NONLINEAR THEORIES FOR THIN SHELLS*

BY

J. LYELL SANDERS, JR.

Harvard University

Abstract. Strain-displacement relations for thin shells valid for large displacements are derived. With these as a starting point approximate strain-displacement relations and equilibrium equations are derived by making certain simplifying assumptions. In particular the middle surface strains are assumed small and the rotations are assumed moderately small. The resulting equations are suitable as a basis for stability investigations or other problems in which the effects of deformation on equilibrium cannot be ignored, but in which the rotations are not too large.

The linearized forms of several of the sets of equations derived herein coincide with small deflection theories in the literature.

Introduction. The literature is not devoid of papers in which some of the effects of finite displacements on the deformation of thin shells are accounted for. This is most obviously the case for papers dealing with the stability of shells, but these have been concerned almost exclusively with cylinders, cones, and spheres. The differential equations governing the phenomenon have been derived specifically for these geometrical shapes. Despite the potential usefulness of a general non-linear theory, the literature on the subject is sparse. It is the purpose of the present paper to derive an exact theory for large deflections of a thin shell with an arbitrary middle surface and then, by making certain simplifying assumptions, to derive from this several approximate theories suitable for applications.

Probably the earliest work of some generality is Marguerre's nonlinear theory of shallow shells [1]. Donnell [2] developed an approximate theory specifically for cylinders and suggested its extension for a general middle surface. The result, a theory for what might be termed "quasi-shallow shells", has been worked out by a number of authors, notably Mushtari and Vlosov [3]. The problem of symmetric deformations of shells of revolution has been reduced to the solution of a pair of equations analogous to the Reissner-Meissner equations by E. Reissner [4]. These several problems are adequately formulated but the general problem presents difficulties not found in the special cases.

The earliest work of a completely general nature appears to be the paper by Synge and Chien [5] followed by a series of papers by Chien [6, 7]. The intrinsic theory of shells developed by Synge and Chien avoids the use of displacements as unknowns in the equations. The theory of shells is deduced from the three-dimensional theory of elasticity and then, by means of series expansions in powers of a small thickness parameter, approximate theories of thin shells are derived. A large number of problem types is found classified according to the relative orders of magnitude of various parameters. Several authors have discussed and criticized this work [8, 9, 10, 11].

An elegant and general formulation of the problem is to be found in the recent paper by Ericksen and Truesdell [8]. In this paper there is a unified treatment of thin shells and curved rods developed as two- and one-dimensional theories respectively without an attempt to deduce them from the three-dimensional theory of elasticity. The consti-

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tutive relations are purposely left out of consideration because they are unnecessary for the description of strain and the establishment of equilibrium conditions. The authors adopt the method of description of deformation originated by the Cosserats and are able to account for transverse shear and normal strains and the rotations associated with couple stresses. They find that in the general case eleven measures of strain are necessary as compared to the six in the usual first approximation theory. The two-dimensional approach to shell theory really evades the question of the approximations involved in the descent from three dimensions, but this seems to be a virtue rather than a defect. Such questions are effectively isolated and shown to belong to the part of the theory in which constitutive relations are established.

An incomplete treatment of the general large deflection theory of thin shells has been given by Novozhilov in [12]. He derives a theory for small middle surface strains, but does not go into detail on further simplifications or discuss approximate equilibrium equations. The author indicates that further simplifications would result if rotations are assumed small. The expressions given for small strain do not vanish for rigid body motions.

A general theory is developed in those chapters of the monograph [3] written by Galimov. This author also begins with a theory based on the assumption of small middle surface strains. The expressions given for the components of bending strain do not form a tensor. In deriving the equilibrium equations the author occasionally neglects middle surface shear strains but retains direct strains. This keeps the coordinate system orthogonal but is at best a questionable procedure. The first system of equations arrived at which is not open to criticism is that belonging to the Donnell-Mushtari-Vlosov approximation.

The developments in the present paper begin with a derivation of an exact system of equations with unrestricted displacements which is undoubtedly a special case of the theory of Ericksen and Truesdell. There is some justification for starting afresh with a simpler approach since we are willing to accept a result of less generality. The present paper is not exclusively concerned with the exact theory but also with the problem of deriving from it approximate systems of equations which may be suitable for application in cases in which the displacements and rotations are restricted in magnitude. The deformations herein are restricted by the Kirchhoff hypotheses and the loading does not include couples distributed over the middle surface.

Since this paper was submitted, the author has learned that two others have arrived independently at nearly the same results. Prof. W. T. Koiter presented his results at a lecture during a recent visit to the United States and Dr. R. W. Leonard of the N.A.S.A. submitted a thesis on the subject to the Virginia Polytechnic Institute [13].

Derivation of Exact Equations for Large Displacements

Geometrical preliminaries. Let the undeformed middle surface of the shell be given by the equations

$$x^i = x^i(\xi^\alpha), \qquad (i = 1, 2, 3; \alpha = 1, 2) \tag{1}$$

where the x^i are cartesian coordinates in space and the ξ^α are curvilinear coordinates on the surface. Let the displacements U^i of material points on the middle surface of the shell be resolved into components tangential and normal to the undeformed middle surface as expressed by the following equation

$$U^i = u^\alpha(\xi)x^i_{,\alpha} + w(\xi)n^i, \tag{2}$$

where $x^i_{,\alpha} \equiv \partial x^i/\partial\xi^\alpha$ are tangent vectors to the coordinate curves on the undeformed middle surface and n^i is the unit normal to the undeformed middle surface. In this paper the coordinates ξ^α will be used to label material particles on both the undeformed middle surface and the deformed middle surface.

Some of the important formulas of the theory of surfaces will be used repeatedly and are reproduced here for convenient reference. For the undeformed middle surface the formula for the squared element of arc in terms of the first fundamental form $g_{\alpha\beta}$ is

$$ds^2 = x^i_{,\alpha}x^i_{,\beta}d\xi^\alpha d\xi^\beta = g_{\alpha\beta}d\xi^\alpha d\xi^\beta; \tag{3}$$

the element of area is

$$da = g^{1/2}d\xi^1 d\xi^2, \tag{4}$$

where g is the determinant of $g_{\alpha\beta}$; and the equations of Gauss, Weingarten and Codazzi are

$$x^i_{,\alpha\beta} = -b_{\alpha\beta}n^i, \tag{5}$$

$$n^i_{,\alpha} = b^\beta_\alpha x^i_{,\beta}, \tag{6}$$

$$b_{\alpha\beta,\gamma} = b_{\alpha\gamma,\beta}, \tag{7}$$

where a comma denotes covariant differentiation with respect to the metric $g_{\alpha\beta}$ and where the second fundamental form $b_{\alpha\beta}$, as here used, differs in sign from the usual definition.

After the displacement U^i given by eq. (2), the material particle originally at x^i will move to the point X^i given by

$$X^i = x^i + U^i = x^i + u^\alpha x^i_{,\alpha} + wn^i. \tag{8}$$

This is the equation of the deformed middle surface in terms of the parameters ξ^α. Tangent vectors to the coordinate curves on the deformed middle surface are given by

$$X^i_{,\alpha} = \lambda^\gamma_\alpha x^i_{,\gamma} + \mu_\alpha n^i, \tag{9}$$

where

$$\lambda_{\alpha\beta} = x^i_{,\alpha}X^i_{,\beta} = g_{\alpha\beta} + u_{\alpha,\beta} + b_{\alpha\beta}w, \tag{10}$$

$$\lambda^\gamma_\alpha \equiv g^{\gamma\delta}\lambda_{\delta\alpha}, \tag{11}$$

$$\mu_\alpha = X^i_{,\alpha}n^i = w_{,\alpha} - b^\beta_\alpha u_\beta . \tag{12}$$

An expression for the unit normal to the deformed middle surface is found by taking the cross-product of the tangent vectors, the result is

$$N^i = \rho^{-1}\epsilon^{\alpha\beta}\epsilon^{\gamma\delta}\left(\frac{1}{2}\lambda_{\gamma\alpha}\lambda_{\delta\beta}n^i + \lambda_{\gamma\beta}\mu_\alpha x^i_{,\delta}\right), \tag{13}$$

where $\rho = (G/g)^{1/2}$. It is convenient to define the following quantities

$$\left.\begin{aligned} \nu_\alpha &= x^i_{,\alpha}N^i = \rho^{-1}(\lambda^\beta_\alpha\mu_\beta - \lambda^\beta_\beta\mu_\alpha), \\ \cos\omega &= n^iN^i = \frac{1}{2}\rho^{-1}(\lambda^\alpha_\alpha\lambda^\beta_\beta - \lambda^\alpha_\beta\lambda^\beta_\alpha). \end{aligned}\right\} \tag{14}$$

Indices on $\lambda_{\alpha\beta}$, μ_α, and ν_α will always be raised or lowered with the metric $g_{\alpha\beta}$. The squared element of arc on the deformed middle surface is given by

$$dS^2 = X^i_{,\alpha} X^i_{,\beta} d\xi^\alpha d\xi^\beta = G_{\alpha\beta} d\xi^\alpha d\xi^\beta. \tag{15}$$

In terms of $\lambda_{\alpha\beta}$ and μ_α

$$G_{\alpha\beta} = \lambda^\gamma_\alpha \lambda_{\gamma\beta} + \mu_\alpha \mu_\beta. \tag{16}$$

The element of area on the deformed middle surface is given by

$$dA = G^{1/2} d\xi^1 d\xi^2 = \rho\, da. \tag{17}$$

The equations of Gauss, Weingarten and Codazzi are

$$X^i_{;\alpha\beta} = -B_{\alpha\beta} N^i, \tag{18}$$

$$N^i_{,\alpha} = B^\beta_\alpha X^i_{,\beta}, \tag{19}$$

$$B_{\alpha\beta;\gamma} = B_{\alpha\gamma;\beta}, \tag{20}$$

where $B_{\alpha\beta}$ is the second fundamental form of the deformed middle surface. A semi-colon is used to denote covariant differentiation with respect to the metric $G_{\alpha\beta}$. An expression for $B_{\alpha\beta}$ in terms of $\lambda_{\alpha\beta}$, μ_α, ν_α and $\cos\omega$ can be derived as follows

$$\begin{aligned} B_{\alpha\beta} &= X^i_{,\alpha} N^i_{,\beta} = -X^i_{,\alpha\beta} N^i \\ &= -[(\lambda^\gamma_{\alpha,\beta} + b^\gamma_\beta \mu_\alpha) x^i_{;\gamma} + (\mu_{\alpha,\beta} - \lambda^\gamma_\alpha b_{\gamma\beta}) n^i] N^i \\ &= (b^\gamma_\beta \lambda_{\gamma\alpha} - \mu_{\alpha,\beta}) \cos\omega - (\lambda_{\gamma\alpha,\beta} + b_{\beta\gamma}\mu_\alpha)\nu^\gamma. \end{aligned} \tag{21}$$

The following identity will prove to be useful

$$X^i_{,\alpha} N^i = \nu^\beta \lambda_{\beta\alpha} + \mu_\alpha \cos\omega = 0. \tag{22}$$

Equilibrium equations. In the coordinate system of the deformed middle surface the equilibrium equations of the shell are the same as in the linear theory and need not be derived here. They are (see [11], for one): force equilibrium

$$N^{\alpha\beta}_{0;\alpha} + B^\beta_\alpha Q^\alpha_0 + P^\beta = 0, \tag{23}$$

$$Q^\alpha_{0;\alpha} - B_{\alpha\beta} N^{\alpha\beta}_0 + P = 0, \tag{24}$$

moment equilibrium

$$M^{\alpha\beta}_{0;\alpha} - Q^\beta_0 = 0, \tag{25}$$

$$e_{\alpha\beta}(N^{\alpha\beta}_0 + B^\alpha_\gamma M^{\gamma\beta}_0) = 0, \tag{26}$$

where $N^{\alpha\beta}_0$ is the membrane stress resultant, $M^{\alpha\beta}_0$ is the bending moment resultant, and Q^α_0 is the transverse shear stress resultant, all defined with respect to the deformed shell. The quantities P^α and P are applied load intensities per unit of area of the deformed middle surface, $e_{\alpha\beta}$ is the covariant permutation tensor in the deformed coordinate system.

The above equilibrium equations are exact but, of course, the ten stress quantities entering into them do not furnish a complete description of the state of stress throughout the thickness of the shell. However, in thin shell theory it is always assumed that the state of stress is adequately described in terms of these quantities.

Finite strains. The strain quantities entering into a thin shell theory are a matter for definition. The literature of the subject shows a wide variety of choices of strain-displacement relations, particularly for the bending strains. Some choices have been shown to be better than others (see [14]) but at the present time no set of conditions sufficient to render the choice unique has been generally agreed upon. In the present paper the choice has been guided by two considerations, the first of which was the desire to derive a theory which admits a principle of virtual work. This requirement forces a close relation between the equilibrium equations and the strain-displacement relations. The second consideration was simplicity. The resultant choice will be shown to furnish an adequate description of the deformation of the shell provided the Kirchhoff hypotheses are accepted as adequate descriptions of the displacements.

Let A be a simply connected region on the deformed middle surface enclosed by the curve C. The following identity follows from eqs. (23) to (26).

$$\int_A [(N_{0;\alpha}^{\alpha\beta} + B_\alpha^\beta Q_0^\alpha + P^\beta)\delta U_\beta + (Q_{0;\alpha}^\alpha - B_{\alpha\beta}N_0^{\alpha\beta} + P)\delta W + (M_{0;\alpha}^{\alpha\beta} - Q_0^\beta)\delta\phi_\beta + e_{\alpha\beta}(N_0^{\alpha\beta} + B_\gamma^\alpha M_0^{\gamma\beta})\delta\phi]dA = 0. \tag{27}$$

By application of the divergence theorem for a curved surface (27) may be transformed into the following identity which is the preliminary form of the principle of virtual work and all subsequent derivations will proceed from it.

$$\oint_C (N_0^{\alpha\beta}\delta U_\beta + Q_0^\alpha\delta W + M_0^{\alpha\beta}\delta\phi_\beta)\eta_\alpha dS + \int_A (P^\beta\delta U_\beta + P\delta W)dA = \int_A [N_0^{\alpha\beta}(\delta U_{\beta;\alpha} + B_{\alpha\beta}\delta W - e_{\alpha\beta}\delta\phi) + Q_0^\alpha(\delta W_{;\alpha} - B_\alpha^\beta\delta U_\beta + \delta\phi_\alpha) + M_0^{\alpha\beta}(\delta\phi_{\beta;\alpha} - e_{\gamma\beta}B_\alpha^\gamma\delta\phi)]dA, \tag{28}$$

where the virtual displacements δU_α , δW and rotations $\delta\phi_\alpha$, $\delta\phi$ refer to components in the directions of the tangents and normal to the deformed middle surface. In (28) the terms on the left hand side are interpretable as the external virtual work of edge loads and surface loads respectively. The right hand side of (28) might be interpreted as internal virtual work if the coefficients of $N_0^{\alpha\beta}$, Q_0^α and $M_0^{\alpha\beta}$ were identified with strain increments. Such an identification will be postponed. First these coefficients will be written in a different form.

By definition

$$\delta U^i = \delta U^\alpha X_{,\alpha}^i + \delta W N^i = \delta u^\alpha x_{,\alpha}^i + \delta w n^i. \tag{29}$$

Now

$$\delta U_{;\beta}^i = (\delta W_{;\beta} - B_{\alpha\beta}\delta U^\alpha)N^i + (\delta U_{;\beta}^\alpha + B_\beta^\alpha\delta W)X_{,\alpha}^i = \delta U_{,\beta}^i = (\delta w_{,\beta} - b_{\alpha\beta}\delta u^\alpha)n^i + (\delta u_{,\beta}^\alpha + b_\beta^\alpha\delta w)x_{,\alpha}^i. \tag{30}$$

From this, it follows that

$$\delta U_{\beta;\alpha} + B_{\alpha\beta}\delta W = \lambda_{\gamma\beta}\delta\lambda_\alpha^\gamma + \mu_\beta\delta\mu_\alpha, \tag{31}$$

$$\delta W_{;\alpha} - B_\alpha^\gamma\delta U_\gamma = \nu_\gamma\delta\lambda_\alpha^\gamma + \cos\omega\,\delta\mu_\alpha. \tag{32}$$

The rotation around the normal $\delta\phi$ is given in terms of displacements by the formula

$$\delta\phi = \frac{1}{2} e^{\alpha\beta} \delta U_{\beta;\alpha}. \tag{33}$$

By the use of (31) and the fact that $B_{\alpha\beta}$ is symmetric (33) becomes

$$\delta\phi = \frac{1}{2} e^{\alpha\beta} (\lambda_{\gamma\beta} \delta\lambda^{\gamma}_{\alpha} + \mu_\beta \delta\mu_\alpha). \tag{34}$$

Also,

$$e_{\alpha\beta}\delta\phi = \frac{1}{2} (\lambda_{\gamma\beta}\delta\lambda^{\gamma}_{\alpha} + \mu_\beta\delta\mu_\alpha - \lambda_{\gamma\alpha}\delta\lambda^{\gamma}_{\beta} - \mu_\alpha\delta\mu_\beta). \tag{35}$$

From (31) and (35) the coefficient of $N_0^{\alpha\beta}$ in (28) is

$$\frac{1}{2} (\lambda_{\gamma\beta}\delta\lambda^{\gamma}_{\alpha} + \mu_\beta\delta\mu_\alpha + \lambda_{\gamma\alpha}\delta\lambda^{\gamma}_{\beta} + \mu_\alpha\delta\mu_\beta) = \frac{1}{2} \delta(\lambda_{\gamma\beta}\lambda^{\gamma}_{\alpha} + \mu_\alpha\mu_\beta) = \frac{1}{2} \delta G_{\alpha\beta}. \tag{36}$$

The natural definition of the finite membrane strain is thus

$$E_{\alpha\beta} = \frac{1}{2} (G_{\alpha\beta} - g_{\alpha\beta}). \tag{37}$$

The coefficient of Q_0^{α} in (28) is

$$\delta\gamma_\alpha = \delta W_{;\alpha} - B^{\beta}_{\alpha}\delta U_\beta + \delta\phi_\alpha = \nu_\gamma\delta\lambda^{\gamma}_{\alpha} + \cos\omega\,\delta\mu_\alpha + \delta\phi_\alpha. \tag{38}$$

Since the intention is to derive a theory in which transverse shear strains are neglected, set $\delta\gamma_\alpha = 0$ which gives

$$\delta\phi_\alpha = -\nu_\gamma\delta\lambda^{\gamma}_{\alpha} - \cos\omega\,\delta\mu_\alpha, \tag{39}$$

which serves to relate rotations to displacements. From (9), (13), (14) and (39)

$$N^i\delta X^i_{,\alpha} = \nu_\gamma\delta\lambda^{\gamma}_{\alpha} + \cos\omega\,\delta\mu_\alpha = -\delta\phi_\alpha \tag{40}$$

or, since $N^iX^i_{,\alpha} = 0$, it follows that

$$\delta\phi_\alpha = X^i_{,\alpha}\delta N^i. \tag{41}$$

A finite transverse shear strain γ_α consistent with (38) and the requirement $\gamma_\alpha = 0$ may be defined as follows

$$\gamma_\alpha = N^iX^i_{,\alpha} = \lambda^{\gamma}_{\alpha}\nu_\gamma + \mu_\alpha\cos\omega \tag{42}$$

The coefficient of $M_0^{\alpha\beta}$ in (28) may be found in terms of $B_{\alpha\beta}$ and $G_{\alpha\beta}$ as follows. From (41)

$$\delta\phi_{\beta;\alpha} = X^i_{;\beta\alpha}\delta N^i + X^i_{,\beta}\delta N^i_{,\alpha} = -B_{\alpha\beta}N^i\delta N^i + X^i_{,\beta}\delta N^i_{,\alpha} = X^i_{,\beta}\delta N^i_{,\alpha} \tag{43}$$

since $N^i\delta N^i = 0$. Now recall that

$$B_{\alpha\beta} = N^i_{,\alpha}X^i_{,\beta}\,,$$

so that

$$\begin{aligned} \delta B_{\alpha\beta} &= X^i_{,\beta}\delta N^i_{,\alpha} + N^i_{,\alpha}\delta X^i_{,\beta} \\ &= \delta\phi_{\beta;\alpha} + B^{\gamma}_{\alpha}X^i_{,\gamma}\delta X^i_{,\beta} \end{aligned} \tag{44}$$

This gives

$$\delta\phi_{\beta;\alpha} = \delta B_{\alpha\beta} - B_\alpha^\gamma X^i_{,\gamma}\delta X^i_{,\beta} . \tag{45}$$

From (9) and (35), it follows that

$$e_{\alpha\beta}\delta\phi = \frac{1}{2}(X^i_{,\beta}\delta X^i_{,\alpha} - X^i_{,\alpha}\delta X^i_{,\beta}). \tag{46}$$

From (45) and (46) the coefficient of $M_0^{\alpha\beta}$ in (28) reduces to

$$\delta B_{\alpha\beta} - \frac{1}{2}B_\alpha^\gamma\delta(X^i_{,\beta}X^i_{,\gamma}) = \delta B_{\alpha\beta} - \frac{1}{2}B_\alpha^\gamma\delta G_{\beta\gamma} = \delta B_{\alpha\beta} - B_\alpha^\gamma\delta E_{\beta\gamma} . \tag{47}$$

Using the foregoing results the right hand side of (28) may be written

$$\int_A [N_0^{\alpha\beta}\delta E_{\alpha\beta} + M_0^{\alpha\beta}(\delta B_{\alpha\beta} - B_\alpha^\gamma\delta E_{\beta\gamma})]dA. \tag{48}$$

There is obviously some difficulty in defining a finite bending strain tensor because the coefficient of $M_0^{\alpha\beta}$ in this expression is not the exact variation of anything. However, a way to proceed suggests itself if (48) is rewritten in the following form

$$\int_A [(N_0^{\alpha\beta} - B_\gamma^\beta M_0^{\gamma\alpha})\delta E_{\alpha\beta} + M_0^{\alpha\beta}\delta B_{\alpha\beta}]dA. \tag{49}$$

Define a finite bending strain tensor by

$$K_{\alpha\beta} = B_{\alpha\beta} - b_{\alpha\beta} . \tag{50}$$

Define a modified membrane stress tensor by

$$N^{\alpha\beta} = N_0^{\alpha\beta} - B_\gamma^\beta M_0^{\gamma\alpha}. \tag{51}$$

Since $B_{\alpha\beta}$ is symmetric there will be no loss in generality by defining a modified bending moment tensor by

$$M^{\alpha\beta} = \frac{1}{2}(M_0^{\alpha\beta} + M_0^{\beta\alpha}). \tag{52}$$

Note that the third moment equilibrium equation (26) is equivalent to the statement that $N^{\alpha\beta}$ is symmetric. In terms of the newly defined quantities (49) becomes

$$\int_A (N^{\alpha\beta}\delta E_{\alpha\beta} + M^{\alpha\beta}\delta K_{\alpha\beta})\, dA. \tag{53}$$

The details will not be shown here but this expression for the internal virtual work may be derived from the three dimensional theory by integration through the thickness of the shell and without approximation provided the displacements are restricted by the Kirchhoff hypotheses.

Modified tensors similar to, or identical to, $N^{\alpha\beta}$ and $M^{\alpha\beta}$ have been introduced by several authors. We emphasize the importance of these quantities more than has been done by using them to replace $N_0^{\alpha\beta}$ and $M_0^{\alpha\beta}$ in all the subsequent equations. The fact that the six unknowns $N^{\alpha\beta}$ and $M^{\alpha\beta}$ are sufficient for the theory in place of the eight unknowns $N_0^{\alpha\beta}$ and $M_0^{\alpha\beta}$ simply means that the equilibrium eqs. (23) to (26) have a slightly more general form than is necessary for a theory with three displacement quantities. The reduction in number of unknowns is not possible in a theory which admits couple stresses and additional modes of deformation.

The two strain tensors $E_{\alpha\beta}$ and $K_{\alpha\beta}$ (or equivalently the two tensors $G_{\alpha\beta}$ and $B_{\alpha\beta}$) furnish an adequate description of the deformation of the shell as shown by the following argument. In the first place the deformation of the shell is completely described in terms of the displacements of points on the middle surface provided the displacements throughout the thickness are restricted by the Kirchhoff hypotheses. Secondly, from the theory of surfaces, a knowledge of $G_{\alpha\beta}$ and $B_{\alpha\beta}$ as functions of ξ^α and subject to the Gauss and Codazzi integrability conditions (which in the present case are equivalent to compatibility conditions) completely determines the deformed middle surface together with a coordinate system (the deformed ξ^α system) except for a rigid body motion.

Modified equilibrium equations. Since new stress quantities have been introduced the equilibrium equations (23) to (26) are no longer quite appropriate for the theory being developed here. Appropriate equations can be derived from the expression (53) for internal virtual work.

$$\begin{aligned} \int_A (N^{\alpha\beta}\delta E_{\alpha\beta} + M^{\alpha\beta}\delta K_{\alpha\beta})\, dA &= \int_A [N^{\alpha\beta}(\delta U_{\beta;\alpha} + B_{\alpha\beta}\, \delta W) \\ &+ M^{\alpha\beta}(-\delta W_{;\alpha\beta} + B^{\gamma}_{\beta;\alpha}\delta U_\gamma + 2B^{\gamma}_{\beta}\delta U_{\gamma;\alpha} + B^{\gamma}_{\alpha}B_{\gamma\beta}\delta W)]\, dA \\ &= \oint_C [(N^{\alpha\beta} + B^{\alpha}_{\gamma}M^{\beta\gamma})\delta U_\alpha + M^{\alpha\beta}_{;\alpha}\delta W + M^{\alpha\beta}\delta\phi_\alpha]\eta_\beta\, dS \\ &- \int [(N^{\alpha\beta}_{;\alpha} + 2B^{\beta}_{\alpha}M^{\alpha\gamma}_{;\gamma} + B^{\beta}_{\gamma;\alpha}M^{\gamma\alpha})\delta U_\beta \\ &+ (M^{\alpha\beta}_{;\alpha\beta} - B_{\alpha\beta}N^{\alpha\beta} - B^{\gamma}_{\alpha}B_{\gamma\beta}M^{\alpha\beta})\delta W]\, dA. \end{aligned} \tag{54}$$

The line integral around C is the external virtual work of the edge forces and moments If a principle of virtual work is required to hold, and if the portion of the shell within C is in equilibrium, then the internal virtual work must equal the external virtual work for arbitrary virtual displacements. Thus the condition of equilibrium is that the last integral in (54) vanishes. This leads to the following equilibrium equations (surface forces have been omitted in this derivation for simplicity).

$$N^{\alpha\beta}_{;\alpha} + 2B^{\beta}_{\alpha}M^{\alpha\gamma}_{;\gamma} + B^{\beta}_{\gamma;\alpha}M^{\gamma\alpha} = 0, \tag{55}$$

$$M^{\alpha\beta}_{;\alpha\beta} - B_{\alpha\beta}N^{\alpha\beta} - B^{\gamma}_{\alpha}B_{\gamma\beta}M^{\alpha\beta} = 0, \tag{56}$$

Equations (55) and (56) are in fact identical to the equations (23) to (25) with Q^α_0 eliminated. Equation (26) is accounted for by the symmetry of $N^{\alpha\beta}$. If Q^α defined by

$$Q^\alpha = M^{\alpha\beta}_{;\beta} \tag{57}$$

is introduced as an approximation to Q^α_0, then the equilibrium equations, in an expanded form, may be written

$$(N^{\alpha\beta} + B^{\beta}_{\gamma}M^{\alpha\gamma})_{;\alpha} + B^{\beta}_{\alpha}Q^\alpha = 0, \tag{58}$$

$$Q^{\alpha}_{;\alpha} - B_{\alpha\beta}(N^{\alpha\beta} + B^{\beta}_{\gamma}M^{\alpha\gamma}) = 0, \tag{59}$$

$$M^{\alpha\beta}_{;\alpha} - Q^\beta = 0. \tag{60}$$

The following equations equivalent to eqs. (58), (59) and (60) express equilibrium of forces and moments in the directions of the tangents and normal to the undeformed middle surface.

$$[\rho\lambda^{\gamma}_{\alpha}(N^{\alpha\beta} + B^{\alpha}_{\delta}M^{\beta\delta})]_{,\beta} + \rho b^{\gamma}_{\beta}\mu_{\alpha}(N^{\alpha\beta} + B^{\alpha}_{\delta}M^{\beta\delta}) + (\rho\nu^{\gamma}Q^{\alpha})_{,\alpha} + \rho b^{\gamma}_{\alpha} \cos\omega Q^{\alpha} + p^{\gamma} = 0, \tag{58$'$}$$

$$(\rho \cos\omega Q^{\alpha})_{,\alpha} - \rho b_{\alpha\gamma}\nu^{\gamma}Q^{\alpha} + [\rho\mu_{\alpha}(N^{\alpha\beta} + B^{\alpha}_{\delta}M^{\beta\delta})]_{,\beta} - \rho b_{\beta\gamma}\lambda^{\gamma}_{\alpha}(N^{\alpha\beta} + B^{\alpha}_{\delta}M^{\beta\delta}) + p = 0, \tag{59$'$}$$

$$(\rho M^{\alpha\beta})_{,\beta} + \rho C^{\alpha}_{\beta\delta}\, M^{\beta\delta} - \rho Q^{\alpha} = 0, \tag{60$'$}$$

where $C^{\delta}_{\alpha\beta}$ is the difference between the Christoffel symbols of the deformed and undeformed coordinate systems. It is expressed in terms of displacements by

$$C^{\delta}_{\alpha\beta} = G^{\delta\epsilon}[\lambda^{\gamma}_{\alpha,\beta}\lambda_{\gamma\epsilon} + \mu_{\alpha,\beta}\mu_{\epsilon} + b^{\gamma}_{\beta}(\lambda_{\gamma\epsilon}\mu_{\alpha} - \lambda_{\gamma\alpha}\mu_{\epsilon})].$$

The load terms in the preceding equilibrium equations are forces per unit of area of the undeformed middle surface.

Small Strain Approximations

Strain-displacement and equilibrium equations. If the shell bends without extension then the metric of the deformed middle surface is the same as the metric of the undeformed middle surface and eqs. (58), (59) and (60) can be simplified by replacing a semi-colon by a comma or eqs. (58)′, (59)′ and (60)′ can be simplified by setting $\rho = 1$ and $C^{\delta}_{\alpha\beta} = 0$. The expression for $E_{\alpha\beta}$ does not simplify but the expression (21) for $B_{\alpha\beta}$ simplifies by setting $\rho = 1$ in the expression (14) for $\cos\omega$ and ν_{α}. The system of equations thus obtained could serve as an approximate theory for small strains provided it is legitimate to neglect the covariant derivative of $E_{\alpha\beta}$ as well as $E_{\alpha\beta}$. In some applications this might not be true in which case another method of approximation may be substituted. One of the groups of terms in the expression for $B_{\alpha\beta}$ (see [3]) may be transformed as follows

$$\lambda_{\gamma\alpha} \cos\omega - \mu_{\alpha}\nu_{\gamma} = (X^{i}_{,\alpha}x^{i}_{,\gamma})(N^{i}n^{i}) - (X^{i}_{,\alpha}n^{i})(N^{i}x^{i}_{,\gamma}). \tag{61}$$

An application of Lagrange's identity shows this to be equal to

$$\rho^{-1}\epsilon^{\sigma\rho}\epsilon_{\delta\gamma}\lambda^{\delta}_{\sigma}G_{\alpha\rho} = \rho^{-1}\epsilon^{\sigma\rho}\epsilon_{\delta\gamma}\lambda^{\delta}_{\sigma}(g_{\alpha\rho} + 2E_{\alpha\rho}). \tag{62}$$

If we neglect $E_{\alpha\beta}$ and set $\rho = 1$, then an approximation to $B_{\alpha\beta}$ is

$$B_{\alpha\beta} = b_{\alpha\beta}\lambda^{\sigma}_{\sigma} - b^{\gamma}_{\beta}\lambda_{\gamma\alpha} - \frac{1}{2}\mu_{\alpha,\beta}(\lambda^{\sigma}_{\sigma}\lambda^{\rho}_{\rho} - \lambda^{\sigma}_{\rho}\lambda^{\rho}_{\sigma}) - \lambda^{\gamma}_{\alpha,\beta}(\lambda^{\sigma}_{\gamma}\mu_{\sigma} - \lambda^{\sigma}_{\sigma}\mu_{\gamma}). \tag{63}$$

Equilibrium equations appropriate to this approximation can be found by use of the principle of virtual work.

Constitutive relations. Consistent constitutive relations for the linear small strain theory of thin elastic shells have been derived in [14, 15 and 16]. These derivations require only minor modifications in the case of finite displacements and small strains so they will not be reproduced here. For a thin shell of uniform thickness h composed of an isotropic hookean material, the constitutive relations are the same as in Love's first approximation, namely the linear relations

$$\begin{aligned} EhE_{\alpha\beta} &= (1+\nu)g_{\alpha\gamma}g_{\beta\delta}N^{\gamma\delta} - \nu g_{\alpha\beta}g_{\gamma\delta}N^{\gamma\delta}, \\ \frac{1}{12}Eh^3K_{\alpha\beta} &= (1+\nu)g_{\alpha\gamma}g_{\beta\delta}M^{\gamma\delta} - \nu g_{\alpha\beta}g_{\gamma\delta}M^{\gamma\delta}. \end{aligned} \tag{64}$$

According to [14] these relations may be used even if the definition of $K_{\alpha\beta}$ in terms of displacements is altered by the addition of terms of the form $B_\alpha^\gamma E_{\gamma\beta}$. A similar argument to that in [14] shows that $N^{\alpha\beta}$ may be altered by addition of terms of the form $B_\gamma^\beta M^{\alpha\gamma}$. In the case of small strain the indices on $N^{\alpha\beta}$ and $M^{\alpha\beta}$ may be raised and lowered with the metric $g_{\alpha\beta}$ instead of $G_{\alpha\beta}$ with negligible error.

If the material of the shell is not elastic and isotropic the relations (64) must be replaced by others appropriate for the material. However, the strain-displacement relations and the equilibrium equations given previously are unaffected by the material so long as transverse shear and normal strains can be neglected.

Approximation of Small Strains and Moderately Small Rotations

The exact theory was somewhat simplified by the assumption that the middle surface strains are small, but the equations are still very complicated. Considerable additional simplification can be achieved if the rotations are assumed small also. This simplification will be carried out in the following.

Approximate strains. For infinitesimal displacements and rotations it is evident from (33) and (38) that the rotations are given by the formulas

$$\phi = \frac{1}{2}\epsilon^{\alpha\beta}u_{\beta,\alpha} \tag{65}$$

and

$$\phi_\alpha = -w_{,\alpha} + b_\alpha^\beta u_\beta = -\mu_\alpha. \tag{66}$$

For small but finite rotations it is convenient to think of the expressions in (65) and (66) as rotations (just as in the linear theory of shells). Purely for convenience, suppose that the coordinates ξ^α have the units of length so that ϕ, ϕ_α and $E_{\alpha\beta}$ are dimensionless. The following order of magnitude assumptions will lead to a theory for small strain and moderately small rotations

$$\phi \quad \text{or} \quad \tfrac{1}{2}(u_{\alpha,\beta} - u_{\beta,\alpha}) = 0(\epsilon) \tag{67}$$

$$\phi_\alpha = 0(\epsilon) \tag{68}$$

$$\tfrac{1}{2}(u_{\alpha,\beta} + u_{\beta,\alpha}) + b_{\alpha\beta}w = 0(\epsilon^2) \tag{69}$$

where ϵ is a number small compared to unity. Write $\lambda_{\alpha\beta}$ in the form

$$\lambda_{\alpha\beta} = g_{\alpha\beta} - \epsilon_{\alpha\beta}\phi + \tfrac{1}{2}(u_{\alpha,\beta} + u_{\beta,\alpha} + 2b_{\alpha\beta}w) \tag{70}$$

From (16) and (37) we find that $E_{\alpha\beta}$ is $0(\epsilon^2)$ and is given approximately by the expression

$$E_{\alpha\beta} = \tfrac{1}{2}(u_{\alpha,\beta} + u_{\beta,\alpha}) + b_{\alpha\beta}w + \tfrac{1}{2}\phi_\alpha\phi_\beta + \tfrac{1}{2}g_{\alpha\beta}\phi^2 \tag{71}$$

The order of magnitude assumption (69) was made so that those terms would not dominate the expression for $E_{\alpha\beta}$; otherwise the linear theory would result.

From $G_{\alpha\beta} - g_{\alpha\beta} = 0\ (\epsilon^2)$ it follows that $\rho = 1 + 0\ (\epsilon^2)$; then from (67) to (70) and (14) it follows that

$$\nu^{\delta} = -\mu^{\delta} + 0(\epsilon^2) \tag{72}$$

$$\cos\omega = 1 + 0(\epsilon^2) \tag{73}$$

From the foregoing and (21) and with due regard for symmetry, $K_{\alpha\beta} = B_{\alpha\beta} - b_{\alpha\beta}$ is given to $O(\epsilon)$ by

$$K_{\alpha\beta} = \tfrac{1}{2}(\phi_{\beta,\alpha} + \phi_{\alpha,\beta}) + \tfrac{1}{2}(\epsilon_{\alpha\gamma}b^{\gamma}_{\beta} + \epsilon_{\beta\gamma}b^{\gamma}_{\alpha})\phi \tag{74}$$

(assuming $\phi_{,\alpha} = O(\epsilon)$). Similarly the expression for γ_α becomes

$$\gamma_\alpha = \phi_\alpha + w_{,\alpha} - b^{\gamma}_{\alpha}\,\mu_\gamma = 0, \tag{75}$$

It is conceivable that the above expression for $K_{\alpha\beta}$ could become $O(\epsilon^2)$, in which case the approximation would be invalid, but in this case the shell would be acting essentially as a membrane and errors in $K_{\alpha\beta}$ would be immaterial. The expression given by (74) for the bending strain is linear and this simplifies the theory considerably. The definition of bending strain given above is the same as the one derived in references [14] and [17].

Approximate equilibrium equations. Approximate equilibrium equations corresponding to the approximate strain-displacement equations may be found by the same method used to derive the equilibrium equations (55) and (56). Since the strains are small dA may be replaced by da and we have

$$\begin{aligned}\int_a \{N^{\alpha\beta}[\delta u_{\alpha,\beta} + b_{\alpha\beta}\delta w - \phi_\alpha(\delta w_{,\beta} - b^{\gamma}_{\beta}\delta u_\gamma) + \tfrac{1}{2}g_{\alpha\beta}\phi\epsilon^{\gamma\delta}\delta u_{\delta,\gamma}]\\ + M^{\alpha\beta}[\delta\phi_{\alpha,\beta} + \tfrac{1}{2}b^{\gamma}_{\alpha}(\delta u_{\gamma,\beta} - \delta u_{\beta,\gamma})] + Q^{\alpha}(\delta\phi_\alpha + \delta w_{,\alpha} - b^{\gamma}_{\alpha}\delta u_\gamma)\}\,da\\ = \oint_c - \int_a \{[N^{\alpha\beta}_{,\alpha} - b^{\beta}_{\gamma}\phi_\alpha N^{\alpha\gamma} + \tfrac{1}{2}\epsilon^{\gamma\beta}(\phi N^{\delta}_{\delta})_{,\gamma} + \tfrac{1}{2}(b^{\beta}_{\alpha}M^{\alpha\gamma})_{,\gamma} - \tfrac{1}{2}(b^{\gamma}_{\alpha}M^{\alpha\beta})_{,\gamma}\\ + b^{\beta}_{\alpha}Q^{\alpha}]\delta u_\beta + [Q^{\alpha}_{,\alpha} - b_{\alpha\beta}N^{\alpha\beta} - (\phi_\alpha N^{\alpha\beta})_{,\beta}]\delta w + [M^{\alpha\beta}_{,\beta} - Q^{\alpha}]\delta\phi_\alpha\}\,da.\end{aligned} \tag{76}$$

By inspection the equilibrium equations are

$$N^{\alpha\beta}_{,\alpha} - b^{\beta}_{\gamma}\phi_\alpha N^{\alpha\gamma} + \tfrac{1}{2}\epsilon^{\alpha\beta}(\phi N^{\gamma}_{\gamma})_{,\alpha} + \tfrac{1}{2}(b^{\beta}_{\alpha}M^{\alpha\gamma})_{,\gamma} - \tfrac{1}{2}(b^{\gamma}_{\alpha}M^{\alpha\beta})_{,\gamma} + b^{\beta}_{\alpha}Q^{\alpha} + P^{\beta} = 0, \tag{77}$$

$$Q^{\alpha}_{,\alpha} - b_{\alpha\beta}N^{\alpha\beta} - (\phi_\alpha N^{\alpha\beta})_{,\beta} + p = 0, \tag{78}$$

$$M^{\alpha\beta}_{,\beta} - Q^{\alpha} = 0, \tag{79}$$

where the load terms p^{α} and p have been supplied. These equations express equilibrium of forces and moments in directions parallel to the tangents and normal of the undeformed middle surface. In the left-hand side of (76) $\delta\phi_\alpha$ could be expressed in terms of displacements and the term $Q^{\alpha}\delta\gamma_\alpha$ could be omitted. The result for the equilibrium equations would be (77) and (78) with Q^{α} eliminated by means of (79).

Boundary conditions. The Kirchhoff boundary conditions may be obtained from the boundary integral in (76) which when written out reads

$$\begin{aligned}\oint_c \{[N^{\alpha\beta} - \tfrac{1}{2}\epsilon^{\alpha\beta}\phi N^{\gamma}_{\gamma} + \tfrac{1}{2}b^{\alpha}_{\gamma}M^{\gamma\beta} - \tfrac{1}{2}b^{\beta}_{\gamma}M^{\alpha\gamma}]\delta u_\alpha\\ + (Q^{\beta} - \phi_\alpha N^{\alpha\beta})\delta w + M^{\alpha\beta}\delta\phi_\alpha\}\,n_\beta\,ds.\end{aligned} \tag{80}$$

Let t^α be the unit tangent to the curve c, then $n_\beta = \epsilon_{\beta\gamma}t^\gamma$ is the unit normal to c in the surface a. Let

$$\phi_\alpha = \phi_s t_\alpha + \phi_n n_\alpha , \tag{81}$$

where ϕ_s and ϕ_n are scalars. From (81)

$$\phi_s = \phi_\alpha t^\alpha = (-w_{,\alpha} + b^\gamma_\alpha u_\gamma)t^\alpha = -\frac{dw}{ds} + b^\gamma_\alpha u_\gamma t^\alpha \tag{82}$$

Obviously ϕ_s is not independent of w and u_α on c. The last term in (80), namely,

$$\oint_c M^{\alpha\beta}\delta\phi_\alpha n_\beta \, ds = \oint_c M^{\alpha\beta}\left[(-\frac{d\delta w}{ds} + b^\gamma_\delta \delta u_\gamma t^\delta)t_\alpha + \delta\phi_n n_\alpha\right]n_\beta \, ds \tag{83}$$

becomes, upon integrating by parts,

$$\oint_c \left[\frac{d}{ds}(M^{\alpha\beta}t_\alpha n_\beta)\delta w + M^{\gamma\beta}b^\alpha_\delta t^\delta t_\gamma n_\beta \delta u_\alpha + M^{\alpha\beta}n_\alpha n_\beta \delta\phi_n\right] ds \tag{84}$$

assuming c has a continuously turning tangent. Altogether (80) becomes

$$\oint_c \Big\{[N^{\alpha\beta} - \tfrac{1}{2}\epsilon^{\alpha\beta}\phi N^\gamma_\gamma + \tfrac{1}{2}b^\alpha_\gamma M^{\gamma\beta} - \tfrac{1}{2}b^\beta_\gamma M^{\alpha\gamma} + b^\alpha_\gamma M^{\delta\beta}t^\gamma t_\delta]n_\beta \, \delta u_\alpha$$

$$+ \left[Q^\beta n_\beta - \phi_\alpha N^{\alpha\beta}n_\beta + \frac{d}{ds}(M^{\alpha\beta}n_\beta t_\alpha)\right]\delta w + M^{\alpha\beta}n_\alpha n_\beta \delta\phi_n\Big\} \, ds. \tag{85}$$

From this the boundary conditions on c may be read off. They are: prescribe

$$[N^{\alpha\beta} - \tfrac{1}{2}\epsilon^{\alpha\beta}\phi N^\gamma_\gamma + \tfrac{1}{2}b^\alpha_\gamma M^{\gamma\beta} - \tfrac{1}{2}b^\beta_\gamma M^{\alpha\gamma} + b^\alpha_\gamma M^{\delta\beta}t^\gamma t_\delta]n_\beta \quad \text{or} \quad u_\alpha \tag{86}$$

$$Q^\beta n_\beta - \phi_\alpha N^{\alpha\beta}n_\beta + \frac{d}{ds}(M^{\alpha\beta}n_\beta t_\alpha) \quad \text{or} \quad w \tag{87}$$

$$M^{\alpha\beta}n_\alpha n_\beta \quad \text{or} \quad \phi_n \tag{88}$$

Further Approximations

Small rotation about the normal. If the rotation about the normal can be neglected compared to the other two rotations, then the equations can be simplified further. The importance of the rotation about the normal is not entirely established at the present time. Several linear theories for thin shells have been constructed which differ from Love's first approximation only by terms in the bending strain proportional to the rotation about the normal. The differences between these theories and Love's are tabulated in [14] where the general validity of these theories is questioned. That the rotation about the normal can sometimes be neglected is evidenced by the fact that these theories lead to very nearly the same results as more accurate theories in some specific applications. See, for example, [18]. On the other hand these theories lead to erroneous results in other applications. See [19 and 20].

For those cases in which the approximation is valid the strains $E_{\alpha\beta}$ and $K_{\alpha\beta}$ given by equations (71) and (74) can be simplified to read

$$E_{\alpha\beta} = \tfrac{1}{2}(u_{\alpha,\beta} + u_{\beta,\alpha}) + b_{\alpha\beta}w + \tfrac{1}{2}\phi_\alpha\phi_\beta , \tag{89}$$

$$K_{\alpha\beta} = \tfrac{1}{2}(\phi_{\alpha,\beta} + \phi_{\beta,\alpha}). \tag{90}$$

The corresponding approximate equilibrium equations (obtained via the virtual work principle) are

$$N^{\alpha\beta}_{,\beta} + b^{\alpha}_{\beta}Q^{\beta} - b^{\alpha}_{\beta}\phi_{\gamma}N^{\beta\gamma} + p^{\alpha} = 0, \tag{91}$$

$$Q^{\alpha}_{,\alpha} - b_{\alpha\beta}N^{\alpha\beta} - (\phi_{\alpha}N^{\alpha\beta})_{,\beta} + p = 0, \tag{92}$$

$$M^{\alpha\beta}_{,\beta} - Q^{\alpha} = 0, \tag{93}$$

and thė boundary conditions are to prescribe

$$[N^{\alpha\beta} + b^{\alpha}_{\delta}t^{\delta}t_{\gamma}M^{\gamma\beta}]n_{\beta} \quad \text{or} \quad u_{\alpha} , \tag{94}$$

$$Q^{\beta}n_{\beta} - \phi_{\alpha}N^{\alpha\beta}n_{\beta} + \frac{d}{ds}(M^{\alpha\beta}t_{\alpha}n_{\beta}) \quad \text{or} \quad w, \tag{95}$$

$$M^{\alpha\beta}n_{\alpha}n_{\beta} \quad \text{or} \quad \phi_{n} . \tag{96}$$

When these equations are linearized they reduce, essentially, to those given in [11].

The Donnell-Mushtari-Vlasov approximation. A further simplification of the above equations is possible under assumptions discussed in [3 and 16]. This consists in neglecting the term containing u_{α} in the expression for μ_{α} with the following results for strains,

$$E_{\alpha\beta} = \tfrac{1}{2}(u_{\alpha,\beta} + u_{\beta,\alpha}) + b_{\alpha\beta}w + \tfrac{1}{2}w_{,\alpha}w_{,\beta} , \tag{97}$$

$$K_{\alpha\beta} = \tfrac{1}{2}(\phi_{\alpha,\beta} + \phi_{\beta,\alpha}) = -w_{,\alpha\beta} , \tag{98}$$

$$\gamma_{\alpha} = \phi_{\alpha} + w_{,\alpha} \tag{99}$$

for equilibrium equations,

$$N^{\alpha\beta}_{,\beta} + p^{\alpha} = 0, \tag{100}$$

$$Q^{\alpha}_{,\alpha} - b_{\alpha\beta}N^{\alpha\beta} + (w_{,\alpha}N^{\alpha\beta})_{,\beta} + p = 0, \tag{101}$$

$$M^{\alpha\beta}_{,\beta} - Q^{\alpha} = 0, \tag{102}$$

and for boundary conditions prescribe

$$N^{\alpha\beta}n_{\beta} \quad \text{or} \quad u_{\alpha} , \tag{103}$$

$$(Q^{\beta} + w_{,\alpha}N^{\alpha\beta})n_{\beta} + \frac{d}{ds}(M^{\alpha\beta}t_{\alpha}n_{\beta}) \quad \text{or} \quad w, \tag{104}$$

$$M^{\alpha\beta}n_{\alpha}n_{\beta} \quad \text{or} \quad \phi_{n} . \tag{105}$$

Marguerre's shallow shell equations. If applied to a shallow shell the preceding equations can be further simplified because of the geometry. Suppose that the shell is nearly flat and parallel to the $x^3 = z = 0$ plane, and that the squares of the slopes of the shell with respect to the $z = 0$ plane may be neglected. Then, approximately:

$$b_{\alpha\beta} = -z_{,\alpha\beta} . \tag{106}$$

Since the displacements u_{α} are considered small compared to w, the horizontal displacements U_α and the vertical displacements W are given approximately in terms of u_α and w by

$$u_\alpha \approx U_\alpha + z_{,\alpha} W, \qquad w \approx W \tag{107}$$

In terms of U_α and W the membrane strain $E_{\alpha\beta}$, eq. (97) becomes:

$$E_{\alpha\beta} = \tfrac{1}{2}(U_{\alpha,\beta} + U_{\beta,\alpha} + z_{,\alpha} W_{,\beta} + z_{,\beta} W_{,\alpha} + W_{,\alpha} W_{,\beta}) \tag{108}$$

The strains $K_{\alpha\beta}$ and γ_α are as in eqs. (98) and (99). The conditions of equilibrium in the horizontal and vertical directions are

$$N^{\alpha\beta}_{,\beta} + q^\alpha = 0, \tag{109}$$

$$Q^\alpha_{,\alpha} + [(z_{,\alpha} + W_{,\alpha})N^{\alpha\beta}]_{,\beta} + q = 0, \tag{110}$$

$$M^{\alpha\beta}_{,\beta} - Q^\alpha = 0, \tag{111}$$

where q^α is the horizontal load intensity and q is the vertical load intensity. The boundary conditions are to prescribe

$$N^{\alpha\beta} n_\beta \quad \text{or} \quad U_\alpha, \tag{112}$$

$$[Q^\beta + (z_{,\alpha} + W_{,\alpha})N^{\alpha\beta}]n_\beta + \frac{d}{ds}(M^{\alpha\beta} t_\alpha n_\beta) \quad \text{or} \quad W, \tag{113}$$

$$M^{\alpha\beta} n_\alpha n_\beta \quad \text{or} \quad \phi_n . \tag{114}$$

These are Marguerre's shallow shell equations in tensor form [1].

Concluding remarks

Several nonlinear theories for thin shells have been derived in increasing stages of approximation. The linearization of these equations, which is more or less obvious, has been omitted but in most cases the resulting linear equations are essentially the same as shell equations already given in the literature. In all cases the theories are first approximation theories in the sense that transverse shear and normal strains are neglected.

In each of the theories derived in this paper the equilibrium equations and strain-displacement relations are related by a principle of virtual work and hence the usual variational principles may be formulated and proved.

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Appendix

Shell equations in ordinary notation. In the ordinary notation with lines of curvature for coordinates (as used in [10] and [17]) the expressions (71) for middle surface strains and (74) for bending strains are:

$$\epsilon_{11} = (\alpha_1\alpha_2)^{-1}[\alpha_2 u_{1,1} + \alpha_{1,2} u_2 + \alpha_1\alpha_2 R_1^{-1} w + 1/2\alpha_1\alpha_2\phi_1^2 + 1/2\alpha_1\alpha_2\phi^{2*}], \tag{A-1}$$

$$\epsilon_{12} = 1/2(\alpha_1\alpha_2)^{-1}[\alpha_2 u_{2,1} + \alpha_1 u_{1,2} - \alpha_{1,2} u_1 - \alpha_{2,1} u_2 + \alpha_1\alpha_2\phi_1\phi_2], \tag{A-2}$$

$$\kappa_{11} = (\alpha_1\alpha_2)^{-1}[\alpha_2\phi_{1,1} + \alpha_{1,2}\phi_2], \tag{A-3}$$

$$\kappa_{12} = 1/2(\alpha_1\alpha_2)^{-1}[\alpha_2\phi_{2,1} + \alpha_1\phi_{1,2} - \alpha_{1,2}\phi_1 - \alpha_{2,1}\phi_2 + \alpha_1\alpha_2(R_2^{-1} - R_1^{-1})\phi^*], \tag{A-4}$$

where a comma means partial differentiation with respect to ξ_1 or ξ_2 as the subscript following the comma indicates.

The rotations are given by

$$\phi_1 = -\alpha_1^{-1}w_{,1} + R_1^{-1}u_1{}^{**}, \tag{A-5}$$

$$\phi = 1/2(\alpha_1\alpha_2)^{-1}[(\alpha_2 u_2)_{,1} - (\alpha_1 u_1)_{,2}]. \tag{A-6}$$

Here, as in the following, missing equations may be obtained by interchanging subscripts 1 and 2 and changing the sign of ϕ. The equilibrium eqs. (77), (78) and (79) now read

$$\begin{aligned}(\alpha_2 N_{11})_{,1} + (\alpha_1 N_{12})_{,2} + \alpha_{1,2}N_{12} - \alpha_{2,1}N_{22} + \alpha_1\alpha_2 R_1^{-1}Q_1{}^{**}\\ + 1/2\alpha_1[(R_1^{-1} - R_2^{-1})M_{12}]^*_{,2} - \alpha_1\alpha_2 R_1^{-1}(\phi_1 N_{11} + \phi_2 N_{12})^{**}\\ - 1/2\alpha_1[\phi(N_{11} + N_{22})]^*_{,2} + \alpha_1\alpha_2 p_1 = 0,\end{aligned} \tag{A-7}$$

$$\begin{aligned}(\alpha_2 Q_1)_{,1} + (\alpha_1 Q_2)_{,2} - \alpha_1\alpha_2(R_1^{-1}N_{11} + R_2^{-1}N_{22}) - (\alpha_2\phi_1 N_{11} + \alpha_2\phi_2 N_{12})_{,1}\\ - (\alpha_1\phi_1 N_{12} + \alpha_1\phi_2 N_{22})_{,2} + \alpha_1\alpha_2 p = 0,\end{aligned} \tag{A-8}$$

$$(\alpha_2 M_{11})_{,1} + (\alpha_1 M_{12})_{,2} + \alpha_{1,2}M_{12} - \alpha_{2,1}M_{22} - \alpha_1\alpha_2 Q_1 = 0. \tag{A-9}$$

The boundary conditions on an edge ξ_1 = constant are to prescribe

$$\left.\begin{array}{rcl} N_{11} & \text{or} & u_1,\\ N_{12} + 1/2(3R_2^{-1} - R_1^{-1})M_{12}^{**} + 1/2(N_{11} + N_{22})\phi^* & \text{or} & u_2,\\ Q_1 + \alpha_2^{-1}M_{12,2} - \phi_1 N_{11} - \phi_2 N_{12} & \text{or} & w,\\ M_{11} & \text{or} & \phi_1.\end{array}\right\} \tag{A-10}$$

The aforementioned rule for interchanging subscripts 1 and 2 applies here as well.

The terms in the preceding equations which drop out when rotations around the normal are neglected have been marked with a single asterisk. The terms which drop out in the Donnell-Mushtari-Vlosov approximation are those with either a single or a double asterisk.

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