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NONLINEAR WAVES IN RODS

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Rods are simple structures that are often regarded as one-dimensional continua, but because of finite transverse dimensions, propagating waves are subject to dispersion, which may mask other effects. A one-dimensional continuum theory with one internal, scalar variable can be used to model a solid rod with longitudinal waves. In this paper the material is assumed to be homogeneous, isotropic, and hyperelastic. First, the linear theory is reviewed to exhibit clearly the multiple wave hierarchy that exists in a rod. Next, expansion and

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scaling techniques are used in the fully nonlinear case to examine the main pulse. Finally, steady nonlinear waves are examined and both solitary waves and periodic waves are found to exist. Results are compared to the simple wave solutions available from the most elementary one-dimensional nonlinear theory for rods.

TABLE OF CONTENTS

I. INTRODUCTION

The simplest theory for nonlinear wave propagation in a straight, uniform rod may be written as follows.

$$S_Z = \rho v_t$$
$$v_Z = \varepsilon_t \qquad (1)$$

The axial stress, S, depends only on the axial strain, ε, the axial particle velocity is v, and the reference density, ρ, is a constant. The axial coordinate along the rod is Z, t is time, and subscripts denote partial differentiation. In this version of rod theory, which is formally the same as one dimensional, inviscid gas dynamics, the characteristic speed of propagation, c, is given by $\rho c^2 = dS/d\varepsilon$, and discontinuities in stress, strain, or velocity all travel at the shock speed given by $\rho c^2 = [S]/[\varepsilon]$, where [.] denotes the jump in a quantity across the shock.

Equation (1) has been used to interpret wave propagation experiments in which the strain-time profile is measured at several stations in the rod.[1,2] If each level of strain propagates at a constant velocity, $c_p(\varepsilon)$, then both stress and particle velocity may be computed as a function of strain as follows.

$$v = v_o - \int_{\varepsilon_o}^{\varepsilon(Z,t)} c_p \, d\varepsilon$$
$$S = S_o + \int_{\varepsilon_o}^{\varepsilon(Z,t)} \rho c_p^{\,2} \, d\varepsilon \qquad (2)$$

The initial values ahead of the wave are v_o, S_o and ε_o. Equation (2) is a simple wave solution to (1), exhibiting amplitude dispersion, but not geometric dispersion.

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[1] J. F. Bell, The Experimental Foundations of Solid Mechanics, Handbuch der Physik, Vol. VIa/1, Springer-Verlag, New York, 1973.

[2] G. E. Hauver, "Penetration with Instrumented Rods," Int. J. Eng. Sci, 16, 1978, pp. 871-877.

A rod, no matter how slender, is actually a three-dimensional object, of course. The dispersive influence of a finite diameter is well known in linear theories, eg.[3,4] It is less well known in nonlinear theories[5,6] where it may be expected to play a role as well, whenever the length scale in a wave pulse is comparable in magnitude to the rod diameter. In this paper the effects of finite lateral dimensions will be considered by modeling a rod as a one-dimensional elastic structure with one internal variable used to represent the transverse, axisymmetric motion. Such a structure is a special case of an intrinsic rod theory as described by Antman[7].

II. ONE-DIMENSIONAL EQUATIONS AND RELATIONSHIP TO THREE-DIMENSIONAL ELASTICITY

A straight cylindrical rod of radius a is assumed to have an elastic stored energy density per unit length

$$\hat{W} = \pi\, a^2\, W(w', u, u'). \tag{3}$$

The axial displacement is w, u is a measure of radial strain, and the dash represents differentiation with respect to the axial coordinate, Z. The kinetic energy density per unit length is given by

$$K = \pi\, a^2\left(\tfrac{1}{2}\rho_1 \dot{w}^2 + \tfrac{1}{2}\rho_2 \dot{u}^2\right), \tag{4}$$

where ρ_1 and ρ_2 are the appropriate mass densities (made definite by equation (13)) for the axial and radial motions respectively, and the dot denotes differentiation with respect to time. The Euler-LaGrange equations corresponding to (3) and (4) are given by (5).

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[3] R. Skalak, "Longitudinal Impact of a Semi-Infinite Circular Elastic Bar," J. Appl. Mech., 24, 1957, pp. 59-64.

[4] W. A. Green, "Dispersion Relations for Elastic Waves in Bars," in Progress in Solid Mechanics, Vol. I, ed. I. N. Sneddon and R. Hill, North-Holland, Amsterdam, 1960.

[5] J. H. Shea, "Propagation of Plastic Strain Pulses in Cylindrical Lead Bars," J. Appl. Phys., 39, 1968, pp. 4004-4011.

[6] G. P. DeVault, "The Effect of Lateral Inertia on the Propagation of Plastic Strain in a Cylindrical Rod," J. Mech. Phys. Sol., 13, 1965, pp. 55-68.

[7] S. S. Antman, The Theory of Rods, Hanbuch der Physik, Vol. VIa/2, Springer-Verlag, New York, 1972.

$$
\begin{aligned}
S' &= \rho_1 \ddot{w} \\
Q' - P &= \rho_2 \ddot{u}
\end{aligned}
\tag{5}
$$

The forces S, P, and Q are obtained from the stored energy as follows.

$$
S = \frac{\partial W}{\partial w'} \, , \quad P = \frac{\partial W}{\partial u} \quad , \quad Q = \frac{\partial W}{\partial u'}
\tag{6}
$$

Equations (5) and (6) may be interpreted in terms of the three-dimensional theory of nonlinear elasticity. For the axisymmetric motions considered here, the deformation from an unstressed reference configuration (R,Θ,Z) to the present configuration (r,θ,z) may be written

$$
r = r(R,Z,t), \; \theta = \Theta, \; z = z(R,Z,t) \; ,
\tag{7}
$$

and the corresponding equations of motion are given by

$$
\begin{aligned}
\frac{\partial T^{rR}}{\partial R} + \frac{\partial T^{rZ}}{\partial Z} + \frac{T^{rR} - T^{\theta\Theta}}{R} &= \rho \ddot{r} \; , \\
\frac{\partial T^{zR}}{\partial R} + \frac{\partial T^{zZ}}{\partial Z} + \frac{T^{zR}}{R} &= \rho \ddot{z} \; .
\end{aligned}
\tag{8}
$$

The stress components used here are obtained from the unsymmetric Piola-Kirchhoff tensor referred to unit basis vectors in the given cylindrical coordinates (r,θ,z) and (R,Θ,Z).

$$
\underset{\sim}{T} = T^{i\alpha} \, \underset{\sim}{e}_i \otimes \underset{\sim}{E}_\alpha
\tag{9}
$$

If $(8)_1$ is multiplied by R, and if the resulting equation and $(8)_2$ are both averaged over the cross-section, equations (10) are obtained for the case of stress-free lateral surfaces.

$$
\begin{aligned}
\frac{1}{\pi a^2} \frac{\partial}{\partial Z} \int T^{zZ} \, dA &= \frac{\rho}{\pi a^2} \int \ddot{z} \, dA \\
\frac{1}{\pi a^2} \frac{\partial}{\partial Z} \int R T^{rZ} \, dA - \frac{1}{\pi a^2} \int (T^{rR} + T^{\theta\Theta}) dA &= \frac{\rho}{\pi a^2} \int R \, \ddot{r} \, dA
\end{aligned}
\tag{10}
$$

Now if the following identifications are made

$$S \leftrightarrow \frac{1}{\pi a^2}\int T^{zZ}\, dA \ ,$$

$$P \leftrightarrow \frac{1}{\pi a^2}\int (T^{rR} + T^{\theta\theta})\, dA \ , \tag{11}$$

$$Q \leftrightarrow \frac{1}{\pi a^2}\int RT^{rZ}\, dA \ ,$$

and if a first approximation for the position functions r and z is assumed to be

$$\begin{aligned} r &= R[1 + u(Z,t)] \ , \\ z &= Z + w(Z,t) \ , \end{aligned} \tag{12}$$

then equations (5) are obtained with

$$\rho_1 = \rho \ , \ \rho_2 = \tfrac{1}{2}\rho a^2 \ . \tag{13}$$

In all of the preceding discussion it has been tacitly assumed that the Z-axis is an axis of material symmetry so that no angular motion will occur anywhere in the cross-section. Furthermore, since the strain energy must be invariant under reversal of the z-axis, and if it is assumed to be invariant under reversal of the Z-axis as well, W must have the property

$$W(w',u,u') = W(w',u,-u') \ . \tag{14}$$

It follows that Q is odd in u' and S and P are even in u'.

It should be remarked that equations (5) and (6) have been used to describe one-dimensional waves in porous materials[8,9,10], and it has

[8] J. W. Nunziato and E. K. Walsh, "On the Influence of Void Compaction and Material Non-uniformity on the Propagation of One-Dimensional Acceleration Waves in Granular Materials," Arch. Rat. Mech. Anal., 64, 1977, pp. 299-316 and Adendum, Arch. Rat. Mech. Anal., 67, 1977, pp. 395-397.

[9] J. W. Nunziato and E. K. Walsh, "One-Dimensional Shock Waves in Uniformly Distributed Granular Materials," Int. J. Solids and Structures, 14, 1978, pp. 681-689.

[10] D. F. Parker and B. R. Seymour, "Finite Amplitude One-Dimensional Pulses in an Inhomogeneous Granular Material," Arch. Rat. Mech. Anal., 72, 1980, pp. 265-284.

also been suggested that perhaps they could be used to describe waves in a layered composite[11]. There are undoubtedly other applications as well.

III. STATIC SOLUTIONS

It is convenient to distinguish two types of static solutions. In the first type u' is identically zero, so equations (5) reduce to

$$
\begin{aligned}
S(w',u,0) &= S_o \ , \\
P(w',u,0) &= 0 \ .
\end{aligned}
\tag{15}
$$

Since the reference configuration has been assumed to be unstressed, it is required that when $w' = u = 0$, (15) is satisfied with $S_o = 0$. It will be further assumed that the slope along the curve defined by $(15)_2$ is always negative and bounded.

$$\infty > -\frac{du}{dw'} = \frac{P_{w'}}{P_u} = \nu(w',u,0) > 0 \tag{16}$$

where the subscripts u and w' denote partial differentiation with respect to the arguments of P. This corresponds to the physically reasonable assumption that the rod contracts (expands) laterally as it is stretched (compressed). Finally, it will be assumed that S increases monotonically as w' increases along the curve $P(w',u,0) = 0$. That is

$$S_{w'} - \nu S_u = E(w',u,0) > 0. \tag{17}$$

The bar modulus, E, and Poisson's ratio, ν, will be considered as defined by (17) and (16) with the inequalities holding as well for all values of their arguments, not just along the curve of $(15)_2$. Since W has a minimum at the unstressed reference configuration ($w' = u = u' = 0$), $P_u > 0$ there, so by (16) $P_{w'} > 0$ and $P_u > 0$ everywhere. This also implies from (17) that $W_{w'w'}W_{uu} - W_{uw'}^2 > 0$ so that necking instabilities have been ruled out (see Antman [12], pp. 97).

In the second type of static solution u' is not identically zero. There are two integrals of (5).

---

[11] M. F. McCarthy, private communication.

[12] S. S. Antman, "Qualitative Theory of the Ordinary Differential Equations of Nonlinear Elasticity," in Mechanics Today, V. 1, ed. S. Nemat-Nasser, Pergamon, New York, 1972.

$$S(w',u,u') = S_o$$
$$w'Q(w',u,u') + w'S_o - W(w',u,u') = B \tag{18}$$

If these may be solved for u' and w' as functions of u and the constants S_o and B, then u may be found by quadrature, and the axial strain w' is given parametrically through u. The study of equations (18) will not be pursued further here except to remark that the version arising from linear elasticity gives boundary layer solutions with Q decaying exponentially from the ends. Phase plane analysis indicates that the behavior of the fully nonlinear equations is similar.

IV. SOLUTIONS OF THE LINEARIZED EQUATIONS

Before considering the dynamic nonlinear equations, it will be useful to examine some of the properties of solutions of the linearized equations. The three-dimensional strain energy for an isotropic material with displacement field given by (12), when averaged over the cross-section, leads to (19) for the forces.

$$\begin{aligned} S &= (\lambda+2\mu)w' + 2\lambda u \ , \\ P &= 4(\lambda+\mu)u + 2\lambda w' \ , \\ Q &= \tfrac{1}{2}a^2\mu u^1 \ , \end{aligned} \tag{19}$$

where λ and μ are the usual Lamé constants. With nondimensional variables $\zeta = Z/a$ and $\tau = ct/a$, where c is a speed that is unspecified for now, and with $\bar{w} = w/a$, equations (5) become

$$\begin{aligned} w_{\zeta\zeta} + 2\,\frac{\lambda}{\lambda+2}\,u_\zeta &= \frac{c^2}{c_1{}^2}\,w_{\tau\tau} \ , \\ u_{\zeta\zeta} - \left[8\,\frac{\lambda+\mu}{\mu}\,u + 4\,\frac{\lambda}{\mu}\,w_\zeta\right] &= \frac{c^2}{c_2^2}\,u_{\tau\tau} \ . \end{aligned} \tag{20}$$

where subscripts again denote partial differentiation. Subscripts indicate partial differentiation, and the overbar has been dropped from $\bar{w}$. The longitudinal and shear wave speeds are $c_1 = \sqrt{(\lambda+2\mu)/\rho}$ and $c_2 = \sqrt{\mu/\rho}$. These equations in dimensional form were first given by Mindlin and Herrmann[13].

---

[13] R. D. Mindlin and G. Herrmann, "A One-Dimensional Theory of Compressional Waves in an Elastic Rod," Proceedings of the First U.S. National Congress of Applied Mechanics, 1950, pp. 187-191.

Since all the coefficients are constants, either w or u may be eliminated from (20) to obtain (21).

$$\left[\left(\frac{\partial^2}{\partial\zeta^2} - \frac{c^2}{c_1^2}\frac{\partial^2}{\partial\tau^2}\right)\left(\frac{\partial^2}{\partial\zeta^2} - \frac{c^2}{c_2^2}\frac{\partial^2}{\partial\tau^2}\right) - 8\,\frac{\lambda+\mu}{\mu}\,\frac{c_b^2}{c_1^2}\left(\frac{\partial^2}{\partial\zeta^2} - \frac{c^2}{c_b^2}\frac{\partial^2}{\partial\tau^2}\right)\right](w,u) = 0 \tag{21}$$

The one-dimensional bar speed is $c_b = \sqrt{E/\rho}$ where E is Young's modulus, $E = \mu(3\lambda + 2\mu)/(\lambda+\mu)$. Equation (21) exhibits what Whitham has called a hierarchy of wave speeds[14,15]. Whereas discontinuities can only propagate with the speeds c_1/c or c_2/c, it is well known that in a thin bar the main disturbance in a pulse travels with speed c_b/c. Whitham has discussed extensively the hierarchical case where the orders of the highest and lowest derivatives differ by one. Wu[16] has pointed out that the case where the orders differ by two, as here, also exhibits a stable hierarchical wave structure, so long as $c_2<c_b<c_1$, which is always the case for linear elastic rods.

In Whitham's case the higher order waves decay exponentially, and the lower order wave, which carries the main pulse, has a diffusing front. In the present case, discontinuities in the higher order waves do not decay with time, but the pulse decays rapidly behind the front. The main pulse travels with the lower order speed, and it too has a diffusing front.

To give substance to the preceding remarks, consider a boundary - initial value problem for (20).

$$\begin{aligned} &\text{B.V.: } u_\zeta(0,\tau) = 0,\ w(0,\tau) = f(\tau) \\ &\text{I.V.: } u(\zeta,0) = w(\zeta,0) = 0 \end{aligned} \tag{22}$$

A straightforward application of the LaPlace transform on τ leads to the following representations for u and w.

---

[14] G. B. Whitham, "Some Comments on Wave Propagation and Shock Wave Structure with Applications to Magnetohydrodynamics," Comm. on Pure and Appl. Math, 12, 1959, pp. 113-158.

[15] G. B. Whitham, Linear and Nonlinear Waves, John Wiley, New York, 1974.

[16] T. T. Wu, "A Note on the Stability Condition for Certain Wave Propagation Problems," Comm. on Pure and Appl. Math., 14, 1961, pp. 745-747.

$$w = \frac{1}{2\pi i} \int_{Br} C(p)\, e^{p\tau + m_1\zeta} dp + \frac{1}{2\pi i} \int_{Br} D(p)\, e^{p\tau + m_2\zeta} dp \tag{23}$$

$$-\frac{2\lambda}{\lambda+2\mu} u = \frac{1}{2\pi i} \int_{Br} \frac{C(p)}{m_1} \left(m_1^2 - p^2 \frac{c^2}{c_1^2}\right) e^{p\tau + m_1\zeta} dp + \frac{1}{2\pi i} \int_{Br} \frac{D(p)}{m_2} \left(m_2^2 - p^2 \frac{c^2}{c_1^2}\right) e^{p\tau + m_2\zeta} dp$$

$C(p)$ and $D(p)$ are given by (24)

$$C(p) = -\frac{\bar{f}}{p} \frac{m_2^2 - p^2 \dfrac{c^2}{c_1^2}}{m_1^2 - m_2^2}, \qquad D(p) = \frac{\bar{f}}{p} \frac{m_1^2 - p^2 \dfrac{c^2}{c_1^2}}{m_1^2 - m_2^2} \tag{24}$$

$\bar{f}(p)$ is the transform of $f(\tau)$, and $m_1(p)$ and $m_2(p)$ are the roots with negative real parts (for outgoing waves) of

$$\left(m^2 - p^2 \frac{c^2}{c_1^2}\right)\left(m^2 - p^2 \frac{c^2}{c_2^2}\right) - 8\, \frac{\lambda+\mu}{\mu}\, \frac{c_b^2}{c_1^2} \left(m^2 - p^2 \frac{c^2}{c_b^2}\right) = 0\ . \tag{25}$$

In each equation of (23) the first integral represents a wave whose front travels at speed c_1 and the second integral represents a wave whose front travels at speed c_2.

To examine the behavior of the solution near the wavefronts, let p be large. Expansions for m_1 and m_2 are

$$m_1 = -\frac{c}{c_1}p - 4\frac{\lambda+\mu}{\lambda+2\mu}\frac{c_1}{c}\frac{c_1^2 - c_b^2}{c_1^2 - c_2^2}\frac{1}{p} + O\left(\frac{1}{p^3}\right) , \tag{26}$$

$$m_2 = -\frac{c}{c_2}p - 4\frac{\lambda+\mu}{\mu}\frac{c_2}{c}\frac{c_b^2 - c_2^2}{c_1^2 - c_2^2}\frac{1}{p} + O\left(\frac{1}{p^3}\right) .$$

The leading terms in the integrals for w_τ may be written asymptotically as

$$w_\tau \sim \frac{1}{2\pi i}\int_{Br}\left[1 + O\left(\frac{1}{p^2}\right)\right]\bar{f}\, e^{p\tau - \frac{c}{c_1}\left(p^2 + r_1^2\right)^{\frac{1}{2}}\zeta} dp$$

$$+ \frac{1}{2\pi i}\int_{Br} O\left(\frac{1}{p^2}\right)\bar{f}\, e^{p\tau - \frac{c}{c_2}\left(p^2 + r_2^2\right)^{\frac{1}{2}}\zeta} dp . \tag{27}$$

In (27) the square root terms are equivalent to m_1 or m_2 within $O\left(\frac{1}{p^3}\right)$ when r_1^2 and r_2^2 are given by

$$r_1^2 = 8\,\frac{c_1^2 - c_b^2}{c^2} , \tag{28}$$

$$r_2^2 = 8\,\frac{c_b^2 - c_2^2}{c^2} .$$

The first integral in (27) vanishes for $\tau < \frac{c}{c_1}\zeta$ and the second vanishes for $\tau < \frac{c}{c_2}\zeta$. Thus, near the leading wavefront only the first integral is required, which may be written, after a little manipulation and use of the convolution theorem, as follows.

$$w_\tau \sim -\frac{c_1}{c}\frac{\partial}{\partial\zeta}\int_{\frac{c}{c_1}\zeta}^{\tau} f(\tau-\tau')\, J_o\left[r_1\sqrt{\tau'^2-\frac{c^2}{c_1^2}\zeta^2}\right]d\tau'$$

$$\sim f\left(\tau-\frac{c}{c_1}\zeta\right) - \frac{c}{c_1}r_1\zeta\int_{\frac{c}{c_1}\zeta}^{\tau} f(\tau-\tau')\,\frac{J_1\left[r_1\sqrt{\tau'^2-\frac{c^2}{c_1^2}\zeta^2}\right]}{\sqrt{\tau'^2-\frac{c^2}{c_1^2}\zeta^2}}\,d\tau' \qquad (29)$$

J_o and J_1 are Bessel functions. Suppose that $f(\tau) = w_\tau^{\,o}\, h(\tau)$ where $h(\tau)$ is the Heavisde step function and $w_\tau^{\,o}$ is a constant. The change of variable $\gamma = r_1\sqrt{\tau'^2-\frac{c^2}{c_1^2}\zeta^2}$ and use of the identity (taken from Reference 17, pg. 64)

$$\int_o^\infty \frac{J_1(\gamma)\, d\gamma}{\sqrt{\gamma^2+\eta^2}} = \frac{1-e^{-\eta}}{\eta} \qquad (30)$$

now reduces (29) to

$$\frac{w_\tau}{w_\tau^{\,o}} \sim e^{-r_1\frac{c}{c_1}\zeta} + r_1\frac{c}{c_1}\zeta\int_{r_1\sqrt{\tau^2-\frac{c^2}{c_1^2}\zeta^2}}^{\infty} \frac{J_1(\gamma)\, d\gamma}{\sqrt{\gamma^2+\left(r_1\frac{c}{c_1}\zeta\right)^2}} \, . \qquad (31)$$

Equation (31) shows that at points removed from the boundary the velocity becomes exponentially small shortly after passage of the wave front. Equation (29) is plotted in Fig. 1 with $f(\tau) = h(\tau)$.

Similar analysis may be applied to the other integrals in (23). The second integral for w_τ represents a wave whose front moves with speed c_2, and is similar in form to (29), but smoothed by a double integration in τ. u_ζ has a contribution at each wavefront that is similar to (29), but smoothed by a single integration in τ.

---

[17]F. Bowman, "Introduction to Bessel Functions," Dover, New York, 1958.

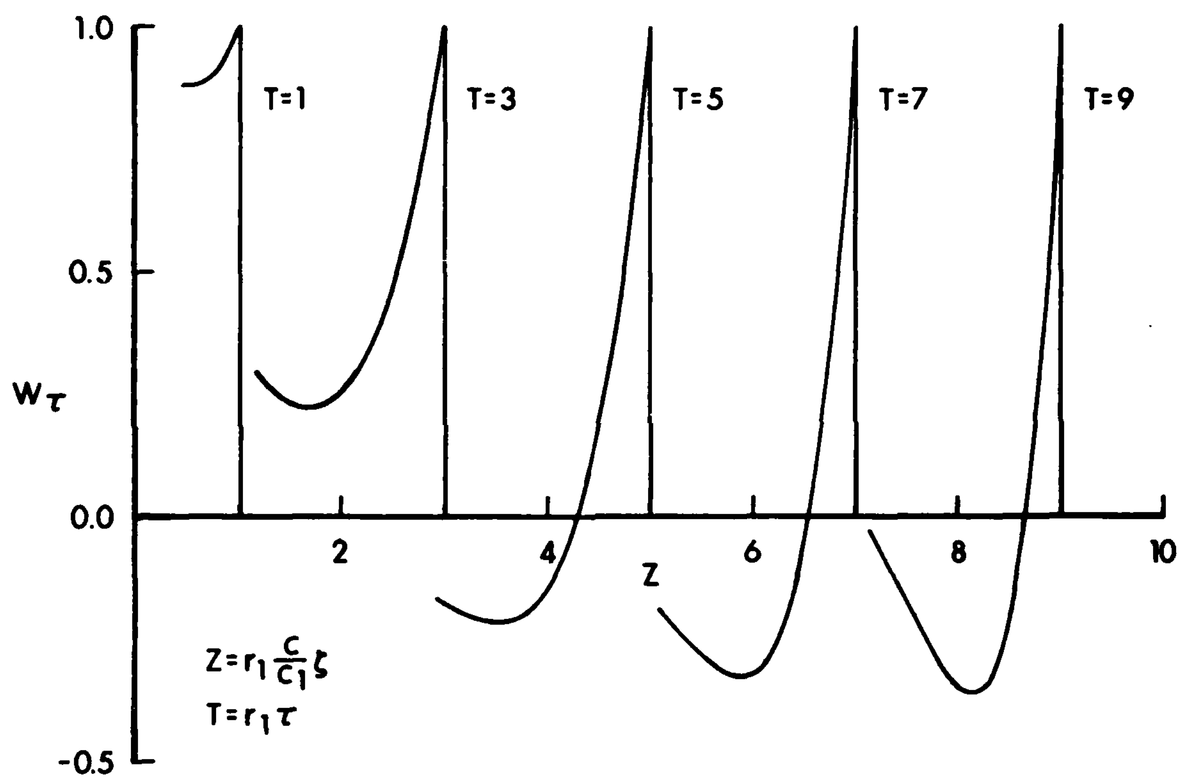

Figure 1: Dispersion in Leading Wave. Linear Case.

To examine the main pulse in (23) it is convenient to follow Whitham's approach to the problem (Reference 15, pg. 345). First, the exponent in one of the integrals is written as $\tau\left(p + m_1(p)\,\frac{\zeta}{\tau}\right)$, and then the saddle point method is used to find an asymptotic expansion valid at large τ. For a fixed value of ζ/τ saddles occur at

$$\frac{d}{dp}\left(p + m_1(p)\,\frac{\zeta}{\tau}\right) = 1 + m_1'(p)\,\frac{\zeta}{\tau} = 0. \tag{32}$$

Solution of (32) gives $p = \tilde{p}\left(\frac{\zeta}{\tau}\right)$, and the exponent is maximized with respect to ζ/τ when $m_1[\tilde{p}(\zeta/\tau)] = 0$ because of (32) and (33).

$$\frac{d}{d(\zeta/\tau)}\left[\tilde{p} + m_1(\tilde{p})\,\frac{\zeta}{\tau}\right] = m_1(\tilde{p}) + \left[1 + m_1'(\tilde{p})\,\frac{\zeta}{\tau}\right]\frac{dp}{d(\zeta/\tau)} = 0 \tag{33}$$

From (25) if $m_1 = 0$, either $p = 0$ or $p^2 = -8\,\frac{\lambda+\mu}{\mu}\,\frac{c_2^{\,2}}{c^2}$. By differentiating (25) once with respect to p, it is clear that the correct choice is $p = 0$, and by differentiating (25) a second time with respect to p, it is found that

$$m_1'(0) = \pm\,\frac{c}{c_b}, \tag{34}$$

and so from (32)

$$\zeta/\tau = \pm\,\frac{c_b}{c}. \tag{35}$$

Equation (35) shows that the main disturbance travels with the bar speed, c_b. The correct asymptotic expansions will be obtained by expanding the integrands in (23) about $p = 0$. The following discussion will be simplified by taking $c = c_b$.

It has been shown that $m_1(0) = 0$, but it does not follow that $m_2(0) = 0$. Therefore, let $m_2^{\,2} = a + bp^2 + dp^4 + \ldots$ in (25). By equating powers of p^2 it is found that

$$\begin{aligned} m_1^{\,2} &= p^2 - \frac{1}{8}\,\frac{\left(c_1^{\,2} - c_b^{\,2}\right)\left(c_b^{\,2} - c_2^{\,2}\right)}{c_b^{\,2}\left(c_1^{\,2} - c_2^{\,2}\right)}\,p^4 + \ldots, \\ m_2^{\,2} &= 8\,\frac{\lambda+\mu}{\mu}\,\frac{c_b^{\,2}}{c_1^{\,2}} + \left(\frac{c_b^{\,2}}{c_1^{\,2}} + \frac{c_b^{\,2}}{c_2^{\,2}} - 1\right)p^2 + \ldots\,. \end{aligned} \tag{36}$$

Choosing roots to correspond to outgoing waves or bounded functions at infinity gives

$$m_1 = -p + \frac{1}{16}\frac{\left(c_1^2 - c_b^2\right)\left(c_b^2 - c_2^2\right)}{c_b^2\left(c_1^2 - c_2^2\right)} p^3 + \dots ,$$

$$m_2 = -\left\{a_2^{\frac{1}{2}} + \frac{1}{2}\frac{b_2}{a_2^{\frac{1}{2}}} p^2 + \dots\right\} . \qquad (37)$$

Since the second integrals for u and w in (23) give no contribution until $\tau > \frac{c}{c_2}\zeta$, near the head of the main pulse the velocity is given asymptotically as follows

$$w_\tau \sim \frac{1}{2\pi i}\int_{Br} \bar{f}\, e^{p(\tau-\zeta) - \frac{1}{2}d_1 p^3 \zeta}\left(1 + 0(p^2)\right) dp , \qquad (38)$$

$$\text{where} \quad d_1 = -\frac{1}{8}\frac{\left(c_1^2 - c_b^2\right)\left(c_b^2 - c_2^2\right)}{c_b^2\left(c_1^2 - c_2^2\right)} .$$

By the convolution theorem, (38) may be written

$$w_\tau = \frac{1}{\left(-\frac{3}{2}d_1\zeta\right)^{1/3}}\int_0^\tau f(\tau-\tau')\, \mathrm{Ai}\left[\frac{\zeta-\tau'}{\left(-\frac{3}{2}d_1\zeta\right)^{1/3}}\right] d\tau' . \qquad (39)$$

where Ai(·) is the Airy function with integral representation given in Abromowitz and Stegun[18], formula 10.4.32. For a step function input, $f(\tau) = h(\tau)$, w is simply the integral of the Airy function and is exponentially small for $\zeta > \tau$ and oscillates about an amplitude of 1 for $\tau > \zeta$. Fig. 2 shows the waveform for a step input. Fig. 3 shows the composite waveform schematically for both the leading wave and the main pulse. Note the progressive separation and decreasing importance of the leading fast wave.

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[18] M. Abromowitz and I. A. Stegun, Handbook of Mathematical Functions, Dover, New York, 1965.

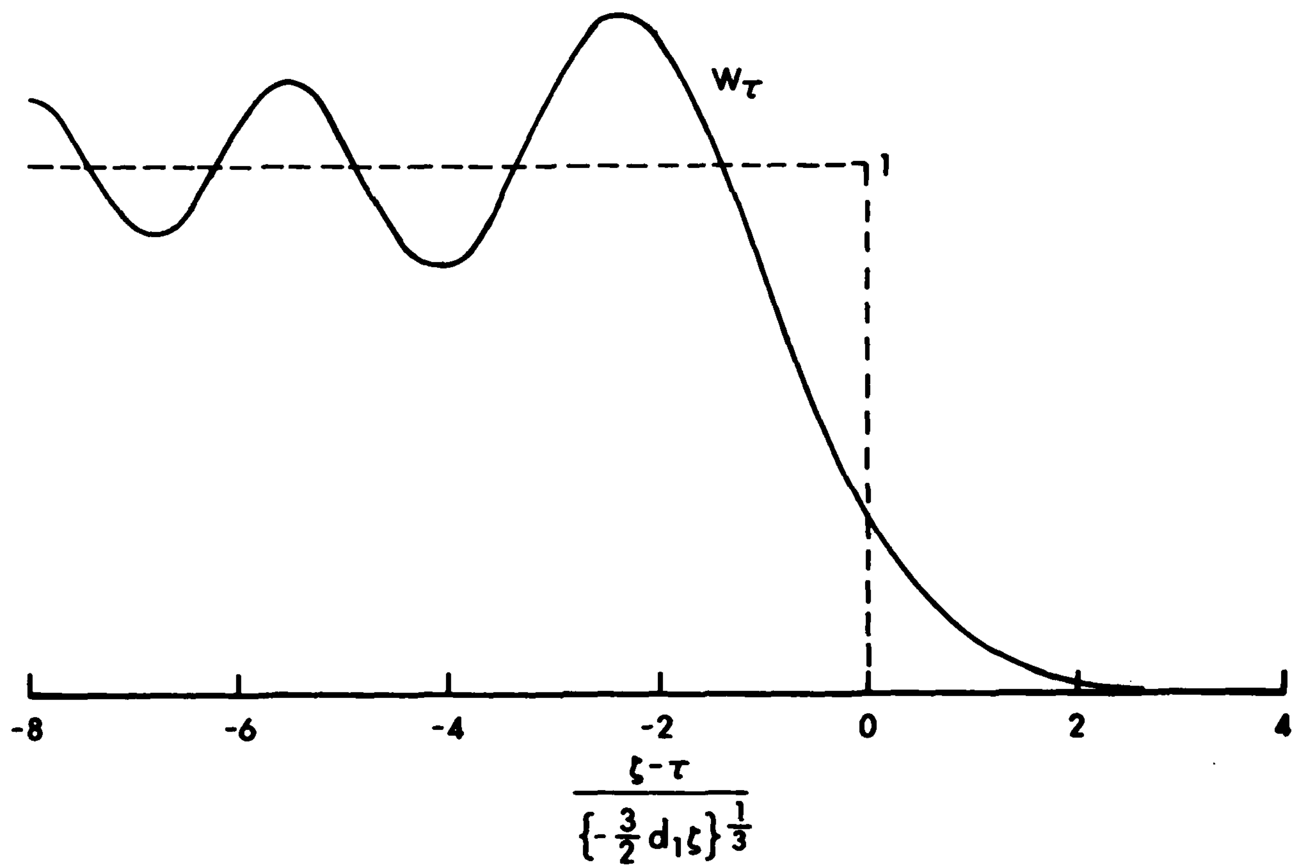

Figure 2: Dispersion in Main Pulse. Linear Case.

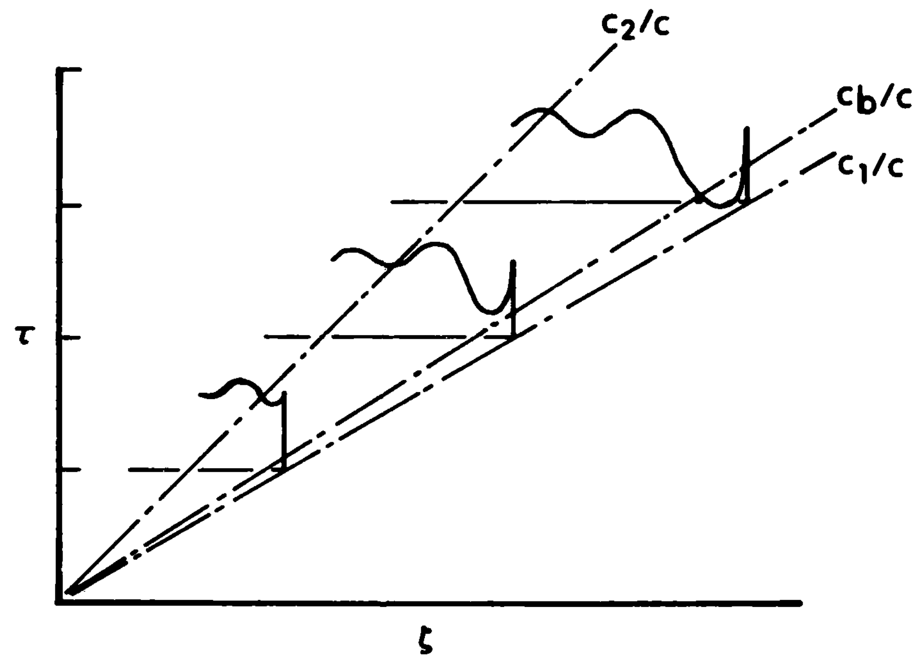

Figure 3: Composite Sketch Showing Separation of Leading Wave and Main Pulse. Linear Case.

The initial development of this separation was shown numerically by Nunziato and Walsh in their consideration of small amplitude waves in porous soils[19] where the equations are formally the same as (20).

V. NONLINEAR WAVE EQUATIONS

The multiple wave structure, which the linearized equations show so clearly, has an exact counterpart in the fully nonlinear theory. The characteristic wave speeds of (5), which carry discontinuities in w'' and $\ddot{w}$ or u'' and $\ddot{u}$, are given by

$$\begin{aligned} \rho_1 c_1^2 &= S_{w'}(w',u,u') \ , \\ \rho_2 c_2^2 &= Q_{u'}(w',u,u') \ . \end{aligned} \tag{40}$$

On the other hand, the simplest one-dimensional theory for a rod, which is given by (1), and which may be obtained from (5) by setting $u' = 0$, $P(w',u,0) = 0$, has characteristic speeds given by

$$\rho c_b^2 = E < S_{w'} \ , \tag{41}$$

where E is defined in (17). In analogy with the linear theory it might be expected that (41) represents a low order approximation for wave speeds in the main pulse, but that dispersive effects due to the finite diameter will modify the pulse shape. Since the full rod theory, (5), is nonlinear, it is to be expected that wave speeds will be modified as well.

Although transform methods cannot be used on the full equations, it is possible, by suitable scaling and choice of independent variables, to find asymptotic results for the various wave orders directly from the equations themselves rather than from representations for exact solutions. The method of relatively undistorted waves, introduced by Varley and Cumberbatch[20] has been used recently by Seymour and Parker[10] to examine the leading fast wave in (5). The same approach, also called the method of modulated simple waves, may be used to investigate the structure of the main pulse.

To simplify the analysis, it will be assumed that the stored energy in (3) has the decomposition

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[20] E. Varley and E. Cumberbatch, "Non-Linear High Frequency Sound Waves," J. Inst. Maths. Applics., 2, 1966, pp. 133-143.

$$W(w',u,u') = W_1(w',u) + W_2(u') \tag{42}$$

Let the dependent variables be u, $v = \dot{w}$, and $\varepsilon = w'$. Equations (5) are rewritten as

$$\begin{aligned} S' &= \rho \dot{v} , \\ \dot{Q}' - \dot{P} &= \rho_2 \ddot{u} , \\ v' &= \dot{\varepsilon} . \end{aligned} \tag{43}$$

First define a new set of independent variables

$$\zeta = \delta^{\frac{1}{2}} Z, \ \tau = \delta^{\frac{1}{2}} t, \tag{44}$$

where δ is a small parameter. This scaling implies that fixed values of ζ, τ will correspond to long times and large distances as δ tends to zero. Next define a second set of independent variables in terms of ζ, τ.

$$\begin{aligned} \alpha &= \alpha(\zeta,\tau) \\ z &= \delta\zeta \end{aligned} \tag{45}$$

In terms of these new variables, derivatives with respect to Z, t are replaced as follows.

$$\begin{aligned} \frac{\partial}{\partial Z} &= \delta^{\frac{1}{2}} \alpha_\zeta \frac{\partial}{\partial \alpha} + \delta^{3/2} \frac{\partial}{\partial z} \\ \frac{\partial}{\partial t} &= \delta^{\frac{1}{2}} \alpha_\tau \frac{\partial}{\partial \alpha} \end{aligned} \tag{46}$$

In (45) α is a "fast" and z a "slow" variable. The reason for the odd combination of scaling in (44) and (45) will become apparent later. It turns out that α has properties like a characteristic variable. Accordingly, let ω and κ be defined as

$$\omega = \alpha_\tau \ , \ \kappa = \alpha_\zeta \ . \tag{47}$$

α is constant along curves with slope

$$\frac{d\zeta}{d\tau} = c = -\frac{\omega}{\kappa} \tag{48}$$

It is convenient, instead of ω and κ to use the slowness, $s = \frac{1}{c}$, and the incremental arrival time $\ell = \frac{1}{\omega}$. In terms of these variables, the compatibility condition $\omega_\zeta = \kappa_\tau$ becomes

$$s_\alpha = \delta\, \ell_z \quad , \tag{49}$$

and, taking (42) into account, equation (43) becomes

$$\begin{aligned} sS_u u_\alpha + \rho_1 v_\alpha + sS_{w'}\varepsilon_\alpha &= \delta\ell \left\{ S_u u_z + S_{w'}\varepsilon_z \right\} , \\ P_u u_\alpha - sP_{w'} v_\alpha &= \delta\ell \left\{ -P_{w'} v_z + Q_{u'} u_{\tau\zeta\zeta} - \rho_2 u_{\tau\tau\tau} \right\} , \\ sv_\alpha + \varepsilon_\alpha &= \delta\ell v_z \quad , \end{aligned} \tag{50}$$

where derivatives in τ and ζ are retained on the right hand side of $(50)_2$ for compactness.

To lowest order the right hand sides of (50) may be set equal to zero. This requires that the determinant of coefficients of u_α, v_α, ε_α must vanish giving

$$\frac{1}{s^2} = c_b{}^2 = \frac{1}{\rho_1} \left(S_{w'} - \frac{P_{w'}}{P_u} S_u \right) , \tag{51}$$

and u_α, v_α, ε_α must occur in the ratio

$$u_\alpha : v_\alpha : \varepsilon_\alpha = -\nu : -c_b : 1 . \tag{52}$$

With the energy partition of (42) neither S nor P depends on u'. Thus (52) may be integrated to find two of u, v, ε in terms of the third. For example

$$\begin{aligned} v &= v(u) \ , \\ \varepsilon &= \varepsilon(u) \ . \end{aligned} \tag{53}$$

The right hand side of (50) must satisfy a compatibility condition. Thus multiplication of (50) by the left proper vector

$$(-1,\ s\nu,\ sS_{w'}) \tag{54}$$

gives the transport equation

$$\begin{aligned} S_z - sEv_z = s\nu\rho_2 \Big\{ c_2{}^2 \frac{1}{\ell}\frac{\partial}{\partial\alpha}\left[\frac{s}{\ell}\frac{\partial}{\partial\alpha}\left(\frac{s}{\ell}\ u_\alpha\right)\right] \\ - \frac{1}{\ell}\frac{\partial}{\partial\alpha}\left[\frac{1}{\ell}\frac{\partial}{\partial\alpha}\left(\frac{1}{\ell}\ u_\alpha\right)\right]\Big\}, \end{aligned} \tag{55}$$

where $\rho_2 c_2{}^2 = Q_{u'}$. Since S,s,E and ν all depend on u through (53), the left hand side of (55) may be written as $\phi(u)u_z$ so that the dependent variable is u alone. However, the mapping $\alpha(\zeta,\tau)$ must be developed simultaneously with the solution $u(\alpha, z)$ since the incremental arrival time ℓ is also unknown.

The properties of (55) will not be studied further here except for the following two remarks. First, the linearized version of (55) can be shown to have solutions like (39), and second, for small (but finite) amplitude waves, the transport equation reduces to the Korteweg-de Vries equation. This latter fact may also be deduced directly from (5) following the prescription of Leibovich and Seebass[21]. Many other authors have also noted that the Korteweg-de Vries equation describes longitudinal waves in thin rods, eg., Nariboli and Sedov.[22]

VI. STEADY WAVES

Equation (49) indicates that if $\ell_z = O(1)$, then s, and hence c_b, is almost constant. Therefore, some solutions of (55) should be nearly steady waves, perhaps approaching them asymptotically in time. Accordingly, the properties of steady solutions to (5) will be examined.

A steady wave is one in which the solution to (5) may be written

$$\begin{aligned} w &= w(Z - ct) , \\ u &= u(Z - ct) , \end{aligned} \tag{56}$$

where c is an unspecified, but constant, wave speed. The equations become

$$\begin{aligned} S' &= \rho_1 c^2 \, w'' , \\ Q' - P &= \rho_2 c^2 \, u'' , \end{aligned} \tag{57}$$

where the prime now denotes differentiation with respect to Z - ct.

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[21] S. Leibovich and A. R. Seebass, "Examples of Dissipative and Dispersive Systems Leading to the Burgers and the Korteweg-de Vries Equations," in Nonlinear Waves, S. Leibovich and A. R. Seebass, eds., Cornell University Press, Ithaca, N.Y., 1974.

[22] G. A. Nariboli and A. Sedov, "Burgers's-Korteweg-de Vries Equation for Visscoelastic Rods and Plates," J. Math. Anal. and Appl., 32, 1970, pp. 661-667.

There are two integrals of the motion.

$$S = \rho_1 c^2 w' + A$$

$$Q u' + A w' - W = \frac{1}{2} c^2 \left(\rho_2 u'^2 - \rho_1 w'^2\right) + B \qquad (58)$$

where A and B are constants. The second integral is found by multiplying $(57)_1$ by w' and $(52)_2$ by u', adding the two equations, and noting that the result may be written as $\left(\frac{\partial W}{\partial u'} u' + \frac{\partial W}{\partial w'} w' - W\right)' = \frac{1}{2} c^2 \left(\rho_1 w'^2 + \rho_2 u'^2\right)'$. After integration $(58)_1$ is substituted into this expression to give $(58)_2$.

If (58) can be solved for w' and u', they may be expressed as follows.

$$\begin{aligned} w' &= F(u;\ A,B,c) \\ u'^2 &= G(u;\ A,B,c) \end{aligned} \qquad (59)$$

Equation $(59)_2$ is written in terms of u'^2 rather than u' since both of (58) are even in u'. Now $(59)_2$ may be solved by quadrature and then $(59)_1$ gives the distribution in longitudinal strain parametrically through u.

To examine the qualitative nature of solutions and the role of the constants A, B, and c, first consider the case A = B = 0. For ease of analysis the strain energy will be assumed to be partitioned as in (42). For small but finite amplitude waves let W be expressed as a power series. First note that

$$W = \frac{1}{\pi a^2} \int \tilde{W}\, dA = \frac{2}{a^2} \int_0^a R\tilde{W}\, dR\ , \qquad (60)$$

where $\tilde{W}$ is the three-dimensional strain energy per unit volume. With (12) used in (60), W may be expressed as

$$\begin{aligned} W = {} & \frac{1}{2} (\lambda+2\mu)\, w'^2 + 2\lambda\mu u w' + 2(\lambda+\mu) u^2 \\ & + \frac{1}{4} a^2\, \mu u'^2 + C_1 u^3 + C_2 u^2 w' \\ & + C_3 u w'^2 + C_4 w'^3 + \ldots\ , \end{aligned} \qquad (61)$$

where the C's are combinations of higher order elastic moduli.

Equation $(58)_1$ with A = 0 gives

$$w' = -\frac{2\lambda}{\lambda+2\mu - \rho c^2} u + 0(u^2) \quad , \tag{62}$$

and $(58)_2$ with A = B = 0 and the use of (62) gives

$$u'^2 = Cu^2 + Du^3 + \ldots \quad , \tag{63}$$

where

$$C = \frac{8(\lambda+\mu)\ (E-\rho c^2)}{a^2(\mu-\rho c^2)(\lambda+2\mu-\rho c^2)} \quad , \tag{64}$$

and D may be either positive or negative, depending on the exact combination of higher order elastic moduli. C may be either positive or negative as well.

$$C > 0 \text{ if } c_b < c < c_1 \quad ,$$
$$C < 0 \text{ if } c_2 < c < c_b \quad , \tag{65}$$

where c_1, c_2, and c_b are the linear elastic wave speeds in this context.

Figure 4 shows sketches of u'^2 vs. u and u' vs. u for the various combinations of signs of C and D. The closed loops in (4e) and (4f) correspond to solitary waves that are symmetrical in Z about the point of maximum amplitude and extend from $Z = -\infty$ to $Z = +\infty$. The open curves in (4e) - (4h) correspond to waves in which the amplitude grows without limit. Clearly, to be useful, solutions of this kind must join onto unsteady solutions before the amplitudes become too extreme.

Now consider the case where A = 0 but $B \neq 0$. Instead of (63) the following holds.

$$u'^2 = \hat{B} + Cu^2 + Du^3 \tag{66}$$

The constant $\hat{B}$ is proportional to B, but has opposite sign. $\hat{B}$ merely shifts the curves in (4a)-(4d) vertically. Figure 5 shows two of the possibilities. The closed loops in (5c) and (5d) correspond to periodic solutions.

Next consider the case when both $A \neq 0$ and $B \neq 0$. The integrals of the motion may be rewritten as follows

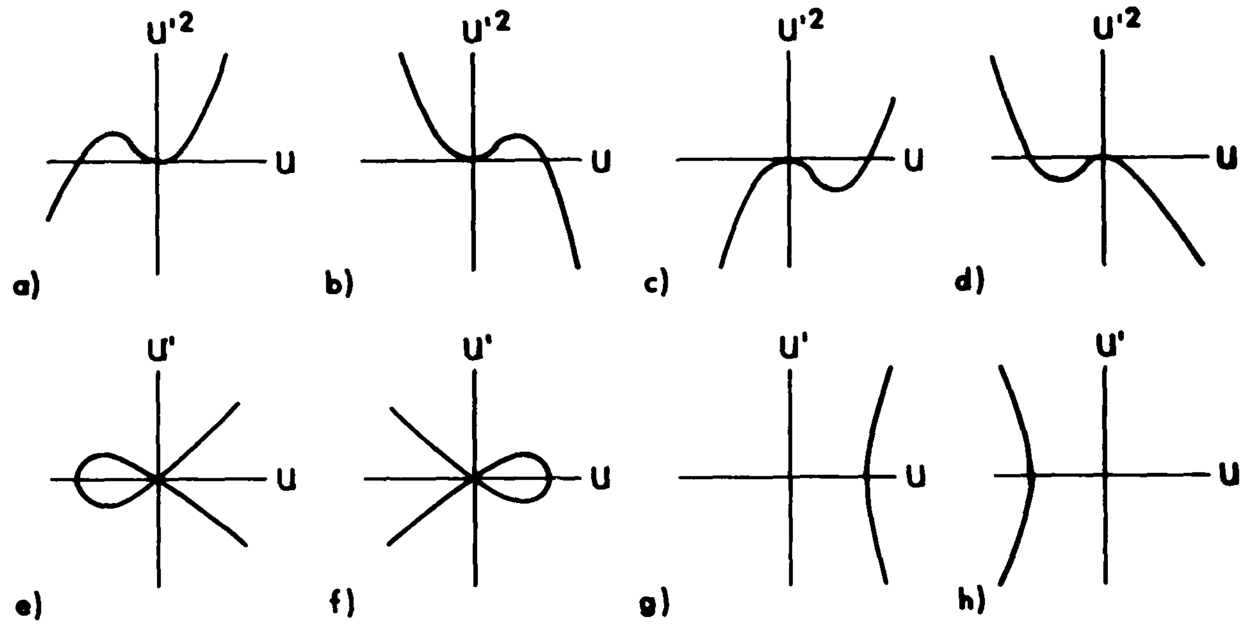

Figure 4: Phase Plane Sketches for Steady Waves, A=B=0 in every case. In a) and e) both C>0, D>0. In b) and f) C>0, D<0. In c) and g) C<0, D>0. In d) and h) both C<0, D<0.

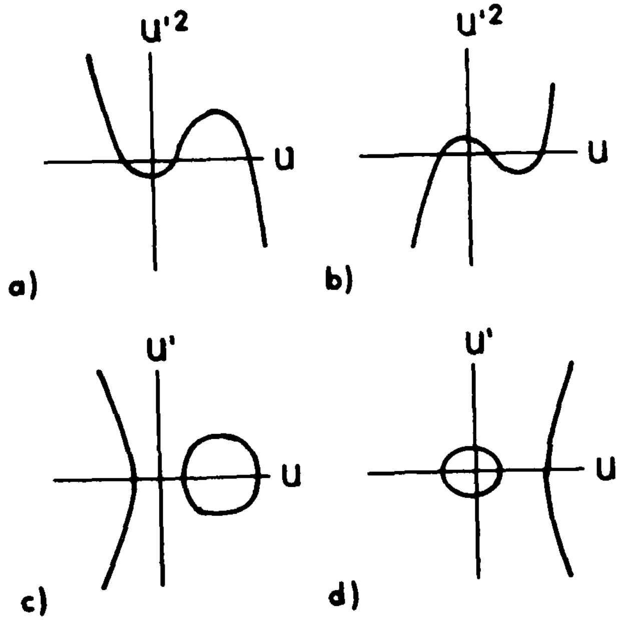

Figure 5: Phase Plane Sketches for Steady Waves, $A=0$, but $B\neq0$. In a) and c) $B>0$, $(\hat{B}<0)$, otherwise like 4b). In b) and d) $B<0$ $(\hat{B}>0)$, otherwise like 4c).

$$S - S_o = \rho_1 c^2 (w' - w'_o)$$

$$Qu' + S_o(w'-w'_o) - (W - W_o) \tag{67}$$

$$= \frac{1}{2} c^2 \left\{ \rho_2 u'^2 - \rho_1 \left(w'-w'_o\right)^2 \right\} + B \ ,$$

where $W_o = W(w'_o, u_o, 0)$, etc. These integrals are the same as before, but now they are centered on the point $(w'_o, u_o, 0)$ and S_o plays the role of A. The character of the solutions is exactly the same as in Figures 4 and 5, but now the curves are shifted to the right $(S_o<0)$ or left $(S_o>0)$.

Finally, suppose that the elastic constants are such that D = 0. Instead of (66), equation (68) must be used.

$$u'^2 = \hat{B} + Cu^2 + Hu^4 \tag{68}$$

This should occur when the expansion is centered at an inflexion point of the static stress-strain curve. C is the same as before and H is determined by higher order elastic moduli. Some of the possibilities are shown in Figure 6. Either positive or negative solitary waves are possible according to (6c), and (6d) is a periodic solution of unusual shape.

The dynamic trajectories, D, of the nonlinear steady waves may deviate considerably from the static curve $P(w',u,0) = 0$ in the w'-u plane. To see this, again suppose that the strain energy is partitioned as in (42) and consider the case A = 0. Figure (7a) shows a contour plot of W_1 on the w'-u plane with the curve $P = W_{1_u} = 0$ and several lines of u = const drawn in. In Figure (7b) several curves of $S = W_{1_{w'}}$ vs. w' with u = constant are shown. Shown as a dashed curve on (7b) is the static stress-strain curve, which has somewhat smaller slope than any of the curves for u = const, according to (17). From reference to $(58)_1$ it can be observed that points of the nonlinear trajectories lie on the intersection of the surfaces $S_1 = S(w',u)$ and $S_2 = \rho_1 c^2 w'$. In the projection of Figure (7b) S_2 is a straight line through the origin with slope $\rho_1 c^2$, for example the line $p_1 p_2$ (for c less than c_b, the bar speed at $w' = 0$). By comparing the line $p_1 p_2$ with the curves for S at u = const and the static stress/strain curve, it can be seen that between points p_1 and p_2 the dynamic trajectory deviates to values of u less than those on P = 0, and at points outside the segment $p_1 p_2$ the trajectory deviates to greater values of u than those on P = 0. Thus, the trajectory, D, is qualitatively like the dashed curve in Figure (7c).

Maximum or minimum amplitudes of u will occur where u' = 0. With u' = 0, equation $(58)_2$ generates curves, C, in the w'-u plane that correspond to the intersection of the surfaces $W_1 = W_1(w',u)$ and $W_2 = \frac{1}{2} \rho_1 c^2 w'^2 - B$. The intersection of curves C and D mark the ends of

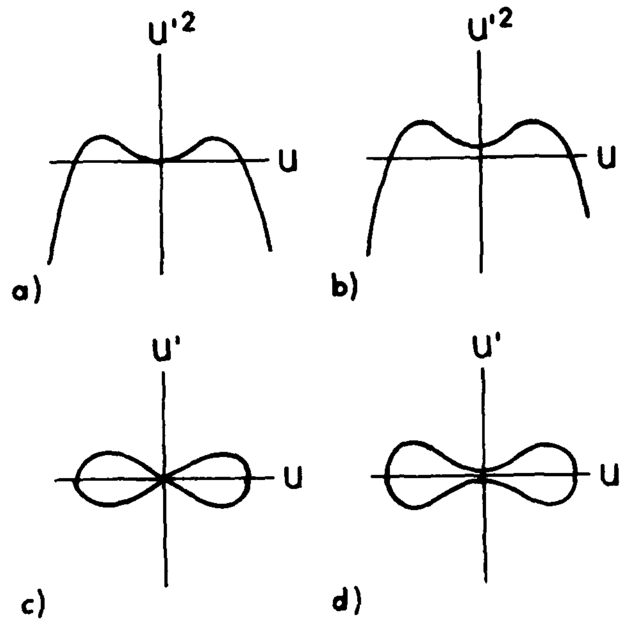

Figure 6: Phase Plane Sketches for Steady Waves, D=0. In a) and c) A=B=0. In b) and d) A=0, but B<0 ($\hat{B}$>0).

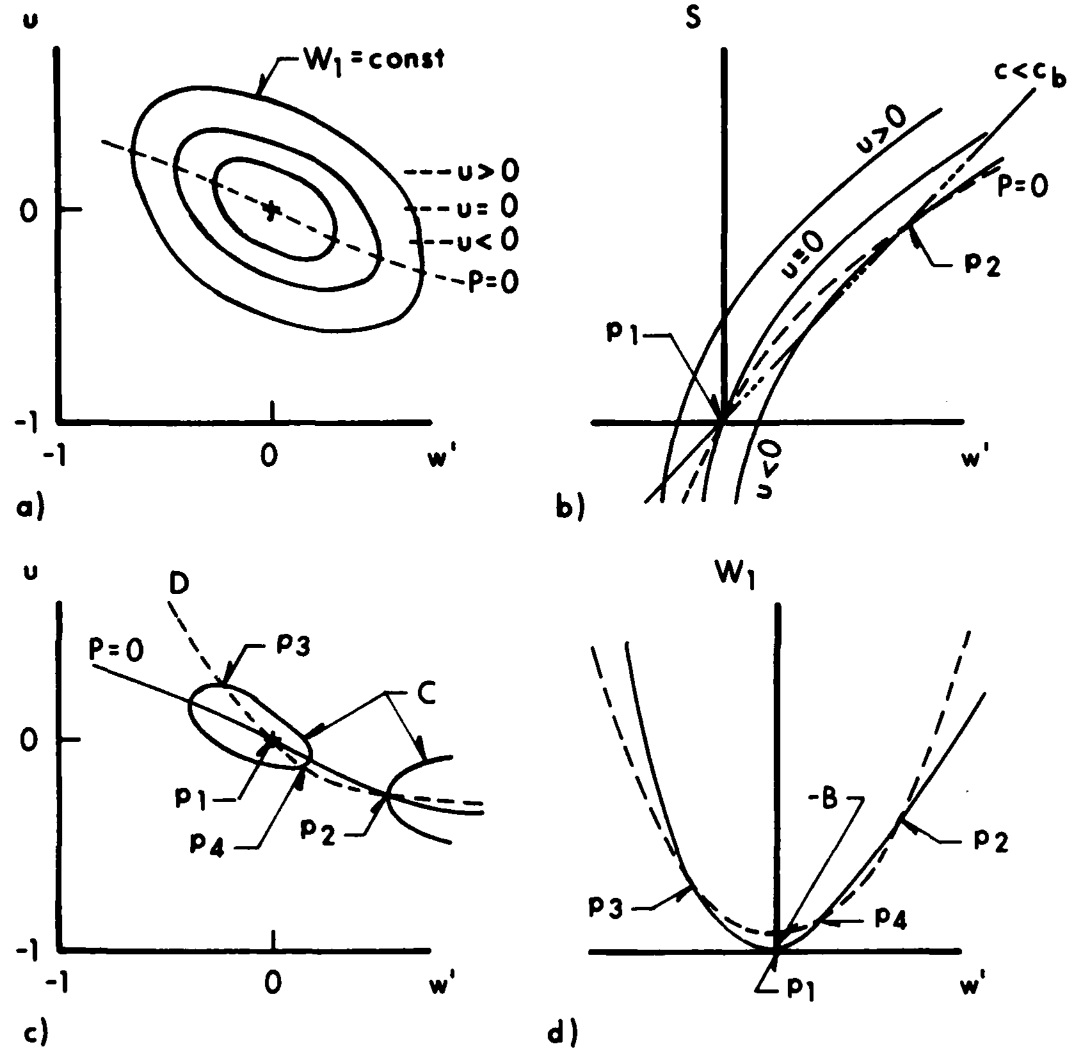

Figure 7: Comparison of Steady Dynamic Trajectory with Static Trajectory. a) Contours of W_1 = const. and Static Curve P=0. b) Intersection of Surfaces $S(w',u)$ and $\rho_1 c^2 w'$. c) Dynamic Trajectory Deduced from 4b). d) Intersection of Cylinder $1/2\rho_1 c^2 w'^2-B$ with Widest Projection of $W_1(w',u)$.

the dynamic trajectory. By differentiating $(58)_2$ with respect to w', it may be seen that the points on C satisfy

$$P \frac{du}{dw'} + S = \rho_1 c^2 w' \quad . \tag{69}$$

At the intersection of C and D, $(58)_1$ holds as well so that at those points

$$P \frac{du}{dw'} = 0 \ . \tag{70}$$

That is, either P = 0 or the slope of C is horizontal.

Now consider further the intersection of surfaces $\hat{W}_1$ and $\hat{W}_2$ with $B < o$ $(\hat{B} > o)$. Fig (7d) shows the projection parallel to the u axis of the widest points of $\hat{W}_1 = W_1(w',u)$. These points correspond to the curve P = 0 in (7a), and, in fact, the projected potential well in (7d) generates the static stress/strain curve. The surface $\hat{W}_2 = \frac{1}{2} \rho_1 c^2 w'^2 - B$ is a parabolic cylinder with generators parallel to the u axis. An end view of the cylinder is shown as the dashed parabola in (7d). Since the level curves of $\hat{W}_1$ are closed, as shown in (7a), and since the level curves of $\hat{W}_2$ are straight lines parallel to the u-axis, the curves C are qualitatively similar to those sketched in (7c). Thus, for the case considered, which corresponds to Figure (5d), the motion lies along D and either oscillates between points p_3 and p_4 or begins at p_2 and moves to the right until interrupted by an unsteady motion. In either case, note the divergence of D from the curve P = 0.

Other cases for different shapes of the potential well $W_1(w',u)$ may be considered in a similar manner. However, from teh preceding qualitative discussion it is clear that te cubic expansion in equation (63) leads to the qualitatively correct curves in the phase plane as shown in Figures (4) and (5), provided that the function

$$K(u'^2) = Q(u')u' - W_2(u') - \frac{1}{2} \rho_2 c^2 u'^2$$

is monotonic and hence single valued in u'^2. If this is not the case, then the curves of u'^2 vs. u in Figures (4), (5), and (6) will have bifurcation points. The analysis would then be somewhat more complicated and will not be attempted here.

VII. REMARKS

This report has been motivated by the need to interpret wave propagation experiments in rods and bars. These experiments are of two types. In one, a cylindrical bar is fired from an air gun so as to impact an identical bar, which is at rest in an evacuated chamber. The bar at rest is instrumented at several stations along its length with foil resistance strain gages. In the other experiment, a long rod penetrator, which is also instrumented with strain gages, is impacted by a target plate.[23] In either case a transient plastic wave propagates into the bar, strain-time data are recorded for the instrumented stations of the bar, and then the data are interpreted with the aid of equations (2). By comparing with streak camera records Hauver has shown that $(2)_1$ gives very accurate results for particle velocity[23], but $(2)_2$ always gives higher stresses than the quasi-static stress-strain curve even though the high strength steels and aluminums used are not rate sensitive in Hopkinson bar tests.[24] The only other source of excess stress that has been suggested is the effect of finite transverse dimensions, hence the present interest in a nonlinear theory that accounts for radial motion.

This report has dealt only with nonlinear elastic materials, not elastic-plastic materials. Nevertheless, there are several results that are at least suggestive of general features that should apply to the elastic-plastic case. These are as follows.

a. The averaging in equation (10) and the identifications made in equations (11) and (12) are independent of constitutive relations. Therefore, the equations of motion (5) hold for the elastic-plastic case as well as the elastic case, but of course equations (6) will not be appropriate then.

b. With $v = \dot{w}$ and $\varepsilon = w'$ equations (1) are entirely consistent with (5). As a consequence, if each level of strain propagates at a constant velocity, experimental data may still be interpreted through the use of (2) for both particle velocity and stress. In this integration no use is made of $(5)_2$ so that radial strain u remains undetermined. In particular, there is no requirement for u to have equilibrium values corresponding to the strain ε. Thus, the computed stress S should be as accurate as the computed particle velocity $\dot{w}$ even though S cannot be measured independently as can $\dot{w}$. In a compressive wave the excess stress may be accounted for by underexpansion in the radial direction caused by inertial lag.

c. In impact experiments of the type described here it is commonly observed that the plastic wave speed becomes nearly constant at the higher values of strain. Thus, steady waves are of interest at least as asymptotic solutions. Because of dissipation it is probably unreasonable to expect that solitary waves or periodic waves will occur in elastic-plastic materials, but pure loading waves should be similar

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[23] G. E. Hauver and A. Melani, "Strain-Gage Techniques for Studies of Projectile Behavior During Penetration," ARBRL-MR-03082, Ballistic Research Laboratory, February 1981.

[24] S. C. Chou, AMMRC, private communication.

in either elastic or elastic-plastic materials. Thus, unbounded waves of the type shown in Figures (4g), (4h), (5c), or (5d) could occur. In an actual physical problem such a wave must be interrupted by an unsteady wave or perhaps a free boundary (as in the penetration experiment).

Future work will attempt to elaborate on some of these issues. Initial-boundary value problems will be posed with an emphasis on asymptotic solutions. The elastic-plastic case will also be formulated and the characteristic features of those solutions examined and compared with elastic solutions.

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