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Nonlinear System Modelling: How to Estimate the Highest Significant Order

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Abstract ***– A new method for estimating the highest significant order of nonlinearity of Volterra type systems is presented. The method is based on the use of multisine signals and the possibility of testing the system at three different amplitudes. The performance of the proposed method is demonstrated in simulation and it is shown that it is possible to estimate the highest order of nonlinearity of Volterra type systems very accurately. The method can be used to provide essential prior knowledge about the nonlinearity and thus aid the accurate representation of the system under test.***

Keywords ***– nonlinear systems, Volterra series, NARMAX models, order of nonlinearity, frequency domain, frequency response, multisine signals.***

I. INTRODUCTION

The Volterra functional series representation constitutes a useful way of representing a nonlinear system since it can be seen as a natural extension of linear system theory. The Volterra kernels have a direct physical significance and can often be given physical interpretation, or be related to the system's constituent elements [1]. Efficient methods for measuring these Volterra kernels have been previously developed, for example [2-5], and the success of these methods depends on the assumption that the highest significant order of nonlinearity of the system under test is known. In [6] practical algorithms for determining the highest significant order of nonlinear systems were presented. These algorithms are based on a number of measurements equal to 2^n to determine if the *nth*-order nonlinear term is significant or not for a given signal amplitude. The number of measurements thus increases with the order of nonlinearity and can become unreasonably large for a higher order nonlinearity. In this paper a simple frequency-domain method is developed to estimate the highest significant order of nonlinearity based on only three tests at different amplitudes. The proposed technique is based on the use of multisine signals and the ability to test the system at three different amplitudes. A complete analysis of the technique is presented followed by computer simulations and an experimental illustration on a practical system.

II. PROBLEM FORMULATION

II. 1. Volterra series

An analytic response function can be represented by an infinite series called the Volterra series. This is a generalisation of the impulse response function of linear systems and is composed of the convolution integral

$$y(t) = \int_{-\infty}^{\infty} h(\tau) \cdot u(t-\tau) d\tau \qquad (1)$$

and a static nonlinearity represented by a Taylor series.

$$y = a_0 + a_1 u + a_2 u^2 + a_3 u^3 + ... = \sum_{n=0}^{\infty} a_n u^n \qquad (2)$$

The Volterra series is then given by

$$y(t) = h_0 + \sum_{n=1}^{\infty} \int_{-\infty}^{\infty} ... \int_{-\infty}^{\infty} h_n(\tau_1, \tau_2, ...\tau_n) \prod_{i=1}^{n} u(t-\tau_i) d\tau_i \qquad (3)$$

which represents a sum of outputs of parallel sub-systems called Volterra functionals illustrated graphically in the schematic diagram in Figure 1.

Nonlinear system identification based on the Volterra representation requires the measurement of the kernels $h_n(\tau_1, \tau_2, ...\tau_n)$. Several approaches exist in the literature for the estimation of these parameters, the most common based on the extension of correlation methods for linear systems and the use of white Gaussian signals. The complexity of this model depends on the highest significant order of nonlinearity *n*, which is dependent on the dynamic range of the input signal. For example, for small signal amplitudes higher-order responses can be neglected and only the lower-order responses are considered as dominant. For sufficiently large amplitudes however, the contribution of the responses generated by higher-order nonlinearities is more significant and they should be taken into account. It is thus necessary to determine the highest significant order of nonlinearity relative to the range of input signal amplitudes that will be applied to the system. The implications to the accuracy of the estimation of the Volterra Kernels is also a major factor for the correct estimation of the highest significant order of nonlinearity as pointed out in [7]. If the order estimate is too small, the resulting Volterra kernels will be highly inaccurate. On the other hand if the order is overestimated an excessive number of measurements will be required which will make the whole process time-consuming.

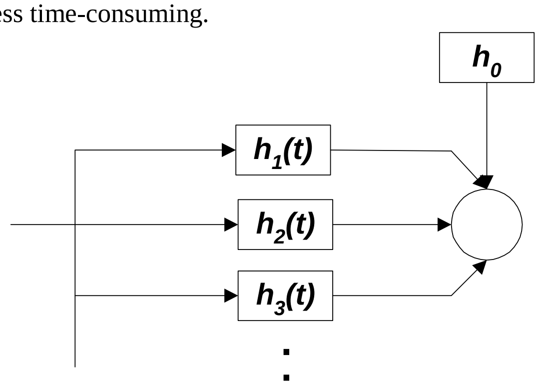

Figure 1. Graphical representation of the Volterra series.

II.2. Narmax Models

The Nonlinear AutoRegressive Moving Average with eXogenous inputs (NARMAX) approach was introduced by Leontaritis and Billings [8, 9] and Chen and Billings [10] as a means of describing the input-output relationship of a nonlinear system. The model represents the extension of the well-known ARMAX model to the nonlinear case, and is defined as

$$y(k) = F\begin{bmatrix} y(k-1),...,y(k-n_y),u(k-1),...,u(k-n_u),... \\ e(k-1),...,e(k-n_e) \end{bmatrix} + e(k) \quad (4)$$

where F is a nonlinear function; $y(k)$, $u(k)$ and $e(k)$ represent the output, input and noise signals respectively; and n_y, n_u, and n_e are their associate maximum lags. The NARMAX representation constitutes a powerful tool for nonlinear modelling and it includes a family of other nonlinear representations such as block-structured models and Volterra series [11]. The most involved task in NARMAX modelling is to select the appropriate regressors (i.e. model terms) to build the model structure. The number of candidate regressors l for a given order of nonlinearity n is given by equation (5), where it can be seen that it increases rapidly with the increase of the order of nonlinearity and maximum input, output and noise lags.

$$l = \sum_{i=1}^{n} [l_{i-1}(n_y + n_u + n_e + i)]/i \quad \text{with} \quad l_0 = n_y + n_u + n_e + 1 \quad (5)$$

Having in mind that the corresponding number of models to choose from is given by $M = 2^l$, the assessment of each possible model is not practical, reinforcing the need to use a structure selection technique to select the most appropriate regressors for inclusion in the model. All structure selection techniques examine a large number of model terms and utilise certain criteria for their inclusion or removal from the model. The whole procedure can be considerably simplified if *a priori* knowledge is utilised, such as knowledge about the highest significant order *n*. This means that the search space of candidate models can be significantly narrowed, and structure selection algorithms can be facilitated with greater confidence, since the possibility of having spurious components entering into the model structure will be minimised. Therefore, it is apparent that knowledge of *n* provides significant advantages since the identification procedure can be greatly simplified, the accuracy of the estimated model can be preserved and thus the estimation time is reduced. This provided the motivation for the development of an analytical technique to estimate the highest significant order of the nonlinearity for Volterra models. The proposed technique can also be applied in NARMAX modelling since the Volterra representation belongs to the NARMAX family.

III. ORDER TEST ALGORITHM

The proposed method is based on Frequency Response Function (FRF) measurements at three different amplitudes. The use of periodic signals is essential in this case since systematic errors arising from FFT leakage problems can be avoided and the signal-to-noise ratios of the data records can be improved by averaging over a number of periods [12]. The use of periodic signals also allows the direct estimation of the FRF as the ratio of the mean values of the output and input coefficients, at the discrete test frequencies ω_k

$$\hat{H}(j\omega_k) = \frac{\frac{1}{M}\sum_{m=1}^{M} Y_m(j\omega_k)}{\frac{1}{M}\sum_{m=1}^{M} U_m(j\omega_k)} = \frac{\overline{Y}(j\omega_k)}{\overline{U}(j\omega_k)} \quad (6)$$

where M is the number of measured periods. The periodic signal used throughout this study is the multisine signal which is a sum of an arbitrary ensemble of harmonically related cosines

$$u(t) = \sum_{k=1}^{F} A(k)\cos(i(k)\omega_0 t + \phi(k)) \tag{7}$$

where **A** is a vector of amplitudes **i** a vector of harmonic numbers and ω_0 the signal funtamental, Φ a vector of phases and F the number of cosines in the signal. The relative phases of the harmonics must be carefully selected in order to minimise the signal crest functor. The lowest crest factor achieved to date is by the l_∞ method proposed in [13]. An example of a ten harmonic multisine signal with a fundamental frequency of 0.05Hz and a harmonic vector **i**=1, 2, …10, is shown in Figure 2.

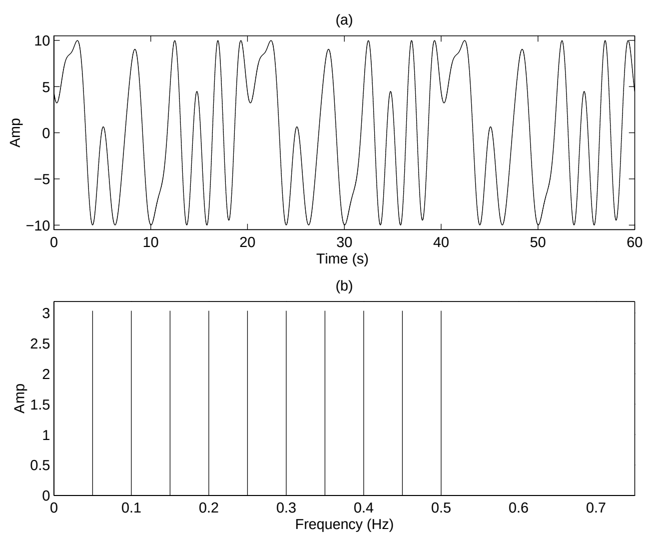

Figure 2. Multisine signal in (a) time domain and (b) frequency domain.

In order to present the proposed methodology in a clear and understandable way the algorithm will be derived using a two-tone consecutive multisine given by

$$u = dc + A_1\cos(\omega_0 t + \phi_1) + A_2\cos(2\omega_0 t + \phi_2)\ . \tag{8}$$

This is used to excite a static nonlinear system containing a single quadratic nonlinearity as shown in Figure 3.

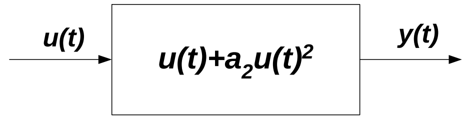

Figure 3. Static quadratic system.

The output $y(t)$ of the system is given by:

$$\begin{aligned} y(t) &= dc - a_2 dc^2 - \frac{(A_1^2 + A_2^2)}{2} a_2 + A_1\cos(\omega_0 t + \phi_1)\ldots \\ &- A_1 A_2 a_2 \cos(\omega_0 t + \phi_2 - \phi_1) - 2dcA_1 a_2\cos(\omega_0 t + \phi_1)\ldots \\ &+ A_2\cos(2\omega_0 t + \phi_2) - \frac{A_1^2}{2} a_2\cos(2\omega_0 t + 2\phi_1)\ldots \\ &- 2dcA_2 a_2\cos(2\omega_0 t + \phi_2) - A_1 A_2 a_2\cos(3\omega_0 t + \phi_2 + \phi_1)\ldots \\ &- \frac{A_2^2}{2} a_2\cos(4\omega_0 t + 2\phi_2) \end{aligned} \tag{9}$$

From the above equation it is clear that the complex amplitudes at the two test frequencies are given by:

$$\begin{aligned} Y_1(j\omega_0) &= A_1(1-2dc)\angle\phi_1 - A_1 A_2 a_2\angle(\phi_2 - \phi_1) \\ Y_1(2j\omega_0) &= A_2(1-2dc)\angle\phi_2 - \frac{A_1^2}{2} a_2\angle(2\phi_1) \end{aligned} \tag{10}$$

If the system is tested using an input signal $K_1\times u(t)$ where K_1 is a constant then the complex amplitudes at the two output frequencies will be given by:

$$\begin{aligned} Y_2(j\omega_0) &= K_1 A_1(1-2dc)\angle\phi_1 - K_1^2 A_1 A_2 a_2\angle(\phi_2 - \phi_1) \\ Y_2(2j\omega_0) &= K_1 A_2(1-2dc)\angle\phi_2 - \frac{K_1^2 A_1^2}{2} a_2\angle(2\phi_1) \end{aligned} \tag{11}$$

Finally, the system is tested using a signal $K_1\times K_2\times u(t)$ where K_2 is another constant. It follows that,

$$\begin{aligned} Y_3(j\omega_0) &= K_1 K_2 A_1(1-2dc)\angle\phi_1 - K_1^2 K_2^2 A_1 A_2 a_2\angle(\phi_2 - \phi_1) \\ Y_3(2j\omega_0) &= K_1 K_2 A_2(1-2dc)\angle\phi_2 - \frac{K_1^2 K_2^2 A_1^2}{2} a_2\angle(2\phi_1) \end{aligned} \tag{12}$$

The corresponding values of the FRFs at the test frequencies are thus given by:

$$\begin{aligned} H_1(j\omega) &= 1 - 2dc - \frac{A_2 a_2\angle(\phi_2 - \phi_1)}{\angle\phi_1} \\ H_1(2j\omega) &= 1 - 2dc - \frac{A_1^2 a_2\angle(2\phi_1)}{2A_2\angle\phi_2} \\ H_2(j\omega) &= 1 - 2dc - \frac{A_2 K_1 a_2\angle(\phi_2 - \phi_1)}{\angle\phi_1} \\ H_2(2j\omega) &= 1 - 2dc - \frac{A_1^2 K_1 a_2\angle(2\phi_1)}{2A_2\angle\phi_2} \\ H_3(j\omega) &= 1 - 2dc - \frac{A_2 K_1 K_2 a_2\angle(\phi_2 - \phi_1)}{\angle\phi_1} \\ H_3(2j\omega) &= 1 - 2dc - \frac{A_1^2 K_1 K_2 a_2\angle(2\phi_1)}{2A_2\angle\phi_2} \end{aligned} \tag{13}$$

The following index can then be calculated

$$r=\left|\frac{\left(H_3(j\omega)-H_1(j\omega)\right)+\left(H_3(2j\omega)-H_1(2j\omega)\right)}{\left(H_2(j\omega)-H_1(j\omega)\right)+\left(H_2(2j\omega)-H_1(2j\omega)\right)}\right|\Rightarrow$$

$$r=\frac{A_2K_1K_2a_2-A_2a_2+\dfrac{A_1{}^2K_1K_2a_2}{2A_2}-\dfrac{A_1{}^2a_2}{2A_2}}{A_2K_1a_2-A_2a_2+\dfrac{A_1{}^2K_1a_2}{2A_2}-\dfrac{A_1{}^2a_2}{2A_2}}\Rightarrow \tag{14}$$

$$r=\frac{a_2(\dfrac{A_1{}^2}{2A_2}+A_2)(K_1K_2-1)}{a_2(\dfrac{A_1{}^2}{2A_2}+A_2)(K_1-1)}=\frac{(K_1K_2-1)}{(K_1-1)}$$

If the same procedure is followed and r is calculated for a system with a single 3rd order nonlinearity then the previous index is given by:

$$r=\frac{(K_1{}^2K_2{}^2-1)}{(K_1{}^2-1)} \tag{15}$$

and if r is calculated for a system with a single 4th order nonlinearity then the index is given by:

$$r=\frac{(K_1{}^3K_2{}^3-1)}{(K_1{}^3-1)} \tag{16}$$

A general formula relating this index with the maximum order of nonlinearity n can thus be derived as

$$r=\frac{(K_1{}^{n-1}K_2{}^{n-1}-1)}{(K_1{}^{n-1}-1)} \tag{17}$$

and n can be calculated since it is the only unknown. It must be stressed here that the above equation is strictly valid for systems with a single nonlinearity. In practice though this is not usually the case since systems will contain other orders of nonlinearity as shown by the generalised static nonlinear system in Figure 4.

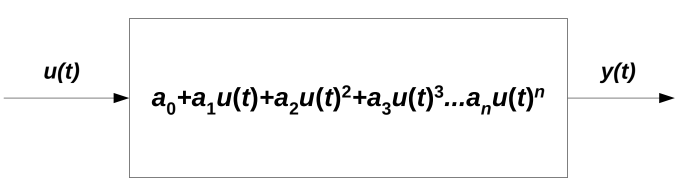

Figure 4. General static nonlinear system.

In this case equation (17) is no strictly valid since the simplifications that occurred in (14) do not take place. The new formula for the index r is a function of the frequency amplitudes A, and the coefficients a_n

$$r=\frac{a_2(K_1K_2-1)f_1(A)+a_3(K_1{}^2K_2{}^2-1)f_2(A)^2+\ldots a_n(K_1{}^{n-1}K_2{}^{n-1}-1)f_n(A)^{n-1}}{a_2(K_1-1)f_1(A)+a_3(K_1{}^2-1)f_2(A)^2+\ldots a_n(K_1{}^{n-1}-1)f_n(A)^{n-1}} \tag{18}$$

From the above equation it can be seen that the dominant term is the term which corresponds to the highest order of nonlinearity n. This of course depends on the value of the coefficient and the choice of K_1 and K_2. Nevertheless it can thus be easily seen that equation (18) is a very close approximation of equation (17) especially for high values of n. It must be stressed here that the method is not dependent on a two-tone multisine as the one used to derive equations (9) to (18). The use of a broadband multisine is recommended since a two–tone multisine could fail in cases where the input to the system is shaped by the dynamics of the system before entering the nonlinear elements as in the Wiener case.

The proposed algorithm is thus summarised as follows:

Step 1: Excite the system under test with a multisine signal at three amplitudes defined by $u(t)$, $K_1\times u(t)$ and $K_1\times K_2\times u(t)$.

Step 2: Calculate the FRFs $H_1(j\omega)$, $H_2(j\omega)$ and $H_3(j\omega)$ at the three amplitudes using equation (6).

Step 3: Calculate the index

$$r=\frac{\sum\left|\left(H_3(j\omega_k)-H_1(j\omega_k)\right)\right|}{\sum\left|\left(H_2(j\omega_k)-H_1(j\omega_k)\right)\right|}$$

Step 4: Obtain an estimate of the maximum order of nonlinearity using equation (17).

IV. EXPERIMENTAL ILLUSTRATIONS

IV.1. Simulated examples

To illustrate the effectiveness of the proposed algorithm the simple Hammerstein and Wiener models shown in Figure 5 were excited using a consecutive multisine containing 120 harmonics with a fundamental frequency of 0.05 Hz and peak amplitude of 10. The linear part of the models is the same as a linear model identified for the High Pressure (HP) shaft of a Rolls-Royce gas turbine engine by Evans *et al.* [14]. The data records were corrupted by additive Gaussian noise of zero mean and unit variance.

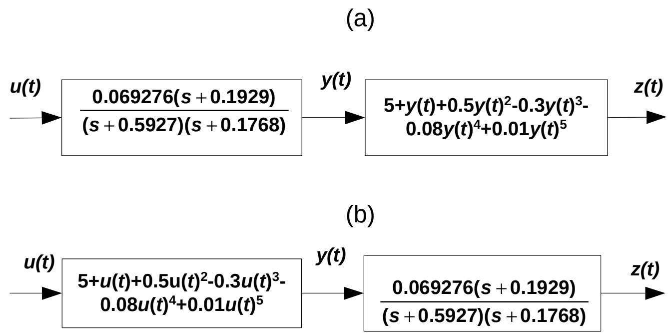

Figure 5. (a) Hammerstein model (b) Wiener model.

The results of the test algorithm for the maximum nonlinearity order for both systems are shown in Table 1 for three different combinations of K_1 and K_2. It can be seen that the proposed algorithm gives a good indication for the maximum order of nonlinearity, even though it is clear that the results obtained for the Wiener model are not as accurate as the results for the Hammerstein model. This was of course expected since in the Wiener case the input is filtered by the linear part of the model before passing through the nonlinearity. The results are more encouraging if a higher-order term, i.e. the 9th-order term $0.0001y(t)^9$ is added to the static polynomials of the two models. It can be seen from Table 2 that even though the coefficient of the 9th-order term is very small compared to the other coefficients, the algorithm detects the contribution of this term very well.

Table 1. Maximum Order of Nonlinearity for Hammerstein and Wiener models (n=5)

Model	K_1=1.5, K_2=1.5	K_1=1.5, K_2=2	K_1=1.5, K_2=4
Hammerstein	4.90	4.90	4.91
Wiener	4.80	4.79	4.79

Table 2. Maximum Order of Nonlinearity for Hammerstein and Wiener models (n=9)

Model	K_1=1.5, K_2=1.5	K_1=1.5, K_2=2	K_1=1.5, K_2=4
Hammerstein	9.00	8.99	8.99
Wiener	8.65	8.75	8.81

Systems with different static polynomials were tested and it was concluded that the algorithm is capable of detecting the maximum order of nonlinearity of systems like Wiener, Hammerstein and Wiener-Hammerstein and Volterra series. It was also observed that the accuracy of the algorithm is improved for high orders of nonlinearity as is clearly suggested by the result in equation (18). It is also clear that further investigation is required on the practical issues concerning the proposed technique such as the optimum selection of the amplitude levels K_1 and K_2.

IV.2. A nonlinear electrical circuit

A nonlinear mechanical resonating system (mass, viscous damping, nonlinear spring) is simulated with an electrical circuit. The displacement $y(t)$ is related to the force $u(t)$ by the following nonlinear, second-order differential equation.

$$m\frac{d^2y(t)}{dt}+d\frac{dy(t)}{dt}+ay(t)+by^3(t)=u(t) \tag{19}$$

As Schoukens *et al*. [15] noted, the actual realized circuit is not in perfect agreement with (19), since a small quadratic term was detected in the measurements. It was also noted that for small excitations the spring becomes almost linear so that the underlying linear system consists of a second-order resonance system. Two different signals, a consecutive multisine ($f_k = kf_0$, k = 1, 2, 3 …, N, N = 601 and $f_0 \approx$ 0.0298 Hz) and an odd multisine ($f_k = kf_0$, k = 1, 3, 5 …, 2N-1, N = 301 and $f_0 \approx$ 0.0298 Hz) where used to test the system at three different amplitudes. Figure 6 shows the amplitude responses of the system at three different amplitudes, obtained using the consecutive multisine. The existence of nonlinearity in the system can be easily visualized as the evolution of the system dynamics with growing input signal amplitude

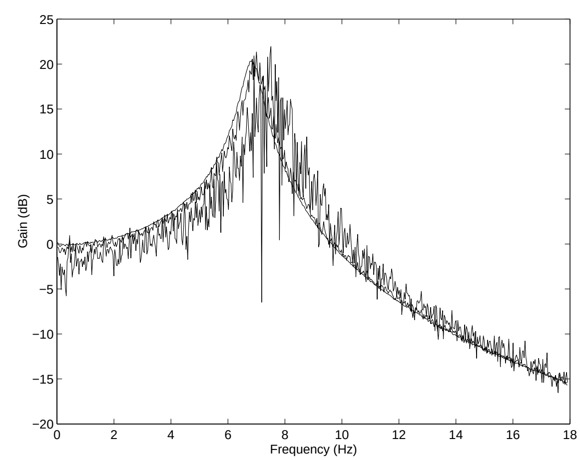

Figure 6. Evolution of system dynamics growing excitation levels: 10,30, 70 mV$_{RMS}$.

Table 3 shows the results of the proposed algorithm when applied to the measured data using the two multisines. It can be seen that the proposed technique gives a good approximation of the maximum order of nonlinearity in the system even though it under predicts the exact order. This is to be expected since nonlinear systems which change dynamics with input excitation level, belong to the family of the Wiener-like structures and more specifically those structures where the input is filtered by the linear part of the model before passing through the nonlinearity. As previously stated, the accuracy of the algorithm improves as the order of the nonlinearity increases. It is clear that in this case the order of the nonlinearity is quite low, which affects the accuracy of the algorithm. Nevertheless the proposed algorithm gives a good indication of the value of the maximum order of nonlinearity.

Table 3. Maximum Order of Nonlinearity for the nonlinear mechanical resonating system

Consecutive multisine Excitation levels: 10, 30, 70 mV_{RMS}.	Odd multisine Excitation levels: 10, 40, 70 mV_{RMS}.
2.31	2.33

V. CONCLUSIONS

A methodology has been presented to estimate the highest order of nonlinearity of Volterra type systems. The proposed algorithm is based on the use of multisine signals and the calculation of the FRFs of the system under test at three amplitudes. The use of multisine signals allows the direct estimation of the FRFs, the systematic errors arising from FFT leakage problems can be avoided and the signal-to-noise ratios of the data records can be improved by averaging over a number of periods. The performance of the proposed algorithm was demostrated in simulation using simple block structure models which belong to the Volterra series family. The proposed technique was also illustrated on a nonlinear mechanical resonating system which is a nonlinear circuit with a maximum 3^{rd} order nonlinearity. It was shown that the proposed algorithm provides a good approximation of the maximum order of nonlinearity in a system. The algorithm is suitable to be used for systems that belong to the Volterra series family as well as a large number of NARMAX structures.

This paper illustrates how frequency-domain techniques and multisine signals can be used in order to provide essential *a priori* knowledge about the nonlinearity in a system and thus aid the identification procedure.

VI. ACKNOWLEDGEMENTS

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