MATEC Web of Conferences **210**, 02021 (2018) https://doi.org/10.1051/matecconf/201821002021
CSCC 2018

Nonlinear systems - describing functions analysis and using

Zdeněk Úředníček[1,*]
[1]Department of Automation and Control, Tomas Bata University in Zlin, Jižní Svahy, Nad Stráněmi 4511, 760 05, Zlin, Czech Republic

Abstract. The paper presents a basic description and examples of the use of so called descriptive functions, allowing analysing the influence of inherent and indispensable components of all mechatronic systems mechanical subsystems - so called hard nonlinearities. These parts "causing" - in addition to the centrifugal and Coriolis generalized forces- the nonlinearity of the system, can be analysed by the above-mentioned method from the point of view of their origin and the estimation of the basic parameters of their frequent consequences - so-called limit cycles. After a short introduction, which introduces and explains describing functions using the example of a nonlinear system taken from the literature, some of the so-called hard nonlinear subsystems (such as the mechanical chain of robots) are shown to be used. The paper is the first part of a more extensive description analysis of nonlinear systems concept using these functions in order to enable analysis and prediction of limit cycles.

1 Introduction

Inherent and inseparable part of the symptomatic subsystems of mechatronic systems (mechanical subsystem, subsystem of actuators, sensory subsystem) - including robotic systems – they are the specific nonlinearities occurring in their mechanical and regulatory subsystems. These specific, so-called **hard nonlinearities**, exemplified by non-viscous friction, saturation, backlash, hysteresis, etc., are often the cause of unwanted behaviour of the system, but in some cases they are used as a tool for introducing specific desired system properties.
Is known that powerful tool for analysing and designing linear control systems is frequency analysis. It is based on the description of a linear system using complex function, **frequency response**, instead of differential equations. However, this tool cannot be directly applied to a non-linear system because the frequency response cannot be defined for a non-linear system. However, for some nonlinear systems, an extended version of the frequency response called **describing functions method** can be used for the approximate analysis and prediction of non-linear behaviour.

Although it is only an approximate method, it takes the necessary feature from the frequency response method, and the lack of other systematic tools for analysing nonlinear systems, makes it an indispensable component of a package for practical engineers. The main use of this method is to predict the **limiting cycles of nonlinear systems**. But the method has a number of other applications such as sub-harmonic prediction, jump phenomenon and non-linear system response to sinusoidal input.

2 Describing functions basics and principle

We begin by introducing an analysis of describing functions using a simple example modified from [Hsu, J. C. and Meyer, A. U., Modern Control Principles and Applications, McGraw-Hill (1968)]:

Consider the classical Van der Pol equation

$$\ddot{x} + \alpha \cdot \left(x^2 - 1\right) \cdot \dot{x} + x = 0 \tag{1}$$

where α is a positive constant.

Let's study it using a technique that will lead us to a describing functions concept. Especially to determine if there is a limit cycle for this system, and if so, give us the possibility to calculate the amplitude and frequency of the limit cycle (pretending we do not know the phase portrait).

Let's first assume the existence of a limit cycle with an undetermined amplitude and frequency and then determine if the system can actually have such a solution[2].

First, let us present the dynamics of the selected system by block diagram scheme shown in Fig. 1.

Because it is valid

$$\ddot{x} + \alpha \cdot \left(x^2 - 1\right) \cdot \dot{x} + x = 0 \Rightarrow \ddot{x} + \alpha \cdot \dot{x} \cdot x^2 - \alpha \cdot \dot{x} + x \Rightarrow$$
$$\Rightarrow \ddot{x} - \alpha \cdot \dot{x} + x = -\alpha \cdot \dot{x} \cdot x^2 = \alpha \cdot \overbrace{\left(-\dot{x} \cdot x^2\right)}^{w},$$

the equation can be drawn as follows:

2 This is similar to the method of the assumed form of the solution function in the theory of ordinary differential equations, where we assume that the solution is of a certain shape, fit it into the differential equation and then determine the coefficients of this solution

* Corresponding author: urednicek@utb.,cz

© The Authors, published by EDP Sciences. This is an open access article distributed under the terms of the Creative Commons Attribution License 4.0 (http://creativecommons.org/licenses/by/4.0/).

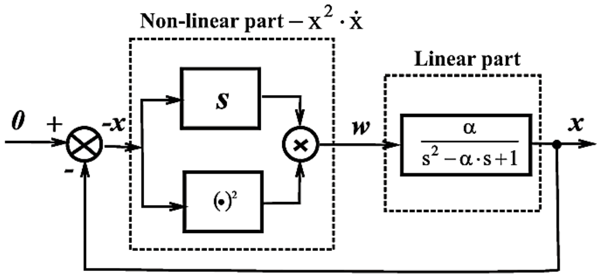


Fig. 1. The feedback interpretation of Van der Pol's oscillator.

We see that the feedback loop in this figure contains a linear and nonlinear block where the linear block, however unstable, has the characteristics of the low pass filter (see Fig. 2).

Suppose now that there is a limit cycle in the system and the oscillating signal is shaped as

$$x(t) = A \cdot \sin(\omega \cdot t)$$

with **A** as the amplitude of the limit cycle and ω as its frequencies.

That is true

$$\dot{x}(t) = A \cdot \omega \cdot \cos(\omega \cdot t)$$

Thus, the output of the nonlinear block in Fig. 1 is

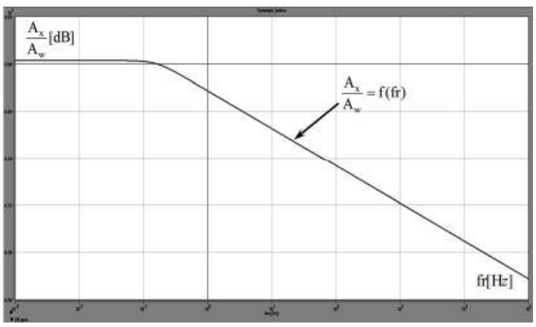


Fig. 2. Linear part of the Van der Pol's oscillator amplitude frequency characteristic for $\alpha = 1.5$.

$$w = -\dot{x}(t) \cdot x^2(t) = -A^2 \cdot \sin^2(\omega \cdot t) \cdot A \cdot \omega \cdot \cos(\omega \cdot t) =$$
$$= -A^3 \cdot \omega \cdot \sin^2(\omega \cdot t) \cdot \cos(\omega \cdot t) =$$
$$= -\frac{A^3 \cdot \omega}{2}[1 - \cos(2\omega \cdot t)] \cdot \cos(\omega \cdot t) =$$
$$= -\frac{A^3 \cdot \omega}{2}[\cos(\omega \cdot t) - \cos(\omega \cdot t) \cdot \cos(2\omega \cdot t)] =$$
$$= -\frac{A^3 \cdot \omega}{4}[\cos(\omega \cdot t) - \cos(3\omega \cdot t)]$$

It can be seen that **w** contains the 3rd harmonic. Because the linear block has low pass filter properties, it can reasonably be assumed that the third harmonic is sufficiently weakened by the linear block and its effect is not apparent. That is, we can approximate **w** as

$$w \approx -\frac{A^3 \cdot \omega}{4}\cos(\omega \cdot t) = \frac{A^2}{4} \cdot \frac{d}{dt}[-A \cdot \sin(\omega \cdot t)] = \frac{A^2}{4} \cdot \frac{d}{dt}[-x(t)] \quad (2)$$

Thus, the nonlinear block in Fig. 1 can be approximated by the equivalent "quasi-linear" block of Fig. 3..

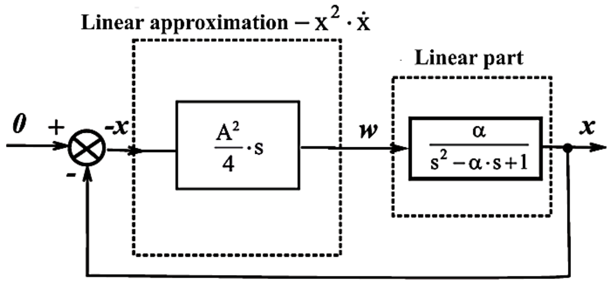


Fig. 3. Quasi linear approximation of the Van der Pol oscillator.

The "transmission function" of the quasi-linear block depends on the amplitude of signal **A**. in contrast to the transmission function of a linear system that is independent of the size of the input signal.

In the frequency domain reads

$$w = N(A, \omega)(-x)$$

where

$$N(A, \omega) = \frac{A^2}{4} \cdot (i \cdot \omega) \quad (3)$$

Thus the nonlinear block can be approximated by the frequency transfer function $N(A, \omega)$. Because we assume sine oscillation of the system, we get it:

$$x(t) = A \cdot \sin(\omega \cdot t) = G(i \cdot \omega) \cdot w = = G(i \cdot \omega) \cdot N(A, \omega)(-x)$$

where $G(i \cdot \omega)$ is the frequency transfer function of the linear part. This means that for the resulting system frequency transmission is valid

$$x(t) \cdot [1 + G(i \cdot \omega) \cdot N(A, \omega)] = 0 \Rightarrow$$
$$\Rightarrow 1 + G(i \cdot \omega) \cdot N(A, \omega) = 0 \Rightarrow$$
$$\Rightarrow 1 + \frac{\alpha}{(i \cdot \omega)^2 - \alpha \cdot (i \cdot \omega) + 1} \cdot \frac{A^2}{4} \cdot (i \cdot \omega) = 0 \Rightarrow$$
$$\Rightarrow \frac{4 \cdot [1 - \omega^2 - i \cdot \alpha \cdot \omega] + i \cdot \alpha \cdot A^2 \cdot \omega}{4 \cdot [1 - \omega^2 - i \cdot \alpha \cdot \omega]} = 0 \Rightarrow$$
$$\Rightarrow \frac{4 \cdot (1 - \omega^2) + i \cdot \alpha \cdot \omega \cdot [A^2 - 4 \cdot]}{4 \cdot [1 - \omega^2 - i \cdot \alpha \cdot \omega]} = 0 \Rightarrow (\omega = 1) \cap (A = 2)$$

Note that the characteristic equation of this negative feedback system is

$$1 + \frac{\alpha}{s^2 - \alpha \cdot s + 1} \cdot \frac{A^2}{4} \cdot s = 0 \quad (4)$$

What are its roots, what are the eigenvalues of this linearized system?

$$1 + \frac{\alpha}{s^2 - \alpha \cdot s + 1} \cdot \frac{A^2}{4} \cdot s = 0 \Rightarrow$$
$$s_{1,2} = -\alpha \cdot \frac{(A^2 - 4)}{8} \pm \sqrt{\alpha^2 \cdot \frac{(A^2 - 4)^2}{64} - 1} \quad (5)$$

For $A = 2$ their eigenvalues are $s_{1,2} = \pm i$. This indicates the limit cycle with amplitude **2** and frequency **1**.

It is interesting to note that neither the amplitude nor the frequency depends on the parameter in the equation (1).

In the phase plane, the previous approximate analysis means that the limit cycle is, regardless of α size, a circle with radius **2**. To verify the credibility of this result, the Fig. 4. draws limit cycles of the complete system for different values α.

It can be seen that the above-mentioned approximation applies only to small α values and the inaccuracy increases with the increase in its value. This is understandable because the increasing degree of non-linearity increases with increasing α and the quasi-linear approximation is less accurate.

Using the above analysis, the stability of the limit cycle can be studied.

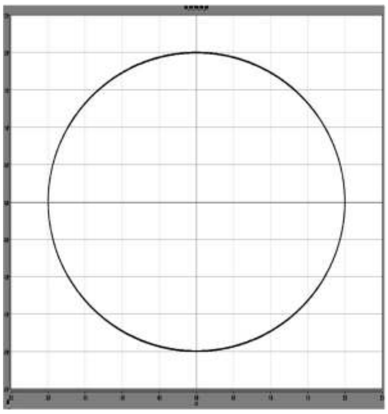
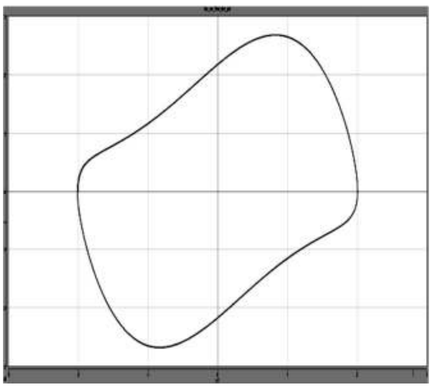
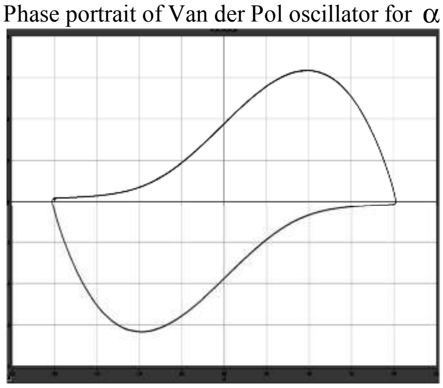


Fig. 4a. Phase portrait of Van der Pol oscillator for $\alpha = 0$.



Obr.4b. Phase portrait of Van der Pol oscillator for $\alpha = 1$.



Obr.4c. Phase portrait of Van der Pol oscillator for $\alpha = 4$.

Let us assume that the amplitude of the limit cycle A is greater than 2. Then, from equation (5) it can be seen that the poles of the characteristic equation of the system will have a negative real part. This means that the system becomes exponentially stable and the signal size is reduced.

What happens when the default state is set to **x(0)** less than 2. The approximate system is unstable initially, the amplitude of the oscillations increases to amplitude 2. Then the situation from the previous one occurs. Thus, it can be concluded that the limiting cycle is stable with amplitude **A=2**.

3 Describing functions applications possibility

Let's first discuss briefly on what types of nonlinear systems the method is applicable and what type of information about the nonlinear system can provide.

3.1. On what types of nonlinear systems the method can be used

Simply put, any system that can be transformed into the arrangement of Fig. 5. can be studied using descriptive functions. There are at least two important classes of systems in this category.

The first important class is "almost" linear systems. By "almost" linear systems we mean systems that contain so-called "hard" nonlinearities in the control loop but are otherwise linear. These systems arise in the design of control law using a linear approach, but its implementation includes "hard" nonlinearities such as motor torque saturation, actuator (or sensor) backlash (dead band), Coulomb friction, or hysteresis in a controlled system.

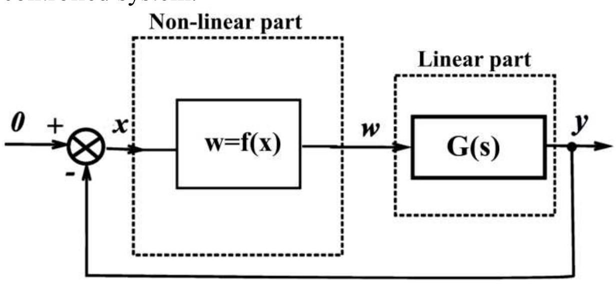


Fig. 5. Non-linear system.

An example is the system of Fig. 6. containing hard non-linearity in the actuator.

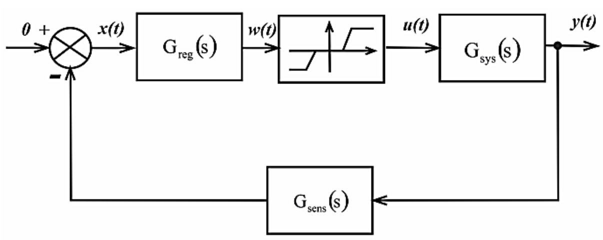


Fig. 6. Control system with one "hard" non-linearity.

The regulated system is linear as well as the controller. But the actuator contains hard nonlinearity. This system can be reconfigured to the form of Fig. 5. with

$$G(s) = G_{reg}(s) \cdot G_{sys}(s) \cdot G_{sens}(s)$$

An "almost" linear system containing non-linearity in a sensor or controlled system can also be reconfigured to the shape of Fig.5..

The second class of systems are systems containing real nonlinear subsystems whose dynamic equations can be converted to the structure of Fig. 5. We have seen an example of such a system in the introductory example.

3.2. Describing function applications

For systems as shown in Fig. 5., the limit cycle may occur as a result of non-linearity. But linear control cannot predict this problem. On the other hand, descriptive functions can conveniently be used to detect the existence of limit cycles and to determine their stability. Regardless of whether they are "hard" or "soft" nonlinearities. The use for limiting cycle analysis is due to the fact that the shape of the system signal on the limiting cycle is commonly approximated by a sinusoidal one.

This can be conveniently explained on the system of Fig. 5. Suppose the linear part in Fig. 5. has the characteristics of the low pass filter (which is the case for many physical systems). If there is a limit cycle in the system, then the system signals must be all periodic. Because the periodic signal, as input to the linear part in Fig. 5., can be decomposed as the sum of many harmonic oscillations, and because the linear member, due to its low pass filter properties, fuses higher frequencies, the output **y(t)** must in most cases consist of the lowest harmonic oscillations. It is therefore reasonable to assume that the signals throughout the system are essentially sinusoidal, thus allowing the technique used in the previous section.

Limit cycle prediction is very important because limit cycles can occur in physical nonlinear systems. Sometimes the limit cycle may be desirable. This is the case for limit cycles in electronic oscillators. Another case is the so-called vibration technique to minimize the negative effect of Coulomb's friction in mechanical systems. On the other hand, in most control systems, the limit cycles are undesirable. This can be for several reasons:

1. The limit cycle is the path to instability, causing poor accuracy of regulation.
2. Constant oscillations associated with the limit cycle may cause increased wear or mechanical failure in the hardware of the control system.
3. The limit cycle may also cause other undesirable effects such as passenger discomfort during autopilot flight.

In general, although precise knowledge of the shape of the limiting cycle curve is not necessary, the knowledge of its existence or non-existence, as well as its approximate amplitude and frequency, is necessary. The method describing functions is applicable for these purposes. Knowledge of this kind can also lead to the design of compensators in order to avoid limiting cycles.

3.3. Basic assumptions of describing functions using

Let us consider the non-linear system in the general form of Fig. 5. In order to be able to use the basic version of the method describing functions, the system must meet the following conditions:

1. **There is only one non-linear member**.
2. **A non-linear member is time-invariant**.
3. In sinusoidal input $x(t) = \sin(\omega \cdot t)$, only the fundamental harmonic can be considered in the output **w**.
4. **Non-linearity is odd function**.

The first condition means that if there are two or more non-linear components in the system, they can either be joined to one (such as parallel pairing of two nonlinearities) or only one nonlinearity is under consideration and the other is neglected.

The second condition means that we only consider **autonomous non-linear systems.** This is sufficient for much practical nonlinearity, such as saturation of amplifiers, transmission backlash, Coulomb friction between surfaces and hysteresis in relay systems. The reason for this assumption is that the Nyquist criterion, on which the describing function is broadly based, requires linear time-invariant systems.

The third condition is essential for the describing function. It represents an approximation because the output from the nonlinear element at the sinus input usually contains, besides the basic, even higher harmonics. The assumption means that higher harmonics can be neglected in the analysis in comparison to the basics one. In order for this assumption to be fulfilled, it is important that the next linear element has the character of the low pass. I.e.

$$|G(i \cdot \omega)| >> |G(n \cdot i \cdot \omega)| \quad \text{for } n = 2,3,\ldots$$

This means that higher harmonics in the output of nonlinearity will be significantly filtered. Thus, the third assumption is often called a **filtering hypothesis**.

The fourth condition means that the graph of the nonlinear relation **f(x)** between the input and the output of this member is symmetrical with respect to the origin of the coordinate system. This assumption is introduced for simplicity, i.e. that Fourier development can neglect the DC component. Note that most of the nonlinearities occurring in our systems (robot motion system) meet this condition.

Failure to meet the above conditions has been widely studied in the literature on the use of general context descriptors such as multiple nonlinearities, time-dependent nonlinearity, or multiple sinusoids. However, these conditions relaxation-based methods are usually much more complicated than basic versions based on the above four conditions.

4 Basic definition

Consider the sinus input of a non-linear element with amplitude A and frequency ω. I.e.

$$x(t) = A \cdot \sin(\omega \cdot t)$$

as is shown in Fig. 7.

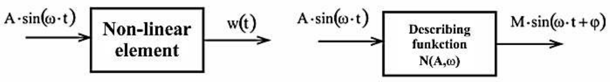


Fig.7. The non-linear element and its representation by the describing component.

Nonlinear element output is often a periodic but generally non-sinusoidal function. Note that this occurs whenever the non-linearity f(x) is a uniquely invertible function because the output is

$$f[A \cdot \sin(\omega \cdot t + 2\pi)] = f[A \cdot \sin(\omega \cdot t)].$$

By using the Fourier series, the periodic function $w(t)$ can be expanded as

$$w(t) = \frac{a_0}{2} + \sum_{k=1}^{\infty}[a_k \cdot \cos(k \cdot \omega \cdot t) + b_k \cdot \sin(k \cdot \omega \cdot t)] \quad (6)$$

where Fourier coefficients are function of **A** and **ω**. It is valid:

$$a_k = \frac{1}{\pi} \cdot \int_{-\pi}^{\pi} w(t) \cdot \cos(k \cdot \omega \cdot t) \cdot d(\omega \cdot t); \quad k = 0, 1, \cdots \infty$$
$$b_k = \frac{1}{\pi} \cdot \int_{-\pi}^{\pi} w(t) \cdot \sin(k \cdot \omega \cdot t) \cdot d(\omega \cdot t); \quad k = 1, \cdots \infty \quad (7)$$

As a result of the fourth of the above assumptions, it is $a_0 = 0$. Furthermore, the third condition means that you only need to consider the basic harmonics $w_1(t)$.

So

$$w(t) \approx w_1(t) = a_1 \cdot \cos(\omega \cdot t) + b_1 \cdot \sin(\omega \cdot t) = M \cdot \sin(\omega \cdot t + \varphi) \quad (8)$$

where

$$M(A, \omega) = \sqrt{a_1^2 + b_1^2} \quad \text{a}$$

$$\sin\varphi(A, \omega) = \frac{a_1}{\sqrt{a_1^2 + b_1^2}}; \cos\varphi(A, \omega) = \frac{b_1}{\sqrt{a_1^2 + b_1^2}}$$

because

$$a_1 \cdot \cos(\omega \cdot t) + b_1 \cdot \sin(\omega \cdot t)$$
$$b_1 = M \cdot \cos\varphi; a_1 = M \cdot \sin\varphi \Rightarrow M = \sqrt{a_1^2 + b_1^2}$$
$$\Rightarrow a_1 \cdot \cos(\omega \cdot t) + b_1 \cdot \sin(\omega \cdot t) =$$
$$= M \cdot [\cos(\omega \cdot t) \cdot \sin\varphi + \sin(\omega \cdot t) \cdot \cos\varphi] =$$
$$= M \cdot \sin(\omega \cdot t + \varphi)$$

The term (8) means that the basic harmonic corresponding to the sinus input is the sinusoidal function of the same frequency. Representing in a complex variable, it is possible to write this sinus as

$$w_1(t) = M \cdot e^{i \cdot (\omega t + \varphi)} = (b_1 + i \cdot a_1) \cdot e^{i \cdot \omega \cdot t}$$

Similarly to the frequency response concept of the linear system, which is the ratio of sinusoidal output to the sinus input in the frequency domain, we define **the describing function** of the nonlinear element as **the complex ratio of the fundamental harmonic output to the sinus input**. I.e.

$$N(A, \omega) = \frac{M \cdot e^{i \cdot (\omega t + \varphi)}}{A \cdot e^{i \cdot \omega \cdot t}} = \frac{M}{A} \cdot e^{i \cdot \varphi} = \frac{1}{A} \cdot (b_1 + i \cdot a_1) \quad (9)$$

By describing function which describes a non-linear element, this element - for the sinusoidal input -can be presented as a linear element with frequency transmission. This is shown in Fig.7..

The describing function concept therefore can be understood as extending of the frequency response term. For linear dynamic system the frequency transition function is independent of the amplitude of the input signal. But describing function of the non-linear element differs from the frequency transition function of the linear element by being dependent on the amplitude of the input signal. Thus, the representation of the non-linear element of Fig. 7 is sometimes called **quasi linearization.**

Let us give the following example of continuous nonlinearity: We will describe the function of the "stiffer" spring. Let the characteristic of the spring be given by the function

$$w = x + \frac{1}{2} \cdot x^3$$

Let the inpute is

$$A \cdot \sin(\omega \cdot t)$$

Then output is

$$w(t) = A \cdot \sin(\omega \cdot t) + \frac{1}{2} \cdot A^3 \cdot \sin^3(\omega \cdot t)$$

Extend it in the Fourier series. The basic harmonic is

$$w_1(t) = a_1 \cdot \cos(\omega \cdot t) + b_1 \cdot \sin(\omega \cdot t)$$

Determine

$$a_1 = \frac{1}{\pi} \cdot \int_{-\pi}^{\pi}\left[A \cdot \sin(\omega \cdot t) + \frac{1}{2} \cdot A^3 \cdot \sin^3(\omega \cdot t)\right] \cdot \cos(\omega \cdot t) \cdot d(\omega \cdot t) = 0$$

$$b_1 = \frac{1}{\pi} \cdot \int_{-\pi}^{\pi}\left[A \cdot \sin(\omega \cdot t) + \frac{1}{2} \cdot A^3 \cdot \sin^3(\omega \cdot t)\right] \cdot \sin(\omega \cdot t) \cdot d(\omega \cdot t) =$$
$$= A + \frac{3}{8} \cdot A^3$$

So the basic harmonic is

$$w_1(t) = b_1 \cdot \sin(\omega \cdot t) = \left(A + \frac{3}{8} \cdot A^3\right) \cdot \sin(\omega \cdot t)$$

Describing function is

$$N(A, \omega) = N(A) = \frac{\left(A + \frac{3}{8} \cdot A^3\right) \cdot \sin(\omega \cdot t)}{A \cdot \sin(\omega \cdot t)} = 1 + \frac{3}{8} \cdot A^2$$

We see that due to the odd behaviour of a function describing this nonlinearity, describing function is real and it is only a function of the sinusoidal input amplitude.

Generally the describing function depends on the frequency and amplitude of the input signal. There are, however, several special cases. If **the non-linearity is a odd function**, describing function is real and does not depend on the input frequency. Real describing function $N(A, \omega)$ is the consequence of $a_1 = 0$ in this case.

5 Examples of discontinuous nonlinearrities

Nonlinearities can be divided into continuous and discontinuous. Since discontinuous nonlinearities cannot be approximated locally by linear functions, they are often referred to as "hard" nonlinearities. These "hard" nonlinearities often occur in regulatory systems, both in small scale and large scale operations. Whether it can be considered as nonlinear or linear- when it is operating in a small scale of activity- the size of the "hard" nonlinearity arbitrates and the also the application of its effect on the performance of the system.

Due to the frequent occurrence of "hard" nonlinearities, let's briefly discuss the characteristics and effects of two important.

5.1 Describing function of the saturation

If the input of the physical device increases, it is often possible to see the following phenomenon. If input is small, its magnification leads (often proportionally) to increasing output. But when it reaches a certain value, its further magnification leads to little or no increase in output. The output simply stays close to its maximum value. We say the **device is saturation** in this state. A simple example is a transistor and a magnetic amplifier. Saturation type of non-linearity is commonly caused by limitations in component size, material properties, and limitation of available power. In Fig.8 there is a typical non-linearity of saturation, where the stronger line represents real non-linearity and thinner is its idealization - partial linearization.

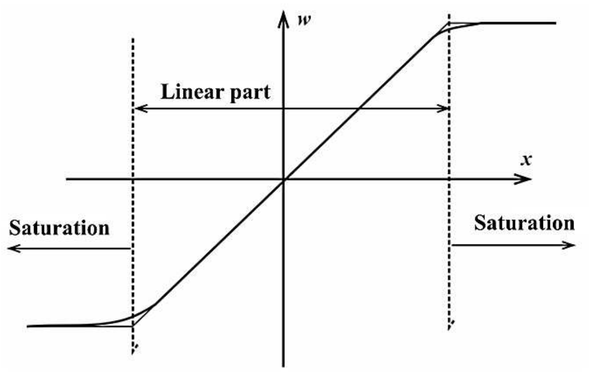


Fig. 8. Saturation nonlinearity

Most actuators show saturation nonlinearity. For example, the output torque of the servo motor can not grow to infinity and exhibits saturation not only due to the properties of the magnetic material. Similarly, the torque (pressure) hydraulic servo motor controlled by valve is limited by the maximum accessible system fluid pressure.

Saturation may have a complicating effect on the properties of the control system. Simply, the occurrence of saturation reduces device gain (e.g.of amplifier) when the input signal increases. As a result, if the system is unstable in its linear part, the divergent behavior can be suppressed to permanent oscilations through signal. On the other hand, in a linear stable saturation system, the system's response drops because saturation reduces effective gain.

The input-output relationship for saturation non-linearity is illustrated in Fig. 9 with **a** and **k** as the determining parameters of non-linearity.

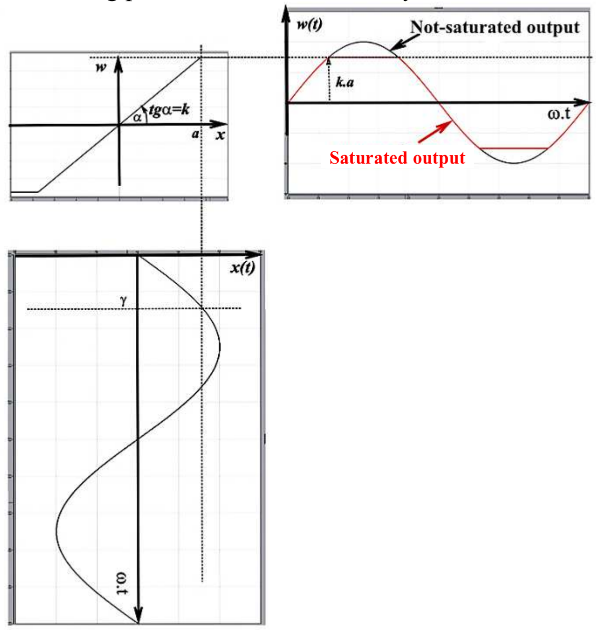


Fig. 9. Non-linearity of saturation and its input-output relationship.

Because this non-linearity is an odd function, we assume that its descriptive function will be real and will be only a function of the input amplitude.

Consider the input

$$x(t) = A \cdot \sin(\omega \cdot t)$$

If $A \le a$, then, the input remains all the time in the linear region and therefore the output is $w(t) = k \cdot A \cdot \sin(\omega \cdot t)$.

The describing function is

$$b_1 = \frac{1}{\pi} \cdot \int_{-\pi}^{\pi} k \cdot A \cdot \sin(\omega \cdot t) \cdot \sin(\omega \cdot t) \cdot d(\omega \cdot t) = k \cdot A$$

And so

$$N(A, \omega) = \frac{1}{A} \cdot (b_1 + i \cdot a_1) = k$$

Let's think that $A > a$. The input and output are then drawn in Fig.9. Output on interval $\langle -\pi, \pi \rangle$ is

$$w(t) = \begin{cases} k \cdot A \cdot \sin(\omega \cdot t) & -\pi \le \omega \cdot t \le -\pi + \gamma \\ k \cdot a & -\pi + \gamma < \omega \cdot t \le -\gamma \\ k \cdot A \cdot \sin(\omega \cdot t) & -\gamma < \omega \cdot t \le \gamma \\ k \cdot a & \gamma < \omega \cdot t \le \pi - \gamma \\ k \cdot A \cdot \sin(\omega \cdot t) & \pi - \gamma < \omega \cdot t \le \pi \end{cases}$$

where for the angle γ we obtain

$$\sin \gamma = \frac{a}{A}$$

Due to the oddity of the function **w = f (x)** is $a_1 = 0$.

Determine $\mathbf{b_1}$.

$$b_1 = \frac{1}{\pi}\cdot\int_{-\pi}^{\pi} f[A\cdot\sin(\omega\cdot t)]\cdot\sin(\omega\cdot t)\cdot d(\omega\cdot t) =$$

$$= \frac{2\cdot k\cdot A}{\pi}\cdot\left[\gamma - \overbrace{\sin(\gamma)}^{a/A}\cdot\overbrace{\cos(\gamma)}^{\sqrt{1-(a/A)^2}}\right] = \frac{2\cdot k\cdot A}{\pi}\cdot\left[\gamma - \frac{a}{A}\cdot\sqrt{1-\left(\frac{a}{A}\right)^2}\right] \Rightarrow$$

$$b_1 = \frac{2\cdot k\cdot A}{\pi}\cdot\left[\gamma - \frac{a}{A}\cdot\sqrt{1-\left(\frac{a}{A}\right)^2}\right] \quad (9)$$

So describing function is

$$N(A,\omega) = N(A) = \frac{1}{A}\cdot(b_1 + i\cdot a_1) = \frac{b_1}{A} = = \frac{2\cdot k}{\pi}\cdot\left[\arcsin\left(\frac{a}{A}\right) - \frac{a}{A}\cdot\sqrt{1-\left(\frac{a}{A}\right)^2}\right] \quad (10)$$

Dividing by **k** we get the so-called normalized description function $N(A)/k$.

$$\frac{N(A)}{k} = \frac{2}{\pi}\cdot\left[\arcsin\left(\frac{a}{A}\right) - \frac{a}{A}\cdot\sqrt{1-\left(\frac{a}{A}\right)^2}\right] \quad (11)$$

In Fig. 10 its shape is plotted according to the ratio $\frac{A}{a}$.

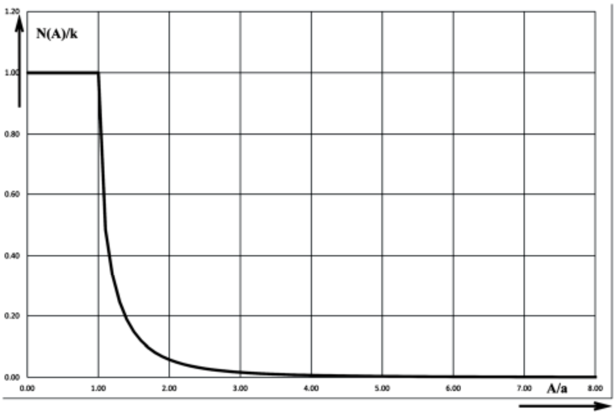


Fig.10. Normalized describing function of saturation non-linearity

It is possible to see the three properties of this describing function:

1. If the input amplitude is in the linear region, then $N(A) = k$.
2. As the input amplitude increases, $N(A)$ decreases.
3. There is no phase shift.

The first feature is evident from the above mentioned. When signal is low, saturation does not occur. The second is also intuitively obvious. Saturation reduces the ratio of output to input. The third property is also understandable. Said symmetric saturation does not cause phase shift in the output.

As a special case of saturation we can consider the saturation with $k = \infty$, i.e. non-linearity on-off-comparator. Its dependence **w = f (x)** is in Fig. 11..

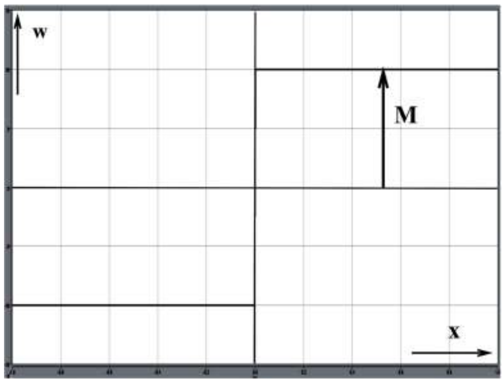


Fig.11. Non-linearity of comparator type

Thus, this case corresponds to the limiting case of a linear saturation function $a \to 0; k \to \infty$, which does not exclude that $a\cdot k = M$. Although $\mathbf{b_1}$ can be obtained from the (9) by limit, it is easier to calculate it directly.

$$b_1 = \frac{1}{\pi}\cdot\int_{-\pi}^{\pi} f[A\cdot\sin(\omega\cdot t)]\cdot\sin(\omega\cdot t)\cdot d(\omega\cdot t) =$$

$$= \frac{1}{\pi}\cdot\left\{-\int_{-\pi}^{0} M\cdot\sin(\omega\cdot t)\cdot d(\omega\cdot t) + \int_{0}^{\pi} M\cdot\sin(\omega\cdot t)\cdot d(\omega\cdot t)\right\} =$$

$$= \frac{1}{\pi}\cdot\left\{-M[-\cos(\omega\cdot t)]_{-\pi}^{0} + M[-\cos(\omega\cdot t)]_{0}^{\pi}\right\} =$$

$$= \frac{1}{\pi}\cdot\left\{-M\left(-1-\overbrace{(-1\cdot -1)}^{1}\right) + M(1-(-1\cdot 1))\right\} =$$

$$= \frac{4}{\pi}\cdot M$$

So, describing function is

$$N(A,\omega) = N(A) = \frac{1}{A}\cdot(b_1 + i\cdot a_1) = = \frac{b_1}{A} = \frac{4}{\pi}\cdot\frac{M}{A} \quad (12)$$

Normalized describing function $N(A)/M = \frac{4}{\pi}\cdot\frac{1}{A} = f(A)$ is on Fig.12..

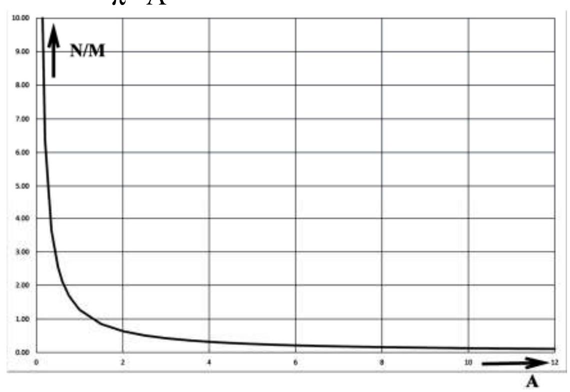


Fig.12. Normalized describing function of on-off non-linearity.

5.2 Describing function of backlash and hysteresis non-linearity

In systems with mechanical force (torque) transmission, the backlash often occurs. This is due to the small airspace in the transmission mechanism. In a transmission chain consisting of, for example, a gearbox with front wheels (with parallel shafts), there is always some airspace between a pair of adjacent wheels. It is not just a consequence of inaccuracies in production and assembly. This is a prerequisite for reasonable transfer efficiency.

Fig.13. shows a typical situation. The result of the dead zone in torque transfer between the teeth is that when the driving wheel is rotated, there is a certain path on the contact wheel circle, within which the torque is not transmitted to the driven wheel. Let **the angle** of rotation of the output wheel corresponding to this path be denoted as **b**. Let the gear ratio be

$$k=\frac{r_2}{r_1}$$

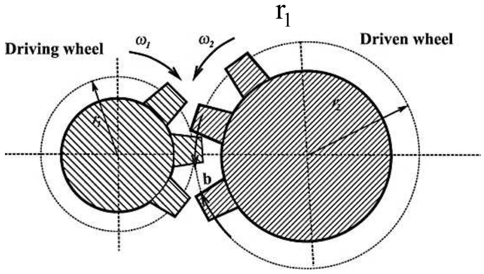


Fig.13. Non-linearity of gear backlash type

Thus, the torque is transmitted only when

$$|r_1\cdot\varphi_1(t)-r_2\cdot\varphi_2(t)|\geq r_2\cdot b\Rightarrow$$

$$\Rightarrow\begin{cases}1)\ r_1\cdot\varphi_1(t)-r_2\cdot\varphi_2(t)\geq 0\Rightarrow\\ \Rightarrow r_2\cdot\left[\frac{r_1}{r_2}\cdot\varphi_1(t)-\varphi_2(t)\right]=r_2\cdot\left[\frac{1}{k}\cdot\varphi_1(t)-\varphi_2(t)\right]\geq 0\Rightarrow\\ \quad\Rightarrow\left[\frac{1}{k}\cdot\varphi_1(t)-\varphi_2(t)\right]\geq 0\Rightarrow\\ \Rightarrow\frac{1}{k}\cdot\varphi_1(t)\geq\varphi_2(t)\ \textit{then}\ \left[\frac{1}{k}\cdot\varphi_1(t)-\varphi_2(t)\right]\geq b\\ 2)\left[\frac{1}{k}\cdot\varphi_1(t)-\varphi_2(t)\right]<0\Rightarrow\\ \frac{1}{k}\cdot\varphi_1(t)<\varphi_2(t)\ \textit{then}\ \left[\frac{1}{k}\cdot\varphi_1(t)-\varphi_2(t)\right]\leq -b\end{cases}$$

Thus, while the angular rotation between the recalculated driving wheel at the output and the driven wheel with different size (gear ratio is **k**) is less than **b**, the torque between the wheels is not transmitted.

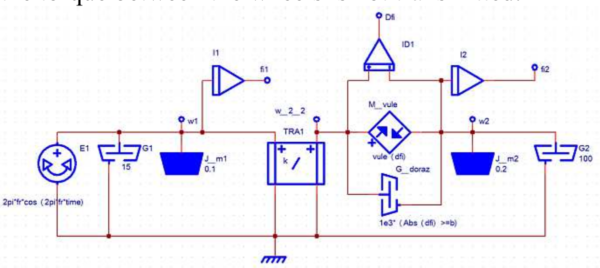


Fig. 14. Torque transfer dynamic model with backlash between teeth

Based on the dynamic model shown in Fig. 14 and the response behaviour of this non-linearity to the sinus input $x(t)=A\cdot\sin\omega\cdot t$ (for $A>k\cdot b$) shown in Fig.15. and Fig.16., can be written

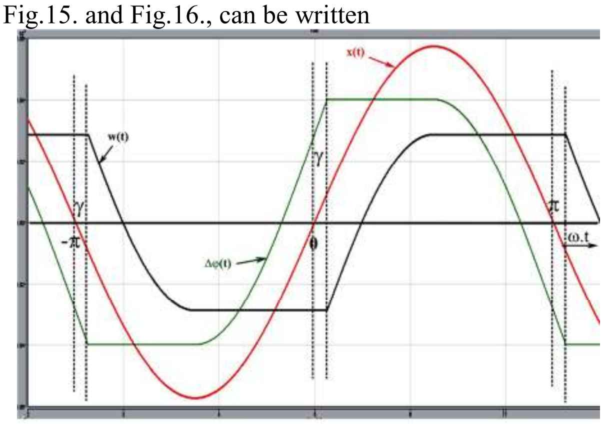


Fig.15. Time course of variables describing the interaction of two wheels with backlash for the sine input angle

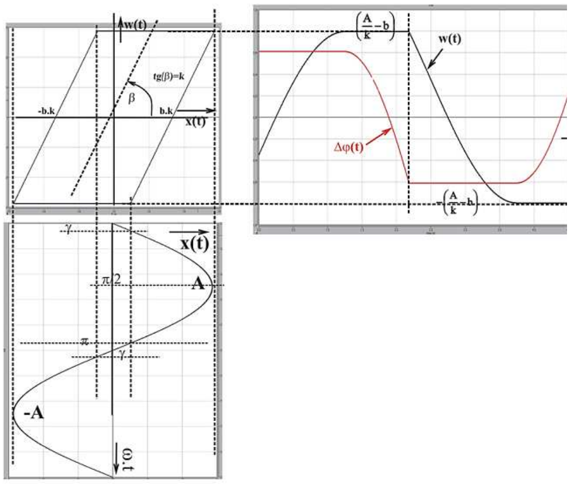


Fig. 16. Input-output context for backlash type non-linearity in gearing

$$w(t)=\begin{cases}\frac{1}{k}\cdot A-b & -\pi\leq\omega\cdot t\leq-\pi+\gamma\\ \frac{1}{k}\cdot A\cdot\sin(\omega\cdot t)+b & -\pi+\gamma<\omega\cdot t\leq-\frac{\pi}{2}\\ -\left(\frac{1}{k}\cdot A-b\right) & -\frac{\pi}{2}<\omega\cdot t\leq\gamma\\ \frac{1}{k}\cdot A\cdot\sin(\omega\cdot t)-b & \gamma<\omega\cdot t\leq\frac{\pi}{2}\\ \frac{1}{k}\cdot A-b & \frac{\pi}{2}<\omega\cdot t\leq\pi\end{cases}$$

For $A\leq k\cdot b\Rightarrow\frac{k\cdot b}{A}\geq 1$ the output is zero. We determine the angle γ, i.e. the angle at which the backlash influence ends and the output wheel begins to move: Is valid

$$\frac{1}{k}\cdot A\cdot\sin\gamma-\left[-\left(\frac{1}{k}\cdot A-b\right)\right]=b\Rightarrow$$

$$\Rightarrow\frac{1}{k}\cdot A\cdot\sin\gamma+\frac{1}{k}\cdot A-b=b\Rightarrow\sin\gamma=\frac{2\cdot k\cdot b}{A}-1$$

Now, unfortunately, **w (t) is not odd**. Thus neither $\mathbf{a_1}$ nor $\mathbf{b_1}$ is zero and we have to determine them. The calculation is lengthy.

$$a_1=-\frac{4\cdot b\cdot\left(1-b\cdot\dfrac{k}{A}\right)}{\pi}$$

$$b_1=\frac{1}{k\cdot\pi}\cdot\left[A\cdot\left(\frac{\pi}{2}-\gamma\right)+2\cdot\left(1-2\cdot\frac{k\cdot b}{A}\right)\cdot\sqrt{k\cdot b\cdot A\cdot\left(1-\frac{k\cdot b}{A}\right)}\right]$$

So

$$|N(A,\omega)|=\frac{1}{A}\cdot|(b_1+i\cdot a_1)|=\frac{1}{A}\cdot\sqrt{a_1^2+b_1^2}$$

$$\angle N(A,\omega)=\mathrm{arctg}\left(\frac{a_1}{b_1}\right)$$

The describing function amplitude for backlash is shown in Fig. 17. and its phase is in Fig. 18..

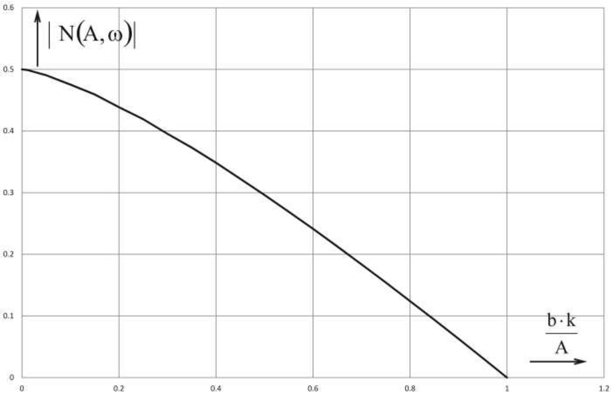


Fig.17. The describing function amplitude for the tooth backlash

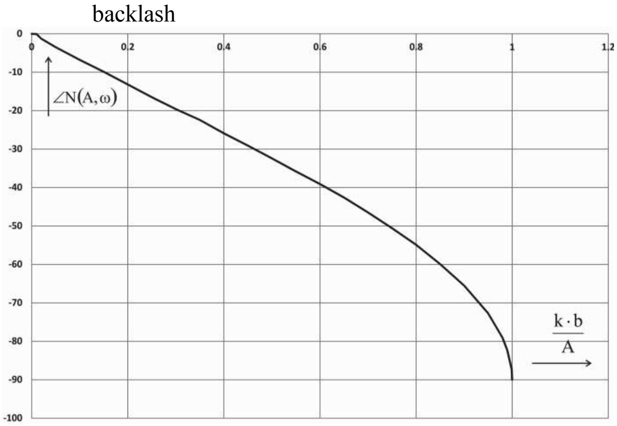


Fig.18. The describing function phase for the tooth backlash

Here they are some interesting facts:

1. $|N(A,\omega)|\rightarrow\frac{1}{k}$ *for* $b\rightarrow 0$
2. $|N(A,\omega)|$ *grows when* $\frac{k\cdot b}{A}$ *decreases*
3. $|N(A,\omega)|\rightarrow 0$ *for* $b\rightarrow\frac{A}{k}$

Phase shift (from 0^o to -90^o) is due to the effect of a given non-linearity. It is the result of the time shift caused by the backlash **b** [rad] on the output side of the gear. Higher **b** leads naturally to greater phase shifting, which may cause a stability problem with the feedback control system.

3 Conclusions

The general mathematical description of the mechatronic systems dynamic behavior as artificial systems with purposeful motion control, in which one part is a subsystem with the motion of interconnected bodies with non-zero resting mass, necessarily leads to a nonlinear system.

The primary cause of its nonlinearity is the existence of the Coriolis type forces (forces dependent on the product of the bonded bodies' motion speeds). But even if in the case of slow movements these elements of the dynamic description are neglected in the design of control laws (we consider these forces as disturbances), in the real systems remain the effects of the so-called hard nonlinearities that are part of both mechanical subsystems (friction, backlash, hysteresis) and the control system (saturation, hysteresis).

These nonlinearities can cause both desirable and undesired phenomena where their most significant manifestation is the existence of limit cycles.

This article describes how to obtain a describing function for a non-linear system containing one such non-linear element. This will allow us to further analyse the existence of limit cycles based on the representation of the non-linear element by describing function.

The basic approach for this prediction is based on the application of the extended version of the criteria based on Cauchy's lemma from complex analysis (Nyquist criterion known from the linear control theory) to the equivalent system obtained by a describing function application.

The paper is one part of a more extensive analysis of non-linear systems, where the next part is the application of descriptive functions–the so-called **frequency linearization**–to the existence and basic parameters of the limit cycles analysing.

References

1. Hsu J. C., Meyer A. U., *Modern Control Principles and Applications*, McGraw-Hill (1968)
2. Lyapunov A. M., The General Problem of Motion Stability, (1892), in Russian. Translated in French, *Ann. Fac. Sci*. Toulouse 9, pp. 203-474 (1907). Reprinted in *Ann. Math. Study* No. 17, Princeton Univ. Press (1949).
3. R. Marino, *Int. J. Control*, **42**, pp. 1369-1385 (1985).

4. J.-J. Slotine, S. S. Sastry, *Int. J. Control*, **39**, 2 (1983).
5. Z. Chen , J. Huang.: *Stabilization and Regulation of Nonlinear Systems*, Springer International Publishing, Switzerland, ISBN 978-3-319-08833-4 (2015)
6. Z. Úředníček, *Robotika*, T. Bata university in Zlin (in Czech language), ISBN 978–80–7454–223- 7 (2012)
7. Z. Úředníček, *International journal of mechanics,* **10***,* (2016)
8. Z. Úředníček, R. Drga, *MATEC Web Conf*. **76**, (2016)