

# Nonlinear Waves, Solitons and Chaos

Eryk Infeld

Institute for Nuclear Studies, Warsaw

George Rowlands

Department of Physics, University of Warwick

**2nd edition**



**CAMBRIDGE**  
UNIVERSITY PRESS

# Contents

	<i>Foreword to the first edition</i>	page xi
	<i>Foreword to the second edition</i>	xiii
<b>1</b>	<b>Introduction</b>	<b>1</b>
1.1	Occurrence of nonlinear waves and instabilities in Nature	1
	1.1.1 Nonlinear phenomena in our everyday experience	1
	1.1.2 Nonlinear phenomena in the laboratory	3
1.2	Universal wave equations	5
	1.2.1 The Korteweg–de Vries and Kadomtsev–Petviashvili equations and a first look at solitons	5
	1.2.2 The nonlinear Schrödinger equation	8
	1.2.3 Nonlinear optics	9
1.3	What is a plasma?	9
1.4	Wave modes on a water surface	12
	1.4.1 Mathematical theory	12
	1.4.2 Comments	14
1.5	Linear stability analysis and its limitations	18
1.6	Nonlinear structures	18
	1.6.1 Coherent structures and pattern formation	22
1.7	Contents of Chapters 2–11	23
<b>2</b>	<b>Linear waves and instabilities in infinite media</b>	<b>25</b>
2.1	Introduction	25
2.2	Plasma waves	25
2.3	CMA diagrams	30
2.4	Instabilities	33
2.5	The Vlasov equation	37
2.6	Weak instabilities	43
	Exercises	47
<b>3</b>	<b>Convective and non-convective instabilities; group velocity in unstable media</b>	<b>48</b>
3.1	Introduction	48
3.2	Kinematics of unstable wavepackets	50

3.3	Moving coordinate systems	54
3.4	Higher dimensional systems	56
3.5	Summary	57
	Exercise	57
<b>4</b>	<b>A first look at surface waves and instabilities</b>	<b>59</b>
4.1	Introduction	59
4.2	Simple surface waves	61
4.3	The Rayleigh–Taylor instability	66
4.4	The Kelvin–Helmholtz instability	68
4.5	Solid–liquid interface instabilities	71
4.6	A first look at gravity wave instabilities	72
	4.6.1 The small amplitude onset of wave instability	73
	4.6.2 Further numerical results	75
4.7	Summary	81
	Exercises	81
<b>5</b>	<b>Model equations for small amplitude waves and solitons; weakly nonlinear theory</b>	<b>82</b>
5.1	Introduction	82
	5.1.1 Some physical equations ask for surgery	82
	5.1.2 Examples	83
5.2	A few model equations as derived by introducing a small parameter	85
	5.2.1 Shallow water, weak amplitude gravity waves	85
	5.2.2 Weak amplitude ion acoustic waves in an unmagnetized plasma	89
	5.2.3 Weak amplitude ion acoustic waves in a magnetized plasma	91
5.3	Weakly nonlinear waves	92
	5.3.1 Spreading, splitting and instabilities	92
	5.3.2 The story of deep water waves	98
	5.3.3 Mystery of the missing term	100
	5.3.4 Dynamics of a wavepacket	102
	5.3.5 Some generalizations	104
5.4	A general look at two families of model equations	107
5.5	A natural extension to finite amplitude waves due to Hayes	111
5.6	Temporal development of instabilities and wave–wave coupling	114
5.7	Concluding remarks	119
	Exercises	119
<b>6</b>	<b>Exact methods for fully nonlinear waves and solitons</b>	<b>123</b>
6.1	Introduction	123
6.2	Phase plane analysis and other methods	124

6.2.1	One stationary wave in a dissipationless medium	124
6.2.2	A two-fluid layer soliton pair	129
6.2.3	Weak ion acoustic shock waves in a collisional plasma	132
6.2.4	Solitons generated by laser fields	134
6.2.5	Solitons and domains in dipole chains	136
6.2.6	Discrete equations	139
6.3	Bernstein–Greene–Kruskal waves	143
6.3.1	Statistical description of a plasma and BGK waves	144
6.3.2	No trapped particles	145
6.3.3	Various limits	146
6.3.4	Trapped particle equilibria	147
6.3.5	Stability; subsequent developments	150
6.4	Lagrangian methods	152
6.5	Lagrangian interpolation	159
	Exercises	164
<b>7</b>	<b>Cartesian solitons in one and two space dimensions</b>	<b>166</b>
7.1	Introduction	166
7.2	The direct method	168
7.3	Constants of motion	171
7.4	Inverse scattering method	173
7.5	Bäcklund transformations	175
7.6	Entr'acte	177
7.7	Breathers and boundary effects	178
7.8	Experimental evidence	180
7.9	Plane soliton interaction in two space dimensions	181
7.9.1	Introducing the trace method	181
7.9.2	One and two soliton solutions	183
7.9.3	Some other developments and summary	188
7.10	Integrable equations in two space dimensions as treated by the Zakharov–Shabat method	192
7.10.1	Lax pairs and the PDEs they represent	193
7.10.2	Extension to $x, y, t$	194
7.10.3	How to proceed from the Lax pair to the general solution	195
7.10.4	An example: the Kadomtsev–Petviashvili equation	196
7.11	Summary	199
	Exercises	200
<b>8</b>	<b>Evolution and stability of initially one-dimensional waves and solitons</b>	<b>202</b>
8.1	A brief historical survey of large amplitude nonlinear wave studies	202
8.1.1	Solitons	204
8.1.2	Water waves are unstable	206
8.1.3	The geometrical optics limit	207

	8.1.4	More recent results	211
	8.1.5	What the remainder of Chapter 8 is about	212
8.2		Four methods as illustrated by the nonlinear Klein–Gordon equation	213
	8.2.1	Whitham I	214
	8.2.2	Whitham II	219
	8.2.3	$K$ expansion	220
	8.2.4	Hayes	223
8.3		Higher dimensional dynamics	224
	8.3.1	Kadomtsev–Petviashvili as analysed by Whitham II	224
	8.3.2	Various limits	230
	8.3.3	Common features of the weak amplitude and soliton limits for $\psi = 0$	231
	8.3.4	Group velocity	233
	8.3.5	Zakharov–Kuznetsov as analysed by $K$ expansion	236
	8.3.6	The variational method	244
8.4		A more physical approach leading to an assessment of models	245
	8.4.1	Form of the waves considered	245
	8.4.2	Unmagnetized plasmas, $\Omega_c = 0$	246
	8.4.3	Magnetized plasmas, $\Omega_c > 0$	249
8.5		Dynamics of nonlinear wave, shock and soliton solutions to the cubic nonlinear Schrödinger equation	252
	8.5.1	Results of a general stability calculation	253
	8.5.2	One-dimensional dynamics: $\psi = 0$	254
	8.5.3	Oblique and perpendicular propagation of perturbations	257
8.6		The direct $K$ method	258
	8.6.1	Transverse instability of Zakharov–Kuznetsov solitons	259
	8.6.2	The Cahn–Hilliard equation	263
8.7		Some general conclusions and possible future lines of investigation	264
		Exercises	265
<b>9</b>		<b>Cylindrical and spherical solitons in plasmas and other media</b>	<b>268</b>
9.1		Interest in higher dimensional plasma solitons	268
9.2		Unidirectional cylindrical and spherical ion acoustic solitons	269
	9.2.1	Model equations in non-Cartesian geometry	269
	9.2.2	Cylindrical soliton equations CI and CII	269
	9.2.3	Spherical solitons	271
	9.2.4	Summary	272
9.3		Properties of unidirectional soliton equations	272
	9.3.1	Integrability by inverse scattering	272
	9.3.2	Conservation laws	273
9.4		Soliton solutions as compared with numerics and experiments	275
	9.4.1	Exact solutions to CI	277
	9.4.2	Initial value problem and experiments	277
	9.4.3	Reflection from the axis (centre)	280
	9.4.4	Models	284

9.4.5	Stability of cylindrical solitons	287
9.5	Langmuir solitons	287
9.5.1	Integrability	288
9.5.2	Stability of Langmuir solitons	288
9.6	Interacting solitons and some conclusions	291
9.7	Epilogue. Some other examples of spherical and cylindrical solitons	292
	Exercises	294
<b>10</b>	<b>Soliton metamorphosis</b>	<b>296</b>
10.1	The next step in investigating soliton behaviour	296
10.2	Decay of line KPI solitons in two dimensions	297
10.3	Decay of 2D solitons in three dimensions	301
10.3.1	2D solitons perturbed perpendicular to the motion	301
10.3.2	2D solitons perturbed parallel to the velocity	302
10.4	Conclusions	303
	Exercises	303
<b>11</b>	<b>Non-coherent phenomena</b>	<b>304</b>
11.1	Introduction	304
11.2	Bifurcation sequences and chaos	311
11.3	Flows and maps	326
11.4	Strange attractors	329
11.5	Effect of external noise	340
11.6	Experimental evidence for strange attractors	341
11.7	Other theories of turbulence	343
11.8	Conclusions	344
	Exercises	345
	<b>Appendices</b>	<b>346</b>
A1	Parameter stretching as suggested by the linear dispersion relations	346
A1.1	Ion acoustic waves in an unmagnetized plasma, $\Omega_c = 0$	346
A1.2	Magnetized plasmas, $\Omega_c > 0$	347
A2	Relation between the trace method and the inverse scattering method	349
A3	Some formulae for perturbed nonlinear ion acoustic waves and solitons	351
A3.1	No magnetic field	351
A3.2	$\Omega_c > 0$	352
A4	Colliding soliton theory	354
A5	A model equation for spherical solitons	356
A6	Stability calculation for 2D KPI soliton in 3D	358
	<i>References</i>	360
	<i>Author index</i>	379
	<i>Subject index</i>	387