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Nonlinear strain gradient elastic thin shallow shells

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Abstract. The governing equilibrium equations for strain gradient elastic thin shallow shells are derived, considering non-linear strains and linear constitutive strain gradient elastic relations. Adopting Kirchhoff's theory of thin shallow structures, the equilibrium equations, along with the boundary conditions, are formulated through a variational procedure. It turns out that new terms are introduced, indicating the importance of the cross-section area in bending of thin plates. Those terms are missing from the existing strain gradient shallow thin shell theories. Those terms highly increase the stiffness of the structures. When the curvature of the shallow shell becomes zero, the governing equilibrium for the plates are derived.

1.Introduction.

Thin plate theory has found a lot of applications in the areas of micromechanics and nano-mechanics. Thin films, micro-electromechanical systems and nano-electromechanical systems are typical applications of the thin beam theory, where size effects have been observed. Many researchers , Papargyri et al [2003], Lazopoulos [2004], have correlated thin beam theory with the strain gradient elasticity theories Mindlin [1965], Altan & Aifantis [1997], Ru & Aifantis [1993], Yang et al [2002]. The theory of gradient strain elasticity has been applied to many mechanics problems in plasticity and dislocation, Aifantis [2003], Fleck et al [1997,1993,1994]. Further applications of the strain gradient elasticity theories have appeared in lifting various singularities in fracture problems, Altan & Aifantis [1997] and around concentrated forces like the Flamant problem, Lazar & Maugin [2006].

In the present work the bending Kirchhoff's plate theory will be discussed into the context of a simplified strain gradient elasticity theory, where new terms, depending not only on the moment of inertia of the cross-section but also on the area of the cross-section are introduced. Those terms highly increase the stiffness of the plate. The author, Lazopoulos [2009], has already studied the behavior of thin strain gradient elastic beams using the proposed procedure. Terms of the same type have been introduced in bending of beams by Yang et al. [2006] and their theory has been applied to various bending problems, Lam et al [1985], Park & Gao [2008], Ma et al [2008]. Nevertheless, that couple stress theory does not include a substantial part of the strain gradient theory that is the increase of the higher order derivatives in the governing equilibrium equations. Those terms are necessary for the development of boundary layers which are characteristic of the strain gradient elasticity applications. Furthermore Yang et al.[2002] ends up with a symmetric stress tensor assuming zero couple moment, Eq.(27). This requirement is an additional condition which is not derived by any principle of mechanics. Further, couple stresses and symmetric stress tensor is not compatible. In fact the present theory bridges the theories bending theories presented by Papargyri et al [2003] and Yang et al. [2002] in a consistent way including not only the higher order derivatives in the governing equilibrium equations, necessary for the development of boundary layers missing from the theory of Yang et al.[2002], but also the terms depending upon the cross-section area missing from the theory of Papargyri et al. [2003], that highly increase the stiffness of the thin beam when the beam thickness reduces. The governing equilibrium equation for the thin plate with the corresponding boundary conditions will be derived through a variational approach for plate bending problems.

2. Geometrically nonlinear deformations of a shallow thin shell.

Adopting Kirchhoff's theory for thin shallow shells along with the nonlinear strain tensor, a simple version of Mindlin's strain gradient elastic constitutive relations is recalled, introducing a geometrically nonlinear theory of elasticity with microstructure, a micro-elasticity theory equipped with two additional constitutive

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coefficients, apart from the Lame´ constants is used. The intrinsic bulk length g and the directional surface energy length l_k are the additional constitutive parameters.

Hence, the strain energy density function, for the present geometrically nonlinear case, is expressed by,

$$W=\frac{1}{2}\lambda e_{mm}e_{nn}+Ge_{mn}e_{nm}+g^2\left(\frac{1}{2}\lambda e_{kmm}e_{knn}+Ge_{kmn}e_{knm}\right)+ \\ l_k\left(\frac{1}{2}\lambda\left(e_{kmm}e_{nn}+e_{mm}e_{knn}\right)+G\left(e_{kmn}e_{nm}+e_{mn}e_{knm}\right)\right) \tag{1}$$

where, e_{ij} denotes Green's (or Lagrangean) strain and e_{ijk} the nonlinear strain gradient respectively, with

$$e_{ij}=e_{ji}=\frac{1}{2}\left(\partial_i u_j+\partial_j u_i+\partial_i u_k\cdot\partial_j u_k\right),\qquad e_{ijk}=e_{ikj}=\partial_i e_{kj} \tag{2}$$

and $u_i=u_i(x_k)$, the finite displacement field. The present form of the strain energy density function is the simplest one for the strain gradient elasticity problems including surface energy density, see Vardoulakis [2004].

If the shallow shell is described by the middle surface in its initial shape by the function $z(x,y)$, recalling Kirchoff's theory of thin shells, the components of the non-linear Green's tensor are expressed by,

$$\begin{aligned} e_{xx}&=u,_x+z,_x\,w,_x-\zeta\,w,_{xx}+\frac{1}{2}w,_x^2 \\ e_{yy}&=v,_y+z,_y\,w,_y-\zeta\,w,_{yy}+\frac{1}{2}w,_y^2 \\ e_{xy}&=\frac{1}{2}\left(u,_y+v,_x\right)+\frac{1}{2}\left(z,_x\,w,_x+z,_y\,w,_y\right)-\zeta w,_{xy}+\frac{1}{2}w,_x\,w,_y \end{aligned} \tag{3}$$

where, (x,y) is the horizontal plane and $w(x,y)$ is the vertical displacement of the point lying on the middle surface. The second Piola-Kirchhoff's stress S_{ij} used in Lagrangean description is defined by,

$$S_{ij}=\frac{\partial W}{\partial e_{ij}}=\lambda e_{kk}\delta_{ij}+2Ge_{ij}+l_k\left(\lambda e_{knn}\delta_{ij}+2Ge_{kij}\right)\text{ , } k=x \text{ or } y \tag{4}$$

and the double second Piola-Kirchhoff stresses by,

$$S_{ijk}=\frac{\partial W}{\partial e_{ijk}}=g^2\left(\lambda e_{inn}\delta_{jk}+2Ge_{ijk}\right)+l_i\left(\lambda e_{nn}\delta_{jk}+2Ge_{jk}\right)\quad . \tag{5}$$

For the present study we consider a thin plate of thickness *h* shown in Fig. 1. The *xy*-plane is the plane of the plate, whereas the z axis is the deflection axis. The region of the plate in the xy plane is S_m and the boundary in the xy plane is *C*. The plate is

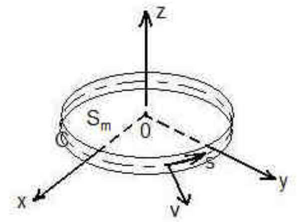

Fig.1 The geometry of the plate

bending under the action of the distributed transversal loads *p(x,y)*, the edge moments $\overline{M}_{cd}$ and the double moments $\overline{m}_{cd}$ where *c,d=ν* or *s* , the edge force $\overline{V}_v$, exhibiting the (additional) deflection $w(x,y)$ in the *ζ*-direction.

Therefore, the variation of the strain energy *δU* of the plate is defined by,

$$\delta U = \iiint_V \left(S_{ij}\delta e_{ij} + S_{ijk}\delta e_{ijk}\right)dv \tag{6}$$

It is pointed out that in the existing theories for thin structures into the context of strain gradient elasticity, the contribution of the e_{zij} terms does not exist Papargyri et al [2003,2008], Park & Gao [2006], Yang et al [2002]. In the present theory, those terms are quite important for thin structures when the thickness of the thin structures is comparable to the bulk intrinsic length of the material. In this case the variation of the strain energy density is expressed by,

$$\begin{aligned}\delta U = \iiint_V \; &((S_{xx}\delta e_{xx} + S_{yy}\delta e_{yy} + 2S_{xy}\delta e_{xy} + (S_{xxx}\delta e_{xxx} + S_{yxx}\delta e_{yxx} + S_{zxx}\delta e_{zxx}) + \\ &(S_{xyy}\delta e_{xyy} + S_{yyy}\delta e_{yyy} + S_{zyy}\delta e_{zyy}) + 2(S_{xxy}\delta e_{xxy} + S_{yxy}\delta e_{yxy} + S_{zxy}\delta e_{zxy}))) \quad dxdyd\zeta\end{aligned} \tag{7}$$

Since the shell is thin and shallow, the transverse nornal stress S_{zz} may be neglected and the the (x,y,ζ) coordinate system can be considered approximately locally rectangular Cartesian. Consequently, we may have

$$S_{xx} = \frac{E}{\left(1-\nu^2\right)}\left(e_{xx} + \nu e_{yy}\right) \quad , \quad S_{yy} = \frac{E}{\left(1-\nu^2\right)}\left(\nu e_{xx} + e_{yy}\right) \quad , S_{xy} = 2Ge_{xy} \tag{8}$$

with *E* Young's modulus, ν Poisson's ratio and *G* shear modulus.

For the thin shallow shell, the external forces are the body forces prescribed per unit area of the (x,y) plane and their components in the x,y, z directions are denoted by, $\overline{X}, \overline{Y}, \overline{Z}$ correspondingly. The traction per unit length of the boundary *C* is composed by the forces R_x, R_y, R_z, acting along the x,y,z directions respectively and the double forces R_{xx}, R_{yy}, R_{xxy}, R_{yxy}. Further, the moments $\overline{M}_{\nu}, \overline{M}_{s}$ are the applied moments per unit boundary length in the normal (ν) and the tangential (s) directions. Non-classical double moments $\overline{m}_{\nu\nu}, \overline{m}_{ss}, \overline{m}_{\nu s}$, due to the gradient elasticity, are also applied to the boundary. Therefore the principle of virtual work gives,

$$
\begin{aligned}
\delta V = \iiint_V & \{ (S_{xx}\delta e_{xx} + S_{yy}\delta e_{yy} + 2S_{xy}\delta e_{xy} + (S_{xxx}\delta e_{xxx} + S_{yxx}\delta e_{yxx} + S_{zxx}\delta e_{zxx}) \\
& + (S_{xyy}\delta e_{xyy} + S_{yyy}\delta e_{yyy} + S_{zyy}\delta e_{zyy}) + 2(S_{xxy}\delta e_{xxy} + S_{yxy}\delta e_{yxy} + S_{zxy}\delta e_{zxy}) \} dx dy d\zeta \\
& - \iint_{S_m} [\overline{X}\delta x + \overline{Y}\delta y + \overline{Z}\delta w] dx dy - \oint_C \{ R_x \delta u + R_y\, \delta v + R_{xx}\delta u,_x + R_{yy}\delta v,_y + R_{xy}\frac{(\delta u,_y + \delta v,_x)}{2} + \\
& R_z \delta w + \overline{M}_{\nu}\delta w,_{\nu} + \overline{M}_{s}\delta w,_{s} + \overline{m}_{\nu\nu}\delta w,_{\nu\nu} + \overline{m}_{ss}\delta w,_{ss} + \overline{m}_{\nu s}\delta w,_{\nu s} \} ds
\end{aligned}
\tag{9}
$$

Further, we introduce the stress resultants,

$$
\begin{aligned}
& N_{xx} = \int S_{xx} d\zeta, \quad N_{yy} = \int S_{yy} d\zeta, \quad N_{xy} = \int S_{xy} d\zeta, \\
& N_{xxx} = \int S_{xxx} d\zeta, \quad N_{xyy} = \int S_{xyy} d\zeta, \quad N_{xxy} = \int S_{xxy} d\zeta, \\
& N_{yxx} = \int S_{yxx} d\zeta, \quad N_{yyy} = \int S_{yyy} d\zeta, \quad N_{yxy} = \int S_{yxy} d\zeta, \\
& N_{zxx} = \int S_{zxx} d\zeta, \quad N_{zyy} = \int S_{zyy} d\zeta, \quad N_{zxy} = \int S_{zxy} d\zeta, \\
& M_{xx} = \int S_{xx} \zeta d\zeta, \quad M_{yy} = \int S_{yy} \zeta d\zeta, \quad M_{xy} = \int S_{xy} \zeta d\zeta \\
& m_{xxx} = \int S_{xxx} \zeta d\zeta, \quad m_{xyy} = \int S_{xyy} \zeta d\zeta, \quad N_{xxy} = \int S_{xxy} \zeta d\zeta, \\
& m_{yxx} = \int S_{yxx} \zeta d\zeta, \quad m_{yyy} = \int S_{yyy} \zeta d\zeta, \quad m_{yxy} = \int S_{yxy} \zeta d\zeta, \\
& m_{zxx} = \int S_{zxx} \zeta d\zeta, \quad m_{zyy} = \int S_{zyy} \zeta d\zeta, \quad m_{zxy} = \int S_{zxy} \zeta d\zeta
\end{aligned}
\tag{10}
$$

Hence the principle of virtual work, Eq.(9), becomes:

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$$\delta V = \iint_{Sm}\Bigg\{\Bigg\{-(\tilde{M}_{xx,xx}+\tilde{M}_{yy,yy}+2\tilde{M}_{xy,xy})-(\frac{\partial}{\partial x}(\tilde{N}^0_{xx}(z,_x+w,_x))-\frac{\partial}{\partial y}(\tilde{N}^0_{yy}(z,_y+w,_y))$$

$$-\frac{\partial}{\partial y}(\tilde{N}^0_{xy}(z,_x+w,_x))-\frac{\partial}{\partial x}(\tilde{N}^0_{xy}(z,_y+w,_y))\Bigg\}\delta w-(\tilde{N}^0_{xx,x}+\tilde{N}^0_{xy,y})\delta u-(\tilde{N}^0_{yy,y}+\tilde{N}^0_{xy,x})\delta v\Bigg\}dxdy$$

$$+\oint\tilde{M}_{xx,x}\delta wdy-\oint\tilde{M}_{xy,x}\delta wdx+\oint\tilde{M}_{xy,y}\delta wdy-\oint\tilde{M}_{yy,y}\delta wdx+\oint(-\tilde{M}_{xx}\delta w,_x-\tilde{M}_{xy}\delta w,_y)dy$$

$$+\oint(\tilde{M}_{yy}\delta w,_y+\tilde{M}_{xy}\delta w,_x)dx+\oint\tilde{N}^0_{xx}\delta udy+\oint\tilde{N}^0_{xx}z_x\delta wdy+\oint\tilde{N}^0_{xx}w,_x\delta wdy-\oint\tilde{N}^0_{yy}\delta vdx$$

$$-\oint\tilde{N}^0_{yy}z,_y\delta wdx-\oint\tilde{N}^0_{yy}w,_y\delta wdx-\oint\tilde{N}^0_{xy}\delta udx+\oint\tilde{N}^0_{xy}\delta vdy-\oint\tilde{N}^0_{xy}z,_x\delta wdx+\oint\tilde{N}^0_{xy}z,_x\delta wdx$$

$$-\oint\tilde{N}^0_{xy}w,_x\delta wdx+\oint\tilde{N}^0_{xy}w,_y\delta wdy-\oint(m_{xxx}\delta w,_{xx})dy+\oint(m_{yxx}\delta w,_{xx})dx-\oint(m_{xyy}\delta w,_{yy})dy$$

$$+\oint(m_{yyy}\delta w,_{yy})dx-\oint 2m_{xxy}\delta w,_{xy}dy+\oint 2m_{yxy}\delta w,_{xy}dx+\oint\tilde{N}^0_{xxx}\delta e^0_{xx}dy$$

$$-\oint\tilde{N}^0_{yxx}\delta e^0_{xx}dx+\oint\tilde{N}^0_{xyy}\delta e^0_{yy}dy-\oint\tilde{N}^0_{yyy}\delta e^0_{yy}dx+2\oint\tilde{N}^0_{xxy}\delta e^0_{xy}dy-2\oint\tilde{N}^0_{yxy}\delta e^0_{xy}dx$$

$$+\oint\left(-\tilde{M}_{xx}\delta w,_x-\tilde{M}_{xy}\delta w,_y\right)dy+\oint\left(-\tilde{M}_{yy}\delta w,_y-\tilde{M}_{xy}\delta w,_x\right)dx+\oint\tilde{N}^0_{xx}\delta udy+\oint\tilde{N}^0_{xx}z,_x\delta wdy$$

$$+\oint\tilde{N}^0_{xx}w,_x\delta wdy-\oint\tilde{N}^0_{yy}\delta vdx-\oint\tilde{N}^0_{yy}z,_y dx-\oint\tilde{N}^0_{yy}w,_y dx-\oint\tilde{N}^0_{xy}\delta u\,dx+\oint\tilde{N}^0_{xy}\delta v\,dy$$

$$-\oint\tilde{N}^0_{xy}z,_x\delta wdx+\oint\tilde{N}^0_{xy}z,_y\delta wdy-\oint\tilde{N}^0_{xy}w,_x\delta wdx+\oint\tilde{N}^0_{xy}w,_y\delta wdy$$

$$-\iint_{S_m}\left[\tilde{X}\delta x+\tilde{Y}\delta y+\tilde{Z}\delta w\right]dxdy-\oint_C\Big\{R_x\delta u+R_y\,\delta v+R_{xx}\delta u,_x+R_{yy}\delta v,_y+\left(R_{xxy}+R_{yxy}\right)\frac{\left(\delta u,_y+\delta v,_x\right)}{2}+$$

$$R_z\delta w+\overline{M}_\nu\delta w,_\nu+\overline{M}_s\delta w,_s+\overline{m}_{\nu\nu}\delta w,_{\nu\nu}+\overline{m}_{ss}\delta w,_{ss}+\overline{m}_{\nu s}\delta w,_{\nu s}\Big\}ds=0$$

(11)

with, $[\ ]^0=[\ ]_{\zeta=0}$ and

$$\tilde{M}_{ij}=M_{xx}-\frac{\partial m_{xij}}{\partial x}-\frac{\partial m_{yij}}{\partial y}+N^0_{zij}\qquad i,j=1,2$$

$$N_{ij}=N_{xx}-\frac{\partial N_{xij}}{\partial x}-\frac{\partial N_{yij}}{\partial y}$$

Hence, the equilibrium equations are expressed by,

$$\tilde{N}^0_{xx,x}+\tilde{N}^0_{xy,y}+\overline{X}=0$$

$$\tilde{N}^0_{yx,x}+\tilde{N}^0_{yy,y}+\overline{Y}=0$$

$$\frac{\partial^2\tilde{M}_{xx}}{\partial x^2}+\frac{\partial^2\tilde{M}_{yy}}{\partial y^2}+2\frac{\partial^2\tilde{M}_{xy}}{\partial x\partial y}+\frac{\partial}{\partial x}\left[\left(\frac{\partial z}{\partial x}+\frac{\partial w}{\partial x}\right)\tilde{N}^0_{xx}+\left(\frac{\partial z}{\partial y}+\frac{\partial w}{\partial y}\right)\tilde{N}^0_{xy}\right]\qquad(12)$$

$$+\frac{\partial}{\partial y}\left[\left(\frac{\partial z}{\partial y}+\frac{\partial w}{\partial y}\right)\tilde{N}^0_{yy}+\left(\frac{\partial z}{\partial x}+\frac{\partial w}{\partial x}\right)\tilde{N}^0_{xy}\right]+\overline{Z}=0$$

Further, proceeding to the description of the boundary conditions, the following change of the cartesian variables (x,y) to the polar ones (r,s) are defined by the geometrical conditions,

$$\frac{\partial}{\partial x} = l\frac{\partial}{\partial r} - m\frac{\partial}{\partial s}; \qquad \frac{\partial}{\partial y} = m\frac{\partial}{\partial r} + l\frac{\partial}{\partial s} \tag{13}$$

where, (l,m) are the direction cosines of the normal vector $\mathbf{v}$. Likewise,

Let us point out that in the already existing theories of thin plates into the context of strain gradient elasticity, the contribution of the e_{zij} terms does not exist. The present theory includes those terms that are quite important for small thickness when the thickness is comparable to the intrinsic length of the material.

Further, the variation of the strain energy of the plate is defined by,

$$\delta U = \iiint_V \left(\tau_{ij}\delta\varepsilon_{ij} + \mu_{ijk}\delta\varepsilon_{ijk}\right)dv \quad . \tag{8}$$

It is recalled that the stresses and the couple stresses after the replacement of the Lame' constants with the modulus of Elasticity E and Poisson's ratio v become,

$$\begin{aligned} \tau_{ij} &= \frac{E}{1-v^2}\left[v\varepsilon_{kk}\delta_{ij} + (1-v)\varepsilon_{ij}\right] + l_l\frac{E}{1-v^2}\left[v\varepsilon_{lkk}\delta_{ij} + (1-v)\varepsilon_{lij}\right] \\ \mu_{ijk} &= \frac{g^2E}{1-v^2}\left[v\varepsilon_{ikk}\delta_{ij} + (1-v)\varepsilon_{ijk}\right] + l_i\frac{E}{1-v^2}\left[v\varepsilon_{kk}\delta_{jk} + (1-v)\varepsilon_{jk}\right] \end{aligned} \tag{9}$$

and the bending moments and hyper-moments become, see Fig.2,

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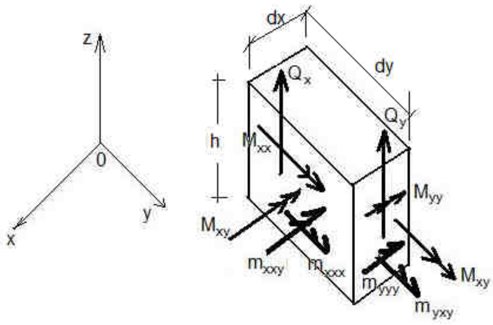

Fig.2 Stress resultants, moments and hyper-moments

$$M_{\alpha\beta} = \int_{-h/2}^{h/2} \tau_{\alpha\beta} z dz$$
$$m_{\alpha\beta\gamma} = \int_{-h/2}^{h/2} \mu_{\alpha\beta\gamma} z dz \qquad \text{for } \alpha,\beta,\gamma = x \text{ or } y \qquad (10a)$$

$$m_{z\alpha\beta} = \int_{-h/2}^{h/2} \mu_{z\alpha\beta} dz \qquad \text{for } \alpha,\beta = x \text{ or } y \qquad (10b)$$

Performing the calculus in Eq.(8) we get,

$$\delta U = -\iint_{S_m} \left(\tilde{M}_{xx}\delta w,_{xx} + \tilde{M}_{yy}\delta w,_{yy} + 2\tilde{M}_{xy}\delta w,_{xy}\right) dxdy + \oint_C m_{xxv}\delta w,_{xx} ds + \oint_C m_{yyv}\delta w,_{yy} ds + \oint_C m_{xyv}\delta w,_{xy} ds \qquad (11)$$

with,

$$\tilde{M}_{xx} = M_{xx} - \frac{\partial m_{xxx}}{\partial x} - \frac{\partial m_{yxx}}{\partial y} + m_{zxx}$$
$$\tilde{M}_{yy} = M_{yy} - \frac{\partial m_{xyy}}{\partial x} - \frac{\partial m_{yyy}}{\partial y} + m_{zyy}$$
$$\tilde{M}_{xy} = M_{xy} - \frac{\partial m_{xxy}}{\partial x} - \frac{\partial m_{yxy}}{\partial y} + m_{zxy} \qquad (12)$$
$$m_{xxv} = m_{xxx} l + m_{yxx} m$$
$$m_{yyv} = m_{xyy} l + m_{yyy} m$$
$$m_{xyv} = m_{xxy} l + m_{yxy} m$$

Further applying again Green's theorem,

$$\iint_{S_m} \left(\tilde{M}_{xx}\delta w,_{xx} + \tilde{M}_{yy}\delta w,_{yy} + 2\tilde{M}_{xy}\delta w,_{xy}\right) dxdy = \int_C \left(\tilde{M}_{xv}\delta w,_x + \tilde{M}_{yv}\delta w,_y\right) ds - \iint_{S_m} \left(\tilde{Q}_x \delta w,_x + \tilde{Q}_y \delta w,_y\right) dxdy \qquad (13)$$

where,

$$\tilde{M}_{xv}=\tilde{M}_{xx}l+\tilde{M}_{xy}m,\qquad \tilde{M}_{yv}=\tilde{M}_{xy}l+\tilde{M}_{yy}m$$
$$\tilde{Q}_x=\frac{\partial\tilde{M}_{xx}}{\partial x}+\frac{\partial\tilde{M}_{xy}}{\partial y},\qquad \tilde{Q}_y=\frac{\partial\tilde{M}_{yx}}{\partial x}+\frac{\partial\tilde{M}_{yy}}{\partial y}\quad . \tag{14}$$

Further, we recall the geometrical conditions,

$$\frac{\partial}{\partial x}=l\frac{\partial}{\partial v}-m\frac{\partial}{\partial s},\qquad \frac{\partial}{\partial y}=l\frac{\partial}{\partial v}+m\frac{\partial}{\partial s}$$
$$\frac{\partial^2}{\partial x^2}=\left(l\frac{\partial}{\partial v}-m\frac{\partial}{\partial s}\right)^2=l^2\frac{\partial^2}{\partial v^2}+m^2\frac{\partial^2}{\partial s^2}-2lm\frac{\partial^2}{\partial v\partial s}$$
$$\frac{\partial^2}{\partial y^2}=\left(l\frac{\partial}{\partial v}+m\frac{\partial}{\partial s}\right)^2=l^2\frac{\partial^2}{\partial v^2}+m^2\frac{\partial^2}{\partial s^2}+2lm\frac{\partial^2}{\partial v\partial s} \tag{15}$$
$$\frac{\partial^2}{\partial x\partial y}=\left(l\frac{\partial}{\partial v}-m\frac{\partial}{\partial s}\right)\left(l\frac{\partial}{\partial v}+m\frac{\partial}{\partial s}\right)=l^2\frac{\partial^2}{\partial v^2}-m^2\frac{\partial^2}{\partial s^2}$$

Moreover, the variation of the work of the external forces is,

$$\delta E=\iint_{S_m}\bar{p}\,\delta w\,dxdy+\iint_{S_1}\bar{F}_z\,\delta w\,dsdz-\oint_C\bar{M}_{vv}\delta w,_v\,ds-\oint_C\bar{M}_{vs}\delta w,_s\,ds-\oint_C\bar{m}_{vv}\delta w,_{vv}\,ds-$$
$$\oint_C\bar{m}_{ss}\delta w,_{ss}\,ds-\oint_C\bar{m}_{vs}\delta w,_{vs}\,ds\quad . \tag{16}$$

The meaning of various external loading is shown on Fig.3. According to the principle of virtual work,

$$\delta V=\delta U-\delta E=0\quad . \tag{17}$$

Hence, Eq.(17) yields,

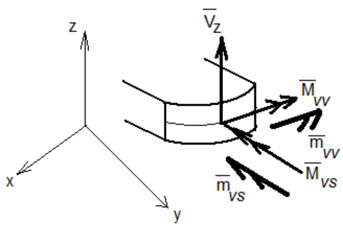

Fig.3 Forces, moments and hyper-moments on the boundary

$$-\iint_{S_m}\left(\tilde{Q}_{x,x}+\tilde{Q}_{y,y}+\bar{p}\right)\delta w\,dxdy+\oint_C\left[\left(V_z-\bar{V}_z\right)\delta w-\left(\tilde{M}_{vv}-\bar{M}_{vv}\right)\delta w,_v-\left(\tilde{M}_{vs}-\bar{M}_{vs}\right)\delta w,_s\right]ds-$$
$$\oint_C\left[\left(m_{vv}-\bar{m}_{vv}\right)\delta w,_{vv}+\left(m_{ss}-\bar{m}_{ss}\right)\delta w,_{ss}+\left(m_{vs}-\bar{m}_{vs}\right)\delta w,_{vs}\right]ds=0 \tag{18}$$

with

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$$\overline{V}_z = \int_{-h/2}^{h/2} \overline{F}_z dz,$$
$$\tilde{M}_{vv} = \tilde{M}_{xv} l + \tilde{M}_{yv} m = \tilde{M}_{xx} l^2 + \tilde{M}_{yy} m^2 + 2lm\tilde{M}_{xy}$$
$$\tilde{M}_{vs} = -\tilde{M}_{xv} m + \tilde{M}_{yv} l = -\left(\tilde{M}_{xx} - \tilde{M}_{yy}\right) lm + \tilde{M}_{xy}\left(l^2 - m^2\right)$$
$$m_{vv} = \left(m_{xxv} + m_{yyv} + m_{xyv}\right) l^2 \quad , \tag{19}$$
$$m_{ss} = \left(m_{xxv} + m_{yyv} - m_{xyv}\right) m^2$$
$$m_{vs} = 2\left(-m_{xxv} + m_{yyv}\right) lm$$
$$V_z = \tilde{Q}_x l + \tilde{Q}_y m$$

The variational equation (18) yields the governing equilibrium equation,

$$\tilde{Q}_{x,x} + \tilde{Q}_{y,y} + \overline{p} = 0 \tag{20}$$

And the corresponding boundary conditions,

$$V_z = \overline{V}_z \quad \text{or} \quad \delta w = 0 \quad \text{on the boundary } C \tag{21a}$$
$$\tilde{M}_{vv} = \overline{M}_{vv} \quad \text{or} \quad \delta w,_v = 0 \quad \text{on the boundary } C \tag{21b}$$
$$\tilde{M}_{vs} = \overline{M}_{vs} \quad \text{or} \quad \delta w,_s = 0 \quad \text{on the boundary } C \tag{21c}$$
$$m_{vv} = \overline{m}_{vv} \quad \text{or} \quad \delta w,_{vv} = 0 \quad \text{on the boundary } C \tag{21d}$$
$$m_{ss} = \overline{m}_{ss} \quad \text{or} \quad \delta w,_{ss} = 0 \quad \text{on the boundary } C \tag{21e}$$
$$m_{vs} = \overline{m}_{vs} \quad \text{or} \quad \delta w,_{vs} = 0 \quad \text{on the boundary } C \tag{21f}$$

Performing the algebra, the equilibrium equation (20) becomes,

$$\left(1 + g^2 \frac{12}{h^2}\right) D\nabla^4 w - g^2 D\nabla^6 w = p \tag{22}$$

Where, the flexural rigidity D of the plate is given by,

$$D = Eh^3/12\left(1 - v^2\right) \quad . \tag{23}$$

The corresponding classical boundary conditions are expressed by,

$$\overline{V}_z = l\left\{-D\frac{\partial\left(\nabla^2 w\right)}{\partial x} + g^2 D\frac{\partial\left(\nabla^4 w\right)}{\partial x} - g^2\frac{12D}{h^2}\left(w,_{xxx} + vw,_{xyy}\right)\right\} +$$
$$m\left\{-D\frac{\partial\left(\nabla^2 w\right)}{\partial y} + g^2 D\frac{\partial\left(\nabla^4 w\right)}{\partial y} - g^2\frac{12D}{h^2}\left(vw,_{xxy} + w,_{yyy}\right)\right\}$$

or $\delta w = 0$ on the boundary C . (24)

$$\overline{M}_{vv} = l^2 \begin{Bmatrix} -D\left(w,_{xx} + \nu w,_{yy}\right) + g^2 D\left(w,_{xxxx} + \nu w,_{yyyy} + (1+v) w,_{xxyy}\right) - \\ g^2 (12D/h^2)\left(w,_{xx} + \nu w,_{yy}\right) \end{Bmatrix} +$$

$$m^2 \begin{Bmatrix} -D\left(v w,_{xx} + w,_{yy}\right) + g^2 D\left(v w,_{xxxx} + w,_{yyyy} + (1+v) w,_{xxyy}\right) - \\ g^2 (12D/h^2)\left(v w,_{xx} + w,_{yy}\right) \end{Bmatrix} +$$

$$2lm\left\{-D(1-v)w,_{xy} + g^2 D(1-v)\left(w,_{xxxy} + w,_{xyyy}\right)\right\}$$

or $\delta w,_v = 0$ on the boundary C. (25)

$$\overline{M}_{vs} = -lm \begin{Bmatrix} \begin{pmatrix} -D\left(w,_{xx} + \nu w,_{yy}\right) + g^2 D\left(w,_{xxxx} + \nu w,_{yyyy} + (1+v) w,_{xxyy}\right) - \\ g^2 (12D/h^2)\left(w,_{xx} + \nu w,_{yy}\right) \end{pmatrix} - \\ \begin{pmatrix} -D\left(v w,_{xx} + w,_{yy}\right) + g^2 D\left(v w,_{xxxx} + w,_{yyyy} + (1+v) w,_{xxyy}\right) - \\ g^2 (12D/h^2)\left(v w,_{xx} + w,_{yy}\right) \end{pmatrix} \end{Bmatrix} +$$

$$\left(l^2 - m^2\right)\left\{-D(1-v)w,_{xy} + g^2 D(1-v)\left(w,_{xxxy} + w,_{xyyy}\right)\right\}$$

or $\delta w,_s = 0$ on the boundary C. (26)

Likewise, the non-classical boundary conditions are,

$$\overline{m}_{vv} = l^2 \begin{Bmatrix} \left[-g^2 D\left(w,_{xxx} + \nu w,_{xyy}\right) - l_x D\left(w,_{xx} + \nu w,_{yy}\right)\right] l + \\ \left[-g^2 D\left(w,_{yxx} + \nu w,_{yyy}\right) - l_y D\left(w,_{xx} + \nu w,_{yy}\right)\right] m + \\ \left[-g^2 D\left(w,_{xyy} + \nu w,_{xxx}\right) - l_x D\left(w,_{yy} + \nu w,_{xx}\right)\right] l + \\ \left[-g^2 D\left(w,_{yyy} + \nu w,_{yxx}\right) - l_y D\left(w,_{yy} + \nu w,_{xx}\right)\right] m + \\ \left[-g^2 D(1-v) w,_{xxy} - l_x D(1-v) w,_{xy}\right] l + \\ \left[-g^2 D(1-v) w,_{yxy} - l_y D(1-v) w,_{xy}\right] m \end{Bmatrix} \tag{27}$$

or $\delta w,_{vv} = 0$ on the boundary C.

$$\overline{m}_{ss} = m^2 \begin{Bmatrix} \left[-g^2 D\left(w,_{xxx} + \nu w,_{xyy}\right) - l_x D\left(w,_{xx} + \nu w,_{yy}\right)\right] l + \\ \left[-g^2 D\left(w,_{yxx} + \nu w,_{yyy}\right) - l_y D\left(w,_{xx} + \nu w,_{yy}\right)\right] m + \\ \left[-g^2 D\left(w,_{xyy} + \nu w,_{xxx}\right) - l_x D\left(w,_{yy} + \nu w,_{xx}\right)\right] l + \\ \left[-g^2 D\left(w,_{yyy} + \nu w,_{yxx}\right) - l_y D\left(w,_{yy} + \nu w,_{xx}\right)\right] m - \\ \left[-g^2 D(1-v) w,_{xxy} - l_x D(1-v) w,_{xy}\right] l - \\ \left[-g^2 D(1-v) w,_{yxy} - l_y D(1-v) w,_{xy}\right] m \end{Bmatrix} \tag{28}$$

or $\delta w,_{ss} = 0$ on the boundary C.

$$\overline{m}_{vs} = 2lm \left\{ \begin{array}{l} -\left[-g^2 D\left(w,_{xxx}+\nu w,_{xyy}\right) - l_x D\left(w,_{xx}+\nu w,_{yy}\right)\right] l - \\ \qquad \left[-g^2 D\left(w,_{yxx}+\nu w,_{yyy}\right) - l_y D\left(w,_{xx}+\nu w,_{yy}\right)\right] m + \\ \left[-g^2 D\left(w,_{xyy}+\nu w,_{xxx}\right) - l_x D\left(w,_{yy}+\nu w,_{xx}\right)\right] l + \\ \qquad \left[-g^2 D\left(w,_{yyy}+\nu w,_{yxx}\right) - l_y D\left(w,_{yy}+\nu w,_{xx}\right)\right] m \end{array} \right\} \tag{29}$$

or $\delta w,_{vs} = 0$ on the boundary *C*.

3. The simply supported rectangular plate.

Consider a simply supported rectangular plate with sides *a* and *b* along the *x* and *y* directions, subjected to lateral distributed load *p*. The present example is similar to the one presented by Papargyri et al [2008]. The classical boundary conditions are,

$w=0, \quad M_{xx}=0$, at $x=0, a$ (30)

$w=0, \quad M_{yy}=0$, at $y=0, b$

Further, the non-classical boundary conditions, Eqs.(21d-f) are defined by ,

$w_{yy}=0$, at $x=0,a$ (31)

$w_{xx}=0$, at $y=0,b$

Finaly the boundary conditions are defined in the compact form by,

$w=0, \quad w_{xx}=w_{yy}=0, \quad$ at $x=0,a$ and $y=0,b$. (32)

Assuming that,

$$w(x, y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b} \tag{33}$$

And expressing the transversal load *p* in a similar form, through Fourier series, we get

$$p(x, y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} p_{mn} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b} \tag{34}$$

and

$$A_{mn} = \frac{p_{mn}}{D\left[\left(1+12(g/h)^2\right)\left(\frac{m^2\pi^2}{a^2}+\frac{n^2\pi^2}{b^2}\right)^2 + g^2\left(\frac{m^2\pi^2}{a^2}+\frac{n^2\pi^2}{b^2}\right)^3\right]} \tag{35}$$

where, *h* is the thickness of the plate. The classical case with *g=0* is given by,

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$$A_{mn}^{c} = \frac{p_{mn}}{D\left(\frac{m^2\pi^2}{a^2} + \frac{n^2\pi^2}{b^2}\right)^2} \quad . \tag{36}$$

For the square plate, we get,

$$\frac{A_{mn}}{A_{mn}^{c}} = \left[\frac{\left(m^2+n^2\right)^2}{\left[\left(1+12(g/h)^2\right)\left(m^2+n^2\right)^2 + \pi^2 (g/a)^2 \left(m^2+n^2\right)^3\right]}\right] \quad . \tag{37}$$

Further, for the case of the first mode, with $m=n=1$, the relative displacement w with respect to the classical one w^c is defined by,

$$\frac{w}{w^c} = \frac{1}{1+12(g/h)^2 + 2\pi^2 (g/a)^2} \tag{38}$$

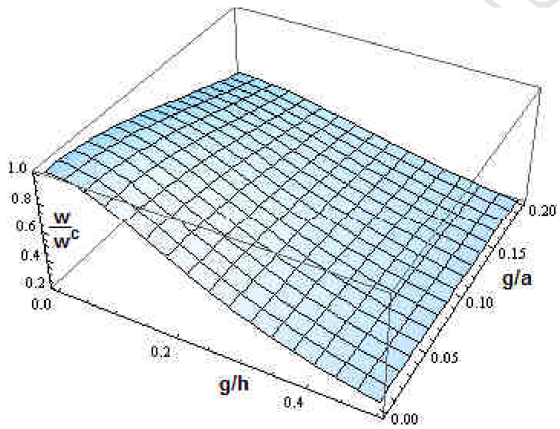

Fig.4. Variation of the relative plate displacement with respect to g/h and g/a

Figure 4. exhibits the variation of the displacement w over the classical displacement w^c when the intrinsic length g, the thickness h and the length of the side of the square plate vary. The term including g/h is new and it is introduced here. It is missing from the work of Papargyri et al [2008]. Nevertheless, its influence is high for small thicknesses, when the intrinsic length *g* is comparable to the thickness.

4.Conclusion-further research

Theory of thin strain gradient elastic plates is presented including new terms involving only the thickness (area of the cross-section) and not the moment of inertia of the cross-section. Those terms are important for thin plates because exhibit high increase of the stiffness of the plates. The present theory might be the basis for the study of behavior of thin films, thin shells and generally the stability of thin structures.

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Appendix

Computation of the in-plane stresses.

$$\begin{aligned}
\tau_{xx} &= -\frac{Ez}{1-\nu^2}\left(w,_{xx}+\nu w,_{yy}\right)-l_x\frac{Ez}{1-\nu^2}\left(w,_{xxx}+\nu w,_{xyy}\right)-l_y\frac{Ez}{1-\nu^2}\left(w,_{yxx}+\nu w,_{yyy}\right)\\
\tau_{yy} &= -\frac{Ez}{1-\nu^2}\left(w,_{yy}+\nu w,_{xx}\right)-l_x\frac{Ez}{1-\nu^2}\left(w,_{xyy}+\nu w,_{xxx}\right)-l_y\frac{Ez}{1-\nu^2}\left(w,_{yyy}+\nu w,_{yxx}\right)\\
\tau_{xy} &= -\frac{Ez}{1+\nu}w,_{xy}-l_x\frac{Ez}{1+\nu}w,_{xxy}-l_y\frac{Ez}{1+\nu}w,_{yxy}
\end{aligned} \tag{A1}$$

Computation of the hyper-stresses.

$$\begin{aligned}
\mu_{xxx} &= -g^2\frac{Ez}{1-\nu^2}\left(w,_{xxx}+\nu w,_{xyy}\right)-l_x\frac{Ez}{1-\nu^2}\left(w,_{xx}+\nu w,_{yy}\right)\\
\mu_{yxx} &= -g^2\frac{Ez}{1-\nu^2}\left(w,_{yxx}+\nu w,_{yyy}\right)-l_y\frac{Ez}{1-\nu^2}\left(w,_{xx}+\nu w,_{yy}\right)\\
\mu_{zxx} &= -g^2\frac{E}{1-\nu^2}\left(w,_{xx}+\nu w,_{yy}\right)\\
\mu_{xyy} &= -g^2\frac{Ez}{1-\nu^2}\left(w,_{xyy}+\nu w,_{xxx}\right)-l_x\frac{Ez}{1-\nu^2}\left(w,_{yy}+\nu w,_{xx}\right)\\
\mu_{yyy} &= -g^2\frac{Ez}{1-\nu^2}\left(w,_{yyy}+\nu w,_{yxx}\right)-l_y\frac{Ez}{1-\nu^2}\left(w,_{yy}+\nu w,_{xx}\right)\\
\mu_{zyy} &= -g^2\frac{E}{1-\nu^2}\left(w,_{yy}+\nu w,_{xx}\right)
\end{aligned} \tag{A2}$$

Computation of the moments and hyper-moments

$$
\begin{aligned}
M_{xx} &= -D\left(w,_{xx}+\nu w,_{yy}\right)-l_x D\left(w,_{xxx}+\nu w,_{xyy}\right)-l_y D\left(w,_{yxx}+\nu w,_{yyy}\right)\\
M_{yy} &= -D\left(w,_{yy}+\nu w,_{xx}\right)-l_x D\left(w,_{xyy}+\nu w,_{xxx}\right)-l_y D\left(w,_{yyy}+\nu w,_{yxx}\right)\\
M_{xy} &= -D(1-\nu)w,_{xy}-l_x D(1-\nu)w,_{xxy}-l_y D(1-\nu)w,_{yxy}\\
m_{xxx} &= -g^2 D\left(w,_{xxx}+\nu w,_{xyy}\right)-l_x D\left(w,_{xx}+\nu w,_{yy}\right)\\
m_{yxx} &= -g^2 D\left(w,_{yxx}+\nu w,_{yyy}\right)-l_y D\left(w,_{xx}+\nu w,_{yy}\right)\\
m_{zxx} &= -g^2 (12D/h^2)\left(w,_{xx}+\nu w,_{yy}\right)\\
m_{xyy} &= -g^2 D\left(w,_{xyy}+\nu w,_{xxx}\right)-l_x D\left(w,_{yy}+\nu w,_{xx}\right)\\
m_{yyy} &= -g^2 D\left(w,_{yyy}+\nu w,_{yxx}\right)-l_y D\left(w,_{yy}+\nu w,_{xx}\right)\\
m_{zyy} &= -g^2 \frac{12D}{h^2}\left(w,_{yy}+\nu w,_{xx}\right)\\
m_{axy} &= -g^2 D(1-\nu)w,_{axy}-l_a D(1-\nu)w,_{xy} \qquad a = x, y\\
m_{zxy} &= -g^2 \frac{12D(1-\nu)}{h^2} w,_{xy}\\
m_{zxy} &= -g^2 \frac{12D(1-\nu)}{h^2} w,_{xy}
\end{aligned}
\tag{A3}
$$

$$
\begin{aligned}
\tilde{M}_{xx} &= -D\left(w,_{xx}+\nu w,_{yy}\right)+g^2 D\left(w,_{xxxx}+\nu w,_{yyyy}+(1+\nu)w,_{xxyy}\right)-\\
&\quad g^2(12D/h^2)\left(w,_{xx}+\nu w,_{yy}\right)\\
\tilde{M}_{yy} &= -D\left(\nu w,_{xx}+w,_{yy}\right)+g^2 D\left(\nu w,_{xxxx}+w,_{yyyy}+(1+\nu)w,_{xxyy}\right)-\\
&\quad g^2(12D/h^2)\left(\nu w,_{xx}+w,_{yy}\right)\\
\tilde{M}_{xy} &= -D(1-\nu)w,_{xy}+g^2 D(1-\nu)\left(w,_{xxxy}+w,_{xyyy}\right)-g^2\frac{12D(1-\nu)}{h^2}w,_{xy}
\end{aligned}
\tag{A4}
$$

$$
\begin{aligned}
\tilde{M}_{xv} &= l\left\{\begin{array}{c}-D\left(w,_{xx}+\nu w,_{yy}\right)+g^2 D\left(w,_{xxxx}+\nu w,_{yyyy}+(1+\nu)w,_{xxyy}\right)-\\ g^2(12D/h^2)\left(w,_{xx}+\nu w,_{yy}\right)\end{array}\right\}+\\
&\quad m\left\{-D(1-\nu)w,_{xy}+g^2 D(1-\nu)\left(w,_{xxxy}+w,_{xyyy}\right)-g^2\frac{12D(1-\nu)}{h^2}w,_{xy}\right\}\\
\tilde{M}_{yv} &= l\left\{-D(1-\nu)w,_{xy}+g^2 D(1-\nu)\left(w,_{xxxy}+w,_{xyyy}\right)-g^2\frac{12D(1-\nu)}{h^2}w,_{xy}\right\}+\\
&\quad m\left\{\begin{array}{c}-D\left(\nu w,_{xx}+w,_{yy}\right)+g^2 D\left(\nu w,_{xxxx}+w,_{yyyy}+(1+\nu)w,_{xxyy}\right)-\\ g^2(12D/h^2)\left(\nu w,_{xx}+w,_{yy}\right)\end{array}\right\}
\end{aligned}
\tag{A5}
$$

Computation of shear forces

$$\tilde{Q}_x = -D\frac{\partial\left(\nabla^2 w\right)}{\partial x} + g^2 D\frac{\partial\left(\nabla^4 w\right)}{\partial x} - g^2\frac{12D}{h^2}\frac{\partial\left(\nabla^2 w\right)}{\partial x} \tag{A6}$$

$$\tilde{Q}_y = -D\frac{\partial\left(\nabla^2 w\right)}{\partial y} + g^2 D\frac{\partial\left(\nabla^4 w\right)}{\partial y} - g^2\frac{12D}{h^2}\frac{\partial\left(\nabla^2 w\right)}{\partial y}$$

Computation of hypermoments along the boundary

$$\begin{aligned} m_{xxv} = {} & \left[-g^2 D\left(w,_{xxx}+\nu w,_{xyy}\right) - l_x D\left(w,_{xx}+\nu w,_{yy}\right)\right]l + \\ & \left[-g^2 D\left(w,_{yxx}+\nu w,_{yyy}\right) - l_y D\left(w,_{xx}+\nu w,_{yy}\right)\right]m \end{aligned} \quad \text{A(7)}$$

$$\begin{aligned} m_{yyv} = {} & \left[-g^2 D\left(w,_{xyy}+\nu w,_{xxx}\right) - l_x D\left(w,_{yy}+\nu w,_{xx}\right)\right]l + \\ & \left[-g^2 D\left(w,_{yyy}+\nu w,_{yxx}\right) - l_y D\left(w,_{yy}+\nu w,_{xx}\right)\right]m \end{aligned}$$

$$\begin{aligned} m_{xyv} = {} & \left[-g^2 D(1-\nu)w,_{xxy} - l_x D(1-\nu)w,_{xy}\right]l + \\ & \left[-g^2 D(1-\nu)w,_{yxy} - l_y D(1-\nu)w,_{xy}\right]m \end{aligned}$$

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