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Nonlinear variational inequalities and fixed point theorems

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§ 1. Introduction.

It was proved by Hartman and Stampacchia [8] in 1966 that if $T: R^n \rightarrow R^n$ is a continuous mapping on a compact, convex subset X of R^n, then there exists $x_0 \in X$ such that $\langle Tx_0, x_0 - x \rangle \geqq 0$ for all $x \in X$. This remarkable result has been investigated and generalized in various points of views by Browder [1], [2], Moré [10] and others. For example, Browder extended this theorem to the case of which our considering mappings T are of a compact convex subset X of a topological vector space E into the dual space E^*; see Theorem 2 of [2]. In § 2 of this paper, we shall obtain two generalizations of this Browder's theorem. One of them is Lemma 1 that has various applications. The other is Theorem 3 that generalizes the Browder's result to closed and convex sets in topological vector spaces. We shall also make use of Theorem 3 to prove Theorem 4 that generalizes Moré's theorem [10, Theorem 2.4]. In § 3, using Lemma 1, we shall prove some fixed point theorems. Theorem 5 and Theorem 9 extend Browder's fixed point theorems [1, Theorem 1], [2, Theorem 3]. In § 4, we shall discuss Sion's minimax theorem and Terkelsen's minimax theorem. At first, we shall show that Sion's theorem follows simply from the fundamental and useful theorem of Browder [2, Theorem 1]. Furthermore, we state a necessary and sufficient condition that a minimax condition holds. Using this, we shall generalize Terkelsen's minimax theorem; see Theorems 16 and 17. In § 5, we give another proof for Fan's theorem [5] concerning systems of convex inequalities. The proof is simple. Furthermore, using this Fan's result, we prove Fan's minimax theorem [4] and also obtain a generalization of the result of Browder [2, Lemma 1]; see Theorems 18, 19 and 20. At last, by Lemma 1, we generalize Browder's theorem [2, Theorem 6] for multi valued mappings; see Theorem 21. By the same methods, we shall also generalize Kakutani's fixed point theorem [9]; see Theorem 22.

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§ 2. Variational inequalities.

Throughout this paper, we assume that a topological space is separated and a topological vector space is real. We also denote by R the set of all real numbers. In [2], Browder proved the following useful theorem.

THEOREM 1 (Browder). *Let X be a nonempty compact convex subset of a topological vector space E (where we assume that E is separated but not necessarily locally convex). Let T be a mapping of X into 2^X, where for each x in X, $T(x)$ is a nonempty convex subset of X [resp. open in X]. Suppose further that for each y in X, $T^{-1}(y)=\{x\in X; y\in T(x)\}$ is open in X [resp. a nonempty convex subset of X]. Then, there exists x_0 in X such that $x_0\in T(x_0)$.*

By using this, we can prove the following Lemma 1. However, we shall directly give a proof.

LEMMA 1. *Let X be a nonempty compact convex subset of a topological vector space E and let F be a real valued function on $X\times X$ satisfying:*

(1) *For each $y\in X$, the function $F(x, y)$ of x is upper semicontinuous;*

(2) *for each $x\in X$, the function $F(x, y)$ of y is convex;*

(3) *$F(x, x)\geqq c$ for all $x\in X$ with some real number $c\in R$. Then, there exists $x_0\in X$ such that $F(x_0, y)\geqq c$ for all $y\in X$.*

PROOF. Suppose that for each $x\in X$, there exists $y\in X$ such that $F(x, y) < c$. Setting $A_y=\{x\in X; F(x, y)<c\}$ for each $y\in X$, we have $X=\bigcup_{y\in X} A_y$. Since X is compact, there exists a finite family $\{y_1, y_2, \cdots, y_n\}$ such that $X=\bigcup_{i=1}^n A_{y_i}$. Let $\{\beta_1, \beta_2, \cdots, \beta_n\}$ be a partition of unity corresponding to this covering, i. e. each β_i is a continuous mapping of X into $[0, 1]$ which vanishes outside of A_{y_i}, while $\sum_{i=1}^n \beta_i(x)=1$ for all x in X. For each i such that $\beta_i(x)\neq 0$, x lies in A_{y_i}, so that $F(x, y_i)<c$. Hence we have that

$$\sum_{i=1}^n \beta_i(x)F(x, y_i)<c$$

for all $x\in X$. Define a continuous mapping p of X into X by setting

$$p(x)=\sum_{i=1}^n \beta_i(x)y_i .$$

By using the Brouwer's fixed point theorem, we obtain an element $x_0\in X$ such that

$$x_0=p(x_0)=\sum_{i=1}^n \beta_i(x_0)y_i .$$

For this point x_0, we have

$$c\leqq F(x_0, x_0)=F(x_0, \sum_{i=1}^n \beta_i(x_0)y_i)$$

$$\leqq \sum_{i=1}^{n} \beta_i(x_0)F(x_0, y_i) < c\,.$$

This is a contradiction. Therefore, there exists $x_0 \in X$ such that $F(x_0, y) \geqq c$ for every $y \in X$.

The following Theorem which has been given in [2] is very useful. We shall prove this by using Lemma 1.

THEOREM 2 (Browder). *Let X be a compact convex subset of a locally convex topological vector space E, T a continuous (single valued) mapping of X into E^*. Then, there exists x_0 in X such that $\langle Tx_0, x_0 - y\rangle \geqq 0$ for all y in X.*

PROOF. Define a real valued function F on $X\times X$ by setting $F(x, y) = \langle Tx, x-y\rangle$. Then, for each $y \in X$, the function $F(x, y)$ of x is continuous and for each $x \in X$, the function $F(x, y)$ of y is affine. Furthermore, $F(x, x) = 0$ for all x in X. Therefore, by Lemma 1, there exists $x_0 \in X$ such that

$$F(x_0, y) = \langle Tx_0, x_0 - y\rangle \geqq 0$$

for all y in X.

We shall generalize Theorem 2 to closed and convex sets X in topological vector spaces. Let H, X be nonempty subsets of a topological vector space E, then we put $B_H X = \bar{X} \cap \overline{H-X}$ and $I_H X = X \cap (B_H X)^c$ where $\bar{A}$ is the closure of $A \subset E$ and A^c is the complement of A.

THEOREM 3. *Let H be a closed convex subset of a locally convex topological vector space E and T be a continuous mapping of H into E^*. If there exists a compact convex subset X of H such that $I_H X \neq \emptyset$ and for each $z \in B_H X$, there is $u_0 \in I_H X$ with $\langle Tz, z-u_0\rangle \geqq 0$, then there exists $x^* \in H$ such that $\langle Tx^*, x-x^*\rangle \geqq 0$ for all $x \in H$.*

PROOF. By Theorem 2, there exists $x^* \in X$ such that $\langle Tx^*, x-x^*\rangle \geqq 0$ for all $x \in X$. If $x^* \in I_H X$, for each $y \in H$, we can choose λ $(0<\lambda<1)$ small enough so that $x = \lambda y + (1-\lambda)x^*$ lies in X. Hence

$$0 \leqq \langle Tx^*, x-x^*\rangle = \lambda\langle Tx^*, y-x^*\rangle\,,$$

and consequently, $0 \leqq \langle Tx^*, y-x^*\rangle$. If $x^* \in B_H X$, by the hypothesis, there exists $u_0 \in I_H X$ such that

$$\langle Tx^*, x^*-u_0\rangle \geqq 0\,.$$

Since $\langle Tx^*, x-x^*\rangle \geqq 0$ for all $x \in X$, it follows that $\langle Tx^*, x-u_0\rangle \geqq 0$ for all $x \in X$. Since $u_0 \in I_H X$, for each $y \in H$ there exists λ $(0<\lambda<1)$ such that $x = \lambda y + (1-\lambda)u_0 \in X$. Hence we obtain $0 \leqq \langle Tx^*, y-u_0\rangle$ for all $y \in H$. Since $u_0 \in X$ implies $0 \leqq \langle Tx^*, u_0 - x^*\rangle$, we obtain $0 \leqq \langle Tx^*, y-x^*\rangle$ for all $y \in H$.

If $H=E$ in Theorem 3, it is obvious that there exists $x^* \in E$ such that $Tx^*=0$. In fact, there exists $x^* \in E$ such that $\langle Tx^*, u-x^*\rangle \geqq 0$ for all $u \in E$ and consequently $Tx^*=0$. Theorem 3 has a very interesting interpretation when H is a cone in E, i.e. a nonempty, closed set H in E such that $\alpha x+\beta y$ belongs to H for all α, $\beta\geqq 0$ and x, $y\in H$. We shall also need to know that the *polar* H^* of a cone H is the cone defined by

$$H^*=\{y\in E^*: \langle y, x\rangle\geqq 0 \quad \text{for all } x\in H\}.$$

THEOREM 4. *Let H be a cone in E and T be a continuous mapping of H into E^*. If there exists a compact convex subset X of H such that $I_H X\neq\emptyset$ and for each $z\in B_H X$, there is $u_0\in I_H X$ with*

$$\langle Tz, z-u_0\rangle\geqq 0,$$

then, there exists $x^*\in H$ such that $Tx^*\in H^*$ and $\langle Tx^*, x^*\rangle=0$.

PROOF. By Theorem 3, there exists $x^*\in H$ such that $\langle Tx^*, y-x^*\rangle\geqq 0$ for all $y\in H$. Since $\langle Tx^*, \alpha y\rangle\geqq\langle Tx^*, x^*\rangle$ for all $\alpha>0$ and $y\in H$, we obtain that $\langle Tx^*, y\rangle\geqq 0$ for all $y\in H$, i.e. $Tx^*\in H^*$. That $\langle Tx^*, x^*\rangle=0$ is obvious from $\langle Tx^*, 0-x^*\rangle\geqq 0$.

The above Theorems 3 and 4 generalize the results proved by Moré [**10**]. These proofs were similar to those of [**10**].

§ 3. Fixed point theorems.

In this section, using Lemma 1, we shall prove some fixed point theorems. In [7], Fan has already obtained the following theorem by continuous seminorms instead of continuous linear functionals.

THEOREM 5. *Let X be a nonempty compact convex subset of a topological vector space E and T be a continuous mapping of X into E. Then, either there exists $y_0\in X$ such that y_0 and Ty_0 can not separated by a continuous linear functional, or there exist $x_0\in X$ and $g\in E^*$ such that*

$$g(x_0-Tx_0)<0\leqq\inf_{y\in X} g(x_0-y).$$

PROOF. Suppose that for each $x\in X$, there exists $f\in E^*$ such that $f(x-Tx)<0$. Setting $A_f=\{x\in X: f(x-Tx)<0\}$ for each $f\in E^*$, we have $X=\bigcup_{f\in E^*} A_f$. Since X is compact, there exists a finite family $\{f_1, f_2, \cdots, f_n\}$ in E^* such that $X=\bigcup_{i=1}^{n} A_{f_i}$. Let $\{\beta_1, \beta_2, \cdots, \beta_n\}$ be a partition of unity corresponding to this covering $\{A_{f_i}\}$ of X. Define a real valued function F on $X\times X$ by setting

$$F(x, y)=\sum_{i=1}^{n}\beta_i(x)f_i(x-y).$$

Then, by Lemma 1, there exists $x_0\in X$ such that

$$F(x_0, y)=\sum_{i=1}^{n}\beta_i(x_0)f_i(x_0-y)\geqq 0$$

for all $y \in X$. On the other hand, we know that

$$F(x_0, x_0) = \sum_{i=1}^{n} \beta_i(x_0) f_i(x_0 - Tx_0) < 0 .$$

By putting $g = \sum_{i=1}^{n} \beta_i(x_0) f_i$, we complete the proof.

As direct consequences of Theorem 5, we have the following two Theorems.

THEOREM 6 (Browder). *Let X be a nonempty compact convex subset of a locally convex topological vector space E and T be a continuous mapping of X into E. If for each $x \in X$, there exist $x_1 \in X$ and $\lambda \geqq 0$ such that $Tx - x = \lambda(x_1 - x)$, then T has a fixed point.*

PROOF. Suppose T has no fixed point. By Theorem 5, there exist $x_0 \in X$ and $g \in E^*$ such that

$$g(x_0 - Tx_0) < 0 \leqq \inf_{y \in X} g(x_0 - y) .$$

For this x_0, we can choose $x_1 \in X$ and $\lambda \geqq 0$ such that $Tx_0 - x_0 = \lambda(x_1 - x_0)$. Since T has no fixed point, $\lambda > 0$. Hence we have

$$g(x_0 - Tx_0) < 0 \leqq \frac{1}{\lambda} g(x_0 - Tx_0) .$$

This is a contradiction. Therefore, we have a fixed point.

THEOREM 7. *Let H be a closed convex subset of a locally convex topological vector space E and T be a continuous mapping of H into H. If there exists a compact convex subset X of H such that for each $x \in B_H X$, there exist $x_1 \in X$ and $\lambda \geqq 0$ with $Tx - x = \lambda(x_1 - x)$, then T has a fixed point in H.*

PROOF. Consider the restriction to X of T. If T has no fixed point in X, by Theorem 5 there exist $x_0 \in X$ and $g \in E^*$ such that

$$g(x_0 - Tx_0) < 0 \leqq \inf_{y \in X} g(x_0 - y) .$$

Let $x_0 \in I_H X$. Since $Tx_0 \in H$, we can choose λ $(0 < \lambda < 1)$ small enough so that $y = \lambda Tx_0 + (1-\lambda)x_0$ lies in X. Hence we obtain

$$g(x_0 - Tx_0) < 0 \leqq \lambda g(x_0 - Tx_0) .$$

This is a contradiction. Similarly, we obtain a contradiction for the case of $x_0 \in B_H X$. Therefore, T has a fixed point.

We shall generalize Theorem 5 to multi valued mappings. Let X and Y be topological spaces. A mapping T of X into 2^Y such that for each $x \in X$, Tx is a nonempty subset of Y is said to be *upper semicontinuous*, if for every point $x_0 \in X$ and any open set G in Y containing $T(x_0)$, there is a neighborhood U of x_0 in X such that $T(x) \subset G$ for all $x \in U$. The following definition is due to Fan [7]. Let E be a topological vector space, and let $X \subset E$. A mapping

T of X into 2^E such that for each $x \in X$, Tx is a nonempty subset of E is said to be *upper demi-continuous*, if for every $x_0 \in X$ and any open half-space H in E containing Tx_0, there is a neighborhood U of x_0 in X such that $Tx \subset H$ for all $x \in U$. An open half-space H in E is a set of the form $\{x \in E : h(x) > r\}$ where h is a continuous linear functional, not identically zero, and r is a real number. It is obvious that if a mapping T of X into 2^E is upper semicontinuous, then T is upper demi-continuous. As usual, we say that two sets A, B in E can be *strictly separated by a closed hyperplane*, if we can find a continuous linear form $h \in E^*$ and a real number r such that $h(x) < r$ for $x \in A$ and $h(y) > r$ for $y \in B$. We can prove the following Theorem. The proof employs suitable modifications of the methods used in [2, Theorem 3] and [7, Theorem 5].

THEOREM 8. *Let X be a nonempty compact convex set in a topological vector space E. Let S, T be two upper demi-continuous set valued mappings defined on X such that for each $x \in X$, Tx and Sx are nonempty subsets of E. Then, there exists $y_0 \in X$ for which Sy_0 and Ty_0 can not be strictly separated by a closed hyperplane, or there exist $x_0 \in X$ and $g \in E^*$ such that $g(x_0 - Tx_0) < g(x_0 - Sx_0)$ and $0 \leqq \inf_{y \in X} g(x_0 - y)$.*

PROOF. Suppose that for each $x \in X$, Sx and Tx can be strictly separated by a closed hyperplane. Thus for each $x \in X$, we can find $g_x \in E^*$ and $r_x \in R$ such that $g_x(Sx) < r_x$ and $r_x < g_x(Tx)$. Because S, T are upper demi-continuous on X, there exists a neighborhood U_x of x in X such that $g_x(Sy) < r_x$ and $r_x < g_x(Ty)$ for all $y \in U_x$. Hence, x is an element of the interior $N(g_x)$ of $\{z \in X : g_x(Sz) < g_x(Tz)\}$. Thus, $X = \bigcup_{x \in X} N(g_x)$. By compactness of X, there exists a finite set $\{x_1, x_2, \cdots, x_n\} \subset X$ such that $X = \bigcup_{i=1}^{n} N(g_{x_i})$. Let $\{\beta_i\}_{i=1}^{n}$ be a partition of unity corresponding to the open covering $\{N(g_{x_i})\}$ of X. Let

$$F(x, y) = \sum_{i=1}^{n} \beta_i(x) g_{x_i}(x - y)$$

for $x, y \in X$. By Lemma 1, we have $x_0 \in X$ such that

$$\sum_{i=1}^{n} \beta_i(x_0) g_{x_i}(x_0 - y) \geqq 0$$

for all $y \in X$. We also know that

$$\sum_{i=1}^{n} \beta_i(x_0) g_{x_i}(Sx_0) < \sum_{i=1}^{n} \beta_i(x_0) g_{x_i}(Tx_0).$$

By putting $g = \sum \beta_i(x_0) g_{x_i}$, we complete the proof.

If S is the identity mapping of X, then Theorem 8 becomes the following result which generalizes Theorem 5.

THEOREM 9. *Let X be a nonempty compact convex subset of a topological* *vector space E and T be a upper semicontinuous mapping of X into 2^E such that for each $x\in X$, Tx is a nonempty subset of E. Then, either there exists $y_0\in X$ such that y_0 and Ty_0 can not be strictly separated by a closed hyperplane, or there exist $x_0\in X$ and $g\in E^*$ such that*

$$g(x_0-Tx_0)<0\leqq \inf_{y\in X} g(x_0-y).$$

As a consequence of Theorem 9, we have

THEOREM 10 (Browder). *Let X be a compact convex subset of a locally convex topological vector space E and T be a upper semicontinuous mapping of X into 2^E such that for each $x\in X$, Tx is a nonempty closed convex set in E. If for each $x\in X$, there exist $x_1\in X$, $w_1\in Tx$ and $\lambda\geqq 0$ such that $w_1-x=\lambda(x_1-x)$, then T has a point $x_0\in X$ such that $x_0\in Tx_0$.*

Using Lemma 1 for normed vector spaces, we have

THEOREM 11. *Let X be a nonempty compact convex subset of a normed vector space E and T be a continuous mapping of X into E. Then, there exists $x_0\in X$ such that*

$$\min_{y\in X} \|Tx_0-y\|\geqq \min_{x\in X} \|Tx-x\|.$$

PROOF. Define a real valued function F on $X\times X$ by $F(x, y)=\|Tx-y\|$. Theorem is obvious from Lemma 1.

Using Theorem 10, we shall prove the following Theorem which generalizes Theorem 17 of [2].

THEOREM 12. *Let X be a nonempty compact convex subset in a locally convex topological vector space E and A be an open subset of $X\times X$ having the following properties:*

(1) *$(x, x)\in A$ for every $x\in X$;*

(2) *for any $x\in X$, the set $\{y\in X: (x, y)\notin A\}$ is convex.*

Then, there exists a point $x_0\in X$ such that $x_0\times X\subset A$.

PROOF. Suppose that for each $x\in X$, there exists $y\in X$ such that $(x, y)\notin A$. We define a set valued mapping T of X into 2^X setting $Tx=\{y\in X: (x, y)\notin A\}$ for each $x\in X$. It is obvious that for each $x\in X$, Tx is nonempty, closed and convex. Since the graph of T, i.e. $G(T)=\{(x, y): x\in X, y\in Tx\}=\{(x, y): (x, y)\notin A\}$ is closed, it follows that T is upper semicontinuous. Now, by using Theorem 10, we obtain an element $x_0\in Tx_0$, i.e. $(x_0, x_0)\notin A$. This completes the proof.

As direct consequences of Theorem 12, we have the following two Theorems.

THEOREM 13 (Browder). *Let X be a nonempty compact convex subset of a locally convex topological vector space E_1, let E_2 be a separated topological vector space, and let g be a continuous mapping of $X\times X$ into E_2. Let C be a closed subset of E_2. Suppose that for each x in X, the set $\{y\in X: g(x, y)\in C\}$*

is nonempty and convex. Then there exists an element u of X such that $g(u, u) \in C$.

PROOF. Let $A=\{(x, y)\in X\times X : g(x, y)\notin C\}$. Then, by Theorem 12, we can obtain an element u of X such that $g(u, u)\in C$.

THEOREM 14. *Let X be a nonempty compact convex subset in a locally convex topological vector space E and let F be a real valued lower semicontinuous function on $X\times X$. Let $c\in R$ and suppose that for each $x\in X$, $\{y\in X : F(x, y)\leqq c\}$ is nonempty and convex. Then, there exists $x_0\in X$ such that $F(x_0, x_0)\leqq c$.*

PROOF. Let $A=\{(x, y)\in X\times X : F(x, y)>c\}$. Then, by Theorem 12, we can obtain an element x_0 of X such that $F(x_0, x_0)\leqq c$.

As a consequence of Theorem 1, we have the following Theorem which generalizes Lemma 1 and [7, Lemma].

THEOREM 15. *Let X be a nonempty compact convex set in a separated topological vector space E. Let A be a subset of $X\times X$ having the following properties:*

(1) *For any $y\in X$, the set $\{x\in X : (x, y)\in A\}$ is closed;*

(2) *$(x, x)\in A$ for every $x\in X$;*

(3) *for any $x\in X$, the set $\{y\in X : (x, y)\notin A\}$ is convex.*

Then, there exists a point $x_0\in X$ such that $x_0\times X\subset A$.

PROOF. Suppose that for each $x\in X$, there exists $y\in X$ such that $(x, y)\notin A$. Setting $Tx=\{y\in X : (x, y)\notin A\}$ for each $x\in X$, it is obvious that for each $x\in X$, Tx is nonempty and convex and for each $y\in X$, $T^{-1}y$ is open. Hence by Theorem 1 we have $x_0\in X$ such that $x_0\in Tx_0$, i. e. $(x_0, x_0)\notin A$.

§ 4. Minimax theorems.

In this section, we discuss minimax theorems. At first, as a direct consequence of Theorem 1, we shall prove Sion's minimax theorem [**12**]. Let X and Y be convex subsets each in a topological vector space and let f be a mapping of $X\times Y$ into R. If for each $(y, a)\in Y\times R$, $\{x : f(x, y)<a\}$ is convex, $f(x, y)$ is said to be *quasi-convex* on X. If for each $(x, a)\in X\times R$, $\{y : f(x, y)>a\}$ is convex, $f(x, y)$ is said to be *quasi-concave* on Y; see Sion's paper [**12**].

THEOREM 16 (Sion). *Let X and Y be compact convex subsets each in a topological vector space and let $f: X\times Y\to R$ be a function satisfying:*

(1) *For each $y\in Y$, $f(x, y)$ is lower semicontinuous and quasi-convex on X;*

(2) *for each $x\in X$, $f(x, y)$ is upper semicontinuous and quasi-concave on Y.*

Then,

$$\max_{y\in Y}\min_{x\in X} f(x, y)=\min_{x\in X}\max_{y\in Y} f(x, y)\,.$$

PROOF. Let us be

$$\max_y \min_x f(x, y) < c < \min_x \max_y f(x, y).$$

We define a set valued mapping T of $X \times Y$ into $2^{X \times Y}$ setting $T(x, y) = B_y \times A_x$, where $A_x = \{y \in Y : f(x, y) < c\}$ and $B_y = \{x \in X : f(x, y) > c\}$. By using Theorem 1, we obtain an element $(x_0, y_0) \in X \times Y$ such that $(x_0, y_0) \in T(x_0, y_0)$. Therefore, we have $c < f(x_0, y_0) < c$. This is a contradiction.

Secondly we state a necessary and sufficient condition that a minimax condition holds and then generalize Terkelsen's minimax theorem by using the method employed in [**13**].

LEMMA 2. *Let X be a compact space, and let F be a family of lower semicontinuous real valued functions on X. If $c = \sup_f \min_x f(x)$, the following are equivalent.*

(1) *The family $\{A(f) : f \in F\}$ has the finite intersection property, where $A(f) = \{x : f(x) \leqq c\}$ for each $f \in F$.*

(2) $$\sup_f \min_x f(x) = \min_x \sup_f f(x).$$

PROOF. (1) $\Rightarrow$ (2). Since the family $\{A(f) : f \in F\}$ has the finite intersection property and X is compact, we have $\bigcap_{f \in F} A(f) \neq \emptyset$. Let $x_0 \in \bigcap_f A(f)$. Since $f(x_0) \leqq c$ for every $f \in F$, we obtain $\sup_f f(x_0) \leqq c$. Hence we have

$$\min_x \sup_f f(x) \leqq \sup_f f(x_0) \leqq \sup_f \min_x f(x).$$

In the other hand, it is obvious that

$$\sup_f \min_x f(x) \leqq \min_x \sup_f f(x).$$

(2) $\Rightarrow$ (1). The equalities

$$\min_x \sup_f f(x) = \sup_f \min_x f(x) = c$$

imply the existence of $x_0 \in X$ with $\sup_f f(x_0) = c$. Hence, since $f(x_0) \leqq c$ for every $f \in F$, it is obvious that the family $\{A(f) : f \in F\}$ has the finite intersection property.

THEOREM 17. *Let X be a compact space, and let F be a family of lower semicontinuous real valued functions on X satisfying:*

(1) *For any $f, g \in F$, there exists $h \in F$ such that $f + g \leqq 2h$;*

(2) *for each $(f, b) \in F \times \{b \in R : c \leqq b\}$, every finite intersection of sets $\{x : f(x) \leqq b\}$ is connected, with $c = \sup \min f(x)$. Then,*

$$\sup_f \min_x f(x) = \min_x \sup_f f(x).$$

PROOF. Suppose that $A(f) \cap A(g) = \emptyset$ for some pair $f, g \in F$. Then, since

$c<\min_x \max[f(x), g(x)]$, it follows that there exists $b\in R$ such that $c<b<\min_x \max[f(x), g(x)]$. By (1), there exists $k\in F$ such that $f+g\leqq 2k$. Let $A=\{x: f(x)\leqq b\}$, $B=\{x: g(x)\leqq b\}$ and $C=\{x: k(x)\leqq b\}$. Since A and B are nonempty closed sets with $A\cap B=\emptyset$ and $C\subset A\cup B$, by (2) we obtain either $C\subset A$ or $C\subset B$. If $C\subset A$, set $f_1=k$, $g_1=g$, and if $C\subset B$, set $f_1=f$, $g_1=k$. Then $b<\min_x \max[f_1(x), g_1(x)]$ in each case. Defining $r=\min f(x)$, $d=\min g(x)$, $r_1=\min f_1(x)$ and $d_1=\min g_1(x)$, it can be verified that we have either $2r_1>r+b$ and $d_1=d$, or $r_1=r$ and $2d_1>d+b$. Similarly, we obtain $f_i, g_i\in F$ for all $i=2, 3, \cdots$ such that either $2r_i>r_{i-1}+b$, $d_i=d_{i-1}$, or $r_i=r_{i-1}$, $2d_i>b+d_{i-1}$, where $r_i=\min f_i(x)$, $d_i=\min g_i(x)$. It is obvious that at least one of the sequences $\{r_i\}$ and $\{d_i\}$ must converge to b. Suppose $r_i\to b$ for $i\to\infty$. Since $c<b$, there exists i_0 such that $c<r_{i_0}<b$. This is a contradiction, because $\{x: f_{i_0}(x)\leqq c\}$ is nonempty. Therefore, we have $A(f)\cap A(g)\neq\emptyset$ for each pair $f, g\in F$.

Now, we can show by the mathematical induction that the family $\{A(f): f\in F\}$ has the finite intersection property. In fact, let $\{f_1, f_2, \cdots, f_n\}\subset F$ and $A=A(f_n)$. Since $A(f)\cap A\neq\emptyset$ for each $f\in F$, we have $\min_{x\in A} f(x)\leqq c$ and hence $\sup_f \min_{x\in A} f(x)\leqq c$. The inequality $\min_{x\in X} f(x)\leqq \min_{x\in A} f(x)$ implies

$$c=\sup_f \min_{x\in X} f(x)\leqq \sup_f \min_{x\in A} f(x)\leqq c\,,$$

i.e. $\sup_f \min_{x\in A} f(x)=c$. Therefore, the family of restrictions $\{f|_A: f\in F\}$ satisfies the assumptions of Theorem with respect to A.

§5. Systems of convex inequalities.

In this section, at first we give another proof for Fan's theorem [5] concerning systems of convex inequalities. The proof is simple.

LEMMA 3. *Let X and Y be topological spaces. Suppose that f is a nonnegative real valued continuous function on X and g is a real valued lower semicontinuous function on Y. Then, setting $F(x, y)=f(x)g(y)$ for each $(x, y)\in X\times Y$, F is lower semicontinuous on $X\times Y$.*

PROOF. For any $c\in R$, let $Z=\{(x, y): f(x)g(y)\leqq c\}$. We shall show that if $\{(x_\alpha, y_\alpha): \alpha\in I\}$ is a generalized sequence such that (x_α, y_α) converges to (x_0, y_0), then $(x_0, y_0)\in Z$. Let $c\geqq 0$. Suppose that $f(x_0)g(y_0)>c\geqq 0$. Then it follows that $f(x_0)>0$ and $g(y_0)>0$ and hence there exists $\beta>0$ such that $g(y_0)>\beta>c/f(x_0)$. Defining $A=\{x: f(x)>c/\beta\geqq 0\}$ and $B=\{y: g(y)>\beta\}$, it is obvious that $A\times B$ is open and $(x_0, y_0)\in A\times B$. Hence, we have that $(x_\alpha, y_\alpha)\in A\times B$ for some $\alpha\in I$. This implies $g(y_\alpha)>\beta>c/f(x_\alpha)$, i.e. $f(x_\alpha)g(y_\alpha)\geqq c$. This is a contradiction. Therefore we have $f(x_0)g(y_0)\leqq c$. Similarly we can prove

Lemma 3 for the case of $c<0$.

THEOREM 18 (Fan). *Let X be a compact convex set in a topological vector space. Let $f_1, f_2, \cdots, f_n$ be n real valued lower semicontinuous convex functions defined on X and $c \in R$. Then there exists a point $x \in X$ satisfying $f_i(x) \leqq c$ $(1 \leqq i \leqq n)$, if and only if, for any n nonnegative numbers α_i with $\sum_{i=1}^{n} \alpha_i = 1$, there is a point $x \in X$ such that*

$$\sum_{i=1}^{n} \alpha_i f_i(x) \leqq c .$$

PROOF. We need only prove the "if" part. Let $A(f_i)=\{x : f_i(x) \leqq c\}$ and $B(f_i)=\{x : f_i(x) > c\}$ for each $i=1, 2, \cdots, n$. Suppose that $\cap A(f_i)=\emptyset$. Then, $\cup B(f_i)=X$. Let $\{\beta_1, \beta_2, \cdots, \beta_n\}$ be a partition of unity corresponding to this covering. Setting

$$F(x, y)=\sum_{i=1}^{n} \beta_i(x) f_i(y)$$

for each $(x, y) \in X \times X$, then there exists a real number α_0 such that $F(x, x) \geqq \alpha_0 > c$ for all $x \in X$. By using Lemma 1, we obtain $x_0 \in X$ such that $F(x_0, y) \geqq \alpha_0 > c$ for all $y \in X$. This completes the proof.

Let X and Y be arbitrary sets. A function $F: X \times Y \to R$ is *convexlike* on X, if for any $x_1, x_2 \in X$ and $\lambda \in R$ with $0<\lambda<1$, there exists $x_0 \in X$ such that

$$F(x_0, y) \leqq \lambda F(x_1, y)+(1-\lambda) F(x_2, y)$$

for all $y \in Y$. Similarly, F is *concavelike* on Y, if for any $y_1, y_2 \in Y$ and $\lambda \in R$ with $0<\lambda<1$, there exists $y_0 \in Y$ such that

$$F(x, y_0) \geqq \lambda F(x, y_1)+(1-\lambda) F(x, y_2)$$

for all $x \in X$. By using Theorem 18, we can prove the following Fan's minimax theorem [4].

THEOREM 19 (Fan). *Let X be a compact convex set in a topological vector space, let Y be a set, and let $F: X \times Y \to R$ be a function satisfying:*

(1) *For each $y \in Y$, $F(x, y)$ is lower semicontinuous and convex on X;*

(2) *for each $x \in X$, $F(x, y)$ is concavelike on Y.*

Then,

$$\sup_y \min_x F(x, y)=\min_x \sup_y F(x, y) .$$

PROOF. Let $c=\sup_y \min_x F(x, y)$. Let $\{y_1, y_2, \cdots, y_n\}$ be a finite subset of Y and $\{\alpha_1, \alpha_2, \cdots, \alpha_n\}$ be nonnegative numbers such that $\sum_{i=1}^{n} \alpha_i=1$. By hypothesis, there exists $y_0 \in Y$ such that

$$\sum_{i=1}^{n} \alpha_i F(x, y_i) \leqq F(x, y_0)$$

for all $x \in X$. Since for $y_0 \in Y$ there exists $x_0 \in X$ such that $F(x_0, y_0) \leqq c$, i.e. $\sum \alpha_i F(x_0, y_i) \leqq c$, by Theorem 18, it follows that there exists $z \in X$ such that $F(z, y_i) \leqq c$ for $i=1, 2, \cdots, n$. Therefore, by Lemma 2, we have $\sup_y \min_x F(x, y) = \min_x \sup_y F(x, y)$.

As a direct consequence of Theorem 19, we obtain a generalization of [2, Lemma 1].

THEOREM 20. *Let X be a compact convex set in a topological vector space, let Y be a set, and let $F: X \times Y \to R$ be a function satisfying:*

(1) *For each $y \in Y$, $F(x, y)$ is lower semicontinuous and convex on X;*

(2) *for each $x \in X$, $F(x, y)$ is concavelike on Y.*

Let $c \in R$. If for each $y \in Y$, there exists $x \in X$ such that $F(x, y) \leqq c$, then there exists $u \in X$ such that $F(u, y) \leqq c$ for all $y \in Y$.

PROOF. By Theorem 19, we have

$$\sup_y \min_x F(x, y) = \min_x \sup_y F(x, y) .$$

Since for each $y \in Y$, there exists $x \in X$ such that $F(x, y) \leqq c$, we have $\sup \min F(x, y) \leqq c$. Therefore, we have $u \in X$ such that $\sup_y F(u, y) \leqq c$.

§ 6. Variational inequalities for multi valued mappings.

By using Lemma 1 we can prove the following Theorem.

THEOREM 21. *Let X be a compact convex subset of a locally convex topological vector space E and T be a upper semicontinuous multi valued mapping of X into 2^{E^*} such that for each $x \in X$, Tx is nonempty and compact. If for each $x \in X$,*

$$\min_{y \in X} \max_{w \in Tx} \langle w, x-y \rangle = \max_{w \in Tx} \min_{y \in X} \langle w, x-y \rangle ,$$

then there exist $x_0 \in X$ and $w_0 \in Tx_0$ such that $\langle w_0, x_0 - y \rangle \geqq 0$ for all $y \in X$.

PROOF. Define a real valued function F on $X \times X$ by

$$F(x, y) = \max_{w \in Tx} \langle w, x-y \rangle .$$

Then, by upper semicontinuity of T, we have that $x \mapsto F(x, y)$ is upper semicontinuous. In fact, let $y \in X$, $a \in R$ and $A = \{x \in X : F(x, y) \geqq a\}$. We shall show that if $\{x_\alpha \in A : \alpha \in I\}$ is a generalized sequence such that x_α converges to x_0, then $x_0 \in A$. For each $x_\alpha \in A$, there exists $w_\alpha \in Tx_\alpha$ such that $\langle w_\alpha, x_\alpha - y \rangle \geqq a$. Since $\bigcup_{x \in X} Tx$ is compact, there exists a subsequence $\{w_{\alpha'}\}$ of $\{w_\alpha\}$ such that $w_{\alpha'} \to w_0$. Since T is upper semicontinuous of X of 2^{E^*}, we have $w_0 \in Tx_0$. We have also

$$a \leqq \lim_{\alpha'} \langle w_{\alpha'}, x_{\alpha'} - y \rangle = \langle w_0, x_0 - y \rangle .$$

Hence A is closed, i.e. $x\mapsto F(x, y)$ is upper semicontinuous. It is obvious that $y\mapsto F(x, y)$ is convex and $F(x, x)=0$ for all $x\in X$. Hence, by Lemma 1, there exists $x_0\in X$ such that $\max_{w\in Tx_0}\langle w, x_0-y\rangle\geqq 0$ for all $y\in X$. Since

$$\min_{y\in X}\max_{w\in Tx_0}\langle w, x_0-y\rangle=\max_{w\in Tx_0}\min_{y\in X}\langle w, x_0-y\rangle,$$

we have $w_0\in Tx_0$ such that $\langle w_0, x_0-y\rangle\geqq 0$ for all $y\in X$.

In Theorem 21, if Tx is convex, we know by Theorem 19 that a minimax condition holds. So, we shall obtain Browder's theorem [**2**, Theorem 6]. Furthermore, we can prove the following Theorem.

THEOREM 22. *Let X be a compact convex subset of a finite dimensional Euclidean space E and T be a upper semicontinuous multi valued mapping of X into 2^E such that for each $x\in X$, Tx is nonempty compact. If for each $x\in X$,*

$$\min_{y\in X}\max_{w\in Tx}\langle w-x, x-y\rangle=\max_{w\in Tx}\min_{y\in X}\langle w-x, x-y\rangle,$$

then, there exist $x_0\in X$ and $w_0\in Tx_0$ such that $\langle w_0-x_0, x_0-y\rangle\geqq 0$ for all $y\in X$.

PROOF. Let $F(x, y)=\max_{w\in Tx}\langle w-x, x-y\rangle$ for $x, y\in X$. Since T is upper semicontinuous of X into 2^E, for each $y\in X$, $x\mapsto F(x, y)$ is upper semicontinuous. It is obvious that for each $x\in X$, $y\mapsto F(x, y)$ is convex. By using Lemma 1, we obtain $x_0\in X$ such that

$$F(x_0, y)=\max_{w\in Tx_0}\langle w-x_0, x_0-y\rangle\geqq 0$$

for all $y\in X$. Since

$$\min_{y\in X}\max_{w\in Tx_0}\langle w-x_0, x_0-y\rangle=\max_{w\in Tx_0}\min_{y\in X}\langle w-x_0, x_0-y\rangle,$$

there exists $w_0\in Tx_0$ such that $\langle w_0-x_0, x_0-y\rangle\geqq 0$ for all $y\in X$.

In Theorem 22, if T is a mapping of X into 2^X, by putting $y=w_0$, we obtain $w_1=x_0$, i.e. $x_0\in Tx_0$. Kakutani's fixed point theorem [**9**] is the case of which for each $x\in X$, Tx is convex; that is, a minimax condition holds.

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