-u_0^2)}{\left[u_0F_{u_0}(u_1))\right]^2} \tag{36}$$

$$\sqrt{\gamma}=\frac{\exp(-\frac{u_1^2}{2})}{\alpha\lambda\sqrt{\frac{\pi}{2}}\left[g\left(\frac{u_0}{\sqrt{2}},\frac{1}{\sqrt{\pi}}\right)-erf\left(\frac{u_1}{\sqrt{2}}\right)\right]} \tag{37}$$

$$M(u_1)=g\left(\frac{u_0}{\sqrt{2}},\frac{1}{\sqrt{\pi}}\right) \tag{38}$$

where

$$M(x)=g\left(\frac{x}{\sqrt{2}},\frac{1}{\sqrt{\pi}}\left(\frac{1}{\alpha\lambda y_1}+1\right)\right) \tag{39}$$

Lemma 2 *The real function F_{u_0} and M satisfy the following properties:*

$$F_{u_0}(u_0)=0\ ,\ F(+\infty)=-\infty \tag{40}$$

$$F'_{u_0}(x)=\frac{\sqrt{\pi}}{2}\left\{erf\left(\frac{x}{\sqrt{2}}\right)-g\left(\frac{u_0}{\sqrt{2}},\frac{1}{\sqrt{\pi}}\right)\right\}<0 \tag{41}$$

$$M(0)=+\infty\ ,\quad M(+\infty)=1\ \text{and}\ M'(x)<0. \tag{42}$$

Proof. See [1] and [2]. ■

Lemma 3 ***(Existence of the solution)***
There exists a solution of the system $(36)-(38)$ given by

$$\tilde{u}_1=M^{-1}\left(g\left(\frac{\tilde{u}_0}{\sqrt{2}},\frac{1}{\sqrt{\pi}}\right)\right) \tag{43}$$

$$\tilde{\gamma} = \frac{\exp(-\tilde{u}_1^2)}{\alpha^2\lambda^2\left(\frac{\exp(-\frac{\tilde{u}_0^2}{2})}{\tilde{u}_0} - \int\limits_{\tilde{u}_0}^{\tilde{u}_1}\exp(-\frac{x^2}{2})dx\right)^2} \tag{44}$$

where $\tilde{u}_0$ *is a solution of*

$$P^{-1}\left(\frac{u_0}{\sqrt{2}h\lambda} + T_m\right) = \frac{\gamma exp(-u_0^2)}{\left[u_0 F_{u_0}\left(M^{-1}\left(g\left(\frac{u_0}{\sqrt{2}}, \frac{1}{\sqrt{\pi}}\right)\right)\right)\right]^2}. \tag{45}$$

Proof. Because M is a decreasing function there exists the inverse function M^{-1} and from (38) for each u_0 there exists a unique u_1 given by

$$u_1(u_0) = M^{-1}\left(g\left(\frac{u_0}{\sqrt{2}}, \frac{1}{\sqrt{\pi}}\right)\right). \tag{46}$$

If we replace (46) in (37) and (36) we have

$$\gamma(u_0) = \frac{\exp(-u_1^2(u_0))}{\alpha^2\lambda^2\left(\frac{\exp(-\frac{u_0^2}{2})}{u_0} - \int\limits_{u_0}^{u_1(u_0)}\exp(-\frac{x^2}{2})dx\right)^2} \tag{47}$$

and

$$P^{-1}\left(\frac{u_0}{\sqrt{2}h\lambda} + T_m\right) = \frac{\gamma(u_0)exp(-u_0^2)}{\left[u_0 F_{u_0}(u_1(u_0))\right]^2}. \tag{48}$$

We define the function

$$G(u_0) := P^{-1}\left(\frac{u_0}{\sqrt{2}h\lambda} + T_m\right)$$

which satisfies $G(0) = \frac{k}{s}(T_m)$ and $G'(u_0) < 0$, and let

$$H(u_0) := \frac{\gamma(u_0)exp(-u_0^2)}{\left[u_0 F_{u_0}(u_1(u_0))\right]^2}.$$

From (24), (46) and (47) it follows that

$$H(u_0) = \frac{2y_1^2 exp(-u_0^2)}{u_0^2\pi\left[erf\left(\frac{M^{-1}\left(g\left(\frac{u_0}{\sqrt{2}}, \frac{1}{\sqrt{\pi}}\right)\right)}{\sqrt{2}}\right) - g\left(\frac{u_0}{\sqrt{2}}, \frac{1}{\sqrt{\pi}}\right)\right]^2},$$

$H(0) = y_1^2$, $H(+\infty) = +\infty$ and $H(u_0) \geq y_1^2, \forall u_0 \geq 0$.

Since $T_m < T_f$ we conclude $G(0) = \dfrac{k}{s}(T_{m)} > \dfrac{k}{s}(T_f) = y_1^2 = H(0)$. Taking into account that the properties of G and H there exists $\tilde{u}_0 < u_0^* = (T_f - T_m)\sqrt{2}h\lambda$ which satisfies (48). Then by (46) and (47) we complete the solution $\tilde{u}_1 = u_1(\tilde{u}_0)$ and $\tilde{\gamma}_{=}\gamma(\tilde{u}_0)$ to the system $(36) - (38)$. ■

Lemma 4 ***(Uniqueness of the solution)***
The solution $(\tilde{u}_0, \tilde{u}_1, \tilde{\gamma})$ *to the system* $(22)-(24)$ *is unique.*

Proof. Suppose the assertion of the lemma is false. That is there exist two solutions $(\tilde{u}_0, \tilde{u}_1, \tilde{\gamma})$ and (u_0^*, u_1^*, γ^*) to $(22)-(24)$.

We assume that $\tilde{u}_0 < u_0^*$, then by (19) we have

$$\xi = \frac{F_{u_0^*}(u)}{F_{u_0^*}(u_1^*)} = \frac{F_{\tilde{u}_0}(u)}{F_{\tilde{u}_0}(\tilde{u}_1)}, \qquad for \qquad u_0^* \leq u \leq min(\tilde{u}_1, u_1^*). \tag{49}$$

For $u = u_0^*$ we have

$$0 = \frac{F_{u_0^*}(u_0^*)}{F_{u_0^*}(u_1^*)} = \frac{F_{\tilde{u}_0}(u_0^*)}{F_{\tilde{u}_0}(\tilde{u}_1)} \tag{50}$$

then $F_{\tilde{u}_0}(u_0^*) = 0$. This is a contradiction because $F_{\tilde{u}_0}(u_0^*) = 0$ if and only if $u = \tilde{u}_0$. ■

Theorem 5 *The problem* $(1)-(6)$ *has a similarity type solution given by*

$$T(x,t) = P\left(\left(\varphi_2\left(\varphi_1^{-1}\left(x/X(t)\right)\right)\right)^2\right), \qquad 0 < x < X(t) \tag{51}$$

where

$$X(t) = \sqrt{2\tilde{\gamma}t}, \quad t > 0 \tag{52}$$

is the free boundary,

$$\varphi_1(u) = \frac{F_{\tilde{u}_0}(u)}{F_{\tilde{u}_0}(\tilde{u}_1)}, \tag{53}$$

$$\varphi_2(u) = \frac{\sqrt{\tilde{\gamma}}\sqrt{\frac{\pi}{2}}\left[erf\left(\frac{u}{\sqrt{2}}\right) - g\left(\frac{\tilde{u}_0}{\sqrt{2}}, \frac{1}{\sqrt{\pi}}\right)\right]}{F_{\tilde{u}_0}(\tilde{u}_1)}, \tag{54}$$

$(\tilde{u}_0, \tilde{u}_1, \tilde{\gamma})$ *is the unique solution of* $(22)-(24)$ *and* $P = \left(\frac{k}{s}\right)^{-1}$ *is the inverse function of the function* $\frac{k}{s}$.

Proof. Fixed the data: $\alpha,\ \lambda,\ h,\ T_f$ of the problem $(1)-(6)$, we obtain the solutions of the equations $(22)-(24)$ given by (43), (44) and $\tilde{u}_0$ is the solution of (45).

Next, we obtain φ_1 and φ_2 given by $(53), (54)$ respectively and the free boundary is $X(t) = \sqrt{2\tilde{\gamma}t}$. Taking into account that φ_1 is an increasing function we determine $\varphi_1^{-1}\left(\frac{x}{X(t)}\right)$. Finally, we invert the relation (13) and from (7) we obtain (51). ■

Remark 1 *Si* $T(0,t) = T_s$ *is constant, the convective condition* (2) *at the fixed face* $x = 0$ *of the problem* $(1)-(6)$ *becomes a Neumann boundary condition given by*

$$k(T(0,t))\frac{\partial T}{\partial x}(0,t) = \frac{q_0}{\sqrt{t}} \tag{55}$$

with

$$q_0 = h[T_s - T_m].$$

The Stefan problem $(1)-(6)$ *with the condition* (55) *instead* (2) *was studied in* [2].

3 Asymptotic behavior of the solution when $h \to +\infty$

Let $h > 0$ and $T = T_h(x,t), X = X_h(t)$ denote the solution to the problem $(1) - (6)$ given by $(51) - (54)$. We will study the behavior of this solution when the transfer coefficient $h \to +\infty$. We will prove that T_h, X_h converges to the solution T_∞, X_∞ of the following parabolic free boundary problem:

$$s(T)\frac{\partial T}{\partial t} = \frac{\partial}{\partial x}\left[k(T)\frac{\partial T}{\partial x}\right] \quad , \quad 0 < x < X(t) \quad , \quad t > 0 \quad , \tag{56}$$

$$T(0,t) = T_m \; , \quad t > 0 \quad , \tag{57}$$

$$T(X(t),t) = T_f \; , t > 0 \quad , \tag{58}$$

$$k(T_f)\frac{\partial T}{\partial x}(X(t),t) = \alpha \overset{\bullet}{X}(t) \; , \quad t > 0 \quad , \tag{59}$$

$$X(0) = 0 \tag{60}$$

with the Storm's condition

$$\frac{\frac{d}{dT}\left(\sqrt{\frac{s(T)}{k(T)}}\right)}{s(T)} = \lambda \; . \tag{61}$$

The problem $(56) - (61)$ was studied in [2]. The solution is given by

$$T_\infty(x,t) = P\left(\left(\varphi_{2\infty}\left(\varphi_{1\infty}^{-1}\left(x/X_\infty(t)\right)\right)\right)^2\right) \tag{62}$$

$$X_\infty(t) = \sqrt{2\gamma_\infty t} \tag{63}$$

where

$$\varphi_{1\infty}(u) = \frac{F_{v_0}(u)}{F_{v_0}(v_1)}, \tag{64}$$

$$\varphi_{2\infty}(u) = \frac{\sqrt{\gamma_\infty}\sqrt{\frac{\pi}{2}}\left[erf\left(\frac{u}{\sqrt{2}}\right) - g\left(\frac{v_0}{\sqrt{2}}, \frac{1}{\sqrt{\pi}}\right)\right]}{F_{v_0}(v_1)} \tag{65}$$

with $v_0 \leq u \leq v_1$. The parameters v_0, v_1 and γ_∞ satisfy the following equations

$$y_1 = \sqrt{\gamma_\infty}\frac{F_{v_0}(v_1) - exp\left(-\frac{v_1^2}{2}\right)}{v_1 F_{v_0}(v_1)} \tag{66}$$

$$\sqrt{\gamma_\infty} = \frac{v_1 y_1}{1 + \alpha\lambda y_1} \tag{67}$$

$$\frac{k}{s}(T_m) = y_0 = -\sqrt{\gamma_\infty}\frac{exp\left(-\frac{v_0^2}{2}\right)}{v_0 F_{v_0}(v_1)} \tag{68}$$

which are equivalent to

$$\frac{k}{s}(T_m) = H(v_0) = \frac{2y_1^2 exp\left(-v_0^2\right)}{v_0^2\pi\left[erf\left(\frac{M^{-1}\left(g\left(\frac{v_0}{\sqrt{2}},\frac{1}{\sqrt{\pi}}\right)\right)}{\sqrt{2}}\right) - g\left(\frac{v_0}{\sqrt{2}},\frac{1}{\sqrt{\pi}}\right)\right]^2} \tag{69}$$

$$\sqrt{\gamma_\infty} = \frac{\exp(-\frac{v_1^2}{2})}{\alpha\lambda\sqrt{\frac{\pi}{2}}\left[g\left(\frac{v_0}{\sqrt{2}},\frac{1}{\sqrt{\pi}}\right) - erf\left(\frac{v_1}{\sqrt{2}}\right)\right]} \tag{70}$$

$$v_1 = M^{-1}\left(g\left(\frac{v_0}{\sqrt{2}},\frac{1}{\sqrt{\pi}}\right)\right). \tag{71}$$

For simplicity of notation, we wright $(u_{0h}, u_{1h}, \gamma_h)$ instead of $(\tilde{u}_{0h}, \tilde{u}_{1h}, \tilde{\gamma}_h)$ which is the solution of $(36)-(38)$. Firstly we will prove that $(u_{0h}, u_{1h}, \gamma_h)$ converges to $(v_0, v_1, \gamma_\infty)$ when $h \to +\infty$. The proof of this statement is based on the following lemma:

Lemma 6 *The sequences* $\{u_{0h}\}$, $\{u_{1h}\}$ *and* $\{\gamma_h\}$ *are increasing and bounded. Moreover*

$$\lim_{h\to+\infty} u_{0h} = v_0, \quad \lim_{h\to+\infty} u_{1h} = v_1, \quad \text{and} \quad \lim_{h\to+\infty} \gamma_h = \gamma_\infty.$$

Proof. From properties of function $G = G_h(x) = P^{-1}\left(\frac{x}{\sqrt{2h\lambda}} + T_m\right)$ we have

a) $h_1 \leq h_2 \Rightarrow G_{h_1}(x) \leq G_{h_2}(x), \quad \forall x \geq 0$

b) $G_h(x) \leq \frac{k}{s}(T_m), \quad \forall x \geq 0, \quad h > 0.$

We consider $h_1 < h_2$, if u_{0h_1} and u_{0h_2} are the solutions of $G_{h_1}(x) = H(x)$ and $G_{h_2}(x) = H(x)$ respectively, by a) and properties of function H we have that $u_{0h_1} < u_{0h_2}$. Moreover from b) results $u_{0h} \leq v_0$ for all $h > 0$. Then, $\{u_{0h}\}$ is an increasing bounded sequence and there exists $\tilde{u}_0$ such that

$$\lim_{h\to+\infty} u_{0h} = \tilde{u_0}.$$

Letting $h \to +\infty$ on $G_h(u_{0h}) = H(u_{0h})$ yields $\frac{k}{s}(T_m) = H(\tilde{u_0})$. By uniqueness of the solution of (69) results $\tilde{u_0} = v_0$.

From (38) we have

$$u_{1h} = M^{-1}\left(g\left(\frac{u_{0h}}{\sqrt{2}},\frac{1}{\sqrt{\pi}}\right)\right) \tag{72}$$

Because $\{u_{0h}\}$ is increasing, M and g are decreasing functions we have that the sequence $\{u_{1h}\}$ is increasing. Moreover taking into account $u_{0h} \leq v_0$ and (71) follows

$$u_{1h} = M^{-1}\left(g\left(\frac{u_{0h}}{\sqrt{2}},\frac{1}{\sqrt{\pi}}\right)\right) \leq M^{-1}\left(g\left(\frac{v_0}{\sqrt{2}},\frac{1}{\sqrt{\pi}}\right)\right) = v_1$$

for all $h > 0$.

By (72) we obtain

$$\lim_{h\to+\infty} u_{1h} = \lim_{h\to+\infty} M^{-1}\left(g\left(\frac{u_{0h}}{\sqrt{2}}, \frac{1}{\sqrt{\pi}}\right)\right) = M^{-1}\left(g\left(\frac{v_0}{\sqrt{2}}, \frac{1}{\sqrt{\pi}}\right)\right) = v_1.$$

Finally, letting $h \to +\infty$ in (37) we have

$$\lim_{h\to+\infty} \gamma_h = \gamma_\infty.$$

It follows easily of (37) and (38) that $\sqrt{\gamma_h} = \dfrac{u_{1h} y_1}{1+\alpha\lambda y_1}$. Taking into account $u_{1h} \le v_1$ we have

$$\sqrt{\gamma_h} = \frac{u_{1h} y_1}{1+\alpha\lambda y_1} \le \frac{v_1 y_1}{1+\alpha\lambda y_1} = \sqrt{\gamma_\infty} \quad \forall h > 0.$$

■

Corollary 7 *For each $t > 0$, the sequence $\{X_h(t)\}$ is monotonically increasing and $\lim_{h\to+\infty} X_h(t) = X_\infty(t)$.*

We can now define an extension $\tilde{T}_h = \tilde{T}_h(x,t) \in C^1\left[0, X_\infty(t)\right]$ of $T_h(x,t)$ as follows

$$\tilde{T}_h(x,t) = \begin{cases} T_h(x,t) \quad if \quad 0 \le x < X_h(t) \\ \\ \dfrac{\alpha\sqrt{2\gamma_h}}{2k(T_f)\sqrt{t}}\left(x - X_h(t)\right) + T_f \quad if \quad X_h(t) \le x \le X_\infty(t) \end{cases} \tag{73}$$

Lemma 8 *The functions $\tilde{T}_h \in C^1\left[0, X_\infty(t))\right]$ satisfy $|\dfrac{\partial \tilde{T}_h}{\partial x}| \le M$ on $[0, X_\infty(t)]$ for all $h > 0$, $t > 0$.*

Proof. Let $t > 0$ and $x \in [0, X_\infty(t)]$.

If $x \in [X_h(t), X_\infty(t)]$ then

$$|\frac{\partial \tilde{T}_h(x,t)}{\partial x}| = \frac{\alpha\sqrt{2\gamma_\infty}}{2k(T_f)\sqrt{t}}.$$

For otherwise, this is $x \in [0, X_h(t))$ according to (7) and (13) we have

$$\frac{\partial \tilde{T}_h}{\partial x}(x,t) = P'\left(y_h^2\left(\tfrac{x}{X_h(t)}\right)\right) 2y_h\left(\tfrac{x}{X_h(t)}\right) y_h'\left(\tfrac{x}{X_h(t)}\right)\frac{1}{X_h(t)}$$

Since $\frac{k}{s}$ is decreasing and $T_m \leq T_h(x,t) \leq T_f$, from (13) we have $y_1 \leq y_h\left(\frac{x}{X_h(t)}\right) \leq y_0$, for all $h > 0$. From (6) follows that

$$|P'\left(y_h^2\left(\tfrac{x}{X_h(t)}\right)\right)| \leq \frac{1}{2\lambda y_1 k_m}$$

where $k_m = min\{k(T), T_m \leq T \leq T_f\}$. Taking into account (29), (30), (53) and Lemma 6 we have

$$|y_h'\left(\tfrac{x}{X_h(t)}\right)\frac{1}{X_h(t)}| \leq \frac{1}{\sqrt{\pi t}\left[1 - erf\left(\frac{v_1}{\sqrt{2}}\right)\right]}.$$

Then for $x \in [0, X_h(t))$ results

$$|\frac{\partial \tilde{T}_h(x,t)}{\partial x}| \leq \frac{y_0}{\lambda y_1 k_m \sqrt{\pi t}\left[1 - erf\left(\frac{v_1}{\sqrt{2}}\right)\right]}.$$

Summaring, for all $h > 0$ and $x \in [0, X_\infty(t)]$ we obtain

$$|\frac{\partial \tilde{T}_h(x,t)}{\partial x}| \leq M = max\left\{\frac{\alpha\sqrt{2\gamma_\infty}}{2k(T_f)\sqrt{t}}, \frac{y_0}{\lambda y_1 k_m \sqrt{\pi t}\left[1 - erf\left(\frac{v_1}{\sqrt{2}}\right)\right]}\right\}$$

and this precisely the assertion of the lemma. ■

Lemma 9 *We have* $\lim_{h\to+\infty} \tilde{T}_h(x,t) = T_\infty(x,t)$ *for each* $t > 0$ *and* $x \in [0, X_\infty(t)]$.

Proof. Let $t > 0$ and $x \in [0, X_\infty(t))$. By Corollary 7 there exists $h_0 = h_0(x) > 0$ such that $x \in [0, X_h(t)]$ for all $h \geq h_0$. We consider $\tilde{T}_h(x,t)$ for $h \geq h_0$ we have

$$\tilde{T}_h(x,t) = T_h(x,t) = P\left(\left(\varphi_{2h}\left(\varphi_{1h}^{-1}\left(x/X_h(t)\right)\right)\right)^2\right). \qquad (74)$$

Taking into account Lemma 6, Corollary 7, (53) and (54) we obtain that the sequence $\{T_h(x,t)\}$ converges to $T_\infty(x,t)$. If $x = X_\infty(t)$ then $\tilde{T}_h(X_\infty(t),t) = T_f = T_\infty(X_\infty(t),t)$.

Hence, the sequence $\{\tilde{T}_h(x,t)\}$ converges to $T_\infty(x,t)$ pointwise on $[0, X_\infty(t)]$ for each $t > 0$.

■

Theorem 10 *For each* $t > 0$ *we have the family of functions* $\{\tilde{T}_h\}$ *converges uniformly to* T_∞ *for* $h \longrightarrow +\infty$ *on* $[0, X_\infty(t)]$.

Proof. By Lemma 8, for any $t > 0$ the functions $\tilde{T}_h(x,t)$ are equicontinuous on $[0, X_\infty(t)]$ and from Lemma 9 converges pointwise to $T_\infty(x,t)$ for $h \longrightarrow +\infty$. Then, by Ascoli Arzela lemma we obtain their uniform convergence on $[0, X_\infty(t)]$. ■

4 Conclusions

One phase nonlinear, one-dimensional Stefan problems for a semi-infinite material $x > 0$, with phase change temperature T_f has been considered with the assumption of a Storm's condition for the heat capacity and thermal conductivity and a convective condition at the fixed face. Existence and uniqueness of a similarity type solution has been obtained. Moreover, the convergence of this problem to problem with temperature condition at the fixed face when $h \to +\infty$ has been proved.

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