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Nonlocal beam models for buckling of nanobeams using state-space method regarding different boundary conditions †

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Abstract

Buckling analysis of nanobeams is investigated using nonlocal continuum beam models of the different classical beam theories namely as Euler-Bernoulli beam theory (EBT), Timoshenko beam theory (TBT), and Levinson beam theory (LBT). To this end, Eringen's equations of nonlocal elasticity are incorporated into the classical beam theories for buckling of nanobeams with rectangular cross-section. In contrast to the classical theories, the nonlocal elastic beam models developed here have the capability to predict critical buckling loads that allowing for the inclusion of size effects. The values of critical buckling loads corresponding to four commonly used boundary conditions are obtained using state-space method. The results are presented for different geometric parameters, boundary conditions, and values of nonlocal parameter to show the effects of each of them in detail. Then the results are fitted with those of molecular dynamics simulations through a nonlinear least square fitting procedure to find the appropriate values of nonlocal parameter for the buckling analysis of nanobeams relevant to each type of nonlocal beam model and boundary conditions.analysis.

Keywords: Nanomechanics; Nanobeams; Nonlocal elasticity parameter; Beam theory; State-space method high

1. Introduction

Due to exceptionally good physical, mechanical, and electrical properties [1-6], nano-sized structures have attracted much investment to develop innovatory applications in a wide range of disciplines. To accomplish the design of nanostructures and systems, an essential study of their mechanical behavior seems necessary. Nanomechanics is a branch of mechanics in which the mechanical properties and behavior of structures at nanoscale are investigated.

Modified continuum models have been the subject of much attention in nanomechanics due to their computational efficiency and the capability to produce accurate results which are comparable to those of atomistic models [7-11]. One approach for including nanoscale size-effects into the classical continuum mechanics is the use of modified continuum models based on the concept of nonlocal elasticity. Nonlocal continuum model has gained much popularity among the researchers because of its efficiency as well as simplicity to analyze the behavior of various nanostructures [11-21]. It has been observed that the mechanical properties of nanostrustures predicted by nonlocal continuum models are different from those previously obtained by the classical continuum mechanics which shows the size-effects on the behavior of structures at nanoscale.

Based on the above introduction, it seems that size-effects consideration in the analysis of nanobeams is necessary. In this work, different nonlocal beam models corresponding to the different classical beam theories [22-24] are presented on the basis of Eringen's equations of nonlocal elasticity [25] to predict the buckling behavior of nanobeams with four commonly used boundary conditions. State-space method is used to solve the governing differential equations for each type of nonlocal beam model with different boundary conditions. Various numerical results are given to show the influences of boundary conditions, aspect ratio, and values of nonlocal constant, separately. Then the results are matched with those of molecular dynamics simulations which are available in the literature to extract the correct values of nonlocal parameter corresponding to each type of nonlocal beam model and boundary conditions.

2. Overview of different beam theories

2.1 Introduction

There are various types of beam theory to describe the behavior of beams. Consider a straight uniform beam with the length L and rectangular cross-section of thickness h which is

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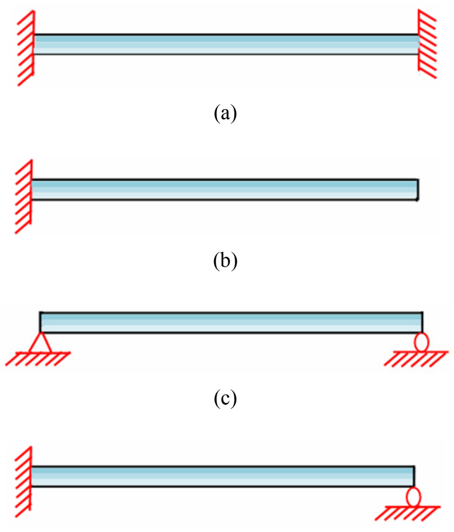

Fig. 1. (a) clamped-clamped; (b) clamped-free; (c) simply supported-simply supported; (d) simply supported-clamped straight uniform nanobeams with rectangular cross-section.

shown in Fig. 1. A coordinate system (x, y, z) is introduced on the central axis of the beam, whereas the x axis is taken along the length of the beam, the y axis in the width direction and the z axis is taken along the depth (height) direction. Also, the origin of the coordinate system is selected at the left end of the beam. It is assumed that the deformations of the beam take place in the x-z plane, so the displacement components (u_1, u_2, u_3) along the axis (x, y, z) are only dependent on the x and z coordinates and time t. In a general form, the following displacement field can be written:

$$
\begin{aligned}
u_1(x, z, t) &= -z\frac{\partial w(x, t)}{\partial x} + \psi(z)\left(\frac{\partial w(x, t)}{\partial x} + \varphi(x, t)\right) \\
u_2(x, z, t) &= 0 \\
u_3(x, z, t) &= w(x, t)
\end{aligned} \tag{1}
$$

where w and φ are the transverse displacement and angular displacement of the beam, respectively, and $\psi(z)$ is the shape function as follows:

For Euler-Bernoulli beam theory (EBT): $\psi(z) = 0$
For Timoshenko beam theory (TBT): $\psi(z) = z$
For Levinson beam theory (LBT): $\psi(z) = z - 4z^3/3h^2$

2.2 Euler-Bernoulli beam theory (EBT)

The simplest and the most well-known beam theory is the Euler-Bernoulli beam theory (the classical beam theory) in which it is assumed that the straight lines which are vertical to the mid-plane will remain straight and vertical to the mid-plane after deformation So, the effects of shear deformation and rotational inertia are not considered in this theory. On the basis of Eq. (2), the strain-displacement relations appropriate to EBT can be obtained as

$$\varepsilon_{xx} = \frac{\partial u_1}{\partial x} = -Z\frac{\partial^2 w}{\partial x^2} \tag{2a}$$

$$\gamma_{xz} = \frac{\partial u_1}{\partial z} + \frac{\partial u_3}{\partial x} = 0\,. \tag{2b}$$

Also, the following Euler-Lagrange equation can be expressed for EBT

$$\frac{\partial^2 M}{\partial x^2} - P\frac{\partial^2 w}{\partial x^2} = 0 \tag{3}$$

where P is the critical buckling load and $M = \int z\sigma_{xx} dA$.

2.3 Timoshenko beam theory (TBT)

The next type of beam theory is the Timoshenko beam theory in which the effects of shear deformation and rotational inertia are taken into account, so the straight lines will no longer remain vertical to the mid-plane of the beam after deformation. However, it is assumed that the transverse shear stress has a linear distribution along the thickness of the beam. Using Eq. (1), the following strain-displacement relations can be obtained as

$$\varepsilon_{xx} = \frac{\partial u_1}{\partial x} = Z\frac{\partial \varphi}{\partial x} \tag{4a}$$

$$\gamma_{xz} = \frac{\partial u_1}{\partial z} + \frac{\partial u_3}{\partial x} = \frac{\partial w}{\partial x} + \varphi\,. \tag{4b}$$

Also, the following Euler-Lagrange equations can be expressed for TBT

$$\frac{\partial Q}{\partial x} - P\frac{\partial^2 w}{\partial x^2} = 0 \tag{5a}$$

$$\frac{\partial M}{\partial x} - Q = 0 \tag{5b}$$

where $Q = \int \sigma_{xz} dA$.

2.4 Levinson beam theory (LBT)

The distribution of transverse shear stress in the Levinson beam theory [24] has a parabolic distribution with respect to the thickness of the beam. Also, there is not any shear correction factor to satisfy the transverse shear stress conditions on the upper and lower layers of the cross-section of the beam. The strain-displacement relations for LBT can be expressed as

$$\varepsilon_{xx} = \frac{\partial u_1}{\partial x} = Z\frac{\partial \varphi}{\partial x} - \frac{4z^3}{3h^2}\left(\frac{\partial \varphi}{\partial x} + \frac{\partial^2 w}{\partial x^2}\right) \tag{6a}$$

$$\gamma_{xz} = \frac{\partial u_1}{\partial z} + \frac{\partial u_3}{\partial x} = \left(1 - \frac{4z^2}{h^2}\right)\left(\varphi + \frac{\partial w}{\partial x}\right). \tag{6b}$$

A vector approach is used to derive the equilibrium equations; therefore, they are the same as those of the TBT. So, the Euler-Lagrange equations of Eq. (5) are the same and can be used for LBT too.

3. Nonlocal beam theories for buckling of nanobeam

3.1 Review of Eringen's nonlocal elasticity

The theory of nonlocal elasticity was first considered by Eringen in the 1970s [25]. In contrast to the classical elasticity, in the nonlocal model the stress at a reference point x in an elastic body depends not only on the strains at x, but also on the strains at all other points of the body [25]. According to the nonlocal elasticity theory, this fact was attributed to the atomic theory of lattice dynamics and experimental measurements of phonon dispersion [26].

For homogenous and isotropic elastic continuum, the linear nonlocal elasticity theory can be expressed as the following set of equations [26]:

$$\sigma_{kl,k} + \rho(f_l - \ddot{u}_l) = 0 \tag{7a}$$

$$\sigma_{kl}(x) = \int_V \alpha(|x - x'|, \tau)\sigma^c_{kl}(x')dV \tag{7b}$$

$$\sigma^c_{kl}(x') = L_1 e_{rr}(x')\delta_{kl} + 2L_2 e_{kl}(x') \tag{7c}$$

$$e_{kl}(x') = \frac{1}{2}\left(\frac{du_k(x')}{dx'_l} + \frac{du_l(x')}{dx'_k}\right) \tag{7d}$$

where Eq. (7a) is the equilibrium relation in which $\sigma_{kl,l}$, ρ, f_l and u_l are the nonlocal stress tensor, mass density, body force density and displacement vector at a reference point x in the body, respectively. Eq. (7b) is the relation between local (σ^c_{kl}) and nonlocal $(\sigma_{kl,l})$ stress tensors using the nonlocal modulus $(\alpha(|x - x'|, \tau)$. Finally, Eqs. (7c) and (7d) are the classical constitutive stress-strain and strain-displacement relationships, respectively. L_1 and L_2 are the Lame constants.

Eringen [26] assumed that α can be Green's function of a linear differential operator as

$$L\alpha(|x - x'|, \tau) = \delta(|x - x'|) . \tag{8}$$

By applying L to the Eq. (7b), he could simplify it to a partial differential equation form as

$$(1 - \tau^2\ell^2\nabla^2)t_{kl}(x) = \sigma_{kl}(x), \quad \tau = e_0\frac{a}{\ell} \tag{9}$$

where $t_{kl} = \sigma_{kl,l}$, a/ℓ is the characteristic length ratio and e_0 is the nonlocal constant which are appropriate for the material. Eringen [26] found that the above assumption gives a perfect match of the description curve of one-dimensional plane waves based on the nonlocal elasticity and the Born-Karman model of the atomic lattice dynamics.

It is worth mentioning that the above differential equation reduces to the classical elasticity one by setting the nonlocal constant e_0 to zero.

3.2 Application of nonlocal elasticity on beam theories

3.2.1 Euler Bernoulli beam theory

By using Eq. (9), the only stress resultant for this beam theory (M) can be expressed in terms of components of displacement as follows:

$$M - \mu\frac{\partial^2 M}{\partial x^2} = -EI\frac{\partial^2 w}{\partial x^2} . \tag{10}$$

By substituting Eq. (10) into Eq. (3), the constitutive relation for nonlocal model of EBT is obtained as

$$(\mu P - EI)\frac{\partial^4 w}{\partial x^4} - P\frac{\partial^2 w}{\partial x^2} = 0 . \tag{11}$$

3.2.2 Timoshenko Bernoulli beam theory

The stress resultants for this beam theory (M and Q) can be obtained in terms of displacements using Eq. (9) as

$$Q - \mu\frac{\partial^2 Q}{\partial x^2} = \kappa GA\left(\varphi + \frac{\partial w}{\partial x}\right) \tag{12a}$$

$$M - \mu\frac{\partial^2 M}{\partial x^2} = EI\frac{\partial \varphi}{\partial x} . \tag{12b}$$

By substituting Eq. (12) into Eq. (5), the constitutive relations for nonlocal model of TBT can be expressed as

$$\mu P\frac{\partial^4 w}{\partial x^4} + (\kappa GA - P)\frac{\partial^2 w}{\partial x^2} + \kappa GA\frac{\partial \varphi}{\partial x} = 0 \tag{13a}$$

$$-\kappa GA\frac{\partial w}{\partial x} + EI\frac{\partial^2 \varphi}{\partial x^2} - \kappa GA\varphi = 0 . \tag{13b}$$

3.2.3 Levinson Bernoulli beam theory

By using Eq. (9), the stress resultants for this beam theory (M and Q) can be expressed in terms of displacements as follows:

$$Q - \mu\frac{\partial^2 Q}{\partial x^2} = \frac{2GA}{3}\left(\varphi + \frac{\partial w}{\partial x}\right) \tag{14a}$$

$$M - \mu\frac{\partial^2 M}{\partial x^2} = \frac{EI}{5}\left(4\frac{\partial \varphi}{\partial x} - \frac{\partial^2 w}{\partial x^2}\right) . \tag{14b}$$

By substituting Eq. (15) into Eq. (5), the constitutive relations for nonlocal model of LBT are obtained as

$$\mu P\frac{\partial^4 w}{\partial x^4} + \left(\frac{2GA}{3} - P\right)\frac{\partial^2 w}{\partial x^2} + \frac{2GA}{3}\frac{\partial \varphi}{\partial x} = 0 \tag{15a}$$

$$-\frac{EI}{5}\frac{\partial^3 w}{\partial x^3} - \frac{2GA}{3}\frac{\partial w}{\partial x} + \frac{4EI}{5}\frac{\partial^2 \varphi}{\partial x^2} - \frac{2GA}{3}\varphi = 0 . \tag{15b}$$

4. State-space method

4.1 Introduction

A very general approach that appears to be applicable for a large area of scientific research is the state-space method which was first presented by Kalman [27]. Although this method was originally developed as a model principally used in aerospace-related research, it has been applied to model in various fields of knowledge [28-33].

In the current study, state variables and matrix algebra are used to solve the coupled constitutive differential equations of equilibrium for different nonlocal elastic beam models and then four commonly used boundary conditions namely as simply supported-simply supported, clamped-clamped, clamped-simply supported and clamped-free are applied to obtain critical buckling loads of nanobeams corresponding to each one.

4.2 Application of state-space method on monlocal beam models

4.2.1 Euler-Bernoulli beam theory

In order to solve the constitutive differential equation of this type of beam theory, we introduce the state variables as below

$$w = Z_1, \quad \frac{\partial w}{\partial x} = Z_2$$
$$\frac{\partial^2 w}{\partial x^2} = Z_3, \quad \frac{\partial^3 w}{\partial x^3} = Z_4 . \tag{16}$$

It is worth mentioning that the order of the equilibrium equation of EBT is four, so we have used four state variables in the corresponding state-space model.

By considering the governing equation of equilibrium closely, it can be observed that it could be integrated with respect to x once, therefore from linear algebra it is known that in such conditions, the characteristic equation will result in repeated roots which makes the answer more complicated. We can use a method to avoid repeated eigenvalues. In this method, a term is added to the governing equation of equilibrium, so the equation cannot be integrated directly. In this case, a term εw is added to the equation to avoid direct integration.

It should be notified that must be small enough compared to the other coefficients in the governing equation so that it would not affect the final result.

Substituting the state variables into governing equilibrium Eq. (11), one can obtain

$$(\mu P - EI)\frac{dZ_4}{dx} - PZ_3 = 0 . \tag{17}$$

Also, we have three other equations as below:

$$Z_2 = \frac{dZ_1}{dx} \tag{18a}$$

$$Z_3 = \frac{dZ_2}{dx} \tag{18b}$$

$$Z_4 = \frac{dZ_3}{dx} \tag{18c}$$

which leads to a set of four coupled differential equations.

In order to solve this set of equations, matrix algebra is used. Eqs. (17) and (18) can be rearranged and rewritten in matrix form as follows:

$$[A]_{4\times4}\{Z'\}_{4\times1} + [B]_{4\times4}\{Z\}_{4\times1} = \{0\}_{4\times1} \tag{19}$$

where

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \mu P - EI \end{bmatrix} \tag{20a}$$

$$B = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ -\epsilon & 0 & -P & 0 \end{bmatrix} . \tag{20b}$$

Now we perform some matrix algebra on Eq. (19) in order to make it diagonal, so the four differential equations become uncoupled.

By multiplying Eq. (19) by $[A]^{-1}$ one can obtain

$$\{Z'\} + [A]^{-1}[B]\{Z\} = \{0\} . \tag{21}$$

By now, our problem is reduced to

$$\{Z'\} + [D]\{Z\} = \{0\} \tag{22}$$

where $[D] = [A]^{-1}[B]$.

Now we define u and λ as eigenvector and eigenvalue matrices of $[D]$. Therefore, we have

$$[D][u] = [u][\lambda] . \tag{23}$$

With the application of ε to Eq. (20b), it is evident that all eigenvalues are non-identical. So, we can multiply Eq. (23) by $[u]^{-1}$ to obtain

$$[u]^{-1}[D][u] = [\lambda] . \tag{24}$$

Therefore, one will have

$$[u]^{-1}[D] = [\lambda][u]^{-1} . \tag{25}$$

Now with multiplying Eq. (21) by $[u]^{-1}$, one can get

$$[u]^{-1}\{Z'\} + [u]^{-1}[D]\{Z\} = \{0\} . \tag{26}$$

According to Eq. (25), we can rewrite Eq. (26) as follows:

$$[u]^{-1}\{Z'\}+[\lambda][u]^{-1}\{Z\}=\{0\}\ . \tag{27}$$

By defining $\{F\}$ as $\{F\}=[u]^{-1}[Z]$, the above equation can be rewritten as

$$\{F'\}+[\lambda]\{F\}=0\ . \tag{28}$$

Thereupon, since $[\lambda]$ is a diagonal matrix, we have four uncoupled differential equations which can be solved easily. Each equation is in the form of the first-order differential equation, so we have

$$F_i(x)=c_i e^{-\lambda_i x} \qquad i=1,2,3,4\ . \tag{29}$$

Therefore, each state variable can be obtained as

$$Z_i=u_{i1}F_1+u_{i2}F_2+u_{i3}F_3+u_{i4}F_4=\sum_{j=1}^{4}u_{ij}c_je^{-\lambda_j x} \qquad i=1,2,3,4\ . \tag{30}$$

4.2.2 Timoshenko beam theory

The state variables corresponding to this type of beam theory are considered as

$$\begin{aligned} &w=Z_1, &&\frac{\partial w}{\partial x}=Z_2 \\ &\frac{\partial^2 w}{\partial x^2}=Z_3, &&\frac{\partial^3 w}{\partial x^3}=Z_4 \\ &\varphi=Z_5, &&\frac{\partial \varphi}{\partial x}=Z_6\ . \end{aligned} \tag{31}$$

It is worth mentioning that the total order of the equilibrium equations of TBT is six, so we have used six state variables in the corresponding state-space model.

Substituting the state variables into governing equilibrium Eq. (13), one can obtain

$$\mu PZ_4'+(\kappa GA-P)Z_3+\kappa GAZ_6=0 \tag{32a}$$

$$-\kappa GAZ_2+EIZ_6'-\kappa GAZ_5=0\ . \tag{32b}$$

Also, we have four other equations as below:

$$Z_2=\frac{dZ_1}{dx} \tag{33a}$$

$$Z_3=\frac{dZ_2}{dx} \tag{33b}$$

$$Z_4=\frac{dZ_3}{dx} \tag{33c}$$

$$Z_6=\frac{dZ_5}{dx} \tag{33d}$$

which leads to a set of six coupled differential equations.

In order to solve this set of equations, matrix algebra is used. Eqs. (32) and (32) can be rearranged and rewritten in the matrix form as follows:

$$[A]_{6\times6}\{Z'\}_{6\times1}+[B]_{6\times6}\{Z\}_{6\times1}=\{0\}_{6\times1} \tag{34}$$

where

$$A=\begin{bmatrix}1&0&0&0&0&0\\0&1&0&0&0&0\\0&0&1&0&0&0\\0&0&0&\mu P&0&0\\0&0&0&0&1&0\\0&0&0&0&0&EI\end{bmatrix} \tag{35a}$$

$$B=\begin{bmatrix}0&-1&0&0&0&0\\0&0&-1&0&0&0\\0&0&0&-1&0&0\\-\epsilon&0&\kappa GA-P&0&0&\kappa GA\\0&0&0&0&0&-1\\0&-\kappa GA&0&0&-\kappa GA&0\end{bmatrix}. \tag{35b}$$

The same matrix algebra is performed on Eq. (34), like in EBT, to make it diagonal which finally leads to the following equation for state variables

$$\begin{aligned} Z_i&=u_{i1}F_1+u_{i2}F_2+u_{i3}F_3+u_{i4}F_4+u_{i5}F_5+u_{i6}F_6 \\ &=\sum_{j=1}^{6}u_{ij}c_je^{-\lambda_j x} \qquad i=1,2,3,4,5,6\ . \end{aligned} \tag{36}$$

4.2.3 Levinson beam theory

Because the total order of equilibrium differential equations of LBT and TBT are the same, aforesaid state variables in TBT are used in LBT too. So, by substituting the state variables into governing equilibrium Eq. (15), we will have a set of six coupled differential equations as

$$\mu PZ_4'+\left(\frac{2GA}{3}-P\right)Z_3+\frac{2GA}{3}Z_6=0 \tag{37a}$$

$$-\frac{EI}{5}Z_4-\frac{2GA}{3}Z_2+\frac{4EI}{5}Z_6'-\frac{2GA}{3}Z_5=0 \tag{37b}$$

$$Z_2=\frac{dZ_1}{dx} \tag{37c}$$

$$Z_3=\frac{dZ_2}{dx} \tag{37d}$$

$$Z_4=\frac{dZ_3}{dx} \tag{37e}$$

$$Z_6=\frac{dZ_5}{dx}\ . \tag{37f}$$

In order to solve this set of equations, matrix algebra is used. Eq. (37) can be rearranged and rewritten in the matrix form as follows:

$$[A]_{6\times6}\{Z'\}_{6\times1}+[B]_{6\times6}\{Z\}_{6\times1}=\{0\}_{6\times1} \tag{38}$$

Table 1. Non-dimensional critical buckling load $(P \times L^2 / EI)$ for simply supported-simply supported nanobeams.

L/h	μ	EBT	TBT	LBT
10	0	9.8696	9.6357	9.6739
	0.5	9.4055	9.1825	9.2189
	1	8.9830	8.7701	8.8049
	1.5	8.5969	8.3931	8.4264
	2	8.2426	8.0472	8.0791
	2.5	7.9163	7.7287	7.7593
	3	7.6149	7.4344	7.4639
	3.5	7.3356	7.1617	7.1901
	4	7.0761	6.9084	6.9358
20	0	9.8696	9.8101	9.8199
	0.5	9.7493	9.6905	9.7003
	1	9.6319	9.5738	9.5835
	1.5	9.5174	9.4599	9.4695
	2	9.4055	9.3487	9.3581
	2.5	9.2962	9.2401	9.2494
	3	9.1894	9.1339	9.1431
	3.5	9.0850	9.0302	9.0393
	4	8.9830	8.9288	8.9378
50	0	9.8696	9.8600	9.8616
	0.5	9.8502	9.8406	9.8422
	1	9.8308	9.8213	9.8228
	1.5	9.8115	9.8020	9.8036
	2	9.7923	9.7828	9.7844
	2.5	9.7731	9.7637	9.7652
	3	9.7541	9.7446	9.7462
	3.5	9.7351	9.7256	9.7272
	4	9.7162	9.7067	9.7083

Table 2. Non-dimensional critical buckling load $(P \times L^2 / EI)$ for clamped-clamped nanobeams.

L/h	μ	EBT	TBT	LBT
10	0	39.4784	37.7718	37.9215
	0.5	37.6219	35.9952	36.1379
	1	35.9318	34.3786	34.5151
	1.5	34.3754	32.9009	33.0314
	2	32.9703	31.5450	31.6700
	2.5	31.6651	30.2963	30.4162
	3	30.4595	29.1426	29.2582
	3.5	29.3422	28.0740	28.1851
	4	28.3043	27.0808	27.1882
20	0	39.4784	39.1815	39.2206
	0.5	39.0176	38.7242	38.7634
	1	38.5478	38.2579	38.2967
	1.5	38.1824	37.8953	37.9338
	2	37.6417	37.3586	37.3962
	2.5	37.4033	37.1220	37.1594
	3	36.7769	36.5003	35.5371
	3.5	36.3590	36.0856	36.1220
	4	35.9508	35.6804	35.7163
50	0	39.4784	39.4393	39.4457
	0.5	39.4225	39.3835	39.3899
	1	39.3448	39.3058	39.3118
	1.5	39.2676	39.2287	39.2351
	2	39.1907	39.1519	39.1583
	2.5	39.1138	39.0751	39.0811
	3	39.0378	38.9992	39.0056
	3.5	38.9618	38.9232	38.9296
	4	38.8886	38.8501	38.8565

where

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mu P & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{4EI}{5} \end{bmatrix} \tag{39a}$$

$$B = \begin{bmatrix} 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ -\epsilon & 0 & \frac{2GA}{3} - P & 0 & 0 & \frac{2GA}{3} \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & -\frac{2GA}{3} & 0 & -\frac{EI}{5} & -\frac{2GA}{3} & 0 \end{bmatrix}. \tag{39b}$$

The same matrix algebra is performed on Eq. (38), like in TBT, to make it diagonal which finally leads to the following equation for state variables

$$Z_i = u_{i1}F_1 + u_{i2}F_2 + u_{i3}F_3 + u_{i4}F_4 + u_{i5}F_5 + u_{i6}F_6 = \sum_{j=1}^{6} u_{ij} c_j e^{-\lambda_j x} \qquad i = 1, 2, 3, 4, 5, 6 . \tag{40}$$

4.3 Boundary conditions

In order to find the unknown constants c_j, we impose boundary conditions, but at first the boundary conditions should be expressed in the form of state variables as follows:
For simply supported-simply supported boundary conditions:

$$\begin{aligned} &Z_1(0) = Z_1(L) = 0 \\ &M(0) = M(L) = 0 \\ &Z_3(0) = Z_3(L) = 0 \end{aligned} \tag{41}$$

For clamped-clamped boundary conditions:

$$\begin{aligned} &Z_1(0) = Z_1(L) = 0 \\ &Z_2(0) = Z_2(L) = 0 \\ &Z_5(0) = Z_5(L) = 0 \end{aligned} \tag{42}$$

For clamped-simply supported boundary conditions:

$$\begin{aligned} &Z_1(0) = Z_1(L) = 0 \\ &Z_3(L) = M(L) = 0 \\ &Z_2(0) = Z_5(0) = 0 \end{aligned} \tag{43}$$

Table 3. Non-dimensional critical buckling load $(P \times L^2 / EI)$ for clamped-simply supported nanobeams.

L/h	μ	*EBT*	*TBT*	*LBT*
10	0	20.1907	19.3179	19.3945
	0.5	19.2412	18.4093	18.4822
	1	18.3769	17.5825	17.6523
	1.5	17.5870	16.8267	16.8934
	2	16.8622	16.1333	16.1972
	2.5	16.1948	15.4946	15.5560
	3	15.5781	14.9047	14.9638
	3.5	15.0068	14.3580	14.4149
	4	14.4759	13.8501	13.9050
20	0	20.1907	20.0595	20.0796
	0.5	19.9545	19.8248	19.8449
	1	19.7142	19.5861	19.6059
	1.5	19.4798	19.3532	19.3728
	2	19.2508	19.1257	19.1449
	2.5	19.0271	18.9034	18.9225
	3	18.8085	18.6862	18.7050
	3.5	18.5948	18.4739	18.4924
	4	18.3860	18.2665	18.2848
50	0	20.1907	20.1707	20.1740
	0.5	20.1620	20.1420	20.1453
	1	20.1223	20.1024	20.1055
	1.5	20.0828	20.0629	20.0662
	2	20.0435	20.0237	20.0269
	2.5	20.0033	19.9835	19.9866
	3	19.9652	19.9454	19.9487
	3.5	19.9265	19.9068	19.9101
	4	19.8878	19.8681	19.8714

Table 4. Non-dimensional critical buckling load $(P \times L^2 / EI)$ for clamped-free nanobeams.

L/h	μ	*EBT*	*TBT*	*LBT*
10	0	2.4749	2.3679	2.3773
	0.5	2.3585	2.2565	2.2655
	1	2.2526	2.1552	2.1637
	1.5	2.1557	2.0626	2.0707
	2	2.0669	1.9775	1.9854
	2.5	1.9851	1.8993	1.9068
	3	1.9095	1.8270	1.8342
	3.5	1.8395	1.7599	1.7669
	4	1.7744	1.6977	1.7043
20	0	2.4749	2.4598	2.4622
	0.5	2.4459	2.4310	2.4334
	1	2.4164	2.4017	2.4041
	1.5	2.3877	2.3731	2.3755
	2	2.3596	2.3452	2.3456
	2.5	2.3322	2.3180	2.3203
	3	2.3054	2.2913	2.2936
	3.5	2.2792	2.2653	2.2675
	4	2.2536	2.2399	2.2422
50	0	2.4749	2.4724	2.4728
	0.5	2.4712	2.4688	2.4692
	1	2.4665	2.4641	2.4645
	1.5	2.4617	2.4593	2.4597
	2	2.4568	2.4544	2.4548
	2.5	2.4520	2.4496	2.4499
	3	2.4473	2.4449	2.4453
	3.5	2.4424	2.4400	2.4403
	4	2.4377	2.4353	2.4357

For clamped-free boundary conditions:

$$\begin{aligned} &Z_1(0) = Z_2(0) = 0 \\ &M(L) = Q(L) = 0 \\ &Z_4(0) = Z_5(0) = 0. \end{aligned} \tag{44}$$

The above conditions result in six equations in terms of c_j s. So all c_j s could be found using matrix algebra. It should be noted that in the case of Euler-Bernoulli beam theory, we need just four equations corresponding to each boundary condition. One can write these equations in the form of

$$[K]\{c_j\} = \{0\} \tag{45}$$
$$j = 1 \ to \ 4 \ \text{(for EBT)}, \quad j = 1 \ to \ 6 \ \text{(for TBT and LBT)}.$$

In order the Eq. (45) to have the non-obvious solution, the determinant of coefficient matrix *K* must vanish. As the components of *K* are in terms of critical buckling load (*P*), so the value of *P* can be found for each boundary conditions.

5. Numerical results and discussion

5.1 Selected numerical results

In this section, the numerical results for different boundary conditions using the analytical solution developed in the previous section are presented. The following parameters are used to derive the critical buckling loads:

$$E = 70GPa, \quad v = 0.23, \quad \rho = 2700Kg/m^3, \quad h = b = 1nm .$$

The non-dimensional buckling loads for simply supported-simply supported, clamped-clamped, simply supported-clamped, and clamped-free boundary conditions are given in Tables 1-4, respectively.

On the basis of these results, it can be observed that the critical buckling loads corresponding to all boundary conditions decrease with increasing the value of nonlocal parameter which shows this fact that with incorporating the nanoscale size-effects, the stiffness of nanobeam decreases. Moreover, the results show that by incorporating the effect of transverse shear strains, the values of critical buckling load will be reduced relevant to all types of boundary conditions.

Also, it can be found from the results that the effect of nonlocality is more significant for lower values of aspect ratio (length of the nanobeam), and this effect is very negligible for long nanobeams.

The effect of nonlocal parameter on the mode-shapes for the transverse displacement Z_1 and angular displacement Z_5 of

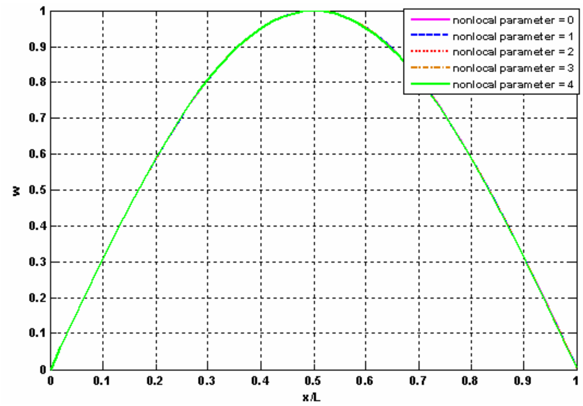

Fig. 2. The effect of nonlocal parameter on the mode-shape of transverse displacement for simply supported-simply supported nanobeam.

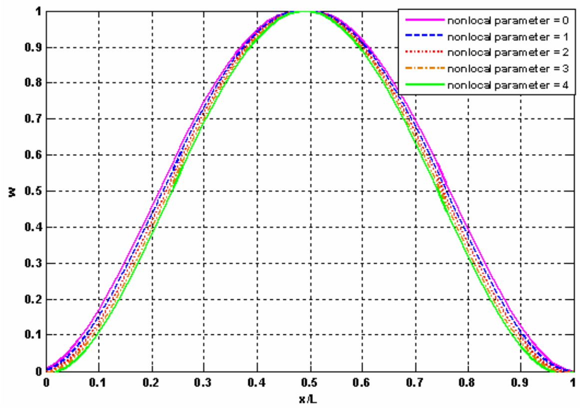

Fig. 3. The effect of nonlocal parameter on the mode-shape of transverse displacement for clamped-clamped nanobeam.

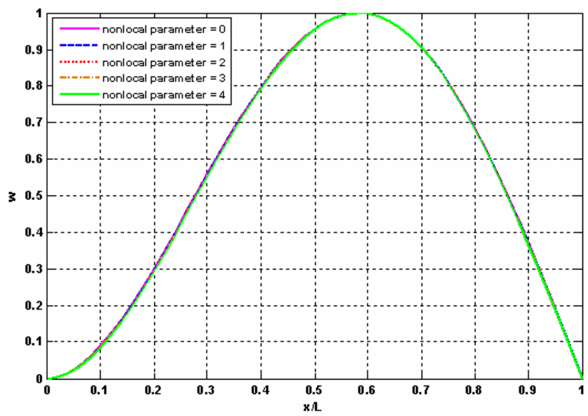

Fig. 4. The effect of nonlocal parameter on the mode-shape of transverse displacement for clamped-simply supported nanobeam.

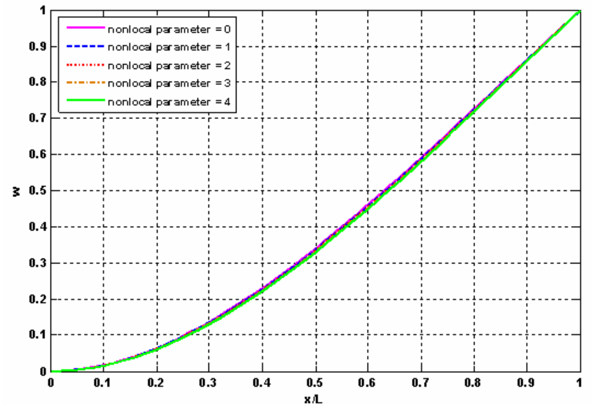

Fig. 5. The effect of nonlocal parameter on the mode-shape of transverse displacement for clamped-free nanobeam.

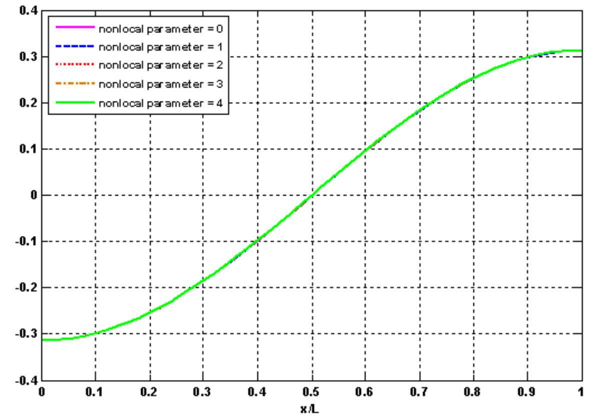

Fig. 6. The effect of nonlocal parameter on the mode-shape of angular displacement for simply supported-simply supported nanobeam.

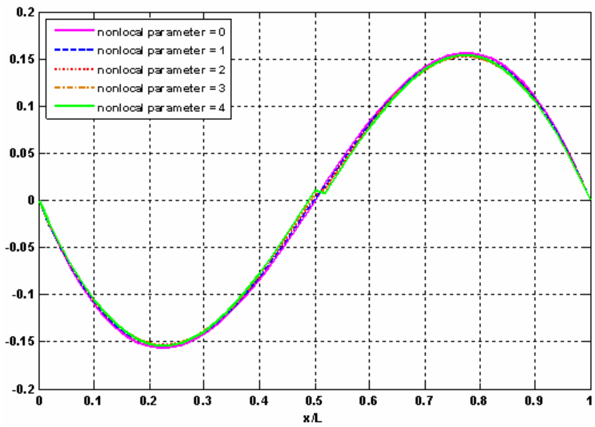

Fig. 7. The effect of nonlocal parameter on the mode-shape of angular displacement for clamped-clamped nanobeam.

Timoshenko nanobeam with $L/h=10$ are depicted in Figs. 2-9 for all sets of boundary conditions with the assumption that they are normalized such as $Z_{1max}=W_{max}=1$.

It can be seen that the nonlocal parameter has a very negligible influence on the mode-shapes of the simply supported-simply supported and clamped-simply supported nanobeams, but it has relatively more considerable effect in the cases of clamped-clamped and clamped-free boundary conditions.

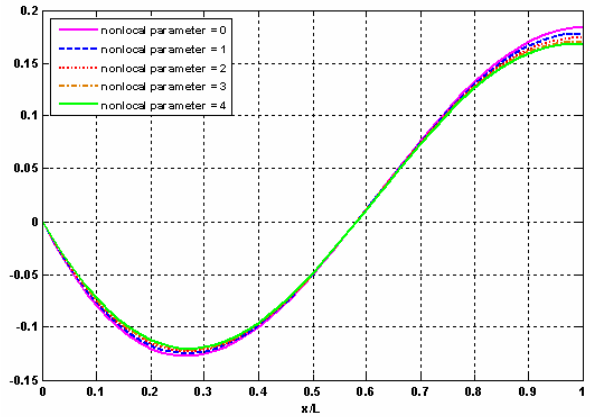

Fig. 8. The effect of nonlocal parameter on the mode-shape of angular displacement for clamped-simply supported nanobeam.

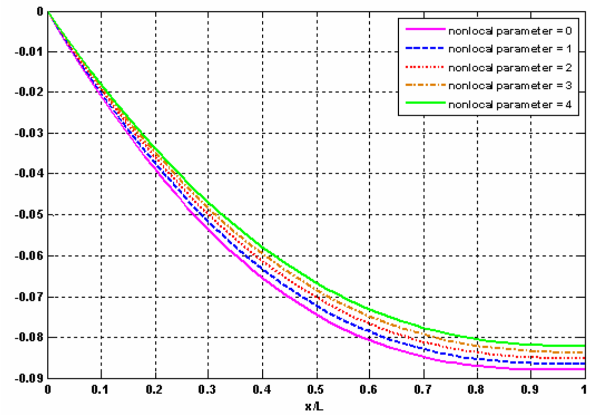

Fig. 9. The effect of nonlocal parameter on the mode-shape of angular displacement for clamped-free nanobeam.

5.2 Appropriate values of nonlocal parameter

An important issue in the application of the nonlocal elasticity models is the determination of the appropriate value of nonlocal elasticity parameter. Use of the nonlocal models is only beneficial if the correct value of nonlocal parameter is available.

In this work, the results of present nonlocal elastic beam models are matched with those of molecular dynamics simulations conducted by Ansari et al. [34] for a series of single-walled carbon nanotubes with different boundary conditions.

Through using a nonlinear least square fitting procedure, the Euclidean norm of the difference between the two series of results is minimized in which the value of nonlocal parameter is set as the optimization variable to obtain the consistent values of it corresponding to each type of nonlocal beam model and boundary conditions. It is assumed that the nanobeams have circular cross-section of diameter D and effective tube thickness h.

The values of μ obtained from the matching procedure for both armchair and zigzag nanotubes modeled as nanobeams are given in Table 8 relevant to various nonlocal beam models and boundary conditions. It can be found that chirality does not have a considerable influence on the appropriate value of nonlocal parameter. However, boundary conditions and the type of nonlocal elastic beam model play important roles in the correct value of nonlocal parameter. Fig. 10 shows that there is a very good agreement between the critical buckling loads predicted by the nonlocal elastic beam models with their recommended values of nonlocal parameter and those of molecular dynamics simulation.

Table 5. Appropriate values of nonlocal parameter corresponding to different nonlocal beam theories and boundary conditions.

Boundary conditions	EBT	TBT	LBT
S-S	1.87	0.72	1.06
C-C	2.03	0.88	1.22
C-S	1.96	0.81	1.15
C-F	2.05	0.89	1.24

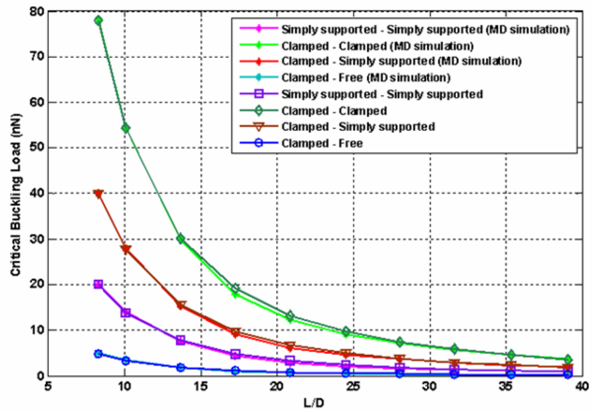

Fig. 10. Comparison of critical axial buckling loads of nanotubes predicted by nonlocal elastic beam models and molecular dynamics simulation corresponding to different boundary conditions.

6. Conclusions

State-space method was used to study the buckling behavior of nanobeams with four commonly used boundary conditions namely as simply supported-simply supported, clamped-clamped, clamped-simply supported, and clamped-free.

Eringen's nonlocal constitutive equations were incorporated into the different classical beam theories to develop nonlocal elastic beam models. The governing differential equations were then solved using state variables and matrix algebra to bring out the effects of boundary conditions, aspect ratio and value of nonlocal parameter on buckling response of nanobeams, separately. The present method appears to be attractive due to its accuracy as well as its extreme simplicity form.

Then the results obtained by the nonlocal elastic beam models were matched with those of molecular dynamics simulation presented by Ansari et al. [34] to find the appropriate values of nonlocal parameter for predicting the critical buckling loads of nanotubes modeled as nanobeams corresponding to each type of nonlocal elastic beam model and boundary conditions. It was observed that in contrast to the chirality, the types of nonlocal elastic beam model and boundary conditions make significant difference between the appropriate values of nonlocal parameter extracted for each one.

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