IN-64

109338

P.26

ICASE

N92-32164

Unclas

G3/64   0109338

NONLINEARLY STABLE COMPACT SCHEMES FOR SHOCK CALCULATIONS

Bernardo Cockburn

Chi-Wang Shu

(NASA-CR-189663) NONLINEARLY
STABLE COMPACT SCHEMES FOR SHOCK
CALCULATIONS Final Report (ICASE)
26 p

NASA

National Aeronautics and
Space Administration

Langley Research Center
Hampton, Virginia 23665-5225

NONLINEARLY STABLE COMPACT SCHEMES
FOR SHOCK CALCULATIONS

Bernardo Cockburn [1]
School of Mathematics
University of Minnesota
Minneapolis, MN 55455

and

Chi-Wang Shu [2]
Division of Applied Mathematics
Brown University
Providence, RI 02912

ABSTRACT

In this paper we discuss the applications of high order compact finite difference methods for shock calculations. The main idea is the definition of a local mean which serves as a reference for introducing a local nonlinear limiting to control spurious numerical oscillations while keeping the formal accuracy of the scheme. For scalar conservation laws, the resulting schemes can be proven total variation stable in one space dimension and maximum norm stable in multiple space dimensions. Numerical examples are shown to verify accuracy and stability of such schemes for problems containing shocks. The idea in this paper can also be applied to other implicit schemes such as the continuous Galerkin finite element methods.

i

1 Introduction

Compact schemes are methods where the derivatives are approximated not by polynomial operators but by rational function operators on the discrete solutions. In this paper we are interested in solving a hyperbolic conservation law

$$
\begin{aligned}
u_t + f(u)_x + g(u)_y &= 0 \\
u(x, y, 0) &= u^0(x, y)
\end{aligned}
\tag{1.1}
$$

using compact schemes. In the semi-discrete form, a compact scheme for solving (1.1) can be written as

$$
\frac{du_{ij}}{dt} = -\frac{1}{\Delta x}(A_x^{-1} B_x f(u))_{ij} - \frac{1}{\Delta y}(A_y^{-1} B_y g(u))_{ij} \equiv L(u)_{ij}
\tag{1.2}
$$

where A and B are both *local*, one dimensional operators. The subscript x or y indicates that the operator is applied in the x or y direction.

For example, a fourth order central compact scheme is given by (1.2) with

$$
\begin{aligned}
(Av)_i &= \frac{1}{6}(v_{i-1} + 4v_i + v_{i+1}) \\
(Bv)_i &= \frac{1}{2}(v_{i+1} - v_{i-1}),
\end{aligned}
\tag{1.3}
$$

a sixth order central compact scheme is given by

$$
\begin{aligned}
(Av)_i &= \frac{1}{5}(v_{i-1} + 3v_i + v_{i+1}) \\
(Bv)_i &= \frac{1}{60}(v_{i+2} + 28v_{i+1} - 28v_{i-1} - v_{i-2}),
\end{aligned}
\tag{1.4}
$$

and two third order upwind compact schemes are given by

$$
\begin{aligned}
(Av)_i &= \frac{1}{3}(-v_{i-1} + 5v_i - v_{i+1}) \\
(Bv)_i &= \frac{1}{2}(3v_i - 4v_{i-1} + v_{i-2})
\end{aligned}
\tag{1.5}
$$

and

$$
\begin{aligned}
(Av)_i &= \frac{1}{3}(-v_{i-1} + 5v_i - v_{i+1}) \\
(Bv)_i &= \frac{1}{2}(-v_{i+2} + 4v_{i+1} - 3v_i)
\end{aligned}
\tag{1.6}
$$

depending upon the wind direction. Notice that (1.5) and (1.6) have the same implicit part A which is symmetric. This fact will be used later in Section 2 to define our local means.

1

The cost of compact schemes, regardless of the number of space dimensions, involves only inversion of the narrowly banded (usually tridiagonal) matrix A and hence is comparable to explicit methods. This is notably different from other implicit methods such as the continuous Galerkin finite element methods in multiple space dimensions, even if they are similar in one space dimension.

The advantages of compact schemes include the relatively high order of accuracy using a compact stencil (for example, the fourth order scheme (1.3) when discretized in time using Euler forward, uses only a three point stencil in each time level), a better (linear) stability, and usually fewer boundary points to handle. In recent years compact schemes have attracted much attention in various fields such as the direct numerical simulations of turbulence. We refer the readers to [2], [3], [4], [12], and [18] for more details.

The objective of this paper is to apply compact schemes for shock calculations. As with any other linear schemes (schemes which are linear when applied to linear equations), compact schemes usually demonstrate nonlinear instability when applied to discontinuous data. We follow the TVD (total variation diminishing) ideas in [9], [13] and try to define a suitable nonlinear local limiting to avoid spurious oscillations while keeping the formal accuracy of the scheme. Notice that the compact scheme, like an implicit scheme, is global. That is, the approximation to $f(u)_x$ at $x = x_i$ involves u_k along the whole line due to the tridiagonal inversion A^{-1}. Our main idea is to define a *local* mean, and use it as a reference for introducing a local limiting. In Section 2 we introduce the limiting for one space dimension and prove total variation stability. In Section 3 we introduce the limiting for multiple space dimensions and prove maximum norm stability. In Section 4 we present numerical examples, and concluding remarks are included in Section 5.

The ideas in this paper were first used by us for continuous Galerkin finite element method in [7]. That is an on-going project. In this paper we restrict our attention to scalar problems in order to obtain provable stability results. The application of the method to systems of hyperbolic conservation laws and to other types of compact schemes (e.g. [1]) is currently under investigation.

In this paper, we use the total variation diminishing (TVD) Runge-Kutta type time discretization, introduced in [14], [17], to discretize the ODE in the method-of-lines formulation (1.2). In the second order case, the time discretization is

$$
\begin{aligned}
u^{(1)} &= u^n + \Delta t L(u^n) \\
u^{n+1} &= \frac{1}{2}u^n + \frac{1}{2}u^{(1)} + \frac{1}{2}\Delta t L(u^{(1)}),
\end{aligned}
\tag{1.7}
$$

and in the third order case it is

$$
\begin{aligned}
u^{(1)} &= u^n + \Delta t L(u^n) \\
u^{(2)} &= \frac{3}{4}u^n + \frac{1}{4}u^{(1)} + \frac{1}{4}\Delta t L(u^{(1)}) \\
u^{n+1} &= \frac{1}{3}u^n + \frac{2}{3}u^{(2)} + \frac{2}{3}\Delta t L(u^{(2)}).
\end{aligned}
\tag{1.8}
$$

These special Runge-Kutta type time discretizations are labelled TVD because it can be proven that under suitable restrictions on the time step Δt (the CFL condition), the full

discretization (1.7) or (1.8) is TVD, or stable under another norm (for example, the L_∞ norm) if the first order Euler forward time discretization for (1.2)

$$u^{n+1} = u^n + \Delta t L(u^n) \tag{1.9}$$

is TVD or stable under the other norm. For details, see [14] and [17].

We thus only need to consider the Euler forward scheme (1.9) for stability analysis in the subsequent sections.

2 One Space Dimension

In one space dimension, equation (1.1) becomes

$$
\begin{aligned}
u_t + f(u)_x &= 0 \\
u(x,0) &= u^0(x),
\end{aligned}
\tag{2.1}
$$

the scheme (1.2) is

$$\frac{du_i}{dt} = -\frac{1}{\Delta x}(A^{-1}Bf(u))_i \equiv L(u)_i, \tag{2.2}$$

and the Euler forward time discretization (1.9) becomes

$$u_i^{n+1} = u_i^n + \Delta t L(u^n)_i. \tag{2.3}$$

Scheme (2.3) can be easily written into a conservation form

$$u_i^{n+1} = u_i^n - \frac{\Delta t}{\Delta x}(h_{i+\frac{1}{2}}^n - h_{i-\frac{1}{2}}^n) \tag{2.4}$$

suitable for shock calculations. However, the numerical flux $h_{i+\frac{1}{2}}^n$ is not a local function of u_k^n due to the tridiagonal inversion A^{-1}. If we define

$$\bar{u}_i \equiv (Au)_i, \tag{2.5}$$

then scheme (2.3) can be left-multiplied by A to become

$$\bar{u}_i^{n+1} = \bar{u}_i^n - \frac{\Delta t}{\Delta x}(Bf(u^n))_i. \tag{2.6}$$

When written into a conservation form,

$$\bar{u}_i^{n+1} = \bar{u}_i^n - \frac{\Delta t}{\Delta x}(\hat{f}_{i+\frac{1}{2}}^n - \hat{f}_{i-\frac{1}{2}}^n), \tag{2.7}$$

this involves a numerical flux $\hat{f}_{i+\frac{1}{2}}^n$ which is a local function of u_k^n. For example,

$$\hat{f}_{i+\frac{1}{2}} = \frac{1}{2}\left(f(u_{i+1}) + f(u_i)\right) \tag{2.8}$$

for the fourth order central scheme (1.3),

3

$$\hat{f}_{i+\frac{1}{2}} = \frac{1}{60} \left(f(u_{i+2}) + 29f(u_{i+1}) + 29f(u_i) + f(u_{i-1}) \right) \tag{2.9}$$

for the sixth order central scheme (1.4), and

$$\hat{f}_{i+\frac{1}{2}} = \frac{1}{2} \left(3f(u_i) - f(u_{i-1}) \right) \tag{2.10}$$

and

$$\hat{f}_{i+\frac{1}{2}} = \frac{1}{2} \left(-f(u_{i+2}) + 3f(u_{i+1}) \right) \tag{2.11}$$

for the two third order upwind schemes (1.5) and (1.6), respectively. Notice that scheme (2.7) resembles a cell-averaged (finite volume) scheme [11]. The $\bar{u}_i$ in (2.5), like a cell average, is a local mean of u, defined by Au in (1.3) through (1.6). Since the computation of the flux $\hat{f}_{i+\frac{1}{2}}$ in (2.7) involves the values of u, a "reconstruction" from $\bar{u}$ to u

$$u_i = (A^{-1}\bar{u})_i \tag{2.12}$$

is needed. This reconstruction is global.

It is now rather straightforward to define the limiting. We first write

$$f(u) = f^+(u) + f^-(u) \tag{2.13}$$

with the requirement that

$$\frac{\partial f^+(u)}{\partial u} \geq 0, \qquad \frac{\partial f^-(u)}{\partial u} \leq 0. \tag{2.14}$$

The purpose of this flux splitting is for easier upwinding at later stages. The simplest such splitting is due to Lax-Friedrichs

$$f^{\pm}(u) = \frac{1}{2} \left(f(u) \pm \alpha u \right), \qquad \alpha = \max_u |f'(u)| \tag{2.15}$$

where the maximum is taken over the range of $u^0(x)$. We then write the flux $\hat{f}_{i+\frac{1}{2}}$ in (2.7) also as

$$\hat{f}_{i+\frac{1}{2}} = \hat{f}^+_{i+\frac{1}{2}} + \hat{f}^-_{i+\frac{1}{2}} \tag{2.16}$$

where $\hat{f}^{\pm}_{i+\frac{1}{2}}$ are obtained by putting superscripts $\pm$ in (2.8) through (2.11).

Next we define

$$d\hat{f}^+_{i+\frac{1}{2}} = \hat{f}^+_{i+\frac{1}{2}} - f^+(\bar{u}_i); \qquad d\hat{f}^-_{i+\frac{1}{2}} = f^-(\bar{u}_{i+1}) - \hat{f}^-_{i+\frac{1}{2}}. \tag{2.17}$$

Here $d\hat{f}^{\pm}_{i+\frac{1}{2}}$ are the differences between the numerical fluxes $\hat{f}^{\pm}_{i+\frac{1}{2}}$ and the first order, upwind fluxes $f^+(\bar{u}_i)$ and $f^-(\bar{u}_{i+1})$. These differences are subject to limiting for nonlinear stability. We define the limiting by

$$
\begin{aligned}
d\hat{f}^{+(m)}_{i+\frac{1}{2}} &= m\left(d\hat{f}^{+}_{i+\frac{1}{2}}, \Delta_+ f^+(\bar{u}_i), \Delta_+ f^+(\bar{u}_{i-1})\right) \\
d\hat{f}^{-(m)}_{i+\frac{1}{2}} &= m\left(d\hat{f}^{-}_{i+\frac{1}{2}}, \Delta_+ f^-(\bar{u}_i), \Delta_+ f^-(\bar{u}_{i+1})\right)
\end{aligned} \tag{2.18}
$$

where $\Delta_+ v_i \equiv v_{i+1} - v_i$ is the usual forward difference operator, and the (now standard) *minmod* function m is defined by

$$
m(a_1, \cdots, a_k) = \begin{cases} s\min_{1\le i\le k}|a_i|, & if \quad sign(a_1) = \cdots = sign(a_k) = s \\ 0, & otherwise. \end{cases} \tag{2.19}
$$

Notice that the limiting defined in (2.18) is upwind biased.

The limited numerical fluxes are then defined by

$$
\hat{f}^{+(m)}_{i+\frac{1}{2}} = f^+(\bar{u}_i) + d\hat{f}^{+(m)}_{i+\frac{1}{2}}; \qquad \hat{f}^{-(m)}_{i+\frac{1}{2}} = f^-(\bar{u}_{i+1}) - d\hat{f}^{-(m)}_{i+\frac{1}{2}} \tag{2.20}
$$

and

$$
\hat{f}^{(m)}_{i+\frac{1}{2}} = \hat{f}^{+(m)}_{i+\frac{1}{2}} + \hat{f}^{-(m)}_{i+\frac{1}{2}}. \tag{2.21}
$$

If we define the total variation of the mean $\bar{u}$ by

$$
TV(\bar{u}) = \sum_i |\bar{u}_{i+1} - \bar{u}_i| \tag{2.22}
$$

we have the following proposition.

Proposition 2.1

Scheme (2.7) with the flux (2.21) is TVDM (total variation diminishing in the means)

$$
TV(\bar{u}^{n+1}) \le TV(\bar{u}^n) \tag{2.23}
$$

under the CFL condition

$$
\max_{\min_i \bar{u}^n_i \le u \le \max_i \bar{u}^n_i} \left(f^{+\prime}(u) - f^{-\prime}(u)\right)\frac{\Delta t}{\Delta x} \le \frac{1}{2}. \tag{2.24}
$$

Proof: We follow Harten [9] and write the flux difference as

$$
\hat{f}^{(m)}_{i+\frac{1}{2}} - \hat{f}^{(m)}_{i-\frac{1}{2}} = -C_{i+\frac{1}{2}}\Delta_+\bar{u}_i + D_{i-\frac{1}{2}}\Delta_+\bar{u}_{i-1} \tag{2.25}
$$

where

$$
\begin{aligned}
C_{i+\frac{1}{2}} &= -\frac{\Delta_+ f^-(\bar{u}_i) - d\hat{f}^{-(m)}_{i+\frac{1}{2}} + d\hat{f}^{-(m)}_{i-\frac{1}{2}}}{\Delta_+\bar{u}_i} \\
D_{i-\frac{1}{2}} &= \frac{\Delta_+ f^+(\bar{u}_{i-1}) + d\hat{f}^{+(m)}_{i+\frac{1}{2}} - d\hat{f}^{+(m)}_{i-\frac{1}{2}}}{\Delta_+\bar{u}_{i-1}}.
\end{aligned} \tag{2.26}
$$

5

The limiting in (2.18) and the properties of $f^{\pm}(u)$ in (2.14) clearly imply

$$C_{i+\frac{1}{2}} \geq 0, \qquad D_{i-\frac{1}{2}} \geq 0 \tag{2.27}$$

and

$$\frac{\Delta t}{\Delta x}\left(C_{i+\frac{1}{2}} + D_{i+\frac{1}{2}}\right) \leq \frac{\Delta t}{\Delta x}\left(\frac{-2\Delta_+ f^-(\bar{u}_i) + 2\Delta_+ f^+(\bar{u}_i)}{\Delta_+ \bar{u}_i}\right) \leq 1 \tag{2.28}$$

The last inequality is due to the CFL condition (2.24). TVDM (2.23) is now immediate according to Harten [9].

□

In order to obtain total variation stability for u, we need the following simple lemma.

Lemma 2.2

If there are two numbers $0 \leq \delta < 1$ and $\alpha > 0$, which are independent of N, such that the $N \times N$ matrix $A = (a_{ij})$ satisfies:

$$\max_{1 \leq j \leq N} \frac{1}{|a_{jj}|} \leq \alpha, \qquad and \qquad \sum_{\substack{i=1 \\ i \neq j}}^{N} |a_{ij}| \leq \delta |a_{jj}|, \qquad j = 1, \cdots, N \tag{2.29}$$

(strongly diagonally dominance for the transpose of A), then the L_1 norm of A^{-1} is bounded independently of N,

$$\|A^{-1}\|_{L_1} \leq \frac{\alpha}{1-\delta}. \tag{2.30}$$

Proof: Let $\Lambda = diag(a_{11}, \cdots, a_{NN})$, $B = A - \Lambda$ and $C = -B\Lambda^{-1}$. We have

$$\|C\|_{L_1} = \max_{1 \leq j \leq N} \sum_{i=1}^{N} |c_{ij}| = \max_{1 \leq j \leq N} \sum_{\substack{i=1 \\ i \neq j}}^{N} \frac{|a_{ij}|}{|a_{jj}|} \leq \delta.$$

Hence it follows that

$$\begin{aligned}
\|A^{-1}\|_{L_1} &= \|[(I-C)\Lambda]^{-1}\|_{L_1} = \|\Lambda^{-1}(I-C)^{-1}\|_{L_1} \\
&\leq \|\Lambda^{-1}\|_{L_1}\|(I-C)^{-1}\|_{L_1} \leq \|\Lambda^{-1}\|_{L_1}\frac{1}{1-\|C\|_{L_1}} \\
&\leq \frac{\alpha}{1-\delta}.
\end{aligned}$$

□

6

For most compact methods, the matrix A satisfies the condition (2.29) for Lemma 2.2. For example, in the schemes defined by (1.3), (1.4), (1.5) and (1.6), A satisfies the condition (2.29) with $\delta = \frac{1}{2}$, $\alpha = 6$; $\delta = \frac{2}{3}$, $\alpha = 5$; $\delta = \frac{2}{5}$, $\alpha = 3$ and $\delta = \frac{2}{5}$, $\alpha = 3$, respectively. For such compact schemes, we can now prove the total variation stability for u.

Proposition 2.3

If a compact scheme (2.7) satisfies the conditions in Proposition 2.1 and Lemma 2.2, then it is TVB (total variation bounded). That is,

$$TV(u^n) = \sum_i \left| u_{i+1}^n - u_i^n \right| \le C \tag{2.31}$$

for all $n \ge 0$ and $\Delta t > 0$. Here C is a constant independent of n and Δt.

Proof:

By (2.12), we have

$$
\begin{aligned}
TV(u^n) &= \sum_i \left| u_{i+1}^n - u_i^n \right| = \sum_i \left| (A^{-1}\bar{u}^n)_{i+1} - (A^{-1}\bar{u}^n)_i \right| \\
&\le \|A^{-1}\|_{L_1} \sum_i \left| \bar{u}_{i+1}^n - \bar{u}_i^n \right| \le \frac{\alpha}{1-\delta} TV(\bar{u}^0).
\end{aligned}
$$

$\square$

This Proposition guarantees convergence of at least a subsequence of the numerical solution.

We now discuss whether the limiting defined in (2.18) maintains the *formal* accuracy of the compact schemes in smooth regions of the solution. For this we need the following assumption.

Assumption 2.4

$$\bar{u}_i = (Au)_i = u_i + O(\Delta x^2) \tag{2.32}$$

for all $u \in C^2$.

$\square$

This Assumption is satisfied by any compact scheme with a *symmetric* A, for example all those listed in (1.3) through (1.6).

Under Assumption 2.4, it is easy to verify by simple Taylor expansions that

$$
\begin{aligned}
\Delta_+ f^\pm(\bar{u}_k) &= f^\pm(\bar{u}_i)_x \Delta x + O(\Delta x^2) \qquad k = i-1, i, i+1 \\
d\hat{f}_{i+\frac{1}{2}}^\pm &= \frac{1}{2} f^\pm(\bar{u}_i)_x \Delta x + O(\Delta x^2).
\end{aligned}
\tag{2.33}
$$

Hence in smooth regions away from critical points (critical points are defined here as points for which $f^+(\bar{u})_x = 0$ or $f^-(\bar{u})_x = 0$), the second and third arguments of the *minmod*

7

functions in (2.18) are asymptotically of the same sign as the first argument and half in magnitude. Hence the first argument will be picked by the *minmod* function (2.19) for sufficiently small Δx, thus yielding

$$d\hat{f}_{i+\frac{1}{2}}^{\pm(m)} = d\hat{f}_{i+\frac{1}{2}}^{\pm}. \tag{2.34}$$

This guarantees the original high order accuracy of the scheme in such smooth, monotone regions. At critical points, the accuracy will degenerate to first order as a generic restriction of all TVD schemes (see, for example, [13]). To overcome this difficulty, we use a modification of the *minmod* function

$$\tilde{m}(a_1, \cdots, a_k) = \begin{cases} a_1, & if \quad |a_1| \le M\Delta x^2 \\ m(a_1, \cdots, a_k), & otherwise \end{cases} \tag{2.35}$$

where M is a constant independent of Δx. This modification is discussed in detail in [5] and [15].

With this modification we can obtain schemes which are formally of uniform high order accuracy and equal the original unlimited scheme in smooth regions including local extrema. Moreover, we can prove the following proposition.

Proposition 2.5

The conclusions of Proposition 2.1 and 2.3 are still valid for any n and Δt such that $0 \le n\Delta t \le T$, with TVDM in (2.23) replaced by TVBM (total variation bounded in the means)

$$TV(\bar{u}^n) \le C \tag{2.36}$$

where C is independent of Δt, if the *minmod* function m in (2.18) is replaced by the *modified minmod* function $\tilde{m}$ defined in (2.35).

Proof:

The proof is similar to that contained in [15] and [5] and is thus omitted.

$\square$

The choice of the constant M in (2.35) is related to the second derivative of the solution near smooth extrema. For details, see [5] and [15]. The numerical result is usually not sensitive to the variation of M in a large range.

In this paper we only consider pure initial value problems. u^0 in (1.1) is assumed to be either periodic or compactly supported. For initial boundary value problems, $\bar{u}$ in (2.5) is defined differently at the boundary, as is the scheme (2.6). The limiting (2.18) can be modified at the boundary so that the scheme remains TVDM (or TVBM) and TVB for initial boundary value problems. We refer the readers to [5] and [16] for more details.

3   Multiple Space Dimensions

For notational simplicity we only consider the two dimensional case (1.1)-(1.2). Three space dimensions do not pose additional conceptional difficulties. As before, we only need to consider the Euler forward time discretization

$$u_{ij}^{n+1} = u_{ij}^n + \Delta t L(u^n)_{ij}. \tag{3.1}$$

We again define

$$\bar{u}_{ij} \equiv (A_y A_x u)_{ij} \tag{3.2}$$

so that scheme (3.1) can be left-multiplied by $A_y A_x$ to become

$$\bar{u}_{ij}^{n+1} = \bar{u}_{ij}^n - \frac{\Delta t}{\Delta x}(A_y B_x f(u^n))_{ij} - \frac{\Delta t}{\Delta y}(A_x B_y f(u^n))_{ij}. \tag{3.3}$$

Here and in what follows we will use the commutativity of A_x, A_y, B_x and B_y so that a product can be written in any order. Scheme (3.3) can be written into a conservation form

$$\bar{u}_{ij}^{n+1} = \bar{u}_{ij}^n - \frac{\Delta t}{\Delta x}(\hat{f}_{i+\frac{1}{2},j}^n - \hat{f}_{i-\frac{1}{2},j}^n) - \frac{\Delta t}{\Delta y}(\hat{g}_{i,j+\frac{1}{2}}^n - \hat{g}_{i,j-\frac{1}{2}}^n) \tag{3.4}$$

which involves numerical fluxes $\hat{f}_{i+\frac{1}{2},j}^n$ and $\hat{g}_{i,j+\frac{1}{2}}^n$ as local functions of u_{kl}^n. For example,

$$\hat{f}_{i+\frac{1}{2},j} = \frac{1}{2}A_y\left(f(u_{i+1,j}) + f(u_{ij})\right)$$
$$\hat{g}_{i,j+\frac{1}{2}} = \frac{1}{2}A_x\left(g(u_{i,j+1}) + g(u_{ij})\right) \tag{3.5}$$

for the fourth order central scheme (1.3), with analogous definitions for the other schemes. Again, scheme (3.4) resembles a cell-averaged (finite volume) scheme [10]. The $\bar{u}_{ij}$ defined by (3.2) is a local mean of u, and a "reconstruction" from $\bar{u}$ to u

$$u_{ij} = (A_x^{-1} A_y^{-1} \bar{u})_{ij} \tag{3.6}$$

is needed to compute the fluxes $\hat{f}_{i+\frac{1}{2},j}$ and $\hat{g}_{i,j+\frac{1}{2}}$ in (3.4).

We remark that the additional costs of implementing scheme (3.4), comparing with the original scheme (3.1), are the two *local* operators A_x and A_y. The major part of the cost still consists of the two tridiagonal inversions.

The limiting to obtain nonlinear stability can now be defined in a dimension by dimension fashion; we can use the one-dimensional flux splitting (2.13), for $f(u)$, to write the flux $\hat{f}_{i+\frac{1}{2},j}$ as

$$\hat{f}_{i+\frac{1}{2},j} = \hat{f}_{i+\frac{1}{2},j}^+ + \hat{f}_{i+\frac{1}{2},j}^- \tag{3.7}$$

where $\hat{f}_{i+\frac{1}{2},j}^\pm$ are again obtained by putting superscripts $\pm$ in, for example (3.5). The remaining definition of the limiting parallels that in Section 2, with a dummy index j added for the reference y value. We still start with the differences between the high order numerical fluxes and the first order upwind fluxes

$$d\hat{f}_{i+\frac{1}{2},j}^+ = \hat{f}_{i+\frac{1}{2},j}^+ - f^+(\bar{u}_{ij}); \qquad d\hat{f}_{i+\frac{1}{2},j}^- = f^-(\bar{u}_{i+1,j}) - \hat{f}_{i+\frac{1}{2},j}^- \tag{3.8}$$

and limit them by

9

$$df_{i+\frac{1}{2},j}^{+(m)} = m\left(df_{i+\frac{1}{2},j}^{+}, \Delta_{+}^{x}f^{+}(\bar{u}_{ij}), \Delta_{+}^{x}f^{+}(\bar{u}_{i-1,j})\right)$$

$$df_{i+\frac{1}{2},j}^{-(m)} = m\left(df_{i+\frac{1}{2},j}^{-}, \Delta_{+}^{x}f^{-}(\bar{u}_{ij}), \Delta_{+}^{x}f^{-}(\bar{u}_{i+1,j})\right) \tag{3.9}$$

where $\Delta_{+}^{x}v_{ij} \equiv v_{i+1,j} - v_{ij}$ is the forward difference operator in the x direction and the *minmod* function m is defined by (2.19). We then obtain the limited numerical fluxes by

$$\hat{f}_{i+\frac{1}{2},j}^{+(m)} = f^{+}(\bar{u}_{ij}) + df_{i+\frac{1}{2},j}^{+(m)}; \qquad \hat{f}_{i+\frac{1}{2},j}^{-(m)} = f^{-}(\bar{u}_{i+1,j}) - df_{i+\frac{1}{2},j}^{-(m)} \tag{3.10}$$

and

$$\hat{f}_{i+\frac{1}{2},j}^{(m)} = \hat{f}_{i+\frac{1}{2},j}^{+(m)} + \hat{f}_{i+\frac{1}{2},j}^{-(m)}. \tag{3.11}$$

The flux in the y-direction is defined analogously.

In light of [8] this scheme cannot be TVD in two space dimensions. However we can obtain maximum norm stability through the following proposition.

Proposition 3.1

Scheme (3.4) with the flux (3.11) satisfies a maximum principle in the means

$$\max_{i,j}\left|\bar{u}_{ij}^{n+1}\right| \le \max_{i,j}\left|\bar{u}_{ij}^{n}\right| \tag{3.12}$$

under the CFL condition

$$\left[\max\left(f^{+'}(u)\right) + \max\left(-f^{-'}(u)\right)\right]\frac{\Delta t}{\Delta x} + \left[\max\left(g^{+'}(u)\right) + \max\left(-g^{-'}(u)\right)\right]\frac{\Delta t}{\Delta y} \le \frac{1}{2} \tag{3.13}$$

where the maximum is taken in $\min_{i,j}\bar{u}_{ij}^{n} \le u \le \max_{i,j}\bar{u}_{ij}^{n}$.

Proof: Similar to the development in Proposition 2.1, we can write the flux differences as

$$\hat{f}_{i+\frac{1}{2},j}^{(m)} - \hat{f}_{i-\frac{1}{2},j}^{(m)} = -C_{i+\frac{1}{2},j}\Delta_{+}^{x}\bar{u}_{ij} + D_{i-\frac{1}{2},j}\Delta_{+}^{x}\bar{u}_{i-1,j}$$

$$\hat{g}_{i,j+\frac{1}{2}}^{(m)} - \hat{g}_{i,j-\frac{1}{2}}^{(m)} = -C_{i,j+\frac{1}{2}}\Delta_{+}^{y}\bar{u}_{ij} + D_{i,j-\frac{1}{2}}\Delta_{+}^{y}\bar{u}_{i,j-1} \tag{3.14}$$

with

$$C_{i+\frac{1}{2},j} \ge 0, \qquad D_{i-\frac{1}{2},j} \ge 0, \qquad C_{i,j+\frac{1}{2}} \ge 0, \qquad D_{i,j-\frac{1}{2}} \ge 0 \tag{3.15}$$

due to the flux splitting (3.7), the limiting (3.9), and

$$\frac{\Delta t}{\Delta x}\left(C_{i+\frac{1}{2},j} + D_{i-\frac{1}{2},j}\right) + \frac{\Delta t}{\Delta y}\left(C_{i,j+\frac{1}{2}} + D_{i,j-\frac{1}{2}}\right) \le 1 \tag{3.16}$$

the CFL condition (3.13).

We then have

10

$$\bar{u}_{ij}^{n+1} = \bar{u}_{ij}^n + \frac{\Delta t}{\Delta x}\left(C_{i+\frac{1}{2},j}\Delta_+^x \bar{u}_{ij}^n - D_{i-\frac{1}{2},j}\Delta_+^x \bar{u}_{i-1,j}^n\right) + \frac{\Delta t}{\Delta y}\left(C_{i,j+\frac{1}{2}}\Delta_+^y \bar{u}_{ij}^n - D_{i,j-\frac{1}{2}}\Delta_+^y \bar{u}_{i,j-1}^n\right)$$

$$= \left[1 - \frac{\Delta t}{\Delta x}\left(C_{i+\frac{1}{2},j} + D_{i-\frac{1}{2},j}\right) - \frac{\Delta t}{\Delta y}\left(C_{i,j+\frac{1}{2}} + D_{i,j-\frac{1}{2}}\right)\right]\bar{u}_{ij}^n$$

$$+ C_{i+\frac{1}{2},j}\bar{u}_{i+1,j}^n + D_{i-\frac{1}{2},j}\bar{u}_{i-1,j}^n + C_{i,j+\frac{1}{2}}\bar{u}_{i,j+1}^n + D_{i,j-\frac{1}{2}}\bar{u}_{i,j-1}^n$$

which implies the maximum principle (3.12) because $\bar{u}_{ij}^{n+1}$ is written as a convex combination of $\bar{u}_{ij}^n$, $\bar{u}_{i\pm1,j}^n$ and $\bar{u}_{i,j\pm1}^n$ with positive coefficients which add up to one.

$\square$

In order to obtain maximum norm stability for u, we need a lemma similar to Lemma 2.2.

Lemma 3.2

If there are two numbers $0 \le \delta < 1$ and $\alpha > 0$, which are independent of N, such that the $N \times N$ matrix $A = (a_{ij})$ satisfies:

$$\max_{1 \le i \le N} \frac{1}{|a_{ii}|} \le \alpha, \quad and \quad \sum_{\substack{j=1 \\ j \ne i}}^{N} |a_{ij}| \le \delta |a_{ii}|, \quad i = 1, \cdots, N \quad (3.17)$$

(strongly diagonally dominance for A), then the L_∞ norm of A^{-1} is bounded independent of N

$$\|A^{-1}\|_{L_\infty} \le \frac{\alpha}{1-\delta}. \quad (3.18)$$

Proof: The proof is similar to that for Lemma 2.2 and is thus omitted.

$\square$

For the compact methods we consider, the matrix A is symmetric. Hence the requirements (2.29) and (3.17) are the same.

We can now use Lemma 3.2 to obtain the maximum norm stability for u.

Proposition 3.3

If a compact scheme (3.4) satisfies the conditions in Proposition 3.1 and Lemma 3.2 for both A_x and A_y, then it is stable in the maximum norm. That is,

$$\max_{i,j} |u_{ij}^n| \le C \quad (3.19)$$

for all $n \ge 0$ and $\Delta t > 0$. Here C is a constant independent of n and Δt.

Proof:

By (3.2), we have

$$\max_{i,j}\left|u_{ij}^n\right| = \max_{i,j}\left|\left(A_y^{-1}A_x^{-1}\bar{u}^n\right)_{ij}\right| \le \|A_y^{-1}\|_{L_\infty}\|A_x^{-1}\|_{L_\infty}\max_{i,j}\left|\bar{u}_{ij}^n\right|$$

$$\le \left(\frac{\alpha}{1-\delta}\right)^2\max_{i,j}\left|\bar{u}_{ij}^0\right|.$$

$\square$

This Proposition does not guarantee convergence, but it at least guarantees that the numerical solution will not blow up due to instability.

Under the Assumption 2.4 for both A_x and A_y, we can again easily verify that the limiting (3.9) maintains formally the original high order accuracy of the scheme in smooth, monotone regions. The degeneracy of accuracy at critical points can once again be overcome by adopting the *modified minmod* function (2.35) in the limiting (3.9).

4 Numerical Examples

To test the behavior of the schemes discussed in Sections 2 and 3, we use the one and two dimensional Burgers equation with the smooth initial conditions

$$u_t + \left(\frac{u^2}{2}\right)_x = 0$$
$$u(x,0) = 0.3 + 0.7\sin(x) \tag{4.1}$$

and

$$u_t + \left(\frac{u^2}{2}\right)_x + \left(\frac{u^2}{2}\right)_y = 0$$
$$u(x,y,0) = 0.3 + 0.7\sin(x+y). \tag{4.2}$$

Both are assumed to have 2π-periodic boundary conditions. The solutions will stay smooth initially, and then develop shocks which move with time. The exact solution to (4.1) can be obtained by following the characteristics and solving the resulting nonlinear equation using Newton iteration. The exact solution to (4.2) is that of (4.1) with x replaced by $x+y$ and t replaced by $2t$. These are standard test problems for scalar nonlinear conservation laws containing shocks. For comparison with finite difference ENO schemes and finite element discontinuous Galerkin methods, see [17], [5] and [6].

The schemes we test are based on the fourth order central scheme (1.3) coupled with a fourth order Runge-Kutta time discretization (henceforth referred to as the central scheme), as well as the third order upwind schemes (1.5)-(1.6) coupled with the third order TVD Runge-Kutta time discretization (1.8) (henceforth referred to as the upwind scheme). For the flux splitting (2.13) we use the Lax-Friedrichs splitting (2.15). The time step Δt^n is taken to satisfy a CFL condition

$$\max_i |\bar{u}_i^n| \frac{\Delta t^n}{\Delta x} \le 0.5 \tag{4.3}$$

in one dimension and

$$\max_{i,j} |\bar{u}_{ij}^n| \left(\frac{\Delta t^n}{\Delta x} + \frac{\Delta t^n}{\Delta y} \right) \le 0.5 \tag{4.4}$$

in two dimensions. When the *modified minmod* limiter (2.35) is used, the constant M is taken as 1.

We first test the effect of limiters when the solution is smooth but not monotone. In Figure 1 we plot the L_1 error versus number of grid points, in a log-log scale, at $t = 0.6$ for the one dimensional case and at $t = 0.3$ for the two dimensional case. In such scales, the error should be a straight line with slope $-k$ for a k-th order method. We can see that the original compact schemes and the schemes with *modified minmod* limiter (2.35) (henceforth referred to as the TVB limiter) give the expected third and fourth order accuracy respectively, while the schemes with the *minmod* limiter (2.19) (henceforth referred to as the TVD limiter) give only second order accuracy due to the degeneracy at the critical points. We can also see that both the central and the upwind schemes work well for this smooth problem.

We then test the effect of limiters when the solution becomes discontinuous. In Figure 2 we show the results of the original compact schemes at $t = 2$ for the one dimensional case. We can see over- and under-shoots as well as oscillations, and in this case the result of the central scheme is much worse than that of the upwind one. In Figures 3 and 4 we show the results with the TVD and the TVB limiters. Apparently the limiters have stabilized the solution, as predicted by the theory. However the result with the central scheme is not quite satisfactory. In Figures 5 and 6, we show the pointwise errors, in a logarithm scale, for the numbers of grid points $N = 10, 20, 40, 80$ and 160. We can see that the central scheme, even with the TVB limiter, shows a reduced accuracy for quite a large region around the shock. This indicates that, for a scheme which is globally oscillatory (like the central compact scheme), limiters can render it stable but may kill accuracy in smooth regions since there are oscillations there to suppress. On the other hand, the upwind compact scheme work well, with bigger errors for the TVD limiter near the smooth extremum which is close to the shock. The errors for the two dimensional case are similar and are not displayed. In the last plot, Figure 7, we show the surface of the two dimensional solution at $t = 1$ with 40×40 points using the third order upwind method with TVB limiting.
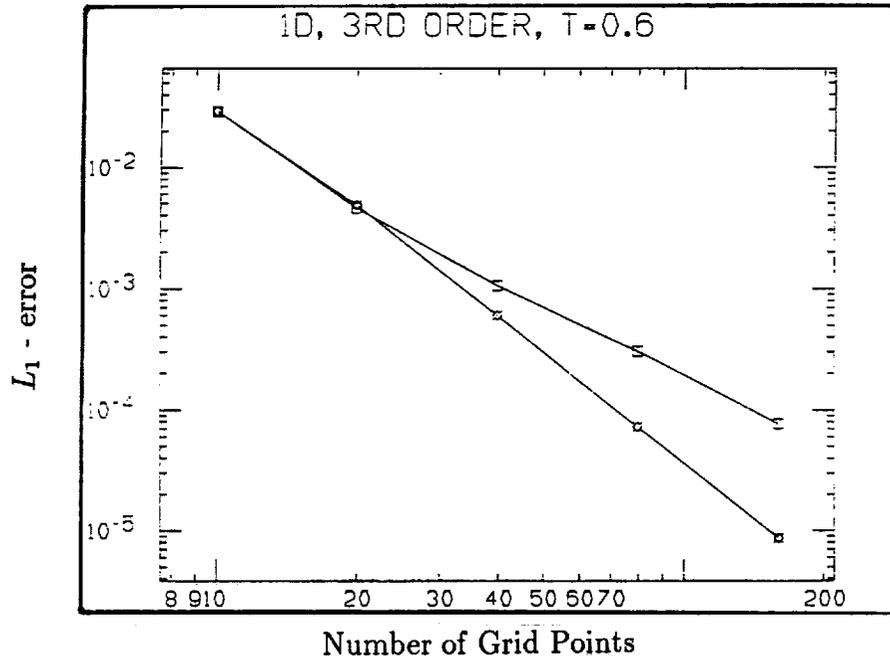
5   Concluding Remarks

We have discussed a general framework to apply local limiters on compact schemes via the definition of a local mean. The resulting schemes are proven TVB (total variation bounded) in one dimension and maximum norm stable for multiple space dimensions. Numerical examples show that the base compact scheme should be upwind-biased in order to obtain high order accuracy after limiting for shocked problems.
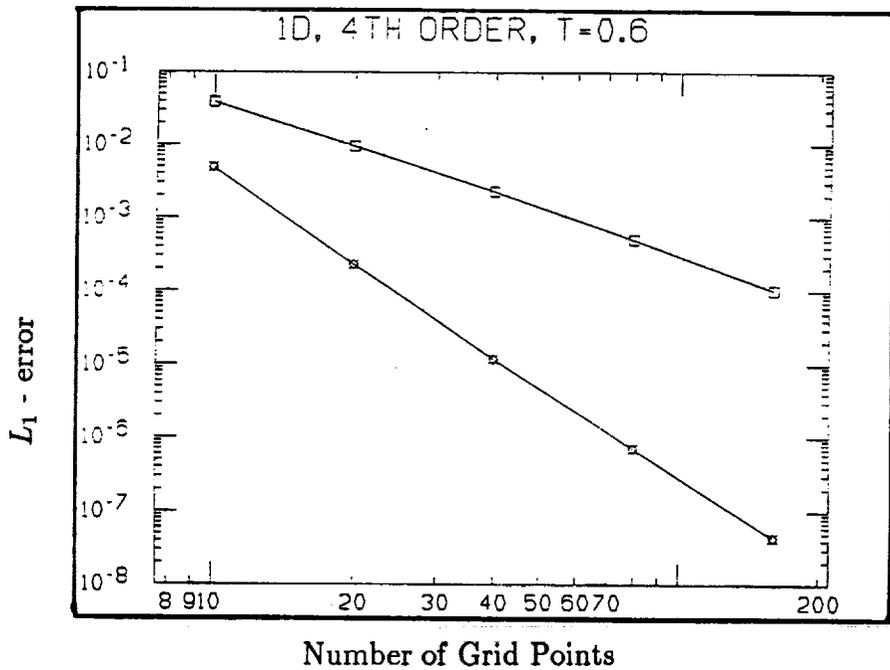
13

References

[1] S. Abarbanel and A. Kumar, *Compact high order schemes for the Euler equations*, J. Sci. Comput., 3 (1988), pp.275-288.

[2] M. Carpenter, *A high-order compact numerical algorithm for supersonic flows*, In Proceedings of the 12th International Conference on Numerical Methods in Fluid Dynamics, K. W. Morton, ed., Lecture Notes in Physics, v371, Springer-Verlag (1990), pp.254-258.

[3] M. Carpenter, D. Gottlieb and S. Abarbanel, *The stability of numerical boundary treatments for compact high-order finite-difference schemes*, ICASE Report 91-71 (1991), NASA Langley Research Center.

[4] M. Ciment and S. Leventhal, *Higher order compact implicit schemes for the wave equation*, Math. Comput., 29 (1975), pp.985-994.

[5] B. Cockburn and C.-W. Shu, *TVB Runge-Kutta local projection discontinuous Galerkin finite element method for conservation laws II: general framework*, Math. Comput., 52 (1989), pp.411-435.

[6] B. Cockburn, S. Hou and C.-W. Shu, *The Runge-Kutta local projection discontinuous Galerkin finite element method for conservation laws IV: the multidimensional case*, Math. Comput., 54 (1990), pp.545-581.

[7] B. Cockburn and C.-W. Shu, *Continuous finite element methods for nonlinear conservation laws: preliminary results*, TICOM Report, University of Texas at Austin, August 1989.

[8] J. Goodman and R. LeVeque, *On the accuracy of stable schemes for 2D scalar conservation laws*, Math. Comput., 45 (1985), pp.15-21.

[9] A. Harten, *High resolution schemes for hyperbolic conservation laws*, J. Comput. Phys., 49 (1983), pp.357-393.

[10] A. Harten, *Preliminary results on the extension of ENO schemes to two dimensional problems* Proc. Internat. Conf. on Hyperbolic Problems, Saint-Etienne, 1986.

[11] A. Harten, B. Engquist, S. Osher and S. Chakravarthy, *Uniformly high order accurate essentially non-oscillatory schemes, III*, J. Comput. Phys., 71 (1987), pp.231-303.

[12] R. Hirsh, *Higher order accurate difference solutions of fluid mechanics problems by a compact differencing technique*, J. Comput. Phys., 19 (1975), pp.90-109.

[13] S. Osher and S. Chakravarthy, *High resolution schemes and the entropy condition*, SIAM J. Numer. Anal., 21 (1984), pp.955-984.

[14] C.-W. Shu, *Total-variation-diminishing time discretizations*, SIAM J. Sci. Stat. Comput., 9 (1988), pp.1073-1084.

[15] C.-W. Shu, *TVB uniformly high-order schemes for conservation laws*, Math. Comput., 49 (1987), pp.105-121.

[16] C.-W. Shu, *TVB boundary treatment for numerical solutions of conservation laws*, Math. Comput., 49 (1987), pp.123-134.

[17] C.-W. Shu and S. Osher, *Efficient implementation of essentially non-oscillatory shock capturing scheme*, J. Comput. Phys., 77 (1988), pp.439-471.

[18] R. Vichnevetsky and J. Bowles, *Fourier Analysis of Numerical Approximations of Hyperbolic Equations*, SIAM, Philadelphia, 1982.

Figure 1: L_1 error versus number of grid points in log-log scale for smooth solutions. Stars: compact schemes without limiter; squares: with TVD limiter; diamonds: with TVB limiter.

1(a): Third order upwind scheme, 1D

Number of Grid Points

1(b): Fourth order central scheme, 1D

Number of Grid Points

16

1(c): Third order upwind scheme, 2D

2D, 3RD ORDER, T=0.3

Number of Grid Points in x

1(d): Fourth order central scheme, 2D

2D, 4TH ORDER, T=0.3

Number of Grid Points in x

17
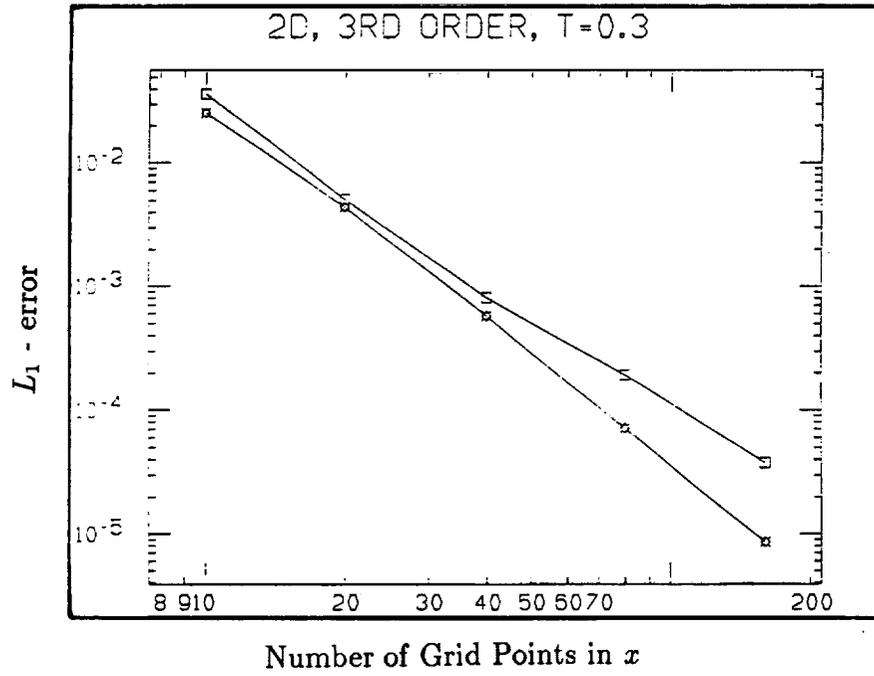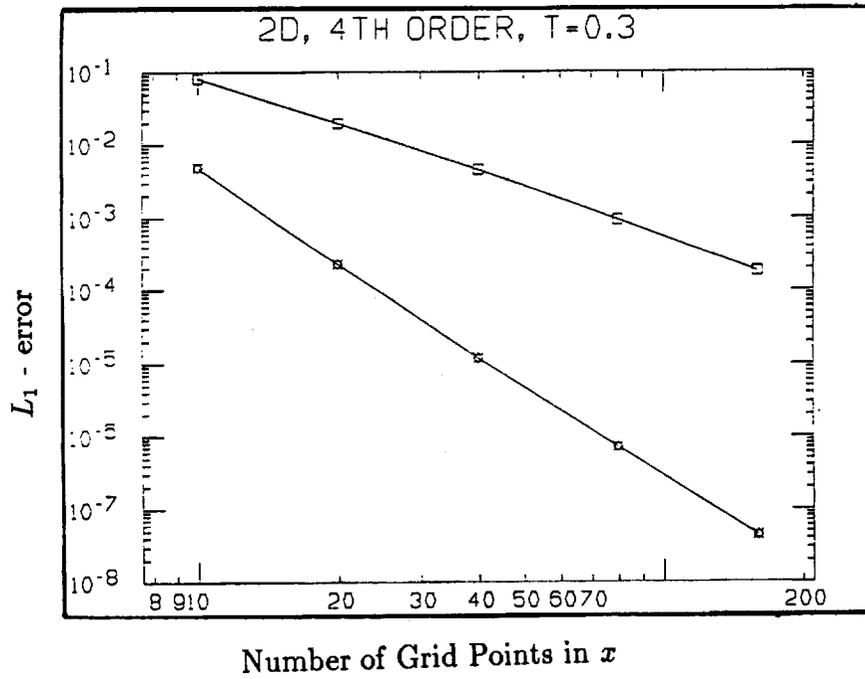
Figure 2: Compact schemes without limiter for shocks. Pluses: computed solution; solid line: exact solution.

2(a): Third order upwind scheme
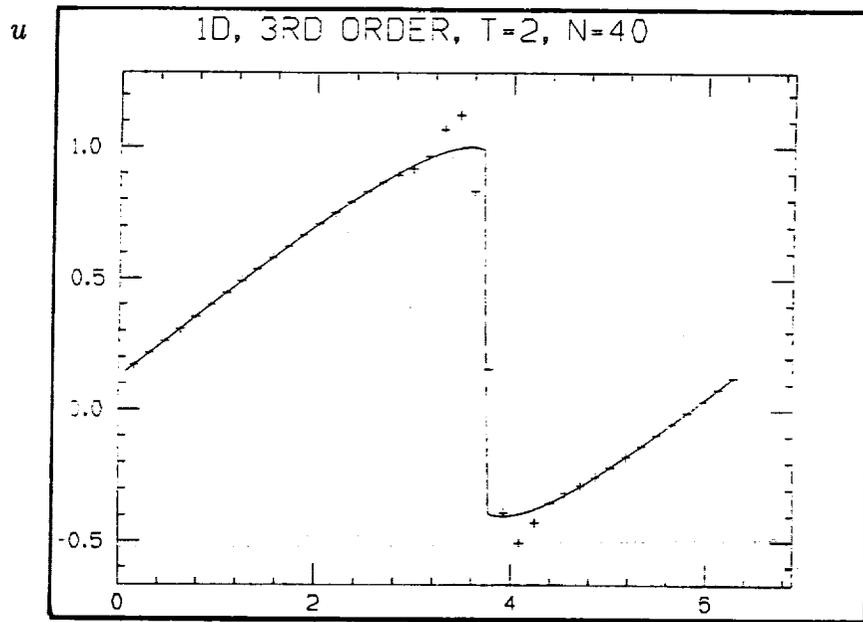
2(b): Fourth order central scheme
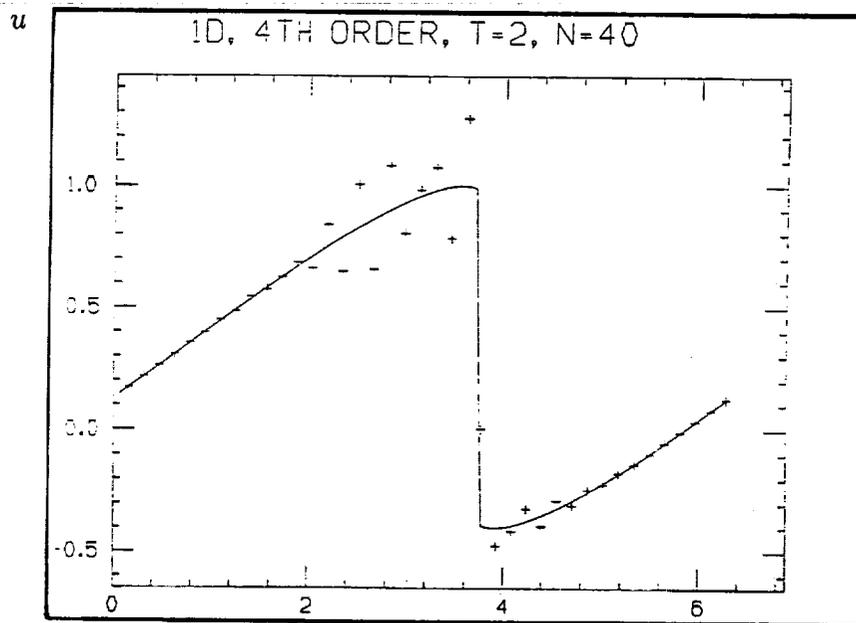
Figure 3: Third order upwind scheme with limiters for shocks. Pluses: computed solution; solid line: exact solution.
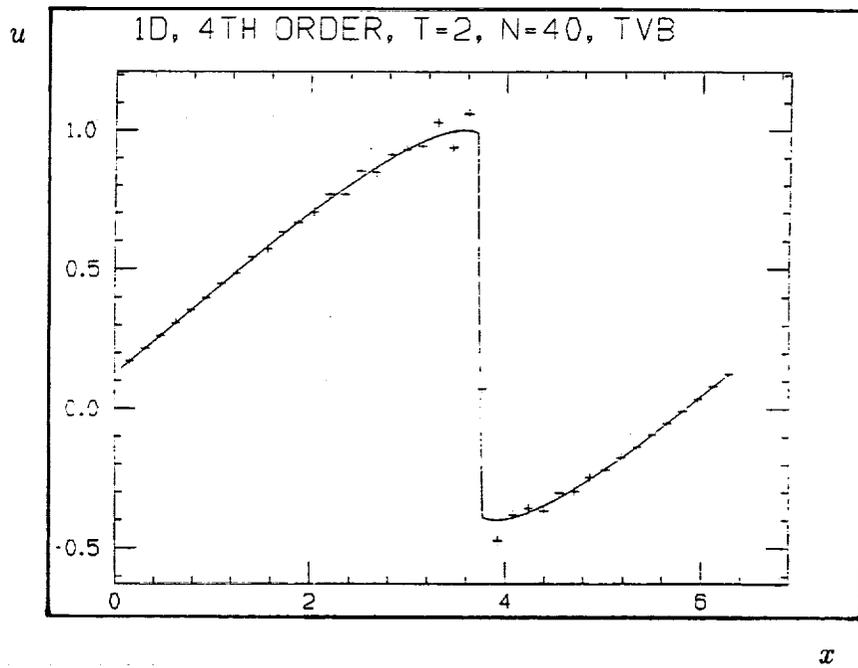
3(a): With TVD limiter
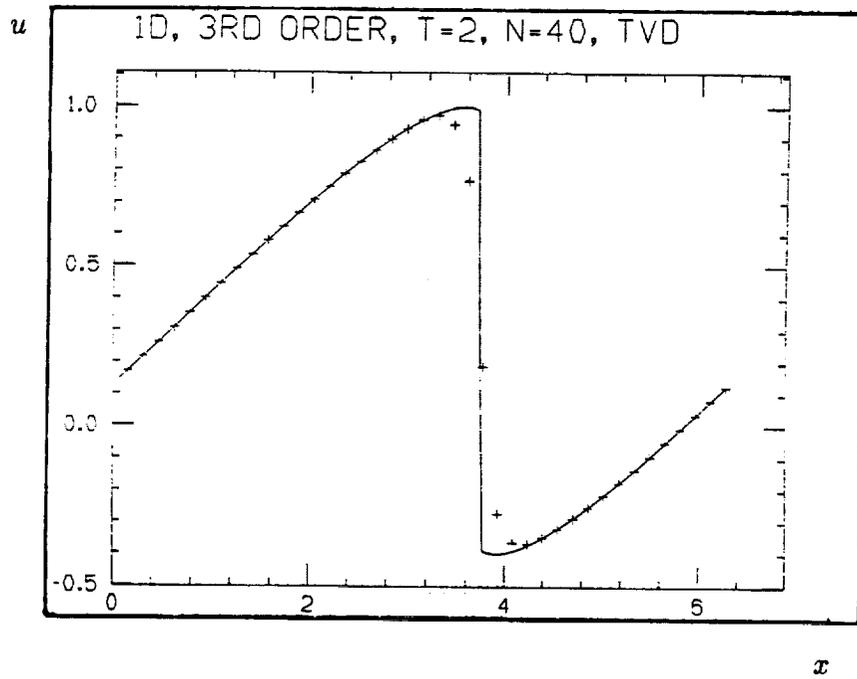
3(b): With TVB limiter
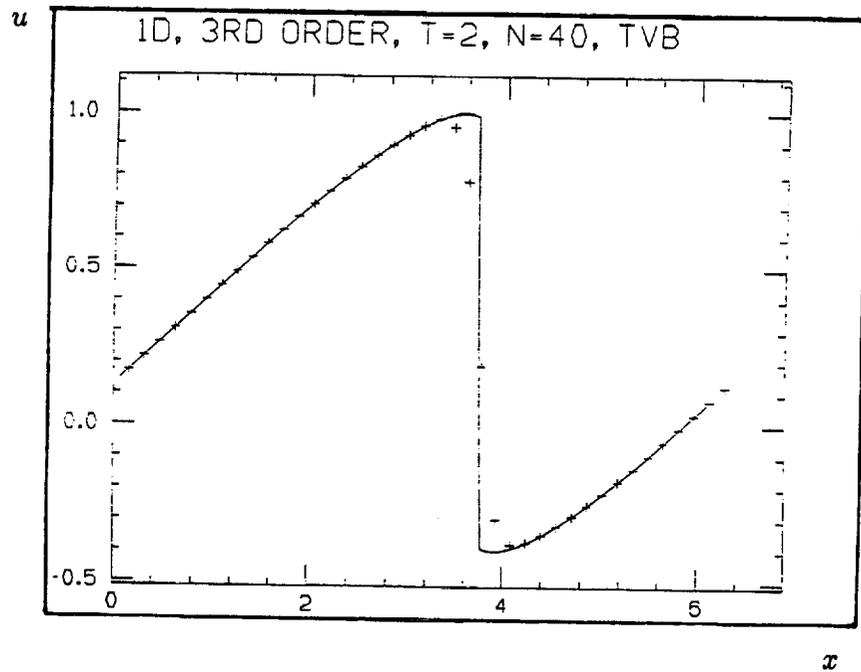
Figure 4: Fourth order central scheme with limiters for shocks. Pluses: computed solution; solid line: exact solution.

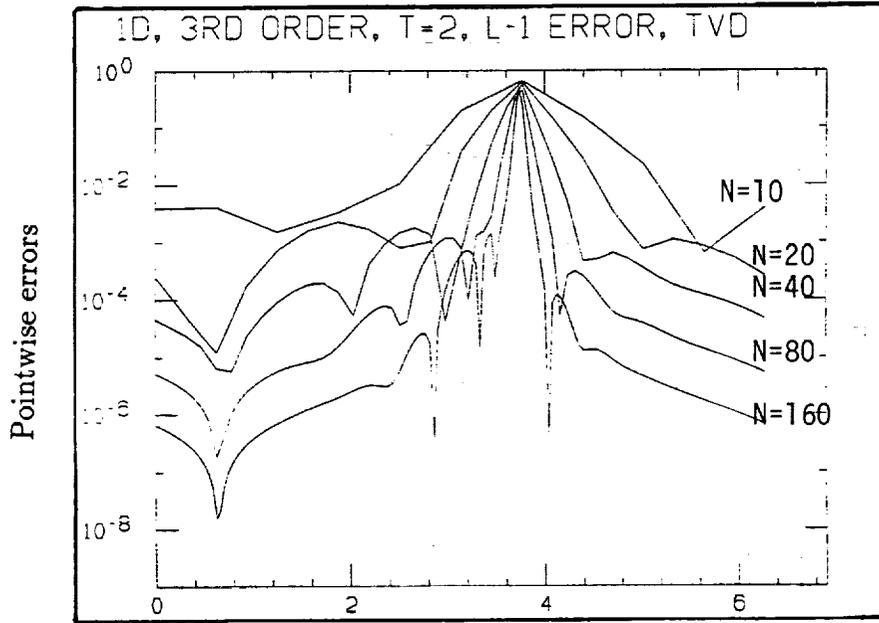4(a): With TVD limiter
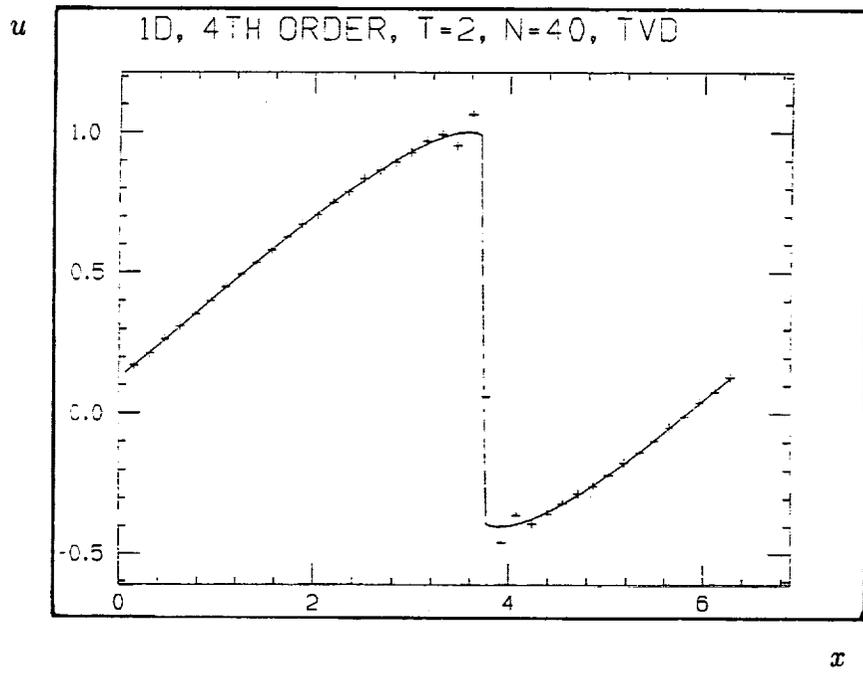
4(b): With TVB limiter

Figure 5: Pointwise error for $N = 10, 20, 40, 80$ and 160 grid points, in a logarithm scale. Third order upwind scheme with limiters for shocks.

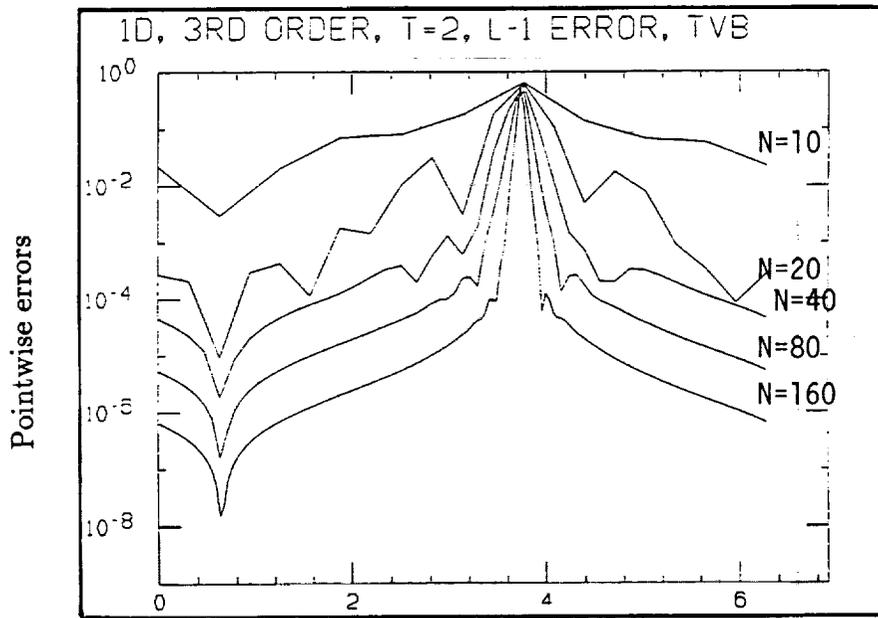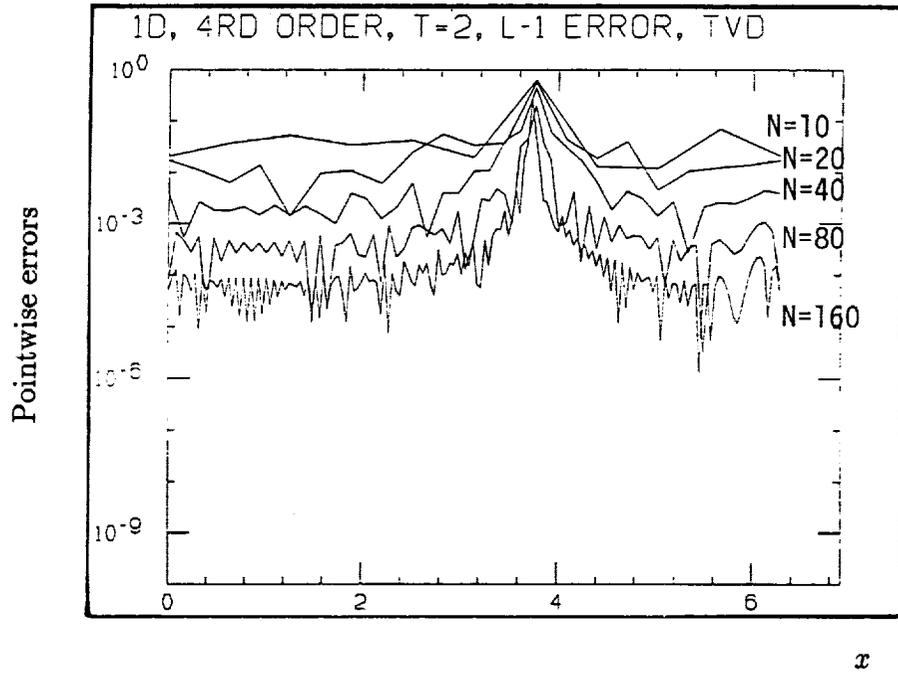5(a): With TVD limiter

5(b): With TVB limiter

Figure 6: Pointwise error for $N = 10, 20, 40, 80$ and 160 grid points, in a logarithm scale. Fourth order central scheme with limiters for shocks.
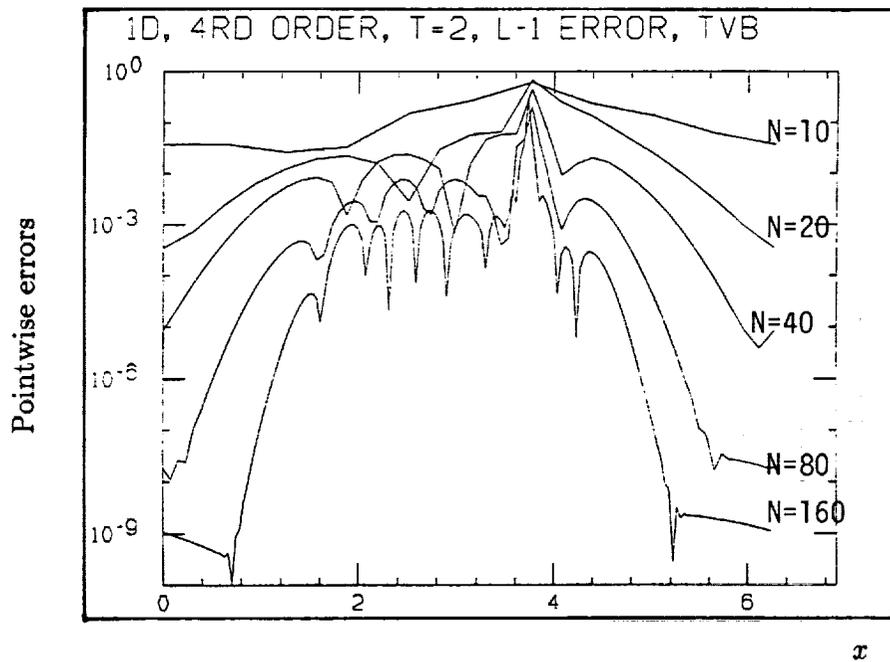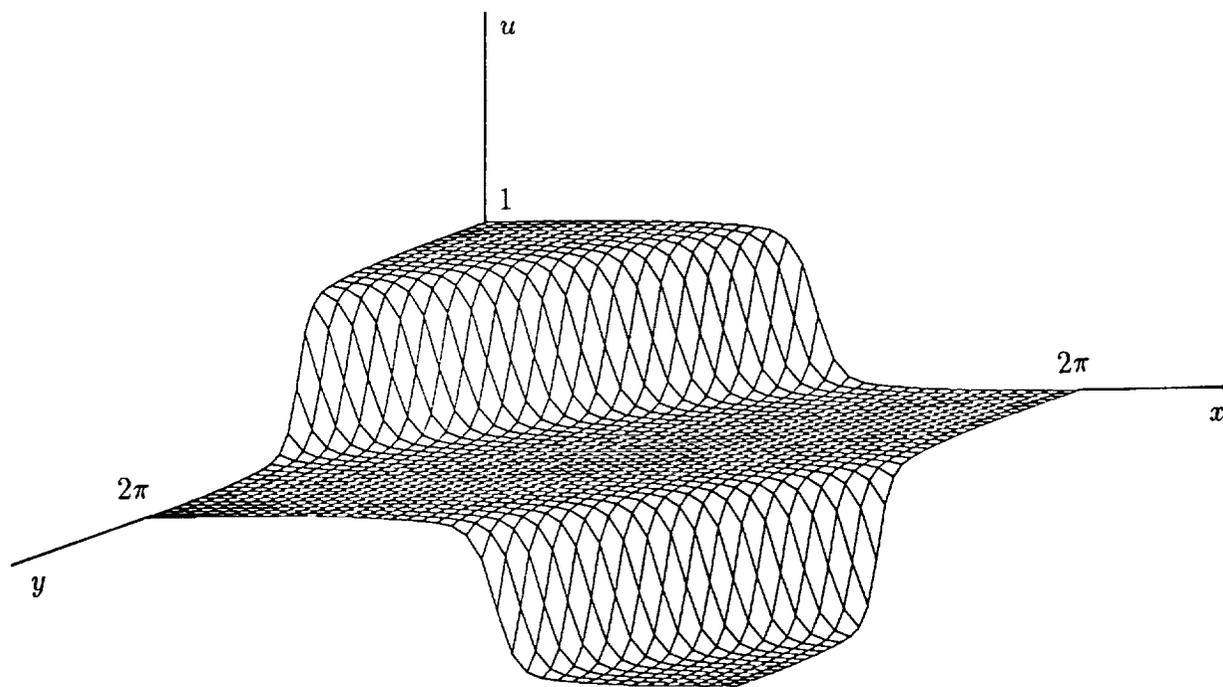
6(a): With TVD limiter

6(b): With TVB limiter

Figure 7: Surface of third order upwind compact scheme with TVB limiter for shocks, with 40 × 40 points.

REPORT DOCUMENTATION PAGE

Public reporting burden for this collection of information is estimated to average 1 hour per response, including the time for reviewing instructions, searching existing data sources, gathering and maintaining the data needed, and completing and reviewing the collection of information. Send comments regarding this burden estimate or any other aspect of this collection of information, including suggestions for reducing this burden, to Washington Headquarters Services, Directorate for information Operations and Reports, 1215 Jefferson Davis Highway, Suite 1204, Arlington, VA 22202-4302, and to the Office of Management and Budget. Paperwork Reduction Project (0704-0188), Washington, DC 20503

1. AGENCY USE ONLY (Leave blank)	2. REPORT DATE	3. REPORT TYPE AND DATES COVERED
	May 1992	Contractor Report

4. TITLE AND SUBTITLE	5. FUNDING NUMBERS
Nonlinearly Stable Compact Schemes for Shock Calculations	C NAS1-18605 WU 505-90-52-01

6. AUTHOR(S)

Bernardo Cockburn
Chi-Wang Shu

7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES)	8. PERFORMING ORGANIZATION REPORT NUMBER
Institute for Computer Applications in Science and Engineering Mail Stop 132C, NASA Langley Research Center Hampton, VA 23665-5225	ICASE Report No. 92-21

9. SPONSORING/MONITORING AGENCY NAME(S) AND ADDRESS(ES)	10. SPONSORING/MONITORING AGENCY REPORT NUMBER
National Aeronautics and Space Administration Langley Research Center Hampton, VA 23665-5225	NASA CR-189663 ICASE Report No. 92-21

11. SUPPLEMENTARY NOTES

12a. DISTRIBUTION/AVAILABILITY STATEMENT	12b. DISTRIBUTION CODE
Unclassified - Unlimited Subject Category 64	

13. ABSTRACT (Maximum 200 words)

In this paper we discuss the applications of high order compact finite difference methods for shock calculations. The main idea is the definition of a local mean which serves as a reference for introducing a local nonlinear limiting to control spurious numerical oscillations while keeping the formal accuracy of the scheme. For scalar conservation laws, the resulting schemes can be proven total variation stable in one space dimension and maximum norm stable in multiple space dimensions. Numerical examples are shown to verify accuracy and stability of such schemes for problems containing shocks. The idea in this paper can also be applied to other implicit schemes such as the continuous Galerkin finite element methods.

14. SUBJECT TERMS		15. NUMBER OF PAGES
compact schemes, total variation stability, maximum principle, shock calculations		25
		16. PRICE CODE A03

17. SECURITY CLASSIFICATION OF REPORT	18. SECURITY CLASSIFICATION OF THIS PAGE	19. SECURITY CLASSIFICATION OF ABSTRACT	20. LIMITATION OF ABSTRACT
Unclassified	Unclassified		