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Nonlocal Cauchy Problem for Impulsive Differential Equations in Banach Spaces

Shaochun Ji[1] *, Shu Wen[2]
[1,2]Faculty of Mathematics and Physics, Huaiyin Institute of Technology, Huaian, Jiangsu 223003, P. R. China.
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Abstract: The paper is concerned with the existence of mild solutions for impulsive differential equations with nonlocal conditions in Banach spaces. The results are obtained under the conditions in respect of the Hausdorff measure of noncompactness. Since we do not assume the compactness of semigroup $T(t)$ and f, our theorems extend some existing results in this area.

Keywords: impulsive differential equations; nonlocal conditions; the Hausdorff measure of noncompactness; fixed point; mild solution

1 Introduction

In this paper we discuss the impulsive Cauchy problem with nonlocal conditions

$$u'(t) = Au(t) + f(t, u(t)), \quad t \in [0, b], t \neq t_i, i = 1, 2, \cdots, p \tag{1.1}$$

$$\Delta u(t_i) = u(t_i^+) - u(t_i^-) = I_i(u(t_i)), \quad i = 1, 2, \cdots, p \tag{1.2}$$

$$u(0) = g(u) + u_0 \tag{1.3}$$

where A is the infinitesimal generator of a strongly continuous semigroup of bounded linear operators $T(t)$ on a Banach space X; $f : [0, b] \times X \to X$; $0 < t_1 < t_2 < \cdots < t_p < t_{p+1} = b$; $I_i : X \to X, i = 1, 2, \cdots, p$ are impulsive functions and $g : PC([0, b]; X) \to X$.

During recent years, the impulsive differential equations have been an object of intensive investigation because of the wide possibilities for their application in various fields of science and technology as theoretical physics, population dynamics, economics,etc. See [1, 4, 16] and the references therein for more comments.

The existence and uniqueness of mild, strong and classical solution of nonlocal abstract Cauchy problem has been established by Byszewski [6, 7]. Subsequently, many authors are devoted to studying of nonlocal problems. Some papers have been written on various classes of differential equations [2, 8, 9, 18–21].

In this paper we will derive some sufficient conditions for the solution of differential equation (1.1)-(1.3),combining impulsive conditions and nonlocal conditions. Our results are achieved by applying the Hausdorff measure of noncompactness and fixed point theorem. Neither the semigroup $T(t)$ nor the function f is needed to be compact in our results. So our work extends and improves many main results such as those in [1, 10, 13, 15].

2 Preliminaries

Let$(X, \|\cdot\|)$ be a real Banach space. We denote by $C([0, b]; X)$ the space of X-valued continuous functions on $[0, b]$ with the norm $\|x\| = \sup\{\|x(t)\|, t \in [0, b]\}$ and by $L^1(0, b; X)$ the space of X-valued Bochner integrable functions on $[0, b]$ with the norm $\|f\|_{L^1} = \int_0^b \|f(t)\| \, dt$.

For the sake of simplicity, we put $J = [0, b]$;$J_0 = [0, t_1]$; $J_i = (t_i, t_{i+1}]$,$i = 1, 2, \cdots, p$. In order to define the mild solution of problem (1.1)-(1.3), we introduce the set $PC([0, b]; X) = \{u : [0, b] \to X : u \text{ is continuous on } J_i, i = 0, 1, 2, \cdots, p \text{ and the right limit } u(t_i^+) \text{ exists}, i = 1, 2, \cdots, p\}$. It is easy to verify that $PC([0, b]; X)$ is a Banach space with the norm $\|u\|_{PC} = \sup\{\|u(t)\|, t \in [0, b]\}$

*Corresponding author. *E-mail address*: jiscmath@yahoo.com.cn

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Definition 2.1 *A function $u \in PC([0,b];X)$ is a mild solution of (1.1)-(1.3) if:*

$$u(t) = T(t)(u_0 + g(u)) + \int_0^t T(t-s)f(s,u(s))\,\mathrm{d}s + \sum_{0<t_i<t} T(t-t_i)I_i(u(t_i))$$

for all $t \in [0,b]$.

The Hausdroff's measure of noncompactness β_Y is defined by

$$\beta_Y(B) = \inf\{r > 0, B\ can\ be\ covered\ by\ finite\ number\ of\ balls\ with\ radii\ r\}$$

for bounded set B in a Banach space Y.

Lemma 2.1 ([3]) *Let Y be a real Banach space and $B, C \subseteq Y$ be bounded, the following properties are satisfied:*
(1) B is pre-compact if and only if $\beta_Y(B) = 0$;
(2) $\beta_Y(B) = \beta_Y(\overline{B}) = \beta_Y(convB)$, where $\overline{B}$ and $convB$ mean the closure and convex hull of B respectively;
(3) $\beta_Y(B) \le \beta_Y(C)$, where $B \subseteq C$;
(4) $\beta_Y(B+C) \le \beta_Y(B) + \beta_Y(C)$, where $B + C = \{x + y : x \in B, y \in C\}$;
(5) $\beta_Y(B \cup C) \le \max\{\beta_Y(B), \beta_Y(C)\}$;
(6) $\beta_Y(\lambda B) \le |\lambda|\beta_Y(B)$ for any $\lambda \in R$;
(7) If the map $Q : D(Q) \subseteq Y \to Z$ is Lipschitz continuous with constant k, then $\beta_Z(QB) \le k\beta_Y(B)$ for any bounded subset $B \subseteq D(Q)$, where Z is a Banach space;
(8) $\beta_Y(B) = \inf\{d_Y(B,C); C \subseteq Y$ is precompact $\} = \inf\{d_Y(B,C); C \subseteq Y$ is finite valued$\}$, where $d_Y(B,C)$ means the nonsymmetric (or symmetric) Hausdorff distance between B and C in Y;
(9) If $\{W_n\}_{n=1}^{+\infty}$ is decreasing sequence of bounded closed nonempty subsets of Y and $\lim_{n\to\infty}\beta_Y(W_n) = 0$, then $\bigcap_{n=1}^{+\infty} W_n$ is nonempty and compact in Y.

The map $Q : W \subseteq Y \to Y$ is said to be a β_Y-contraction if there exists a positive constant $k < 1$ such that $\beta_Y(Q(B)) \le \beta_Y(B)$ for any bounded closed subset $B \subseteq W$, where Y is a Banach space.

Lemma 2.2 (Darbo-Sadovskii [3]) *If $W \subseteq Y$ is bounded closed and convex, the continuous map $Q : W \to W$ is a β_Y-contraction, then the map Q has at least one fixed point in W.*

In this paper we denote by β the Hausdorff measure of noncompactness of X and denote β_{PC} by the Hausdorff measure of noncompactness of $PC([0,b];X)$. To discuss the existence, we need the following lemmas in this paper.

Lemma 2.3 ([12]) *If $\{u_n\}_{n=1}^{\infty} \subset L^1(a,b;X)$ is uniformly integrable, then $\beta(\{u_n(t)\}_{n=1}^{\infty})$ is measurable and*

$$\beta\Big(\Big\{\int_0^t u_n(s)ds\Big\}_{n=1}^{\infty}\Big) \le 2\int_0^t \beta(\{u_n(s)\}_{n=1}^{\infty})ds.$$

Lemma 2.4 ([3]) *If $W \subseteq C([0,b];X)$ is bounded, then $\beta(W(t)) \le \beta_C(W)$ for all $t \in [0,b]$, where $W(t) = \{u(t); u \in W\} \subseteq X$. Furthermore if W is equicontinuous on $[a,b]$, then $\beta(W(t))$ is continuous on $[a,b]$ and $\beta_C(W) = \sup\{\beta(W(t)), t \in [a,b]\}$.*

Let $C((a,b];X) = \{u : (a,b] \to X : u\ is\ continuous\ on\ (a,b]\ and\ the\ right\ limit\ u(a^+)\ exists\}$. Similarly to the proof of Lemma 2.4(see [3] Theorem 11.3), we can obtain

Lemma 2.5 ([3]) *If $W \subseteq C((a,b];X)$ is bounded, then $\beta(W(t)) \le \beta_C(W)$ for all $t \in (a,b]$, where $W(t) = \{u(t); u \in W\} \subseteq X$. Furthermore if W is equicontinuous on $(a,b]$, then $\beta_C(W) = \sup\{\beta(W(t)),\ t \in (a,b]\}$.*

Now we shall show that the result is also true in $PC([0,b];X)$. It plays an important role in the proof of the existence of mild solutions for the impulsive differential equation.

Lemma 2.6 *If $D \subseteq PC([0,b];X)$ is bounded, then $\beta(D(t)) \le \beta_{PC}(D)$ for all $t \in [0,b]$, where $D(t) = \{u(t); u \in W\} \subseteq X$.*

Furthermore if D is equicontinuous on each interval J_i of $[0,b]$, $i = 1,2,\cdots,p$, then $\beta_{PC}(D) = \sup\{\beta(D(t)),\ t \in [0,b]\}$.

Proof. For arbitrary $\varepsilon > 0$, there exists $D_i \subseteq PC([0,b];X)$, $1 \le i \le p$, such that $D = \cup_{i=1}^{n} D_i$ and

$$diam_{PC}(D_i) \le 2\beta_{PC}(D) + 2\varepsilon, \quad i = 1,\ 2,\ \cdots,\ n.$$

In this proof we denote $diam_{PC}(\ \cdot\ )$ by the diameter of a bounded set in $PC([0,b];X)$ and denote $diam(\ \cdot\ )$ by the diameter of a bounded set in X.

Now we have $D(t) = \bigcup_{i=1}^{n} D_i(t)$ for each $t \in [a,b]$, and

$$\|x(t) - y(t)\| \le\ \|x - y\|_{PC} \le\ diam_{PC}(D_i)$$

for $x, y \in D_i$. From the above two inequalities, it follows that

$$2\beta(D(t)) \le\ diam(D_i(t)) \le\ diam_{PC}(D_i) \le\ 2\beta_{PC}(D) + 2\varepsilon$$

By the arbitrariness of ε, we get that $\beta(D(t)) \le \beta_{PC}(D)$. Therefore, we have

$$\sup_{t\in[0,b]} \beta(D(t)) \le \beta_{PC}(D).$$

Next, if D is equicontinuous on each J_i, $i = 1, 2, \cdots, p$, we show that $\beta_{PC}(D) \le \sup_{t\in[0,b]} \beta(D(t))$. We denote $D|_{J_i}$ by the restriction of D on J_i, $i = 1,\ 2,\ \cdots,\ p$. By the Lemma 2.4 and Lemma 2.5, it follows that $\beta_C(D|_{J_i}) = \sup_{t\in J_i} \beta(D(t))$. We only need to prove that $\beta_{PC}(D) \le \max_{0\le i\le p} \beta_C(D|_{J_i})$.

For arbitrary $\varepsilon > 0$, there exists $D_{ij} \subseteq C(J_i;X)$, $j = 1, 2, \cdots, n_i$, such that $D|_{J_i} = \bigcup_{j=1}^{n_i} D_{ij}$ and $diam_C(D_{ij}) \le 2\beta_C(D|_{J_i}) + 2\varepsilon$, for each $i = 0, 1,\ \cdots,\ p$. Let

$$K = \{\mu : \mu(i) \in \{1, 2, \cdots, n_i\}, i = 0, 1, 2, \cdots, p\}.$$

Obviously K is a finite set with element number $n = n_0 \times n_1 \times \cdots \times n_p$. Denoted by

$$D_\mu = \{x \in PC([0,b];X) :\ x|_{J_i} = D_{i,\mu(i)},\ i = 0, 1, 2, \cdots, p\},$$

which provides $D \subseteq \bigcup_\mu D_\mu = \bigcup_{i=1}^{n} D_i$. Notice that

$$\begin{aligned} diam_{PC} D_i &= \sup_{x,y\in D_i} \|x - y\|_{PC} = \sup_{x,y\in D_i} \max_{i=0,1,2,\cdots,p} \|x|_{J_i} - y|_{J_i}\| \\ &= \max_{i=0,1,2,\cdots,p} \sup_{x,y\in D_i} \|x|_{J_i} - y|_{J_i}\| \le \max_{i=0,1,2,\cdots,p} (2\beta(D|_{J_i}) + 2\varepsilon). \end{aligned}$$

By the arbitrariness of ε, we get

$$diam_{PC}(D_i) \le 2 \max_{i=0,1,2,\cdots,p} \beta(D|_{J_i}).$$

So we have

$$\beta_{PC}(D) \le \beta_{PC}(\bigcup_{i=1}^{n} D_i) \le \max_{i=0,1,2,\cdots,p} \beta(D|_{J_i}) = \sup_{t\in[a,b]} \beta(D(t).$$

This completes the proof. ■

The C_0-semigroup $T(t)$ is said to be equicontinuous if $t \to \{T(t)x :\ x \in B\}$ is equicontinuous for $t > 0$ and for all bounded subset B in X.

The following lemma is obvious.

Lemma 2.7 *If the semigroup $T(t)$ is equicontinuous and $\eta \in\ L(0,b;R^+)$, then the set $\{\int_0^t T(t-s)u(s)ds, \|u(s)\| \le \eta(s)$ for a.e. $s \in [0,b]\}$ is equicontinuous for $t \in [0,b]$.*

In section 3, we give some existence results when g is compact and f satisfies the conditions with the Hausdorff measure of noncompactness. In section 4, we use the different method to discuss the case when g is Lipschitz continuous and f satisfies the conditions with the Hauadorff measure of noncompactness. In this paper, we denote by $M = \sup\{\|T(t)\| : t \in [0,b]\}$. Without loss of generality, we let $u_0 = 0$.

3 g is compact

In this Section, we give the existence of the mild solutions for nonlocal Cauchy problem (1.1)-(1.3).

We first give the following hypotheses:

(A) The C_0 semigroup $T(t)$, $0 \leq t \leq b$, generated by A is equicontinuous.

$(g1)$ $g : PC([0,b];X) \to X$ is continuous and compact.

$(g2)$ There exists a constant N such that $\|g(x)\| \leq N$, for all $x \in PC([0,b];X)$.

(I) Let $I_i : X \to X$, be continuous, compact map and there are nondecreasing functions $l_i : R^+ \to R^+$,satisfying $\|I_i(x)\| \leq l_i(\|x\|)$, $i = 1, 2, \cdots, p$.

$(f1)$ $f : [0,b] \times X \to X$, for a.e. $t \in [0,b]$, the function $f(t,\cdot) : X \to X$ is continuous and for all $x \in X$, the function $f(\cdot, x) : [0,b] \to X$ is measurable.

$(f2)$ There exists a function $\theta :\ [0,b]\times R^+ \to R^+$ such that $\theta(\cdot, s) \in L(0,b;R^+)$ for every $s \geq 0$, $\theta(t,\cdot)$ is continuous and increasing for a.e. $t \in [0,b]$, and $\|f(t,x)\| \leq \theta(t, \|x\|)$ for a.e. $t \in [0,b]$ and all $x \in X$. And there exists at least one mild solution to the following scalar equation

$$m(t) = MN + M\int_0^t \theta(s, m(s))ds + M\sum_{i=1}^{p} l_i(m(t)),\ \ t \in [0,b]. \tag{3.1}$$

$(f3)$ there exists a function $h \in L^1(0,b;R^+)$ such that for every bounded $D \subset X$,

$$\beta(f(t,D)) \leq h(t)\beta(D),$$

for a.e. $t \in [0,b]$.

Theorem 3.1 *Assume that the hypotheses (A), $(g1)-(g2)$, (I), $(f1)-(f3)$, are satisfied, then the nonlocal impulsive problem (1.1)-(1.3) has at least one mild solution.*

Proof. Let $m(t)$ be a solution of the scalar equation (3.1), the map $K : PC([0,b];X) \to PC([0,b];X)$ defined by

$$(Ku)(t) = (K_1u)(t) + (K_2u)(t),$$

with

$$(K_1u)(t) = T(t)g(u) + \int_0^t T(t-s)f(s,u(s))ds,$$

$$(K_2u)(t) = \sum_{0<t_i<t} T(t-t_i)I_i(u(t_i)),$$

for all $t \in [0,b]$.

It is easy to see that the fixed point of K is the mild solution of nonlocal impulsive problem (1.1)-(1.3). Subsequently, we will prove that K has a fixed point by using the Schauder fixed point theorem.

From our hypotheses we can get K is continuous on $PC([0,b];X)$. For this purpose, we assume that $u_n \to u$ in $PC([0,b];X)$. By $(f1)$ we have that

$$f(s,u_n(s)) \to f(s,u(s)),\ \ (n \to +\infty),\quad for\ all\ s \in [0,b].$$

Then by hypotheses (g) and $(I1)$, we have

$$\begin{aligned}\|Ku_n - Ku\|_{PC} \ \ &\leq\ \ M\|g(u_n) - g(u)\| + M\int_0^b \|f(s,u_n(s)) - f(s,u(s))\|ds \\ &\quad + \sum_{i=1}^{p} M\|I_i(u_n(t_i)) - I_i(u(t(t_i)))\| \to 0, \qquad as\ \ n \to +\infty,\end{aligned}$$

i.e., K is continuous.

We denote by $W_0 = \{u \in PC([0,b];X), \|u(t)\| \leq m(t)$ for all $t \in [0,b]\}$. Then $W_0 \subseteq PC([0,b];X)$ is bounded and convex.

Define $W_1 = \overline{\text{conv}}K(W_0)$, where $\overline{\text{conv}}$ means the closure of the convex hull in $PC([0,b];X)$. For any $u \in K(W_0)$, we know

$$\|u(t)\| \leq MN + M\int_0^t \theta(s,m(s))ds + M\sum_{i=1}^{p} l_i(m(t)) = m(t)$$

for $t \in [0,b]$. From $(f2)$, it follows that $W_1 \subset W_0$.

W_1 is equicontinuous on each interval J_i of $[0,b]$. In fact, as $T(t)$ is equicontinuous, g is compact and $W_0 \subseteq PC([0,b];X)$ is bounded, due to hypotheses $(f2)$ and Lemma 2.7, $\{K_1 u : u \in W_0\}$ is equicontinuous. Next, for $t_i \leq t < t+h \leq t_{i+1}$, $i = 1,2,\cdots,p$, we have, using the semigroup properties,

$$\begin{aligned}\|(K_2u)(t+h) - (K_2u)(t)\| &\leq \|\sum_{0<t_i<t+h} T(t+h-t_i)I_i(x(t_i)) - \sum_{0<t_i<t} T(t+h-t_i)I_i(x(t_i))\| \\ &\quad + \|\sum_{0<t_i<t} T(t+h-t_i)I_i(x(t_i)) - \sum_{0<t_i<t} T(t-t_i)I_i(x(t_i))\| \\ &\leq \|\sum_{t<t_i<t+h} T(t+h-t_i)I_i(x(t_i))\| \\ &\quad + \sum_{0<t_i<t} \|T(t+h-t_i)I_i(x(t_i)) - T(t-t_i)I_i(x(t_i))\|,\end{aligned}$$

which follows that $\{K_2 u : u \in W_0\}$ is equicontinuous on each J_i due to the equicontinuous of $T(t)$ and hypotheses (I). Therefore, $W_1 \subset PC([0,b];X)$ is bounded closed convex nonempty and equicontinuous on each interval J_i, $i = 0,1,2,\cdots,p$.

We define $W_{n+1} = \overline{\text{conv}}K(W_n)$, for $n = 1,2,\ldots,p$. Form above we know that $\{W_n\}_{n=1}^{\infty}$ is a decreasing sequence of bounded, closed, convex, nonempty subsets in $PC([0,b];X)$ and equicontinuous on each J_i, $i = 0,1,2,\cdots,p$.

Now for $n \geq 1$ and $t \in [0,b]$, $W_n(t)$ and $K(W_n(t))$ are bounded subsets of X. Hence, for any $\varepsilon > 0$, there is a sequence $\{u_k\}_{k=1}^{\infty} \subset W_n$ such that (see, e.g. [5], pp.125)

$$\begin{aligned}\beta(W_{n+1}(t) &= \beta(KW_n(t)) \\ &\leq 2\beta(T(t)g(\{u_k\}_{k=1}^{\infty})) + 2\beta(\int_0^t T(t-s)f(s,\{u_k(s)\}_{k=1}^{\infty})\mathrm{d}s) \\ &\quad +2\sum_{i=1}^{p}\beta(T(t-t_i)I_i(\{u_k(t_i)\}_{k=1}^{\infty})) + \varepsilon\end{aligned}$$

for $t \in [0,b]$. From the compactness of g and I_i, by Lemma 2.1, Lemma 2.3 and $(f3)$, we have that

$$\begin{aligned}\beta(W_{n+1}(t) &\leq 2\beta(\int_0^t T(t-s)f(s,\{u_k(s)\}_{k=1}^{\infty})\mathrm{d}s) + \varepsilon \\ &\leq 4\int_0^t \beta(T(t-s)f(s,\{u_k(s)\}_{k=1}^{\infty}))\mathrm{d}s + \varepsilon \\ &\leq 4M\int_0^t \beta(f(s,W_n(s)))\mathrm{d}s + \varepsilon \\ &\leq 4M\int_0^t h(s)\beta(W_n(s))\mathrm{d}s + \varepsilon\end{aligned}$$

for $t \in [0,b]$. Since $\varepsilon > 0$ is arbitrary, it follows from the above inequality that

$$\beta(W_{n+1}(t)) \leq 4M\int_0^t h(s)\beta(W_n(s))ds, \tag{3.2}$$

for all $t \in [0,b]$. Because W_n is decreasing for n, we define

$$\eta(t) = \lim_{n\to\infty}\beta(W_n(t))$$

for all $t \in [0,b]$. We obtain from (3.2) that

$$\eta(t) \leq 4M\int_0^t h(s)\eta(s)ds,$$

for all $t \in [0, b]$, which implies that $\eta(t) = 0$ for all $t \in [0, b]$.

By Lemma 2.6, we know that

$$\lim_{n\to\infty} \beta_{PC}(W_n) = 0.$$

Using Lemma 2.1, we know that $W = \bigcap_{n=1}^{\infty} W_n$ is convex compact and nonempty in $PC([0, b]; X)$ and $K(W) \subset W$. By the famous Schauder's fixed point theorem, there exists at least one mild solution u of the problem (1.1)-(1.3), where $u \in W$ is a fixed point of the continuous map K. ■

Remark 3.1 *In the case where f is compact or Lipschitz continuous(see, e.g., [4, 13, 21]), the hypotheses $(f3)$ is automatically satisfied.*

In some of the early related results we suppose that the map g is uniformly bounded when g is compact. Here we give the existence under another growth conditions of f when g is not uniformly bounded. We replace the hypothesis $(f2)$ by

$(f2')$ There exists a function $\theta \in L^1(0, b; R^+)$ and an increasing continuous function $\Omega : R^+ \to R^+$ such that $\|f(t, x)\| \le \theta(t)\Omega(\|x\|)$ for a.e. $t \in [0, b]$ and all $x \in X$.

Theorem 3.2 *Assume that the hypotheses (A), $(g1)$, (I), $(f1)(f2')(f3)$, are satisfied, then the nonlocal impulsive problem (1.1)-(1.3) has at least one mild solution if*

$$\limsup_{\lambda\to\infty} \frac{M\gamma(\lambda) + M\Omega(\lambda)\int_0^b \theta(s)ds + M\sum_{i=1}^{p} l_i(\lambda)}{\lambda} < 1, \tag{3.3}$$

where $\gamma(\lambda) = \sup\{\|g(u)\| : \|u\| \le \lambda\}$.

Proof. The inequality(3.3) implies that there exists a constant $\lambda > 0$ such that

$$M\gamma(\lambda) + M\Omega(\lambda)\int_0^b \theta(s)ds + M\sum_{i=1}^{p} l_i(\lambda) \le \lambda$$

Just as the proof of Theorem 3.1, let $W_0 = \{u \in PC([0, b]; X),\ \|u(t)\| \le \lambda \text{ for all } t \in [0, b]\}$ and $W_1 = \overline{conv}K(W_0)$. Then for any $u \in W_1$, we know

$$\|u(t)\| \le M\gamma(\lambda) + M\Omega(\lambda)\int_0^b \theta(s)ds + M\sum_{i=1}^{p} l_i(\lambda) \le \lambda,$$

for $t \in [0, b]$. It means that $W_1 \subset W_0$. So we can complete the proof similarly to Theorem 3.1. ■

4 g is Lipschitz

In the previous section, we obtained the existence results when g is compact but without the compactness of $T(t)$ or f. In this section, we discuss the problem (1.1)-(1.3) when g is Lipschitz continuous and I_i, $i = 1, 2, \cdots, p$ is not compact. Precisely, we replace hypotheses $(g1)$, $(I)$by

$(g1')$ There is a constant $L \in (0, 1/M)$ such that

$$\|g(u) - g(v)\| \le L\|u - v\|_{PC},$$

for all $u, v \in PC([0, b]; X)$.

(I') There exists $L_i > 0$, $i = 1, 2, \cdots, p$, such that $\|I_i(x) - I_i(y)\| \le L_i\|x - y\|$, for all $x, y \in X$.

Theorem 4.1 *Assume that the hypotheses (A), $(g1') - (g2)$, (I'), $(f1) - (f3)$ are satisfied. Then the nonlocal impulsive problem (1.1)-(1.3) has at least one mild solution on $[0, b]$ provided that*

$$ML + 4M\int_0^b h(s)ds + 2M\sum_{i=1}^{p} L_i < 1. \tag{4.1}$$

Proof. We define $K_1, K_2 : PC([0,b];X) \to PC([0,b];X)$ by

$$(K_1x)(t) = T(t)g(x),$$

$$(K_2x)(t) = \int_0^t T(t-s)f(s,u(s))ds + \sum_{0<t_i<t} T(t-t_i)I_i(u_{t_i}).$$

On account of Theorem 3.1, we know that operator $K = K_1 + K_2$ is continuous on $PC([0,b];X)$. We define $W_0 = \{u \in PC([0,b];X) : \|u(t)\| \leq m(t) \ \ for\ all\ t \in [0,b]\}$, and let $W = \overline{\text{conv}} KW_0$. Then from the proof of Theorem 3.1 we know that W is a bounded closed convex and equicontinuous subset of $PC([0,b];X)$ and $KW \subset W$. We shall prove that K is β_{PC}-contraction on W. Then Darbo-Sadovskii fixed point theorem can be used to get a fixed point of K in W, which is a mild solution of (1.1)-(1.3).

We first show that K_1 is Lipschitz on $PC([0,b];X)$. In fact, take $x, y \in PC([0,b];X)$ arbitrary. Then by $(g1')$ we have

$$\|(K_1u)(t) - (K_1v)(t)\| \leq M\|g(u) - g(v)\| \leq ML\|u - v\|_{PC}$$

for $t \in [0,b]$. It follows that

$$\|K_1u - K_1v\|_{PC} \leq ML\|u - v\|_{PC} \tag{4.2}$$

for all $u, v \in PC([0,b];X)$, i.e., K_1 is Lipschitz with Lipschitz constant ML.

Next,let $B \subset W$ be bounded. For any $\varepsilon > 0$, there is a sequence $\{u_k\}_{k=1}^{\infty} \subset B$ such that

$$\beta(K_2B(t)) \leq 2\beta(\{K_2u_k(t)\}_{k=1}^{\infty}) + \varepsilon$$

for $t \in [0,b]$. Noticing that B and K_2B are equicontinuous, we can get, from Lemma 2.1, Lemma 2.3 and $(f3)$, that

$$\begin{aligned}
\beta((K_2B)(t)) \ &\leq \ 2\beta(\int_0^t T(t-s)f(s,\{u_k(s)\}_{k=1}^{\infty})\mathrm{d}s) \\
&\quad +2\beta(\{\sum_{i=1}^{p} T(t-t_i)I_i(u_k(t_i))\}_{k=1}^{\infty}) + \varepsilon \\
&\leq \ 4M\int_0^b \beta(f(s,\{u_k(s)\}_{k=1}^{\infty}))\mathrm{d}s \\
&\quad +2M\sum_{i=1}^{p} \beta(I_i(\{u_k(t_i)\}_{i=1}^{\infty})) + \varepsilon \\
&\leq \ 4M\int_0^b h(s)\beta_{PC}(\{u_k\}_{k=1}^{\infty})\mathrm{d}s \\
&\quad +2M\sum_{i=1}^{p} L_i\beta_{PC}(\{u_k\}_{k=1}^{\infty}) + \varepsilon \\
&\leq \ (4M\int_0^b h(s)\mathrm{d}s + 2M\sum_{i=1}^{p} L_i)\beta_{PC}(B) + \varepsilon
\end{aligned}$$

for $t \in [0,b]$. Since $\varepsilon > 0$ is arbitrary, we have

$$\beta_{PC}(K_2B) \leq (4M\int_0^b h(s)\mathrm{d}s + 2M\sum_{i=1}^{p} L_i)\beta_{PC}(B) \tag{4.3}$$

for any bounded subset $B \subset W$.

Now,for any subset $B \subset W$, due to Lemma 2.1, (4.2) and (4.3), we have

$$\begin{aligned}
\beta_{PC}(KB) \ &\leq \ \beta_{PC}(K_1B) + \beta_{PC}(K_2B) \\
&\leq \ (ML + 4M\int_0^b h(s)\mathrm{d}s + 2M\sum_{i=1}^{p} L_i)\beta_{PC}(B).
\end{aligned}$$

From (4.1) we know that K is β_{PC}-contraction on W. By Lemma 2.2, there is a fixed point u of K in W, which is a mild solution of problem (1.1)-(1.3). This completes the proof. ∎

Remark 4.1 *Clearly if the hypothesis* $(f3)$ *is replaced by one of the following conditions: (1) The semigroup* $T(t)$ *is compact and the functions* $I_i,\ i = 0, 1, \cdots, p$ *are compact; or (2)* $f(t, \cdot)$ *is compact for a.e.* $t \in [0, b]$ *and the functions* $I_i,\ i = 0, 1, \cdots, p$ *are compact, Theorem 4.1 is true when* $ML < 1$.

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