Mathematical and Computational Applications

Article

Nonlinear Vibration of a Nanobeam on a Pasternak Elastic Foundation Based on Non-Local Euler-Bernoulli Beam Theory

Necla Togun [1,*] and Süleyman Murat Bağdatlı [2]

[1] Vocational School of Technical Sciences in Gaziantep, University of Gaziantep, Gaziantep 27310, Turkey
[2] Department of Mechanical Engineering, Faculty of Engineering, Celal Bayar University, Yunusemre 45140, Manisa, Turkey; murat.bagdatli@cbu.edu.tr
* Correspondence: nkara@gantep.edu.tr; Tel.: +90-342-317-1791

Academic Editor: Mehmet Pakdemirli
Received: 16 October 2015; Accepted: 1 March 2016; Published: 7 March 2016

Abstract: In this study, the non-local Euler-Bernoulli beam theory was employed in the nonlinear free and forced vibration analysis of a nanobeam resting on an elastic foundation of the Pasternak type. The analysis considered the effects of the small-scale of the nanobeam on the frequency. By utilizing Hamilton's principle, the nonlinear equations of motion, including stretching of the neutral axis, are derived. Forcing and damping effects are considered in the analysis. The linear part of the problem is solved by using the first equation of the perturbation series to obtain the natural frequencies. The multiple scale method, a perturbation technique, is applied in order to obtain the approximate closed solution of the nonlinear governing equation. The effects of the various non-local parameters, Winkler and Pasternak parameters, as well as effects of the simple-simple and clamped-clamped boundary conditions on the vibrations, are determined and presented numerically and graphically. The non-local parameter alters the frequency of the nanobeam. Frequency-response curves are drawn.

Keywords: vibration; nanobeam; elastic foundation; perturbation method; nonlocal elasticity

1. Introduction

Nanotechnology is the manipulation of matter on a supramolecular, molecular, and atomic scale. Many new devices and materials used in consumer products, electronics, biomaterials, medicine, energy production, *etc.*, may be created with the help of nanotechnology. The exclusive properties of nanoscale materials are due to their very small size. The size effect of nano structures has an important role in their static and dynamic analysis. The classical continuum mechanics is not able to take into account the size effect in modeling of the material behavior at the nanoscale. Therefore, various size-dependent continuum theories, which are the non-local elasticity theory, strain gradient theory, the modified couple stress theory, the micropolar theory, and the surface elasticity theory, have been developed to include the small-scale effect. Among these theories, Eringen's non-local elasticity theory [1,2] is a major subject among scientists. Peddison, *et al.* [3] were the first pioneers applying the non-local elasticity theory to nanostructures.

Vibration analysis of nanostructures is necessary for the ideal design of nanoelectromechanical systems (NEMS) and new nanodevices. The Winkler model is studied as a one-parameter model, namely Winkler-type elastic foundation, consists of a series of closely-spaced elastic springs, where as the Pasternak model studied as a two-parameter model, namely Pasternak-type elastic foundation, consists of a Winkler-type elastic spring and transverse shear deformation. In contrast, the nonlinear elastic foundation model studied as a three-parameter model, in which the layer is indicated by linear elastic springs, shear deformation, and cubic nonlinearity elastic springs. The work of Niknam and

Aghdam [4] deals with the Eringen's non-local elasticity theory for the evaluation of a closed-form solution of the buckling load and natural frequency of non-local functionally-graded (^{FG}) beams on a nonlinear-type elastic foundation. Fallah and Aghdam [5] carried out post buckling and free vibration analysis of FG beams resting on an elastic foundation and subjecting axial force. Additionally, this author and its coauthors [6] investigated nonlinear free vibration and thermo-mechanical buckling analysis of a FG beam resting on a nonlinear type elastic foundation. Kanani, *et al.* [7] investigated the free and force vibration of a FG beam in the presence of large amplitude resting on a nonlinear elastic type foundation including shearing layer and cubic nonlinearity. Şimşek [8] developed a non-classical beam theory for the static and nonlinear vibration analysis of microbeams resting on a nonlinear elastic foundation on the base of the modified couple stress theory and Euler-Bernoulli beam theory.

Mustapha and Zhong [9] presented a mathematical model associated with single-walled carbon nanotube (SWCNT) vibration analysis. The SWCNT taken as a non-local Rayleigh beam is assumed to be axially loaded and embedded in a two parameter elastic medium. Mehdipour, *et al.* [10] employed continuum mechanics and elastic beam model. Their study aims to analyze the transverse vibration of a SWCNT having curved shape and embedded in a Pasternak elastic foundation. Work of Shen and Zhang [11] deals with the post-buckling nonlinear vibration and nonlinear bending of a SWCNT. The SWCNT modeled as a non-local beam including small-scale effect and resting on a two parameter elastic foundation in thermal environments. Arani, *et al.* [12] carried out a study related with the vibration behavior of single-walled boron nitride nanotubes in the presence of von Kármán geometric nonlinearity effects modeled with non-local piezoelasticity. Its nanotube surrounded by an elastic medium was assumed to be Winkler and Pasternak foundation model. Murmu and Pradhan [13] applied an existing method to a well-known Eringen non-local elasticity theory to analyze the stability response of SWCNT surrounded by Winkler- and Pasternak-type foundation models. Yas and Samadi [14] were presented buckling and free vibration analysis of nanocomposite Timeshenko beams reinforced by SWCNT resting on the two parameter medium.

Kazemi-Lari, *et al.* [15] considered the influence of viscoelastic foundation in the presence of interaction between surrounding viscoelastic medium and carbon nanotubes (CNTs) considering the action of a concentrated follower force. Surrounded medium is taken as the Kelvin–Voight, Maxwell, and standard linear solid types of viscoelastic foundation. Ghanvanloo, *et al.* [16] applied an existing method to a well-known classical Euler-Bernoulli beam model considering the instability and vibration response of CNT resting on a linear viscoelastic Winkler foundation. Refiei, *et al.* [17] applied an existing method to a well-known non-local Euler-Bernoulli beam theory to analyze the vibration characteristics of non-uniform SWCNT conveying fluid and also embedded in viscoelastic medium. It was concluded from their study that the nonlocal parameter, small-scale effect, may influence extremely the natural frequency and mode shape of the system. The main motivation for Arani and Amir's [18] work is to develop an analytic model for the electro-thermal vibration of boron nitride nanotubes by using strain gradient theory, in which nanotubes coupled by visco-Pasternak medium. Wang and Li [19] carried out the study of the nonlinear free vibration of a nanotube in the presence of small-scale effect embedded in viscous matrix modeled with non-local elasticity theory and Hamilton principle. The work of Mahdawi, *et al.* [20] deals with the nonlinear free vibrational behavior of a double walled carbon nanotube (DWCNT) in the presence of compressive axial load. DWCNT was surrounded by a polymer matrix. The results of their study indicate that the surrounding medium may influence profoundly the vibrational behavior of the embedded CNT.

Most existing studies in the literature examine the vibrational behavior of nanostructures surrounded by an elastic medium. The natural frequency of a SWCNT conveying a viscous fluid and are also embedded in an elastic medium [21], free transverse vibration of an elastically-supported DWCNT embedded in an elastic matrix in the presence of initial axial force [22], axial vibration of SWCNT embedded in an elastic medium [23], vibration of nanotubes embedded in an elastic matrix [24], nonlinear free vibration of embedded DWCNT including the von Kármán geometric nonlinearity [25], nonlinear free vibration of clamped-clamped DWCNT surrounded by an elastic medium with consideration of the von Kármán geometric nonlinearity and the nonlinear van der Waals forces [26], forced vibration of an elastically-connected DWCNT carrying a moving nanoparticle [27], nonlinear vibration of embedded multiwalled carbon nanotubes (MWCNT) in thermal environments [28], vibration analysis of embedded MWCNT at an elevated temperature with considering the small-scale effect on the large amplitude [29], free transverse vibration of SWCNT embedded in elastic matrix under various boundary conditions [30], thermal vibration of SWCNT embedded in an elastic medium [31], thermal-mechanical vibration and buckling instability of a SWCNT conveying fluid and resting on an elastic medium [32], electro-thermo-mechanical vibration analysis of non-uniform and non-homogeneous boron nitride nanorod embedded in elastic medium [33], buckling behavior of SWCNT on a Winkler foundation under various boundary conditions [34], critical buckling temperature of SWCNT embedded in a one parameter elastic medium [35], and buckling analysis of SWCNT including the effect of temperature change and surrounding elastic medium [36] were studied with the aid of nonlocal elasticity theory. The surrounding elastic medium related with the above studies was described as the Winkler model with spring constant k.

Another class of size-dependent continuum theories that deal with the electro-thermal transverse vibration behavior of double-walled boron nitride nanotubes which are surrounded by an elastic medium was presented with the aid of non-local piezoelasticity cylindirical shell theory [37]. Free vibrations of SWCNT embedded in non-homogenous elastic matrix were studied with the aid of the non-local continuum shell theory [38]. The nonlinear free vibration of embedded MWCNT was investigated by using the multiple elastic beam models and continuum mechanics [39]. Nonlinear thermal stability and vibration of pre/post buckled temperature and microstructure-dependent FG beams resting on an elastic medium was investigated on the base of the modified couple stress theory [40]. The method of multiple scales (a perturbation method) is an efficient technique to solve the nonlinear differential equations. Free vibration analysis of beams resting on elastic foundation [41,42] and nonlinear free vibration behavior of simply supported DWCNT with considering the geometric nonlinearity were presented by using multiple scale method [43]. Nonlinear vibration of tensioned nanobeam and nanobeam with different boundary condition was studied by using non-local elasticity theory [44,45].

Lots of the work presented in the literature includes the vibration behavior of a nanobeam embedded in an elastic medium, whereas investigations on the two-parameter medium are rather limited. We examine the literature presented in the above, and it can be seen clearly that an elastic medium surrounded by a Pasternak-type model is limited in literature. Most of the above work is mainly related with the amplitude–frequency response of the nanotube. However, damping and forcing effect included studies on the nonlinear vibration properties of nanosystems are also rather limited. In the present study, the non-linear free vibration of the nanobeam resting on a two-parameter medium is studied by the non-local continuum theory. The small scale and damping effects are taken into account and nonlinear vibration behaviors of the nanobeam are illustrated.

2. Governing Equations

2.1. Non-Local Effects

In the classical (local) continuum theory, the stress at a point X depends only on the strain at the same point, while the non-local elasticity theory proposed by Eringen [1,2], regards the stress at a point as a function of strains at all points in the continuum. Therefore, the nonlocal stress tensor σ at point X can be written as:

$$\sigma(X) = \int_V K\left(\left|X' - X\right|, \tau\right) T\left(X'\right) dV \tag{1}$$

$$T(X) = C(X) : \varepsilon(X) \tag{2}$$

where $T(X)$ is the classical macroscopic stress tensor at point X, $K\left(\left|X' - X\right|, \tau\right)$ is the non-local modulus, $\left|X' - X\right|$ is the Euclidian distance and τ is a material constant, $C(X)$ is the fourth order elasticity tensor, and $\sigma(X)$ and $\varepsilon(X)$ are the second order tensors representing stress and strain fields, respectively. A simplified equation of differential form is used as a non-local constitutive relation, the reason being is that solving of the integral constitutive Equation (2) is complicated.

$$T = \left(1 - \mu \nabla^2\right) \sigma, \ \mu = \tau^2 l^2 \tag{3}$$

where ∇^2 is the Laplacian operator. Here, the non-dimensional non-local nanoscale parameter τ is defined as $e_0 a/l$, in which e_0 is constant appropriate to each material and a is internal characteristic length and l is external characteristic length. The constitutive equation of nonlocal elasticity for a beam takes the following form:

$$\sigma(X) - \mu \frac{\partial^2 \sigma(X)}{\partial X^2} = E\varepsilon(X) \tag{4}$$

where E is the elasticity modulus.

2.2. Nonlocal Euler-Bernoulli Beam

This study is carried out on the basis of the non-local Euler-Bernoulli beam model. Two types of boundary conditions, which are simple-simple and clamped-clamped, are considered in this work and shown in Figure 1. The nanobeam is resting on a two parameter elastic foundation with the spring constants k_L and k_p of the Winkler elastic medium and Pasternak elastic medium, respectively. The equation of motion is obtained by using Hamilton's principle. For the Euler-Bernoulli beam model, the displacement field is given as:

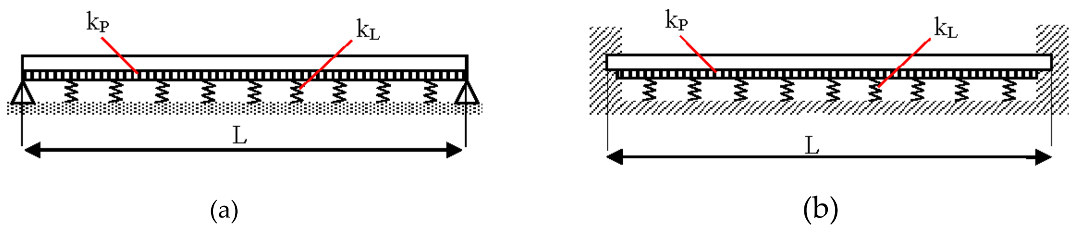


Figure 1. Boundary conditions for different beam supports. (**a**) Simple-Simple case and (**b**) Clamped-Clamped case.

$$u_x(x,z,t) = u(x,t) - z\frac{\partial w}{\partial x}, \ u_y = 0, \ u_z(x,z,t) = w(x,t) \tag{5}$$

where u *and* w are the axial and transverse displacements, respectively. The axial force and resultant bending moment for the beam model are:

$$N = \int_A \sigma_x dA, \ M = \int_A z\sigma_x dA \tag{6}$$

where A is the area of the cross-section for the nanobeam. Taking into account the large amplitude nonlinear vibration, the von Kármán nonlinear strain (*i.e.*, ε_{non}) should be considered and strain-displacement relationship is given by:

$$\varepsilon_0 = \frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2, \ \varepsilon_1 = -z\kappa, \ \kappa = \frac{\partial^2 w}{\partial x^2} \tag{7}$$

where ε_0 is the nonlinear extensional strain and κ is the bending strain. Then the von Kármán nonlinear strain (*i.e.*, ε_{non}) can be expressed as:

$$\varepsilon_{non} = \varepsilon_0 + \varepsilon_1 = \frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 - z\frac{\partial^2 w}{\partial x^2} \tag{8}$$

The force–strain and the moment–strain relations of the nonlocal beam theory can be obtained from Equations (4)–(8):

$$N-(e_0a)^2\frac{\partial^2 N}{\partial x^2}=EA\left[\frac{\partial u}{\partial x}+\frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2\right] \tag{9}$$

$$M-(e_0a)^2\frac{\partial^2 M}{\partial x^2}=-EI\frac{\partial^2 w}{\partial x^2} \tag{10}$$

where I is the moment of inertia. The kinetic energy T can be written as:

$$T=\frac{1}{2}\rho A\int_0^L\left[\left(\frac{\partial u}{\partial t}\right)^2+\left(\frac{\partial w}{\partial t}\right)^2\right]dx \tag{11}$$

where ρA is the mass per unit length. The strain energy U can be written as:

$$U=\frac{1}{2}\int_0^L\left\{N\left[\frac{\partial u}{\partial x}+\frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2\right]-M\frac{\partial^2 w}{\partial x^2}\right\}dx \tag{12}$$

In addition, the virtual work by the external load from the elastic medium of the Pasternak type is given by:

$$\delta W_{ext}=\int_0^L q\delta w dx \tag{13}$$

where $q=-\left[k_L w-k_p\frac{\partial^2 w}{\partial x^2}\right]$ is the load exerted by the Pasternak-type elastic medium. The stiffness and the shear modulus parameters of the deformable medium are represented by k_L and k_p. Hamilton's principle can be represented analytically by the following formula:

$$\delta\int_0^t\left[T-(U-W_{ext})\right]dt=0 \tag{14}$$

Inserting Equations (11)–(13) into Equation (14) and integrating by parts, and collecting the coefficients of δu and δw, the following equation of motion are obtained:

$$\frac{\partial N}{\partial x}=\rho A\frac{\partial^2 u}{\partial t^2} \tag{15}$$

$$\frac{\partial^2 M}{\partial x^2}+N\frac{\partial^2 w}{\partial x^2}+k_L w-k_p\frac{\partial^2 w}{\partial x^2}=\rho A\frac{\partial^2 w}{\partial t^2} \tag{16}$$

Substituting Equation (16) into Equation (10), one obtains the expressions of the non-local force N and non-local moment M as follows:

$$N=EA\left[\frac{\partial u}{\partial x}+\frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2\right]+(e_0a)^2\frac{\partial^3 u}{\partial x\partial t^2} \tag{17}$$

$$M=-EI\frac{\partial^2 w}{\partial x^2}+(e_0a)^2\left[\rho A\frac{\partial^2 w}{\partial t^2}-N\frac{\partial^2 w}{\partial x^2}-k_L w+k_p\frac{\partial^2 w}{\partial x^2}\right] \tag{18}$$

The longitudinal inertia $\frac{\partial^2 u}{\partial t^2}$ can be neglected based on the discussion about the nonlinear vibration of continuous systems [46,47], then the axial normal force N can be represented as:

$$N=\frac{EA}{2L}\int_0^L\left(\frac{\partial w}{\partial x}\right)^2 dx \tag{19}$$

The nonlinear vibration equation of motion for the nanobeam resting on the Pasternak-type elastic foundation can be obtained by substituting Equations (17)–(19) into Equations (15) and (16) as follows:

$$\begin{aligned} & EI\frac{\partial^4 w}{\partial x^4} + \rho A\frac{\partial^2}{\partial t^2}\left(w - (e_0a)^2\frac{\partial^2 w}{\partial x^2}\right) + k_L\left(w - (e_0a)^2\frac{\partial^2 w}{\partial x^2}\right) - k_p\frac{\partial^2}{\partial x^2}\left(w - (e_0a)^2\frac{\partial^2 w}{\partial x^2}\right) \\ & = \frac{EA}{2L}\left[\int_0^L\left(\frac{\partial w}{\partial x}\right)^2 dx\right]\frac{\partial^2}{\partial x^2}\left[w - (e_0a)^2\frac{\partial^2 w}{\partial x^2}\right] \end{aligned} \tag{20}$$

The following non-dimensional quantities aims to study problem under general form are considered:

$$\bar{x} = \frac{x}{L},\ \bar{w} = \frac{w}{L},\ \bar{t} = \frac{t}{L^2}\sqrt{\frac{EI}{\rho A}},\quad \gamma = \frac{e_0a}{L},\ \ K_L = \frac{k_L L^4}{EI},\ \ K_p = \frac{k_p L^2}{EI} \tag{21}$$

In the non-dimensional form considering the Equations (20) and (21) can be expressed as:

$$\begin{aligned} & \frac{\partial^4\bar{w}}{\partial\bar{x}^4} + \frac{\partial^2\bar{w}}{\partial\bar{t}^2} - \gamma^2\frac{\partial^4\bar{w}}{\partial\bar{t}^2\partial\bar{x}^2} + K_L\bar{w} - K_L\gamma^2\frac{\partial^2\bar{w}}{\partial\bar{x}^2} - K_p\frac{\partial^2\bar{w}}{\partial\bar{x}^2} + K_p\gamma^2\frac{\partial^4\bar{w}}{\partial\bar{x}^4} \\ & = \frac{1}{2}\left[\int_0^L\left(\frac{\partial\bar{w}}{\partial\bar{x}}\right)^2 d\bar{x}\right]\left[\frac{\partial^2\bar{w}}{\partial\bar{x}^2} - \gamma^2\frac{\partial^4\bar{w}}{\partial\bar{x}^4}\right] \end{aligned} \tag{22}$$

The non-dimensional form of boundary conditions can be expressed as;

$$\begin{array}{cc} \underline{\textbf{Simple} - \textbf{Simple Case}} & \underline{\textbf{Clamped} - \textbf{Clamped Case}} \\ \bar{w}(0) = 0, \bar{w}(1) = 0 & \bar{w}(0) = 0,\ \bar{w}(1) = 0, \\ \bar{w}''(0) = 0,\ \bar{w}''(1) = 0 & \bar{w}'(0) = 0,\ \bar{w}'(1) = 0 \end{array} \tag{23}$$

The multiple scale method will be able to employ to the partial differential equations and boundary conditions to obtain the approximate solution for the problem [46,47]. Then, the introduction of the forcing and damping term in Equation (22) can also be seen as the nonlinear exact solution:

$$\begin{aligned} & \frac{\partial^4\bar{w}}{\partial\bar{x}^4} + \frac{\partial^2\bar{w}}{\partial\bar{t}^2} - \gamma^2\frac{\partial^4\bar{w}}{\partial\bar{t}^2\partial\bar{x}^2} + K_L\bar{w} - K_L\gamma^2\frac{\partial^2\bar{w}}{\partial\bar{x}^2} - K_p\frac{\partial^2\bar{w}}{\partial\bar{x}^2} + K_p\gamma^2\frac{\partial^4\bar{w}}{\partial\bar{x}^4} \\ & = \frac{1}{2}\left[\int_0^L\left(\frac{\partial\bar{w}}{\partial\bar{x}}\right)^2 d\bar{x}\right]\left[\frac{\partial^2\bar{w}}{\partial\bar{x}^2} - \gamma^2\frac{\partial^4\bar{w}}{\partial\bar{x}^4}\right] + \bar{F}\cos\Omega\bar{t} - 2\bar{\mu}\frac{\partial\bar{w}}{\partial\bar{t}} \end{aligned} \tag{24}$$

In order to include stretching and damping effects at order ε, deflection w is transformed $\bar{w} = \sqrt{\varepsilon}\ y$ to obtain a weak nonlinear system. The following transformation is performed for the damping and forcing terms based on the multiple scale method:

$$\bar{F} = \varepsilon\sqrt{\varepsilon}\,F \tag{25}$$

$$\bar{\mu} = \varepsilon\,\mu \tag{26}$$

Substituting Equations (25) and (26) into Equation (24) and performing some necessary simplifications, the simplified equations takes the following form:

$$\begin{aligned} & \frac{\partial^4 y}{\partial\bar{x}^4} + \frac{\partial^2 y}{\partial\bar{t}^2} - \gamma^2\frac{\partial^4 y}{\partial\bar{t}^2\partial\bar{x}^2} + K_L y - K_L\gamma^2\frac{\partial^2 y}{\partial\bar{x}^2} - K_p\frac{\partial^2 y}{\partial\bar{x}^2} + K_p\gamma^2\frac{\partial^4 y}{\partial\bar{x}^4} \\ & = \frac{1}{2}\varepsilon\left[\int_0^L\left(\frac{\partial y}{\partial\bar{x}}\right)^2 d\bar{x}\right]\left[\frac{\partial^2 y}{\partial\bar{x}^2} - \gamma^2\frac{\partial^4 y}{\partial\bar{x}^4}\right] + \varepsilon F\cos\Omega\bar{t} - 2\varepsilon\mu\frac{\partial y}{\partial\bar{t}} \end{aligned} \tag{27}$$

The non-dimensional form of boundary conditions can be expressed as:

$$\begin{array}{cc} \underline{\textbf{Simple} - \textbf{Simple Case}} & \underline{\textbf{Clamped} - \textbf{Clamped Case}} \\ y(0) = 0, y(1) = 0 & y(0) = 0,\ y(1) = 0, \\ y''(0) = 0, y''(1) = 0 & y'(0) = 0,\ y'(1) = 0 \end{array} \tag{28}$$

A straight forward asymptotic expansion can be introduced, which is why there is no quadratic non-linearity:

$$y\left(\overline{x}, \overline{t}; \varepsilon\right) = \varepsilon^0 y_0(\overline{x}, T_0; T_1) + \varepsilon y_1(\overline{x}, T_0; T_1) \tag{29}$$

where ε is a small parameter to denote the deflections. Hence, a weakly non-linear system can be investigated by this procedure. New independent variables are introduced and the fast and slow time scales are written as:

$$T_0 = \varepsilon^0 \overline{t} = \overline{t}, \quad T_1 = \varepsilon^1 \overline{t} = \varepsilon \overline{t} \tag{30}$$

Denoting $D_0 = \partial/\partial T_0$, $D_1 = \partial/\partial T_1$, the ordinary time derivatives can be transformed into partial derivatives as:

$$\frac{\partial}{\partial \overline{t}} = D_0 + \varepsilon D_1 + \cdots, \quad \frac{\partial^2}{\partial \overline{t}^2} = {D_0}^2 + 2\varepsilon D_0 D_1 + \cdots \tag{31}$$

Inserting Equations (29) and (31) into Equation (27), we can get the following relation for the equation of motion and boundary conditions at different orders:

Order (ε^0)

$$\left(1 + K_p \gamma^2\right) \mathrm{y}_0^{\mathrm{iv}} - \gamma^2 {D_0}^2 \mathrm{y}_0'' - K_L \gamma^2 \mathrm{y}_0'' - K_p \mathrm{y}_0'' + {D_0}^2 \mathrm{y}_0 + K_L \mathrm{y}_0 = 0 \tag{32}$$

<u>*Order*(ε)</u>:

$$\begin{aligned} &\left(1 + K_p \gamma^2\right) \mathrm{y}_1^{\mathrm{iv}} + {D_0}^2 \mathrm{y}_1 + K_L \mathrm{y}_1 - \gamma^2 {D_0}^2 \mathrm{y}_1'' - K_L \gamma^2 \mathrm{y}_1'' - K_p \mathrm{y}_1'' \\ &= -2\, D_0 D_1 \mathrm{y}_0 + 2\gamma^2\, D_0 D_1 \mathrm{y}_0'' + \frac{1}{2}\left(\int_0^1 {\mathrm{y}_0'}^2 d\overline{x}\right) \mathrm{y}_0'' - \frac{1}{2}\gamma^2 \left(\int_0^1 {\mathrm{y}_0'}^2 d\overline{x}\right) \mathrm{y}_0^{\mathrm{iv}} + \mathrm{F}\cos\Omega \mathrm{t} - 2\mu D_0 \mathrm{y}_0 \end{aligned} \tag{33}$$

Fundamental frequencies are obtained by solving the first order of expansions, whereas the solvability condition is obtained by solving the second order of expansion. The first order of perturbation is linear, as given in Equation (12); the solution may be represented by:

$$y_0\left(\overline{x}, T_0, T_1\right) = \left[A\left(T_1\right) e^{i\omega \mathrm{T}_0} + cc\right] Y\left(\overline{x}\right) \tag{34}$$

where cc represents the complex conjugate of the preceding terms. Substituting Equation (34) into Equation (32), one obtains:

$$\left(1 + K_p \gamma^2\right) Y^{iv}(\overline{x}) + \left(\omega^2 \gamma^2 - K_L \gamma^2 - K_p\right) Y''\left(\overline{x}\right) + \left(K_L - \omega^2\right) Y(\overline{x}) = 0 \tag{35}$$

The following shape function for any beam segment can be considered for the solution of the equations:

$$Y(\overline{x}) = c_1 e^{i\beta_1 \overline{x}} + c_2 e^{i\beta_2 \overline{x}} + c_3 e^{i\beta_3 \overline{x}} + c_4 e^{i\beta_4 \overline{x}} \tag{36}$$

The boundary conditions are applied and the frequency equations can be obtained. Using the functions in Equation (36) will give the dispersion relation shown below:

$$\left(1 + K_p \gamma^2\right) {\beta_n}^4 - \left(\omega^2 \gamma^2 - K_L \gamma^2 - K_p\right) {\beta_n}^2 + \left(K_L - \omega^2\right) \ (n = 1, 2, 3, 4) \tag{37}$$

$$y_1\left(\overline{x}, T_0, T_1\right) = \varphi\left(\overline{x}, T_1\right) e^{i\omega \mathrm{T}_0} + cc + W\left(\overline{x}, T_0, T_1\right) \tag{38}$$

and substituting Equation (38) into Equation (33), we eliminate the terms producing secularities. Here $W(\overline{x}, T_0, T_1)$ stands for the solution related with non-secular terms. One obtains:

$$\begin{aligned}\left(1+K_p\gamma^2\right)\varphi^{\text{iv}}+\left(\gamma^2\omega^2-K_L\gamma^2-K_p\right)\varphi''+\left(K_L-\omega^2\right)\varphi\\ =-2i\omega A'Y(\overline{x})+2i\omega\gamma^2A'Y''(\overline{x})\\ +\frac{3}{2}A^2\overline{A}\left(\int_0^1 Y'^2(\overline{x})\mathrm{d}\overline{x}\right)Y''(\overline{x})-\frac{3}{2}\gamma^2A^2\overline{A}\left(\int_0^1 Y'^2(\overline{x})\mathrm{d}\overline{x}\right)Y^{iv}(\overline{x})\\ +\frac{1}{2}\mathrm{F}\mathrm{e}^{\mathrm{i}\sigma\mathrm{T}_1}-4i\mu\omega AY(\overline{x})\end{aligned} \tag{39}$$

where cc represents the complex conjugate of preceding terms and NST represents the non-secular terms. Excitation frequency is assumed to close to one of the natural frequencies of the system; that is:

$$\Omega=\omega+\varepsilon\sigma \tag{40}$$

where σ is a detuning parameter of order 1, the solvability condition for Equations (39) and (40) is obtained as follows:

$$2i\omega\left(\mathrm{D}_1A+2\mu A\right)+2i\omega\gamma^2\mathrm{D}_1Ab+\frac{3}{2}A^2\overline{A}\left(b^2+\gamma^2bc\right)-\frac{1}{2}\mathrm{e}^{\mathrm{i}\sigma\mathrm{T}_1}f=0 \tag{41}$$

where $\int_0^1 Y^2(\overline{x})d\overline{x}=1, \int_0^1 Y'^2(\overline{x})d\overline{x}=b, \int_0^1 Y''^2(\overline{x})d\overline{x}=c, \int_0^1 FY(\overline{x})d\overline{x}=f.$

Taking into account the real amplitude a and phase θ, the complex amplitude A in Equation (41) can be written as the following form:

$$\mathrm{A}=\frac{1}{2}\mathrm{a}\left(\mathrm{T}_1\right)\mathrm{e}^{\mathrm{i}\theta(\mathrm{T}_1)} \tag{42}$$

Then amplitude and phase modulation equations are:

$$\begin{aligned}\omega\mathrm{a}\mathrm{D}_1\psi=\omega\mathrm{a}\sigma+\omega\gamma^2\mathrm{a}b\sigma-\omega\gamma^2\mathrm{a}b\mathrm{D}_1\psi-\frac{3}{16}\mathrm{a}^3\left(b^2+\gamma^2bc\right)+\frac{1}{2}f\cos\psi,\\ \omega\mathrm{D}_1\mathrm{a}\left(1+\gamma^2\right)+2\mu\omega\mathrm{a}=\frac{1}{2}f\sin\psi\end{aligned} \tag{43}$$

where $\theta=\sigma T_1-\psi$. In the steady-state case, Equation (43) will be solved in the following section and variation of nonlinear amplitude will be discussed.

3. Numerical Results

Numerical examples for the simple-simple and clamped-clamped end condition beam frequencies are presented in this section. The linear fundamental frequencies for both types of boundary conditions will be evaluated, and the nonlinear frequencies for free, undamped vibrations will also be evaluated. In the case of the $\mu=f=\sigma=0$, one obtains:

$$\mathrm{D}_1\mathrm{a}=0 \text{ and } \mathrm{a}=\mathrm{a}_0 \ \text{(constant)} \tag{44}$$

from Equation (44). The steady-state real amplitude is represented by a_0. The frequency of non-linear is:

$$\omega_{\mathrm{n}1}=\omega+\mathrm{a}_0^2\lambda \tag{45}$$

where $\lambda=\frac{3}{16}\frac{\left(b^2+\gamma^2bc\right)}{\omega\left(1+\gamma^2b\right)}$ is the nonlinear correction terms.

At the steady state, $a' = 0, \quad \psi' = 0$ become zero. The detuning parameter of frequency is as follows:

$$\sigma = \frac{3}{16}\frac{a^2\left(b^2+\gamma^2 bc\right)}{\omega\left(1+\gamma^2 b\right)} \mp \sqrt{\frac{f^2}{4\omega^2 a^2\left(1+\gamma^2 b\right)^2} - \mu^2} \tag{46}$$

The linear frequencies and nonlinear correction terms with different small scale effect (nonlocal parameter) γ, the Winkler parameter (K_L) and the Pasternak parameter (K_p) are given in Tables 1 and 2 for the first five frequencies for simple-simple (S-S) and clamped-clamped (C-C) supported case, respectively. The similar conclusions are derived from these tables for the effect of non-local parameter and the stiffness coefficients of the Winkler and Pasternak foundation on the natural frequencies. It can be seen in Tables 1 and 2 that non-dimensional natural non-local frequency of the nanobeam is smaller than the classical (local) natural frequency. Note that the non-local parameter $\gamma = 0$ corresponds to the classical nanobeams without the non-local effect. This is attributed to the effect of small scale effect. It is evident that an increase in the nonlocal parameter leads to the decrease in the natural frequency although correction term increases with nonlocal parameter. This situation can be interpreted that the non-local effect reduces the stiffness of the material and, hence, the comparative lower natural frequencies. The effect of the coefficients of the two parameter foundation on the frequency value of nanobeam is also seen in Tables 1 and 2 that show the linear frequency with the Winkler parameter and the Pasternak parameter (K_p). In these tables, the dimensionless parameter of Winkler K_L = 10, 100, 200 and of Pasternak K_p = 0, 5, 25, 50 are taken. It can be deduced from Tables 1 and 2 that the linear frequencies increase when the Winkler and the Pasternak parameters increase with regardless of the type of boundary condition. Furthermore, for the considered values of the foundation parameters, the effect of both foundation parameters on the linear frequency is more prominent for C-C end condition.

Table 1. The first five frequencies and correction term due to nonlinear terms for different γ, K_L, and K_p values for simple-simple support conditions.

K_L	K_P	γ	ω_1	ω_2	ω_3	ω_4	ω_5	λ
10	0	0	10.3638	39.6049	88.8827	157.945	246.76	1.76231
		0.1	9.93271	33.5769	64.7187	98.38	132.544	1.83879
		0.2	8.93522	24.7849	41.7484	58.4659	74.9066	2.04407
		0.3	7.84771	18.7699	29.7864	40.611	51.3169	2.32733
		0.4	6.91145	14.9337	22.9928	30.9739	38.9105	2.6426
		0.5	6.17194	12.3849	18.7082	25.021	31.3244	2.95923
	5	0	12.5203	42.0231	91.347	160.425	249.248	1.45877
		0.1	12.1658	36.3978	68.0635	102.314	137.119	1.50127
		0.2	11.366	28.49	46.7661	64.8678	82.7327	1.60692
		0.3	10.5325	23.4457	36.4878	49.3844	62.1862	1.73408
		0.4	9.85475	20.5039	31.1898	41.8205	52.4188	1.85334
		0.5	9.35098	18.7291	28.1803	37.6247	47.0629	1.95319
	25	0	18.8189	50.552	100.602	169.984	258.958	0.97052
		0.1	18.5849	45.9823	80.0573	116.732	154.067	0.98274
		0.2	18.0715	40.0156	62.9571	85.826	108.533	1.01066
		0.3	17.5592	36.596	55.7485	74.8137	93.8186	1.04015
		0.4	17.1612	34.7847	52.4341	70.0516	87.65	1.06427
		0.5	16.877	33.769	50.7016	67.6306	84.556	1.08219
	50	0	24.513	59.5186	111.092	181.225	270.606	0.74508
		0.1	24.3339	55.6896	92.8969	132.568	172.931	0.75057
		0.2	23.9441	50.8745	78.64	106.367	133.97	0.76279
		0.3	23.5599	48.231	72.9969	97.6982	122.354	0.77522
		0.4	23.2647	46.8715	70.4982	94.1014	117.69	0.78506
		0.5	23.0559	46.1227	69.2193	92.3132	115.405	0.79217

Table 1. *Cont.*

K_L	K_P	γ	ω_1	ω_2	ω_3	ω_4	ω_5	λ
100	0	0	14.0502	40.7252	89.3876	158.23	246.943	1.29992
		0.1	13.7353	34.8914	65.4103	98.8364	132.883	1.32973
		0.2	13.0322	26.5385	42.8127	59.2306	75.505	1.40147
		0.3	12.312	21.0311	31.2607	41.7044	52.1864	1.48345
		0.4	11.7375	17.6923	24.8731	32.3942	40.0503	1.55606
		0.5	11.3178	15.6008	20.9761	26.7591	38.7995	1.61376
	5	0	15.7085	43.0806	91.8383	160.706	249.428	1.1627
		0.1	15.4275	37.6139	68.7215	102.753	137.447	1.18387
		0.2	14.8049	30.028	47.7186	65.5579	83.2749	1.23366
		0.3	14.1751	25.2923	37.701	50.2874	62.9056	1.28847
		0.4	13.679	22.5922	32.6007	42.883	53.2703	1.3352
		0.5	13.3207	20.9947	29.7343	38.8023	48.0095	1.37111
	25	0	21.0748	51.4345	101.048	170.249	259.132	0.86664
		0.1	20.8662	46.9507	80.6175	117.117	154.358	0.8753
		0.2	20.4102	41.1248	63.6678	86.3487	108.947	0.89486
		0.3	19.9581	37.8057	56.5499	75.4128	94.297	0.91513
		0.4	19.6089	36.0552	53.2854	70.6911	88.162	0.93142
		0.5	19.3606	35.0763	51.5815	68.2927	85.0865	0.94337
	50	0	26.2848	60.2699	111.496	181.473	270.772	0.69486
		0.1	26.1178	56.4919	93.3801	132.907	173.191	0.6993
		0.2	25.755	51.7514	79.2102	106.789	134.306	0.70915
		0.3	25.3982	49.1551	73.6108	98.1577	122.721	0.71911
		0.4	25.1247	47.8219	71.1336	94.5784	118.072	0.72694
		0.5	24.9314	47.0883	69.8664	92.7994	115.794	0.73258
200	0	0	17.2456	41.935	89.9452	158.546	247.145	1.05906
		0.1	16.99	36.2961	66.1703	99.341	133.259	1.075
		0.2	16.4267	28.36	43.9651	60.0688	76.1643	1.11186
		0.3	15.8615	23.2875	32.8212	42.8865	53.1359	1.15148
		0.4	15.4197	20.3228	26.808	33.9026	41.2799	1.18447
		0.5	15.1027	18.5307	23.2378	28.5666	34.223	1.20933
	5	0	18.6214	44.226	92.3811	161.016	249.628	0.98082
		0.1	18.385	38.9205	69.4453	103.239	137.81	0.99343
		0.2	17.8658	31.6494	48.7551	66.3162	83.8731	1.0223
		0.3	17.3475	27.1974	39.0046	51.2721	63.6955	1.05284
		0.4	16.9445	24.7064	34.0999	44.0335	54.2008	1.07788
		0.5	16.6566	23.2546	31.3708	40.0702	49.0399	1.09651
	25	0	23.327	52.3976	101.542	170.542	259.324	0.78296
		0.1	23.1387	48.0039	81.2353	117.544	154.682	0.78934
		0.2	22.7284	42.3232	64.4484	86.9259	109.405	0.80359
		0.3	22.3232	39.1059	57.4273	76.073	94.8257	0.81817
		0.4	22.0115	37.4163	54.2156	71.3949	88.7273	0.82976
		0.5	21.7907	36.4739	52.5419	69.021	85.6722	0.83817
	50	0	28.1228	61.0939	111.944	181.748	270.957	0.64944
		0.1	27.9667	57.3701	93.914	133.283	173.479	0.65307
		0.2	27.6282	52.7087	79.8389	107.256	134.677	0.66107
		0.3	27.2959	50.162	74.287	98.6658	123.128	0.66912
		0.4	27.0416	48.8563	71.8331	95.1056	118.495	0.67541
		0.5	26.8621	48.1384	70.5784	93.3367	116.225	0.67992

Table 2. The first five frequencies and correction term due to nonlinear terms for different γ, K_L, and K_P values for clamped-clamped support conditions.

K_L	K_P	γ	ω_1	ω_2	ω_3	ω_4	ω_5	λ
10	0	0	22.5957	61.7538	120.945	199.884	298.572	1.87211
		0.1	21.3446	51.0811	85.7747	121.389	156.772	2.05167
		0.2	18.5608	36.5609	54.6156	71.6824	88.5434	2.42999
		0.3	15.6759	27.1862	38.9625	49.7775	60.8526	2.90516
		0.4	13.2865	21.375	30.1288	37.9597	46.2584	3.87926
		0.5	11.4372	17.5601	24.537	30.6482	37.2987	6.69827
	5	0	23.9143	63.5888	122.972	202.019	300.775	1.65438
		0.1	23.0822	53.8346	89.2907	125.642	161.759	1.82242
		0.2	21.2855	41.035	60.6274	79.1969	97.5636	2.17425
		0.3	19.5297	33.3506	47.3877	60.3333	73.6037	2.51772
		0.4	18.1866	29.0183	40.6783	51.1438	62.2405	2.80017
		0.5	17.2362	26.4531	36.892	46.048	56.0107	3.02728
	25	0	28.5299	70.4196	130.757	210.337	309.426	1.18673
		0.1	28.9764	63.6572	102.155	141.384	180.334	1.34457
		0.2	29.7794	55.4292	80.2951	103.958	127.411	1.66884
		0.3	30.3974	51.0386	71.8034	91.0459	110.794	1.96823
		0.4	30.7834	48.8132	68.137	85.5259	103.971	2.18714
		0.5	31.0191	47.5964	66.3087	82.7336	100.604	2.33713
	50	0	33.3266	78.0638	139.862	220.285	319.907	0.92027
		0.1	34.9378	74.1135	116.255	158.885	201.157	1.0774
		0.2	37.7969	69.3394	99.5596	128.361	156.931	1.37241
		0.3	40.0275	66.8639	93.7733	118.747	144.388	1.62261
		0.4	41.4571	65.6397	91.529	114.841	139.572	1.79539
		0.5	42.3505	64.9805	90.5076	112.918	137.3	1.90954
100	0	0	24.5064	62.4783	121.316	200.109	298.723	1.50573
		0.1	23.3579	51.9546	86.2978	121.759	157.059	1.63646
		0.2	20.8447	37.7717	55.4334	72.3074	89.0502	1.91101
		0.3	18.323	28.7939	40.1008	50.6734	61.5876	2.3019
		0.4	16.3258	23.3857	31.5871	39.1272	47.2212	3.26741
		0.5	14.8597	19.9588	26.3071	32.0829	38.4863	6.27336
	5	0	25.7273	64.2926	123.337	202.242	300.924	1.37867
		0.1	24.9557	54.6641	89.7933	126	162.037	1.50948
		0.2	23.304	42.1173	61.3651	79.763	98.0238	1.7793
		0.3	21.712	34.6737	48.328	61.0746	74.2126	2.04099
		0.4	20.5123	30.5297	41.7699	52.0162	62.9594	2.26079
		0.5	19.6745	28.1028	38.0923	47.0151	56.8084	2.44452
	25	0	30.0659	71.0558	131.101	210.551	309.571	1.06129
		0.1	30.4898	64.3603	102.595	141.702	180.583	1.1989
		0.2	31.254	56.2352	80.8536	104.39	127.764	1.48043
		0.3	31.8434	51.9128	72.4274	91.5388	111.2	1.73861
		0.4	32.2121	49.7265	68.7942	86.0504	104.403	1.92589
		0.5	32.4374	48.5326	66.9839	83.2758	101.05	2.05326
	50	0	34.6506	78.6381	140.183	220.489	320.047	0.85284
		0.1	36.2029	74.7182	116.641	159.168	201.381	0.99659
		0.2	38.9693	69.9854	100.011	128.711	157.218	1.26757
		0.3	41.1364	67.5336	94.252	119.126	144.699	1.49748
		0.4	42.5288	66.3217	92.0193	115.232	139.894	1.65584
		0.5	43.4	65.6694	91.0034	113.315	137.627	1.76014

Table 2. The first five frequencies and correction term due to nonlinear terms for different γ, K_L, and K_p values for clamped-clamped support conditions.

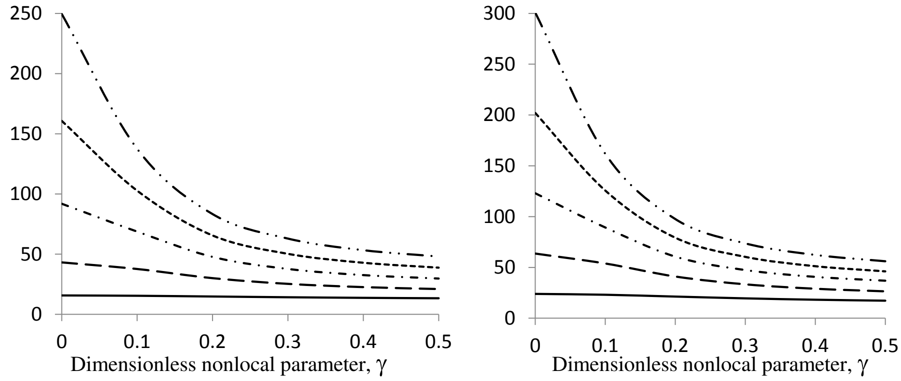
K_L	K_P	γ	ω_1	ω_2	ω_3	ω_4	ω_5	λ
200	0	0	26.4682	63.2735	121.728	200.359	298.89	1.76017
		0.1	25.4085	52.9083	86.8752	122.169	157.377	1.97028
		0.2	23.1193	39.073	56.3282	72.9956	89.6099	2.41183
		0.3	20.8742	30.481	41.3289	51.6507	62.3942	2.83125
		0.4	19.145	25.4341	33.1323	40.3849	48.2685	3.14686
		0.5	17.9112	22.3239	28.1437	33.6052	39.7643	3.36847
	5	0	27.6024	65.0657	123.742	202.489	301.09	1.59829
		0.1	26.8847	55.5713	90.3484	126.396	162.345	1.79521
		0.2	25.3589	43.2882	62.1746	80.3875	98.5325	2.20304
		0.3	23.9042	36.0869	49.3517	61.8879	74.8833	2.58175
		0.4	22.82	32.1257	42.9502	52.9687	63.7486	2.86038
		0.5	22.07	29.829	39.383	48.0668	57.6819	3.05255
	25	0	31.6853	71.756	131.482	210.788	309.732	1.19923
		0.1	32.0879	65.1325	103.081	142.054	180.86	1.37629
		0.2	32.8149	57.1174	81.4697	104.868	128.155	1.71325
		0.3	33.3767	52.8672	73.1145	92.0834	111.649	2.0006
		0.4	33.7286	50.7221	69.5172	86.6295	104.881	2.20012
		0.5	33.9439	49.5521	67.7262	83.874	101.544	2.33272
	50	0	36.0647	79.2714	140.539	220.716	320.203	0.94342
		0.1	37.5586	75.3844	117.069	159.482	201.629	1.11319
		0.2	40.2319	70.6962	100.509	129.099	157.536	1.40332
		0.3	42.3344	68.2699	94.781	119.545	145.044	1.63477
		0.4	43.6886	67.0713	92.5611	115.665	140.251	1.79052
		0.5	44.5372	66.4264	91.5512	113.756	137.99	1.89222

Studies related to the nonlocal beams resting on the Pasternak type elastic foundation in the existing literature are rather limited for the analysis of fundamental and nonlinear frequency. However, the study of Yokoyama [48] includes the first few values of the classical EB beam resting on a Pasternak-type foundation. Also, the study of Mustapha and Zhong [9] includes the non-uniform SWCNT depended on a non-local Rayleigh beam resting on Pasternak-type foundation. A comparison study is performed to check the reliability of the present method. For this purposes, linear frequency of a local EB beam embedded Pasternak foundation for the S-S case are compared with those of the work of Mustapha and Zhong [9] and the work of Yokoyama [48]. It can be seen from the Table 3 that they only studied the first second values of the non-dimensional natural frequencies of a local EB beam embedded on a Pasternak foundation, which takes the value of 25 and 36. However, in this paper extensive natural frequency analyses were performed for the first five frequencies. It is obvious from Table 3 that there is good harmony between the three results.

Table 3. Non-dimensional natural frequencies of a local EB beam embedded on a Pasternak foundation ($\gamma = 0$) for the simple-simple support conditions.

Mode	**Non-Dimensional Natural Frequencies**					
	K_L = 25 and K_p = 25			**K_L = 36 and K_p = 36**		
	Present	**Ref. [9]**	**Ref. [48]**	**Present**	**Ref. [9]**	**Ref. [48]**
1	19.2133	19.2178	19.21	22.1069	22.1112	——
2	50.7002	50.7804	50.71	54.916	55.1873	——
3	100.677	——	——	105.47	——	——
4	170.028	——	——	175.093	——	——
5	258.987	——	——	264.196	——	——

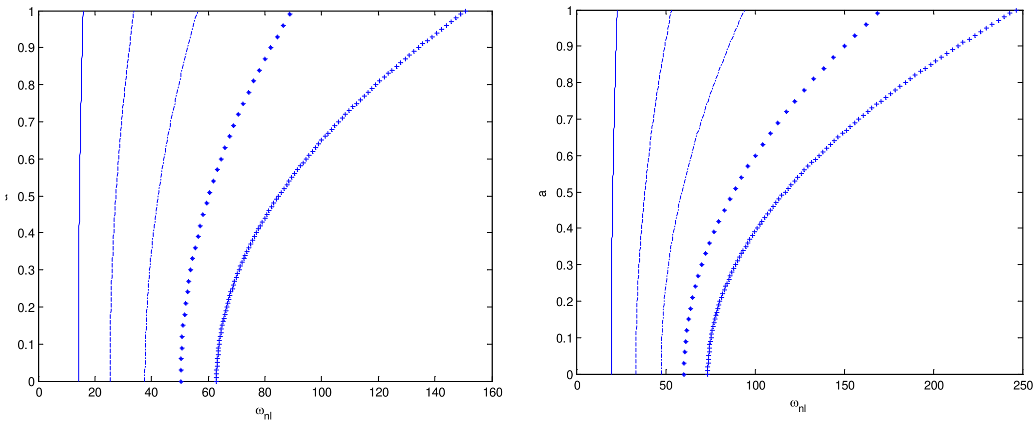
The effect of the non-local parameter on the natural frequency is examined and scrutinized in Figure 2 that plots the variation of the natural frequency (ω) with the non-local parameter (γ) for the S-S and CC nanobeam, respectively. It can be deduced from Figure 2 that the natural frequency decreases when the non-local parameter increases. Regardless of the type of boundary condition, it is observed that the non-local parameter has an influence on the natural frequency.



(**a**) (**b**)

Figure 2. Variation of the natural frequency with the dimensionless nonlocal parameter for $K_L = 10$, $K_p = 5$. (**a**) S-S nanobeam and (**b**) C-C nanobeam (___ ω_1, _ _ ω_2, _._ ω_3, - - ω_4, _ . . ω_5).

Variation of the nonlinear frequency with amplitude is shown for the first five modes of vibration in Figure 3, the frequencies are calculated taking into account the non-local parameter ($\gamma = 0.3$). It can be seen from Figure 3 that the nonlinear frequencies increase with an increase in the mode number.



(**a**) (**b**)

Figure 3. Nonlinear frequency *versus* amplitude curves of nanobeam for different modes for the $K_L = 100$, $K_p = 5$ and $\gamma = 0.3$. (**a**) simple-simple; (**b**) clamped-clamped (___ ω_1, _ _ ω_2, _._ ω_3, ●●●ω_4, +++ ω_5).

In Figures 4–6 the nonlinear frequency *versus* amplitude curves of nanobeam are shown for the first mode and S-S and C-C boundary condition. One can observe a hardening behavior. The frequency response bending to the left side is called the softening nonlinearity, but to the right side is called the hardening nonlinearity. So, the behaviors in Figures 4–6 are of hardening type, *i.e.*, the nonlinear frequency increases as the vibration amplitude increases. Figure 4 shows the effect of the Winkler parameter K_L on the nonlinear frequency *versus* amplitude curves with $\gamma = 0.3$ and $K_p = 5$. It can be seen in Figure 4 that the nonlinear frequency of nanobeam increases with the increment of the K_L values. The Winkler parameter K_L has a significant effect on the nonlinear frequency value. In Figures 5 and 6 $\gamma = 0.3$ is fixed and K_p is increased. The nonlinear frequencies increase in both figures. From Figures 5 and 6 it is noted that the Pasternak parameter K_p has a pronounced effect on the nonlinear frequency amplitude curves of nanobeam. It can be readily observed that the value of nonlinear natural frequency have a direct relation with the Winkler and Pasternak parameter value. The C-C nanobeam has the highest natural frequency and nonlinear frequencies since the end condition is the strongest for the C-C nanobeam.

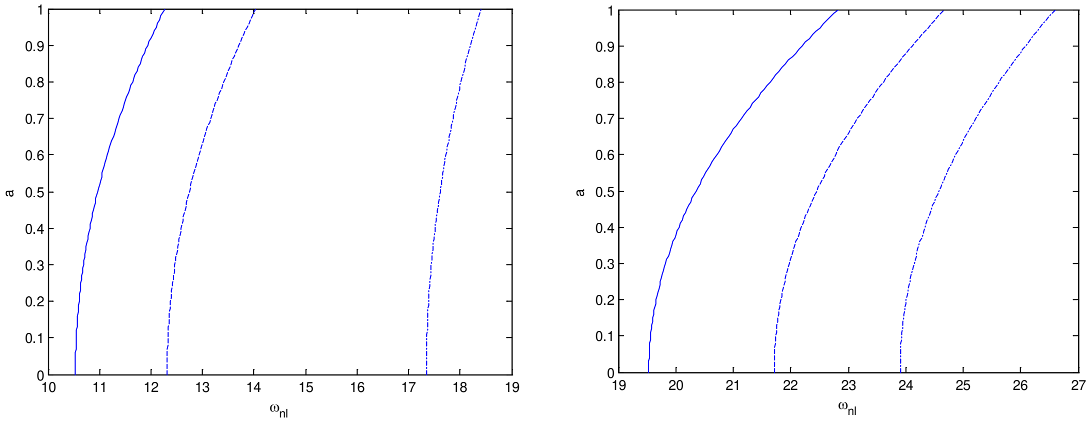


Figure 4. Nonlinear frequency *versus* amplitude curves of nanobeam for the first mode and $\gamma = 0.3$. (**a**) simple-simple; (**b**) clamped-clamped ($K_p = 5$: ___ $K_L = 10$, _ _ $K_L = 100$, _._ $K_L = 200$).

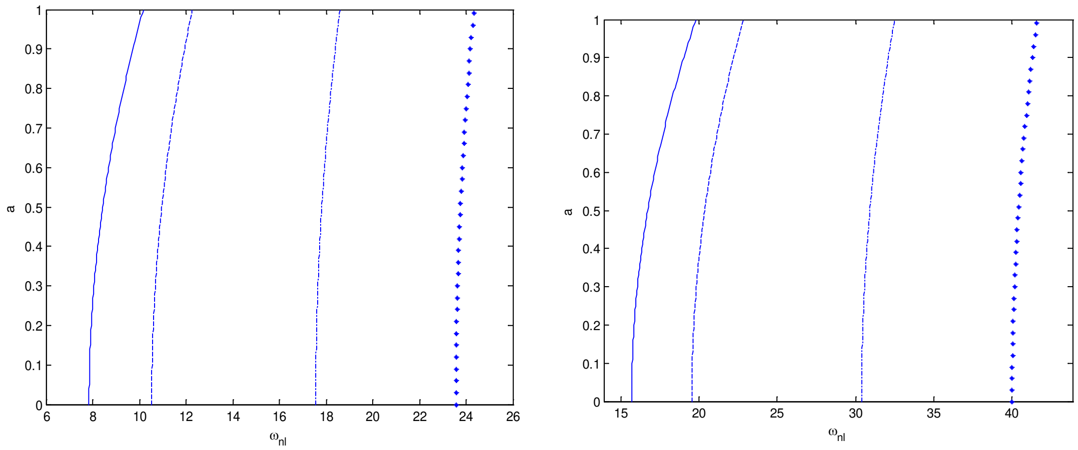


Figure 5. Nonlinear frequency *versus* amplitude curves of nanobeam for the first mode $K_L = 10$ and $\gamma = 0.3$. (**a**) simple-simple; (**b**) clamped-clamped (___ $K_p = 0$, _ _ $K_p = 5$, _._ $K_p = 25$, ●●●$K_p = 50$).

Frequency response curves are presented in Figures 7–10. The detuning parameter σ shows the nearness of the external excitation frequency to the natural frequency of system. Several figures are drawn using Equation (46) assuming f = 1 and damping coefficient μ = 0.1. Increasing the forcing term increase amplitudes when $\sigma < 0$ and decreases the amplitudes when $\sigma > 0$ at different values. The maximum amplitudes happen when $\sigma > 0$. In Figure 7, the influence of the mode number on the hardening nonlinear properties is shown both types of boundary condition. Five different mode numbers are considered and compared. It can be seen that for the first mode or fundamental mode, the resonant amplitude is larger and the corresponding width is broader. Figure 8 presents the frequency response curves of S-S and C-C case for the first mode in order to discuss the influence of the Winkler parameter K_L. It can be observed that, for S-S and C-C end condition, the amplitude decreases with the Winkler parameter increasing. Figures 9 and 10 present the frequency response curves of S-S and C-C case for the first mode in order to discuss the influence of the Pasternak parameter K_p. It can be seen that for the fundamental mode, the amplitude decreases with the Pasternak parameter increasing.

Both observations denote that the Winkler and Pasternak parameter has significant influences on the primary resonance of the nanobeam.

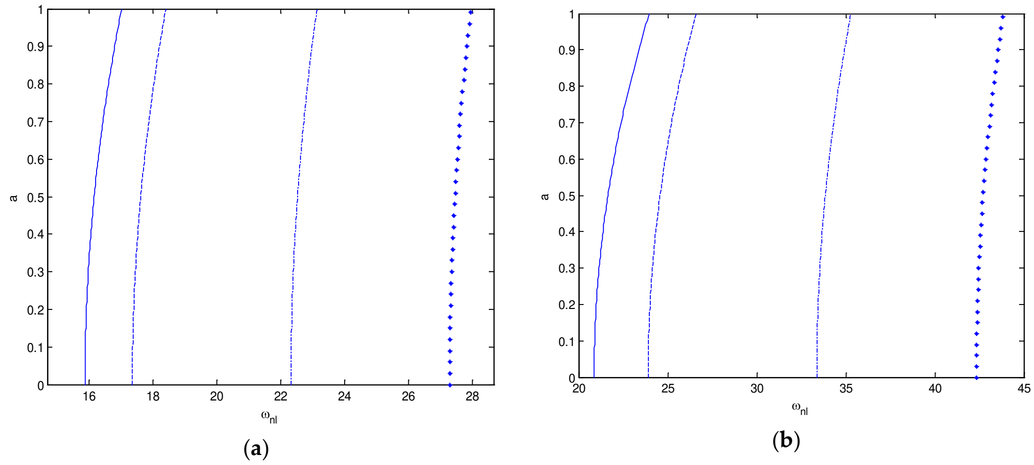


Figure 6. Nonlinear frequency *versus* amplitude curves of nanobeam for the first mode K_L = 200 and γ = 0.3. (**a**) simple-simple; (**b**) clamped-clamped (___ K_p = 0, _ _ K_p = 5, _._ K_p = 25, ●●●K_p = 50).

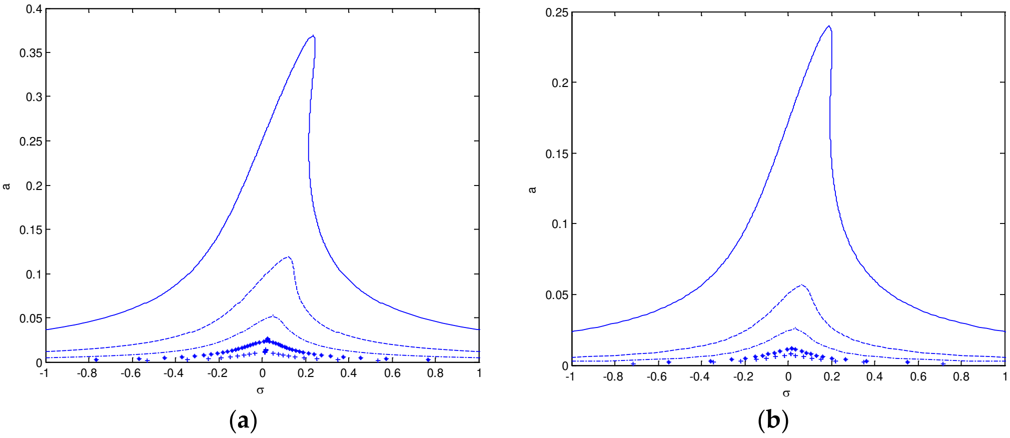


Figure 7. Frequency-response curves for nanobeam with different modes for K_L = 100, K_p = 5 and γ = 0.3. (**a**) simple-simple; and (**b**) clamped-clamped (___ω_1, _ _ω_2, _._ ω_3, ●●●ω_4, +++ω_5).

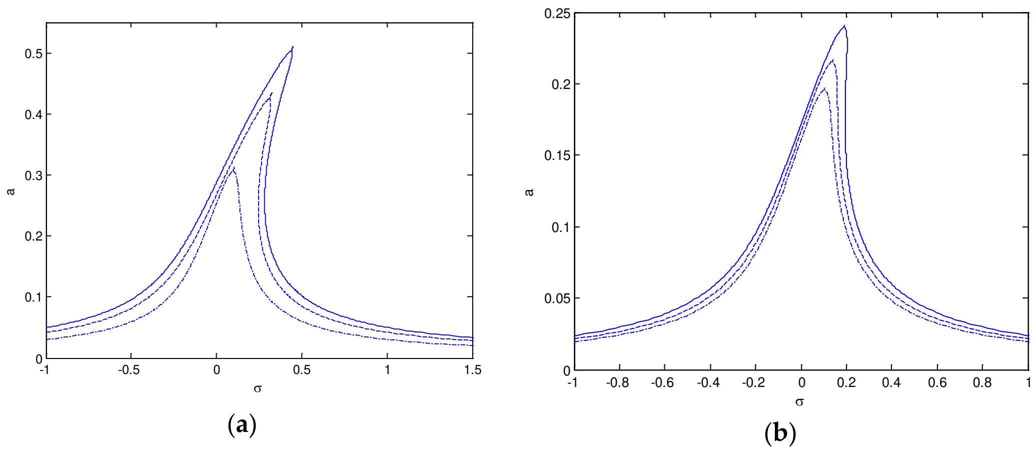
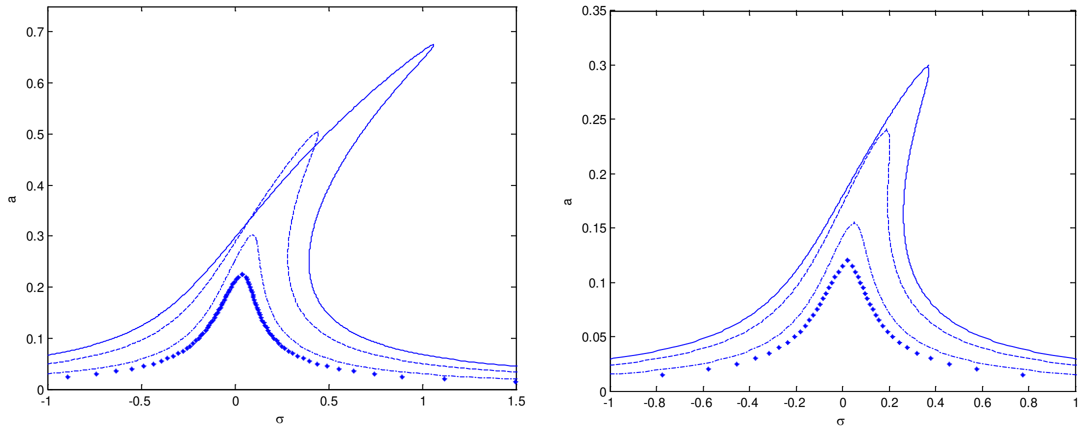
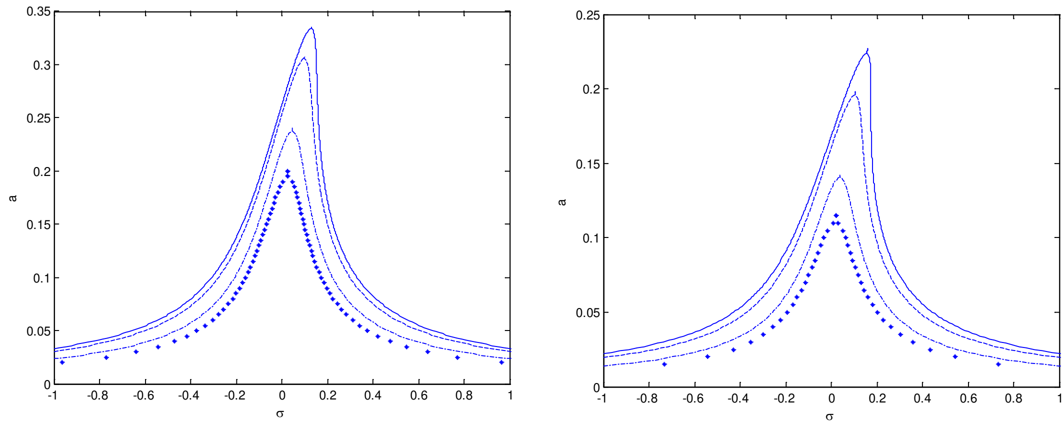


Figure 8. Effects of Winkler parameter on frequency-response curves for the first mode and γ = 0.3. (**a**) simple-simple; (**b**) clamped-clamped (K_p = 5: ___ K_L = 10, _ _ K_L = 100, _._ K_L = 200).



(**a**) (**b**)

Figure 9. Effects of the Pasternak parameter on frequency-response curves for the first mode and $\gamma = 0.3$. (**a**) simple-simple; (**b**) clamped-clamped ($K_L = 10$: ___ $K_p = 0$, _ _ $K_p = 5$, _._ $K_p = 25$, ●●●$K_p = 50$).



(**a**) (**b**)

Figure 10. Effects of the Pasternak parameter on frequency-response curves for the first mode, $K_L = 200$ and $\gamma = 0.3$. (**a**) simple-simple; (**b**) clamped-clamped (___ $K_p = 0$, _ _ $K_p = 5$, _._ $K_p = 25$, ●●●$K_p = 50$.).

4. Conclusions

In the present study, the free and force vibration of a nanobeam resting on an elastic foundation of the Pasternak type is investigated based on the non-local Euler Bernoulli beam theory. The non-linear equations of motion, including stretching of the neutral axis, are derived. The governing equations and boundary conditions are derived by using Hamilton's principle. The multiple scale method is used to solve the governing differential equation of the nanobeam. The effect of different parameters, such as Winkler modulus, Pasternak shear modulus, and the non-local factor on frequencies is investigated for the nanobeam with simple-simple and clamped-clamped boundary conditions. The extensive numerical data is given in tabular form for various values of the parameters so that these results can be used as a reference for future studies. Results revealed that increasing the non-local parameters lead to decreasing the linear and nonlinear frequencies and to increasing the correction terms. Furthermore, increasing the Winkler and Pasternak parameters increase the values of both linear and nonlinear frequencies. Observed non-linearity is of the hardening type because of the stretching of the neutral axis.

Author Contributions: All authors contributed extensively to the work presented in this paper. N.T. and S.M.B. obtained the equations, drawn the figures and wrote the main paper. All authors discussed the results and implications and commented on the manuscript at all stages.

Conflicts of Interest: The authors declare no conflict of interest.

Abbreviations

The following abbreviations are used in this manuscript:

CNT	Carbon Nanotube
SWCNT	Single walled carbon nanotube
DWCNT	Double walled carbon nanotube
MWCNT	Multi walled carbon nanotube

References

1. Eringen, A.C. On differential-equations of nonlocal elasticity and solutions of screw dislocation and surface-waves. *J. Appl. Phys.* **1983**, *54*, 4703–4710. [CrossRef]
2. Eringen, A.C. *Nonlocal Continuum Field Theories*; Springer-Verlag: New York, NY, USA, 2002.
3. Peddieson, J.; Buchanan, G.R.; McNitt, R.P. Application of nonlocal continuum models to nanotechnology. *Int. J. Eng. Sci.* **2003**, *41*, 305–312. [CrossRef]
4. Niknam, H.; Aghdam, M.M. A semi analytical approach for large amplitude free vibration and buckling of nonlocal FG beams resting on elastic foundation. *Compos. Struct.* **2015**, *119*, 452–462. [CrossRef]
5. Fallah, A.; Aghdam, M.M. Nonlinear free vibration and post-buckling analysis of functionally graded beams on nonlinear elastic foundation. *Eur. J. Mech. A Solids* **2011**, *30*, 571–583. [CrossRef]
6. Fallah, A.; Aghdam, M.M. Thermo-mechanical buckling and nonlinear free vibration analysis of functionally graded beams on nonlinear elastic foundation. *Compos. B Eng.* **2012**, *43*, 1523–1530. [CrossRef]
7. Kanani, A.S.; Niknam, H.; Ohadi, A.R.; Aghdam, M.M. Effect of nonlinear elastic foundation on large amplitude free and forced vibration of functionally graded beam. *Compos. Struct.* **2014**, *115*, 60–68. [CrossRef]
8. Şimşek, M. Nonlinear static and free vibration analysis of microbeams based based on the nonlinear elastic foundation using modified couple stress theory and He's variational method. *Compos. Struct.* **2014**, *112*, 264–272. [CrossRef]
9. Mustapha, K.B.; Zhong, Z.W. Free transverse vibration of an axially loaded non-prismatic single-walled carbon nanotube embedded in a two-parameter elastic medium. *Comput. Mater. Sci.* **2010**, *50*, 742–751. [CrossRef]
10. Mehdipour, I.; Barari, A.; Kimiaeifar, A.; Domairry, G. Vibrational analysis of curved single-walled carbon nanotube on a Pasternak elastic foundation. *Adv. Eng. Softw.* **2012**, *48*, 1–5. [CrossRef]
11. Shen, H.S.; Zhang, C.L. Nonlocal beam model for nonlinear analysis of carbon nanotubes on elastomeric substrates. *Comput. Mater. Sci.* **2011**, *50*, 1022–1029. [CrossRef]
12. Arani, A.G.; Atabakhshian, V.; Loghman, A.; Shajari, A.R.; Amir, S. Nonlinear vibration of embedded SWBNNTs based on nonlocal Timoshenko beam theory using DQ method. *Physica B* **2012**, *407*, 2549–2555. [CrossRef]
13. Murmu, T.; Pradhan, S.C. Buckling analysis of a single-walled carbon nanotube embedded in an elastic medium based on nonlocal elasticity and Timoshenko beam theory and using DQM. *Physica E* **2009**, *41*, 1232–1239. [CrossRef]
14. Yas, M.H.; Samadi, N. Free vibrations and buckling analysis of carbon nanotube-reinforced composite Timoshenko beams on elastic foundation. *Int. J. Press. Vessels Pip.* **2012**, *98*, 119–128. [CrossRef]
15. Kazemi-Lari, M.A.; Fazelzadeh, S.A.; Ghavanloo, E. Non-conservative instability of cantilever carbon nanotubes resting on viscoelastic foundation. *Physica E* **2012**, *44*, 1623–1630. [CrossRef]
16. Ghanvanloo, E.; Daneshmand, F.; Rafiei, M. Vibration and instability analysis of carbon nanotubes conveying fluid and resting on a linear viscoelastic Winkler foundation. *Physica E* **2010**, *42*, 2218–2224. [CrossRef]
17. Rafiei, M.; Mohebpour, S.R.; Daneshmand, F. Small-scale effect on the vibration of non-uniform carbon nanotubes conveying fluid and embedded in viscoelastic medium. *Physica E* **2012**, *44*, 1372–1379. [CrossRef]
18. Arani, A.G.; Amir, S. Electro-thermal vibration of visco-elastically coupled BNNT systems conveying fluid embedded on elastic foundation via strain gradient theory. *Physica B* **2013**, *419*, 1–6. [CrossRef]
19. Wang, Y.Z.; Li, F.M. Nonlinear free vibration of nanotube with small scale effects embedded in viscous matrix. *Mech. Res. Commun.* **2014**, *60*, 45–51. [CrossRef]

20. Mahdavi, M.H.; Jiang, L.Y.; Sun, X. Nonlinear vibration of a double-walled carbon nanotube embedded in a polymer matrix. *Physica E* **2011**, *43*, 1813–1819. [CrossRef]
21. Lee, H.L.; Chang, W.J. Vibration analysis of a viscous-fluid-conveying single-walled carbon nanotube embedded in an elastic medium. *Physica E* **2009**, *41*, 529–532. [CrossRef]
22. Kiani, K. Vibration analysis of elastically restrained double-walled carbon nanotubes on elastic foundation subject to axial load using nonlocal shear deformable beam theories. *Int. J. Mech. Sci.* **2013**, *68*, 16–34. [CrossRef]
23. Aydogdu, M. Axial vibration analysis of nanorods (carbon nanotubes) embedded in an elastic medium using nonlocal elasticity. *Mech. Res. Commun.* **2012**, *43*, 34–40. [CrossRef]
24. Wang, B.L.; Wang, K.F. Vibration analysis of embedded nanotubes using nonlocal continuum theory. *Compos. B Eng.* **2013**, *47*, 96–101. [CrossRef]
25. Ke, L.L.; Xiang, Y.; Yang, J.; Kitipornchai, S. Nonlinear free vibration of embedded double-walled carbon nanotubes based on nonlocal Timoshenko beam theory. *Comput. Mater. Sci.* **2009**, *47*, 409–417. [CrossRef]
26. Fang, B.; Zhen, Y.X.; Zhang, C.P.; Tang, Y. Nonlinear vibration analysis of double-walled carbon nanotubes based on nonlocal elasticity theory. *Appl. Math. Model.* **2013**, *37*, 1096–1107. [CrossRef]
27. Şimşek, M. Nonlocal effects in the forced vibration of an elastically connected double-carbon nanotube system under a moving nanoparticle. *Comput. Mater. Sci.* **2011**, *50*, 2112–2123. [CrossRef]
28. Ansari, R.; Ramezannezhad, H. Nonlocal Timoshenko beam model for the large-amplitude vibrations of embedded multiwalled carbon nanotubes including thermal effects. *Physica E* **2011**, *43*, 1171–1178. [CrossRef]
29. Ansari, R.; Ramezannezhad, H.; Gholami, R. Nonlocal beam theory for nonlinear vibrations of embedded multiwalled carbon nanotubes in thermal environment. *Nonlinear Dyn.* **2012**, *67*, 2241–2254. [CrossRef]
30. Kiani, K. A meshless approach for free transverse vibration of embedded single walled nanotubes with arbitrary boundary conditions accounting for nonlocal effect. *Int. J. Mech. Sci.* **2010**, *52*, 1343–1356. [CrossRef]
31. Murmu, T.; Pradhan, S.C. Thermo-mechanical vibration of a single-walled carbon nanotube embedded in an elastic medium based on nonlocal elasticity theory. *Comput. Mater. Sci.* **2009**, *46*, 854–859. [CrossRef]
32. Chang, T.P. Thermal–mechanical vibration and instability of a fluid-conveying single-walled carbon nanotube embedded in an elastic medium based on nonlocal elasticity theory. *Appl. Math. Model.* **2012**, *36*, 1964–1973. [CrossRef]
33. Rahmati, A.H.; Mohammadimehr, M. Vibration analysis of non-uniform and non-homogeneous boron nitride nanorods embedded in elastic medium under combined loadings using DQM. *Physica B* **2014**, *440*, 88–98. [CrossRef]
34. Pradhan, S.C.; Reddy, G.K. Buckling analysis of single walled carbon nanotube on Winkler foundation using on nonlocal elasticity theory and DTM. *Comput. Mater. Sci.* **2011**, *50*, 1052–1056. [CrossRef]
35. Narender, S.; Gopalakrishnan, S. Critical buckling temperature of single walled carbon nanotubes embedded in a one-parameter elastic medium based on nonlocal continuum mechanics. *Physica E* **2011**, *43*, 1185–1191. [CrossRef]
36. Murmu, T.; Pradhan, S.C. Thermal effects on the stability of embedded carbon nanotubes. *Comput. Mater. Sci.* **2010**, *47*, 721–726. [CrossRef]
37. Arani, A.G.; Amir, S.; Shajari, A.R.; Mozdianfard, M.R.; Maraghi, Z.K.; Mohammadimehr, M. Electro-thermal nonlocal vibration analysis of embedded DWBNNTs. *Proc. Inst. Mech. Eng. C J. Mech. Eng. Sci.* **2011**, *224*, 745–756. [CrossRef]
38. Mikhasev, G. On localized modes of free vibrations of single walled carbon nanotubes embedded in nonhomogeneous elastic medium. *ZAMM* **2014**, *94*, 130–141. [CrossRef]
39. Fu, Y.M.; Hong, J.W.; Wang, X.Q. Analysis of nonlinear vibration for embedded carbon nanotubes. *J. Sound Vib.* **2006**, *296*, 746–756. [CrossRef]
40. Komijani, M.; Esfahani, S.E.; Reddy, J.N.; Liu, Y.P.; Eslami, M.R. Nonlinear thermal stability and vibration of pre/post-buckled temperature and microstructure dependent functionally graded beams resting on elastic foundation. *Compos. Struct.* **2014**, *112*, 292–307. [CrossRef]
41. Ozturk, B.; Coskun, S.B.; Koc, M.Z.; Atay, M.T. Homotopy perturbation method for free vibration analysis of beams on elastic foundation. *IOP Conf. Ser. Mater. Sci. Eng.* **2010**, *10*, 012158. [CrossRef]
42. Öz, H.R.; Pakdemirli, M.; Özkaya, E.; Yılmaz, M. Nonlinear vibrations of a slightly curved beam resting on a nonlinear elastic foundation. *J. Sound Vib.* **1998**, *212*, 295–309. [CrossRef]

43. Yan, Y.; Wang, W.; Zhang, L. Applied multiscale method to analysis of nonlinear vibration for double walled carbon nanotubes. *Appl. Math. Model.* **2011**, *35*, 2279–2289. [CrossRef]
44. Bağdatlı, S.M. Non-linear vibration of nanobeams with various boundary condition based on nonlocal elasticity theory. *Compos. B Eng.* **2015**, *80*, 43–52. [CrossRef]
45. Bağdatlı, S.M. Non-linear transverse vibrations of tensioned nanobeams using nonlocal beam theory. *Struct. Eng. Mech.* **2015**, *55*, 281–298. [CrossRef]
46. Nayfeh, A.H. *Introduction to Perturbation Techniques*; John Wiley: New York, NY, USA, 1981.
47. Nayfeh, A.H.; Mook, D.T. *Nonlinear Oscillations*; John Wiley: New York, NY, USA, 1979.
48. Yokoyama, T. Vibrations and transient responses of Timoshenko beams resting on elastic foundations. *Arch. Appl. Mech.* **1987**, *57*, 81–90. [CrossRef]