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Nonlocal diffusion problems
Problemas de difusión no local

MEMORIA PARA OPTAR AL GRADO DE DOCTORA

PRESENTADA POR

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presentada por:

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Bajo la dirección de

Aníbal Rodríguez Bernal

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A mis padres.

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Contents

Resumen

Introducción

La difusión es un proceso natural por el que, por ejemplo, la materia es transportada de un lugar a otro como resultado del movimiento molecular aleatorio. El experimento clásico que ilustra este proceso es aquel en el que se coloca una gota de tinta en un recipiente lleno de agua, y la tinta tiende a extenderse por todo el recipiente y la solución aparece coloreada de manera uniforme. (Existen experimentos más refinados para asegurar que no haya convección).

Los modelos de difusión aparece en diferentes áreas como biología, termodinámica, medicina, e incluso economía. En biología, existen modelos que estudian la dinámica poblacional, i.e., cambios a corto y largo plazo, en el tamaño y edad de la población, y procesos medioambientales y biológicos que influyen en esos cambios. La dinámica poblacional se enfrenta con la forma en la que la población se ve afectada por la tasas de natalidad y mortalidad, y por la inmigración y la emigración. En medicina, los modelos de difusión se usan, por ejemplo, para describir crecimientos tumorales. En termodinámica, la ecuación del calor modela la conducción del calor, esto es cuando un objeto está a diferente temperatura que otro cuerpo, o que a su alrededor, el calor fluye de manera que el cuerpo y sus alrededores alcanzan la misma temperatura. En economía, la difusión modela las fluctuaciones del mercado de valores, usando movimientos brownianos.

Existen dos manera de introducir la noción de difusión: con la aproximación fenomenológica, comenzando con las leyes de difusión de Fick, o con la aproximación física o atómica, considerando movimientos aleatorios de la difusión de partículas.

Primero, introducimos las leyes que rigen los procesos de difusión: Las leyes de Fick. Esta leyes relacionan el flujo difusivo con la concentración bajo la hipótesis de estado estacionario. Ésta postula que el flujo se mueve de regiones con alta concentración hacia regiones con baja concentración, con una magnitud que es proporcional al gradiente de concentración. Entonces en dimensión 1, tenemos

$$F = -D\frac{\partial u}{\partial x}, \tag{1}$$

donde F es el "flujo de difusión", u es la concentración de la substancia que se difunde, y D es el coeficiente de difusión.

Por otro lado, por la ley de Fick y la conservación de la masa en ausencia de reacciones químicas:

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}F = 0. \tag{2}$$

Entonces, por (1) y (2), obtenemos la segunda ley de Fick que predice cómo la difusión provoca cambios en la concentración con el tiempo:

$$\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2}. \tag{3}$$

Para el caso de la difusión en dos o más dimensiones, la segunda ley de Fick viene dada por

$$\frac{\partial u}{\partial t} = D\Delta u. \tag{4}$$

Los modelos de difusión local vienen dados por (4) más condiciones iniciales y de frontera, necesarias para completar el modelo.

Desde el punto de vista atómico, la difusión es considerada el resultado del movimento aleatorio, (random walk) de partículas difusivas. En difusión molecular, las moléculas que se mueven, son propulsadas por energía térmica. El movimiento aleatorio de pequeñas partículas en suspensión en un fluido fue descubierto en 1827 por Robert Brown, y la teoría de movimiento Browniano y el punto de vista atómico de la difusión, fue desarrollado por Albert Einstein en 1905.

Otro tipo de modelos de difusión son los modelos de difusión no local. Estos modelos pueden derivarse de variaciones de procesos de salto (ver por ejemplo [35]). Consideremos una única especie en un hábitat N-dimensional donde se asume que la población se puede modelar por una función $u(x,t)$, que es la densidad en x en tiempo t. Un modelo continuo para la dinámica poblacional de especies se puede derivar considerando con detalle una discretización en espacio y tiempo, y después haciendo tender los intervalos de espacio y tiempo a cero. En particular, la derivación clásica del laplaciano, , (4) por movimientos aleatorios, se tiene asumiendo una distribución binomial. Sin embargo, en el caso de la difusión no local, consideramos cualquier tipo de distribución.

A continuación reproducimos la derivación del modelo no local para el caso $N = 1$. Consideramos que el habitat es $\Omega \subset \mathbb{R}$. Primero, dividimos Ω en intervalos contiguos, cada uno de longitud Δx, y discretizamos el tiempo en pasos de tamaño Δt. Sea $u(i,t)$ la densidad de individuos en la posición i en tiempo t. Queremos derivar el cambio en el número de individuos en esta posición durante el siguiente intervalo de tiempo. La primera hipótesis es que la tasa a la que los individuos salen de i para llegar a j es constante. Por tanto, el número total de individuos saliendo de i a j debería ser proporcional a: la población en el intervalo i, que es $u(i,t)\Delta x$; el tamaño del lugar al que llegan, que es Δx; y la cantidad de tiempo durante el cuál el tránsito se está midiendo, Δt. Sea $J(j,i)$ la constante proporcional, entonces, el número de individuos saliendo de i durante el intervalo de tiempo $[t, t+\Delta t]$ es

$$\sum_{\substack{j=-M \\ j\neq i}}^{M} J(j,i)u(i,t)(\Delta x)^2\Delta t. \tag{5}$$

Durante este mismo intervalo de tiempo, el número de llegadas a i desde otros lugares es

$$\sum_{\substack{j=-M \\ j\neq i}}^{M} J(i,j)u(j,t)(\Delta x)^2\Delta t. \tag{6}$$

Combinando (5) y (6), deducimos que la densidad de población en i en tiempo $t + \Delta t$ viene dado por

$$u(i, t + \Delta t)\Delta x = u(i, t)\Delta x + \sum_{\substack{j=-M \\ j \neq i}}^{M} J(i, j)u(j, t)(\Delta x)^2 \Delta t - \sum_{\substack{j=-M \\ j \neq i}}^{M} J(j, i)u(i, t)(\Delta x)^2 \Delta t, \quad (7)$$

entonces, dividiendo (7) entre Δx, obtenemos

$$u(i, t + \Delta t) = u(i, t) + \sum_{\substack{j=-M \\ j \neq i}}^{M} J(i, j)u(j, t)\Delta x \Delta t - \sum_{\substack{j=-M \\ j \neq i}}^{M} J(j, i)u(i, t)\Delta x \Delta t. \quad (8)$$

Entonces, tendiendo $\Delta t \to 0$ y $\Delta x \to 0$ en (8), tenemos

$$u_t(x, t) = \int_{\Omega} \big(J(x, y)u(y, t) - J(y, x)u(x, t)\big)dy. \quad (9)$$

Ahora, reinterpretamos (9), con $\Omega \subset \mathbb{R}$. Asumimos que $J(x, y)$ es una función positiva definida en $\Omega \times \Omega$ que representa la densidad de probabilidad de saltar de y a x, y $u(x, t)$ es la densidad de población en el punto $x \in \Omega$ en tiempo t, entonces $\int_{\Omega} J(x, y)u(y, t)dy$ es la tasa a la que los individuos llegan a x desde otros lugares $y \in \Omega$. Como hemos asumido que J es la densidad de probabilidad, y J está definida en $\Omega \times \Omega$, entonces $\int_{\Omega} J(x, y)dy = 1$. En particular, $-u(x, t) = -\int_{\Omega} J(x, y)dy\, u(x, t)$ es la tasa a la que los individuos salen de x a otras posiciones $y \in \Omega$. Entonces, podemos escribir la ecuación (9) con condición inicial u_0, como

$$\begin{cases} u_t(x, t) &= \displaystyle\int_{\Omega} J(x, y)u(y, t)dy - u(x, t), \quad x \in \Omega, \\ u(x, 0) &= u_0(x), \qquad\qquad\qquad\qquad\quad x \in \Omega. \end{cases} \quad (10)$$

Este problema y variantes de él, ha sido previamente usado para modelar procesos de difusión, por ejemplo en [2], [18], [27], y [35]. Este modelo permite tener en cuenta interacciones a corta (short-range) y larga (long-range) distancia, y es posible generalizar el problema (10), para $\Omega \subset \mathbb{R}^N$, o incluso espacios medibles Ω más generales, (ver Capítulo 1).

El modelo (10) se llama modelo de difusión no local, pues la difusión de la densidad u en x en tiempo t no depende únicamente de $u(x, t)$, sino que depende de todos los valores de u en un entorno de x, a través del término de "convolución" $\int_{\Omega} J(x, y)u(y, t)dy$.

Objetivos

Ahora, fijemos un conjunto abierto $\Omega \subset \mathbb{R}^N$. Para problemas locales como (4), las dos condiciones de contorno más habituales son la de Neumann y la de Dirichlet. La ecuación del calor local con condición frontera Neumann, viene dada por

$$\begin{cases} u_t(x, t) &= \Delta u(x, t), \quad x \in \Omega,\ t > 0, \\ \dfrac{\partial u}{\partial \nu}(x, t) &= 0, \qquad\qquad x \in \partial\Omega,\ t > 0, \\ u(x, 0) &= u_0(x), \qquad x \in \Omega, \end{cases} \quad (11)$$

donde ν denota la normal exterior a la frontera $\partial\Omega$, y $\frac{\partial u}{\partial \nu} = 0$ modela que los individuos no entren ni salgan de Ω. Un problema no local análogo definido en el abierto $\Omega \subset \mathbb{R}^N$, propuesto en [18], viene dado por

$$\begin{cases} u_t(x,t) = \displaystyle\int_\Omega J(x,y)\left(u(y,t) - u(x,t)\right) dy = \int_\Omega J(x,y)u(y,t)dy - h_0(x)u(x,t) \\ u(x,0) = u_0(x), \end{cases} \tag{12}$$

donde $J : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$, con $\int_{\mathbb{R}^N} J(x,y)dy = 1$, y denotamos

$$h_0(x) = \int_\Omega J(x,y)dy, \quad \forall x \in \Omega.$$

En (12), la integral está definida sobre Ω, entonces, este modelo asume que los individuos no entran ni salen de Ω, y la difusión tiene lugar sólo dentro de Ω. Además, (12) comparte con el problema local (11), que las constantes son equilibrios.

Pro otro lado, la ecuación del calor local con condición de frontera Dirichlet homogénea es

$$\begin{cases} u_t(x,t) &= \Delta u(x,t), & x \in \Omega,\ t > 0, \\ u(x,t) &= 0, & x \in \partial\Omega,\ t > 0, \\ u(x,0) &= u_0(x), & x \in \Omega. \end{cases}$$

En este caso, u es cero en la frontera del hábitat. Un problema no local análogo propuesto en [18] con $\Omega \subset \mathbb{R}^N$ abierto, viene dado por

$$\begin{cases} u_t(x,t) = \displaystyle\int_{\mathbb{R}^N} J(x,y)u(y,t)dy - u(x,t), & x \in \Omega,\ t > 0, \\ u(x,t) = 0, & x \notin \Omega, t > 0, \\ u(x,0) = u_0(x), & x \in \Omega. \end{cases} \tag{13}$$

En este modelo la difusión tiene lugar en todo $\mathbb{R}^N$, y $u = 0$ fuera de Ω. Entonces, este problema modela el caso en que los individuos mueren cuando salen del hábitat Ω, y $\int_{\mathbb{R}^N} J(x,y)u(y,t)dy = \int_\Omega J(x,y)u(y,t)dy$. Entonces, la ecuación (13), es

$$u_t(x,t) = \int_\Omega J(x,y)u(y,t)dy - u(x,t).$$

Los problemas (9), (10), (12) y (13) se pueden unificar considerando el problema no local

$$\begin{cases} u_t(x,t) = \displaystyle\int_\Omega J(x,y)u(y,t)dy - h(x)u(x,t), & x \in \Omega,\ t > 0, \\ u(x,0) = u_0(x), & x \in \Omega, \end{cases} \tag{14}$$

con h definida en Ω. Éste es el tipo de problemas lineales no locales que vamos a estudiar.

Sobre problemas no lineales, introducimos los modelos no locales de reacción-difusión, añadiendo un término de reacción local $f(x, u(x, t))$ al modelo de difusión (14),

$$\begin{cases} u_t(x,t) = \displaystyle\int_\Omega J(x,y)u(y,t)dy - h(x)u(x,t) + f(x,u(x,t)), & x \in \Omega,\ t > 0, \\ u(x,0) = u_0(x), & x \in \Omega, \end{cases} \tag{15}$$

donde $f : \Omega \times \mathbb{R} \to \mathbb{R}$. Éste modelo fue considerado en [35], donde (15) modela la dinámica poblacional de las especies, y f denota la tasa de reproducción en x de una densidad de población $u(x, t)$, que tiene en cuenta el número de individuos nuevos en x en tiempo t.

También consideramos modelos de reacción-difusión, con difusión no local y reacción no local. El problema viene dado por

$$\begin{cases} u_t(x,t) = \displaystyle\int_\Omega J(x,y)u(y,t)dy - h(x)u(x,t) + f(x,u)(\cdot,t), & x \in \Omega,\ t > 0, \\ u(x,0) = u_0(x), & x \in \Omega, \end{cases} \tag{16}$$

pero ahora $f : \Omega \times L^1(\Omega) \to \mathbb{R}$ es un término no local. Éste modelo ha sido previamente considerado en [30].

Los problemas con difusión local y reacción no local han sido considerados en [11], donde el término de reacción no local tiene en cuenta la saturación no local o los efectos de competición no local.

Otro tipo de modelos de difusión no local, es el que aparece en [12, 19], dado por

$$u_t(x,t) = \int_{\mathbb{R}^N} J(x,y)(\Gamma(u(y,t)) - \Gamma(u(x,t)))dy, \quad x \in \mathbb{R}^N,\ t > 0, \tag{17}$$

donde $\Gamma(u) = \mathrm{sign}(u)\big(|u| - 1\big)_+$. Este problema se llama problema de Stefan no local. Modela la distribución de la temperatura y la entalpía en una fase de transición entre diferentes estados, por ejemplo, el cambio de fases entre hielo y agua.

Actualmente, existe un gran interés en el estudio de la difusión en dominios no regulares. Existen varios intentos de generalizar el operador laplaciano a espacios no regulares: las formas de Dirichlet (Dirichlet forms), ayudan a describir procesos de salto que se pueden definir en espacios no regulares. Pro tanto, es posible definir ecuaciones diferenciales en espacios no regulares, como pueden ser los fractales. Con esta teoría, llamada Análisis en fractales, se extienden conceptos como el laplaciano, las funciones de Green, núcleos de calor, (ver [9, 37, 50]).

Por otro lado, los modelos de difusión no local, como (14), (15), (16) se pueden definir en espacios métricos de medida (ver Capítulo 1), pues simplemente necesitamos considerar la densidad de probabilidad de saltar de un punto a otro de Ω, que viene dada por $J(x, y)$. Y este tipo de densidad se puede definir en un espacio métrico de medida general. Lo que nos permite estudiar la difusión en espacios muy diferentes como: grafos, (usados para modelar estructuras complicadas en química, biología molecular o electrónica, incluso pueden representar circuitos eléctricos en computadoras digitales); variedades compactas; multiestructuras compuestas por conjuntos compactos de diferentes dimensiones, (por ejemplo un conjunto de Dumbbell, donde es necesario considerar una perturbación del dominio para estudiar problemas de difusión local, como se puede ver en [3], mientras que en los problemas de difusión no local podremos estudiar el problema directamente en el dominio); o incluso conjuntos fractales como el triángulo de Sierpinski.

Resultados

Centrémonos en lo que será hecho a lo largo de este trabajo. Como mencionamos arriba, en esta tesis estudiamos problemas de difusión no locales generales. Sea μ una medida, y d una métrica definida en Ω, consideramos un espacio métrico de medida (Ω, μ, d), que se introduce en el Capítulo 1.

Primero, consideramos el problema de difusión lineal no local dado por

$$\begin{cases} u_t(x,t) = (K - hI)(u)(x,t), & x \in \Omega,\ t > 0 \\ u(x,0) = u_0(x), & x \in \Omega, \end{cases} \tag{18}$$

donde

$$K(u)(x,t) = \int_\Omega J(x,y)u(y,t)dy$$

es el operador integral, y

$$hI(u)(x,t) = h(x)u(x,t)$$

es el operador multiplicación con $h \in L^\infty(\Omega)$ o en $\mathcal{C}_b(\Omega)$, donde $\mathcal{C}_b(\Omega)$ son las funciones continuas y acotadas definidas en Ω. No asumiremos, a no ser que se diga explícitamente, que $\int_\Omega J(x,y)dy = 1$. Una función que será importante a lo largo de este trabajo es

$$h_0(x) = \int_\Omega J(x,y)dy,$$

que no es necesariamente igual a la identidad.

Para estudiar el problema lineal (18), en el Capítulo 2, primeramente realizaremos un estudio completo del operador lineal $K - hI$, estudiando los espacios donde el operador está definido, la compacidad y el espectro de K y hI de manera separada.

Después en el Capítulo 3, nos concentramos en la existencia y unicidad de soluciones de (18); en las propiedades de monotonía de las soluciones en $X = L^p(\Omega)$, con $1 \leq p \leq \infty$ o $X = \mathcal{C}_b(\Omega)$. Recuperamos y generalizamos el estudio de existencia y unicidad de soluciones de (18), con $h = h_0$ o $h = Id$. Lo cuál ha sido hecho en $L^1(\Omega)$ en [2, 18], considerando un dominio $\Omega \subset \mathbb{R}^N$ abierto.

A continuación, estudiamos el comportamiento asintótico de las soluciones cuando el tiempo se va a infinito. Probamos que si $\sigma_X(K - hI)$ es la unión de dos conjuntos cerrados disjuntos σ_1 y σ_2 con $\mathrm{Re}(\sigma_1) \leq \delta_1$, $\mathrm{Re}(\sigma_2) \leq \delta_2$, con $\delta_2 < \delta_1$, entonces el comportamiento asintótico de la solución de (18) en X está descrito por la proyección de Riesz de $K - hI$ correspondiente a σ_1. Probamos también que la proyección de Riesz y la proyección de Hilbert son iguales. Además, aplicamos este resultado a los casos particulares del problema de difusión no local (18) con h constante y $h = h_0$. En particular, recuperamos y generalizamos el resultado en [18], para $X = L^p(\Omega)$, con $1 \leq p \leq \infty$ o $X = \mathcal{C}_b(\Omega)$, mientras que en [18], los autores obtienen el resultado en $L^2(\Omega)$ si el dato inicial está en $L^2(\Omega)$, y en $L^\infty(\Omega)$ si el dato inicial está en $\mathcal{C}(\overline{\Omega})$, considerando $\Omega \subset \mathbb{R}^N$ un conjunto abierto.

El estudio del problema (18) nos lleva a la conclusión de que la ecuación (18) comparte algunas propiedades con la ecuación clásica del calor. En particular, ambas tienes Principio Débil y Fuerte del Máximo, cuando J satisface hipótesis de positividad, pero no comparten el efecto regularizante, como se indica en [27], para el caso $\Omega = \mathbb{R}^N$. Esto ocurre porque la solución de (18) conserva las singularidades de los datos iniciales. Sin embargo, hemos podido probar que el semigrupo $S(t)$ de (18) satisface que $S(t) = S_1(t) + S_2(t)$, con $S_1(t)$ que converge a 0 mientras t va a infinito en X, y $S_2(t)$ es compacta, entonces $S(t)$ es asintóticamente compacta, (asymptotically smooth), de acuerdo con la definición en [32, p. 4].

En el Capítulo 4, consideramos una ecuación de reacción-difusión no local, con término de reacción no lineal, y trabajamos con el problema

$$\begin{cases} u_t(x,t) = (K - hI)(u)(x,t) + f(x, u(x,t)), & x \in \Omega,\ t > 0 \\ u(x,t_0) = u_0(x), & x \in \Omega, \end{cases} \tag{19}$$

con $f : \Omega \times \mathbb{R} \to \mathbb{R}$, y dato inicial $u_0 \in L^p(\Omega)$. La función $f(x,s)$ se asumirá que es localmente Lipschitz en la variable $s \in \mathbb{R}$, uniformemente con respecto a $x \in \Omega$.

Existe una amplia literatura sobre el estudio de problemas de reacción-difusión locales

$$\begin{cases} u_t(x,t) = \Delta u(x,t) + f(x, u(x,t)), & x \in \Omega,\ t > 0, \\ u(x,t_0) = u_0(x), & x \in \Omega. \end{cases} \tag{20}$$

Existencia, unicidad y resultados de comparación de la soluciones de (43) con término de reacción no lineal localmente Lipschitz, f, como en (19) satisfaciendo condiciones de signo son conocidas, ver por ejemplo [47, 4]. Los argumentos usados para el problema (20) son esencialmente argumentos de punto fijo, pero no podemos usar estos argumentos para el problema no local (19), porque el semigrupo lineal $S(t)$ asociado a (18) no regulariza.

Probamos primero la existencia para la ecuación (19) con f globalmente Lipschitz, y después probamos la existencia con f localmente Lipschitz satisfaciendo condiciones de signo con argumentos de sub-supersolución, en $X = L^p(\Omega)$, con $1 \leq p \leq \infty$ o $X = \mathcal{C}_b(\Omega)$. Por tanto, recuperamos y generalizamos los resultados de existencia y unicidad de las soluciones de (19) en [8], donde $\Omega \subset \mathbb{R}^N$ y el dato inicial está en $\mathcal{C}(\overline{\Omega})$. Observamos que en [30], los autores estudian los exponentes de Fujita para (19), que coinciden con los clásicos de (20).

También estudiamos el comportamiento asintótico de las soluciones de (19). En [44], bajo condiciones de signo en el término no lineal, los autores prueban la existencia de dos equilibrios maximales de (20), con $\Omega \subset \mathbb{R}^N$ un dominio acotado y diferentes tipos de condiciones de contorno. También prueban que la dinámica asintótica de las soluciones entra entre estos equilibrios maximales, uniformemente en espacio, para conjuntos acotados de datos iniciales. Como consecuencia, obtienen una cota del atractor global para las ecuaciones de reacción-difusión locales, (20).

Por otro lado, nosotros probamos la existencia de dos equilibrios maximales ordenados φ_m y φ_M (uno minimal y otro maximal), para el problema (19), y toda la dinámica asintótica de las soluciones de (19) con datos iniciales acotados, entra entre los dos equilibrios maximales φ_m y φ_M, cuando el tiempo va a infinito en $L^p(\Omega)$, para todo $1 \leq p < \infty$. Además éstos mismos equilibrios extremales, φ_m y φ_M, son cotas de cualquier límite débil en $L^p(\Omega)$, con $1 \leq p < \infty$, de las soluciones de (19) con datos iniciales u_0 en $L^p(\Omega)$. Observamos que para el problema no local (19), obtenemos resultados más débiles que para el problema local (20) de nuevo, por la falta de regularización del semigrupo asociado al problema lineal no local.

Después de estudiar el comportamiento asintótico, discutimos la existencia y estabilidad de equilibrios del problema

$$\left\{ \begin{array}{lll} u_t(x,t) & = \displaystyle\int_\Omega J(x,y)u(y,t)dy - h_0(x)\,u(x,t) + f(u(x,t)), & x \in \Omega,\ t > 0 \\ u(x,0) & = u_0(x), & x \in \Omega, \end{array} \right. \tag{21}$$

Sea F el operador de Nemitcky asociado a la función f, tal que $F(u)(x,t) = f(u(x,t))$. Como $F : L^p(\Omega) \to L^p(\Omega)$ no es diferenciable (ver Apéndice B), y el semigrupo asociado al problema lineal no local (18) no regulariza, entonces el Principio de Estabilidad linealizada falla. Sin embargo, bajo hipótesis en la convexidad de la función f, probamos que la estabilidad/inestabilidad respecto a la linealización, implica la estabilidad/inestabilidad de los equilibrios del problema no lineal (21).

También probamos que cualquier equilibrio no constante de (21) es, si existe, inestable cuando f es convexa. En [16], [14] y [40], los autores prueban resultados similares para el problema de reacción-difusión local (20) con condición de frontera Neumann. En [14] y [40], los autores también prueban que si Ω es un dominio convexo, entonces cualquier equilibrio no constante, es inestable, es decir, no existen patrones (patterns). Hasta donde nosotros sabemos, este resultado no ha sido probado para el problema no local (21), y las técnicas que se usan para el problema local, no parecen ser útiles para probar la no existencia de patrones si el dominio es convexo.

Existe un gran interés en el estudio de existencia y estabilidad de equilibrios del problema (19). En [8], los autores estudian la estabilidad de equilibrios positivos con dato inicial en $\mathcal{C}(\overline{\Omega})$. En particular, prueban que bajo hipótesis en el espectro del operador lineal K_J, existe un único equilibrio no negativo asintóticamente estable en $\mathcal{C}(\overline{\Omega})_+$.

En el Capítulo 5, estudiamos problemas de reacción-difusión con ambos términos no locales, i.e., consideramos el problema

$$\left\{ \begin{array}{ll} u_t(x,t) & = (K - hI)(u)(x,t) + f(x,u)(\cdot,t), \qquad x \in \Omega,\ t > 0 \\ u(x,0) & = u_0(x), \quad x \in \Omega, \end{array} \right. \tag{22}$$

donde $(K - hI)(u)$ es el término de difusión no local, y $f : \Omega \times L^1(\Omega) \to \mathbb{R}$ es el término de reacción no local, y está definido como sigue

$$f = g \circ m,$$

donde $g : \mathbb{R} \to \mathbb{R}$ es una función no lineal, y $m : \Omega \times L^1(\Omega) \to \mathbb{R}$ es la media de u en la bola de radio $\delta > 0$ y centro x, definido como

$$m(x, u(\cdot, t)) = \frac{1}{\mu(B_\delta(x))} \int_{B_\delta(x)} u(y, t) dy.$$

Primero derivamos una teoría completa de existencia y unicidad para el problema (22), en $X = L^p(\Omega)$, con $1 \leq p \leq \infty$ o $X = \mathcal{C}_b(\Omega)$, con g globalmente Lipschitz.

El problema (22) no tiene propiedades de comparación en general. Por tanto, damos resultados de comparación para el problema (22), con g globally Lipschitz, con constante Lipschitz suficientemente pequeña en comparación con J, usando argumentos de punto fijo.

Si g es localmente Lipschitz, y satisface condiciones de signo, entonces probamos existencia y unicidad de solución para el problema (22), con término no lineal g, tal que la constante de Lipschitz de g_{k_0} es suficientemente pequeña en comparación con J, donde g_{k_0} es la función truncada en k_0 asociada a g. De hecho, la existencia y unicidad será probada para datos iniciales en $L^\infty(\Omega)$, tales que $\|u_0\|_{L^\infty(\Omega)} \leq k_0$. Además, probaremos propiedades de comparación para las soluciones de (22) con g y u_0 satisfaciendo las condiciones de arriba.

Probamos también que la dinámica asintótica de las soluciones de (22) con g globalmente Lipschitz, entra entre los dos equilibrios extremales φ_m y φ_M, como hacemos para los problemas de reacción-difusión no local (19). Además, si suponemos que el promedio en la bola de radio delta es continuo, entonces probamos que los equilibrios $\varphi_m, \varphi_M \in \mathcal{C}_b(\Omega)$ y la dinámica asintótica de las soluciones de (22) entra entre φ_m y φ_M uniformemente en conjuntos compactos de Ω.

Otra ventaja de este modelo (22), con reacción no local respecto al problema de reacción difusión no local (19), es que el término de reacción no local $F : L^p(\Omega) \to L^p(\Omega)$ es compacto, y probamos que el semigrupo asociado a (22) es asintóticamente compacto, y entonces usamos [32, Theorem 3.4.6.], para probar la existencia de un atractor global para el semigrupo de (22).

En el Capítulo 6, estudiamos el problema de Stefan de dos-fases no local en $\mathbb{R}^N$

$$\begin{cases} u_t = \displaystyle\int_{\mathbb{R}^N} J(x-y)v(y)dy - v, \quad \text{where } v = \Gamma(u), \\ u(\cdot, 0) = f, \end{cases} \tag{23}$$

donde J es un núcleo de convolución no negativo suave, u es la entalpía y

$$\Gamma(u) = \text{sign}(u)\big(|u| - 1\big)_+.$$

El problema de Stefan es un problema no lineal de frontera móvil cuyo objetivo es describir la distribución de la temperatura y la entalpía en una fase de transición entre diferentes estados. La historia del problema comenzó con Lamé y Clapeyron [39], y después con Stefan, en [49]. Para el modelo local se puede ver por ejemplo las monografías [17] y [54] para las fenomenológicas y modelización, [23], [41], [45] y para los aspectos matemáticos del modelo [53].

El modelo principal usa la ecuación local $u_t = \Delta v$, $v = \Gamma(u)$, pero recientemente, una versión no local del problema de Stefan de una-fase fue introducido en [12], que es equivalente a (23) en el caso de soluciones no negativas, y $\Gamma(u)$ viene dada por $\Gamma(u) = \big(u - 1\big)_+$.

Este nuevo modelo matemático es interesante desde el punto de vista de la física, pues a escala intermedia (mesoscópica), explica por ejemplo la evolución de *mushy regions* (regiones que son un estado intermedio entre hielo y agua).

Nosotros estudiamos la existencia, unicidad y comparación en la linea de los capítulos anteriores, y estudiamos el comportamiento asintótico en el espíritu de [12], pero para soluciones que cambian de signo, lo que presenta retos muy difíciles sobre el comportamiento asintótico. Aunque no damos un estudio completo del comportamiento asintótico, que parece ser bastante difícil, damos condiciones suficientes que garanticen la identificación del límite cuando el tiempo tiende a infinito.

Conclusiones

- Los modelos de difusión no local, se pueden platear en espacios métricos de medida (ver Capítulo 1). Lo que nos permite estudiar procesos de difusión en espacios muy diferentes como: grafos, multiestructuras compuestas por conjuntos compactos de diferentes dimensiones, o incluso conjuntos fractales como el triángulo de Sierpinski.

- El estudio del problema lineal (18) nos lleva a la conclusión de que la ecuación (18) comparte algunas propiedades con la ecuación clásica del calor. En particular, ambas tienen Principio Débil y Fuerte del Máximo, cuando J satisface hipótesis de positividad, pero no comparten el efecto regularizante. Esto ocurre porque la solución de (18) carga con las singularidades de los datos iniciales. Sin embargo, hemos podido probar que el semigrupo $S(t)$ de (18) satisface que $S(t) = S_1(t) + S_2(t)$, con $S_1(t)$ que converge a 0 mientras t va a infinito en X, y $S_2(t)$ es compacta, entonces $S(t)$ es asintóticamente compacta, (asymptotically smooth), de acuerdo con la definición en [32, p. 4].

- Para el problema no local (19), probamos la existencia de dos equilibrios maximales ordenados φ_m y φ_M, y toda la dinámica asintótica de las soluciones de (19) con datos iniciales acotados, entra entre los dos equilibrios maximales φ_m y φ_M, cuando el tiempo va a infinito en $L^p(\Omega)$, para todo $1 \leq p < \infty$. Además éstos mismos equilibrios extremales, φ_m y φ_M, son cotas de cualquier límite débil en $L^p(\Omega)$, con $1 \leq p < \infty$, de las soluciones de (19) con datos iniciales u_0 en $L^p(\Omega)$. Estos resultados son más débiles que para el problema local (20), debido a la falta de regularización del semigrupo asociado al problema lineal no local.

- Como $F : L^p(\Omega) \to L^p(\Omega)$ no es diferenciable (ver Apéndice B), y el semigrupo asociado al problema lineal no local (18) no regulariza, entonces el Principio de Estabilidad linealizada falla. Sin embargo, bajo hipótesis en la convexidad de la función f, probamos que la estabilidad/inestabilidad respecto a la linealización, implica la estabilidad/inestabilidad de los equilibrios del problema no lineal (21). Además si f es cóncava, probamos que no existen patrones para el problema (21).

- El problema (22) puede no cumplir las propiedades de comparación. Por tanto, damos resultados de comparación para el problema (22), donde g tiene una constante Lipschitz suficientemente pequeña en comparación con J.

- Probamos la existencia de equilibrios maximales φ_m y φ_M para el problema (22). Y como el término de reacción no local regulariza, probamos que la dinámica asintótica de las soluciones de (22) entra entre φ_m y φ_M uniformemente en conjuntos compactos de Ω. Además podemos probar que el semigrupo asociado al problema (22) es asintóticamente regular, y por tanto, probamos la existencia de un atractor global para el semigrupo del problema.

- Para el problema no local de Stefan de dos fases (23) estudiamos el comportamiento asintótico de las soluciones que cambian de signo en tres casos diferentes: cuando la parte positiva y negativa de las soluciones no interactúan para ningún tiempo $t \geq 0$; cuando la parte positiva y negativa de la temperatura $\Gamma(u)$ no interactúan para ningún tiempo $t \geq 0$, y cuando la parte positiva y negativa de la temperatura $\Gamma(u)$ interactúan pero el comportamiento de las soluciones viene dado por el del problema de Stefan de una fase después de cierto tiempo.

Introduction

Diffusion is the natural process by which, for example matter is transported from one part of a system to another as a result of random molecular motions. The classical experiment that illustrates this is the one in which a drop of ink is leaved in a vessel full of water, and it eventually spreads out around the container and all the whole solution appears uniformly coloured. (There exist more refined experiments to make sure no convection is present).

Diffusion models appear in sciences as diverse as biology, thermodynamics, medicine, and even economics. In biology, population models study the population dynamics, i.e., short-term and long-term changes in the size and age composition of populations, and the biological and environmental processes influencing those changes. Population dynamics deals with the way populations are affected by birth and death rates, and by immigration and emigration. In medicine, the diffusion models are used to describe the growth of cancerous tumors, for example. In thermodynamics, the heat equation models the heat conduction, this is when an object is at a different temperature from another body or its surroundings, heat flows so that the body and the surroundings reach the same temperature. In economics, the diffusion models fluctuations in the stock market, by using Brownian motion.

There are two ways to introduce the notion of diffusion: either a phenomenological approach starting with Fick's laws of diffusion, or a physical and atomistic one, by considering the random walk of the diffusing particles.

First, let us introduce the laws that rule the diffusion processes: The Fick's laws. Fick's first law relates the diffusive flux to the concentration under the assumption of steady state. It postulates that the flux goes from regions of high concentration to regions of low concentration, with a magnitude that is proportional to the concentration gradient. Then in 1-dimension we have

$$F = -D\frac{\partial u}{\partial x}, \tag{24}$$

where F is the "diffusion flux", u is the concentration of the diffusing substance, and D is the diffusion coefficient.

On the other hand, from Fick's first law and the mass conservation in absence of any chemical reaction:

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}F = 0. \tag{25}$$

Then form (24) and (25), we obtain Fick's second law that predicts how diffusion causes the concentration to change with time:

$$\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2}. \tag{26}$$

For the case of diffusion in two or more dimensions Fick's second law is given by

$$\frac{\partial u}{\partial t} = D\Delta u. \tag{27}$$

The local diffusion model is given by (27) plus some boundary and initial conditions which are needed to complete the model.

From the atomistic point of view, diffusion is considered as a result of the random walk of the diffusing particles. In molecular diffusion, the moving molecules are self-propelled by thermal energy. Random walk of small particles in suspension in a fluid was discovered in 1827 by Robert Brown. The theory of the Brownian motion and the atomistic backgrounds of diffusion were developed by Albert Einstein in 1905.

Another kind of diffusion models are the nonlocal diffusion models. These models can be derived from a variation of a position-jump process (see for example [35]). Consider a single specie in an N-dimensional habitat where it is presumed that the population can be adequately modeled by a single function $u(x,t)$, which is the density at position x at time t. A **continuous model** for the population dynamics for species can be derived by considering in detail a situation discrete in both space and time, and then letting the size and time intervals become small. The classic derivation of the Laplacian, (27) via a random walk is given assuming a binomial distribution.

We reproduce the derivation of the nonlocal model for the case $N = 1$. The habitat will be $\Omega \subset \mathbb{R}$. First, divide Ω into contiguous sites, each of length Δx. Discretize time into steps of size Δt. Let $u(i,t)$ be the density of individuals in site i at time t. We wish to derive the change in the number of individuals in this site during the next time interval. The first assumption is that the rate at which individuals are leaving site i and going to site j is constant. Thus the total number of individual leaving location i to location j should be proportional to: the population in the interval i, which is $u(i,t)\Delta x$; the size of the target site, which is Δx; and the amount of time during which the transit is being measured, Δt. Let $J(j,i)$ be the proportionality constant. Then, the number of individuals leaving site i during the interval $[t, t + \Delta t]$ is

$$\sum_{\substack{j=-M\\ j\neq i}}^{M} J(j,i)u(i,t)(\Delta x)^2\Delta t. \tag{28}$$

During the same time interval, the number of arrivals to site i from elsewhere is

$$\sum_{\substack{j=-M\\ j\neq i}}^{M} J(i,j)u(j,t)(\Delta x)^2\Delta t. \tag{29}$$

Combining (28) and (29), we deduce that the populations density at location i and time $t+\Delta t$

is given by

$$u(i,t+\Delta t)\Delta x = u(i,t)\Delta x + \sum_{\substack{j=-M \\ j\neq i}}^{M} J(i,j)u(j,t)(\Delta x)^2\Delta t - \sum_{\substack{j=-M \\ j\neq i}}^{M} J(j,i)u(i,t)(\Delta x)^2\Delta t, \quad (30)$$

then, dividing (30) by Δx, we obtain

$$u(i,t+\Delta t) = u(i,t) + \sum_{\substack{j=-M \\ j\neq i}}^{M} J(i,j)u(j,t)\Delta x\Delta t - \sum_{\substack{j=-M \\ j\neq i}}^{M} J(j,i)u(i,t)\Delta x\Delta t. \quad (31)$$

Thus, allowing $\Delta t \to 0$ and $\Delta x \to 0$ in (31), we obtain

$$u_t(x,t) = \int_\Omega \big(J(x,y)u(y,t) - J(y,x)u(x,t)\big)dy. \quad (32)$$

Now, let us reinterpret equation (32), with $\Omega \subset \mathbb{R}$. We assume $J(x,y)$ is a positive function defined in $\Omega\times\Omega$ that represents the density of probability of jumping from a location y to x, and $u(x,t)$ is the density of population at the point $x\in\Omega$ at time t, then $\int_\Omega J(x,y)u(y,t)dy$ is the rate at which the individuals arrive to location x from all other locations $y\in\Omega$. Since we have assumed that J is the density of probability, and J is defined in $\Omega\times\Omega$, then $\int_\Omega J(x,y)dy = 1$. In particular, $-u(x,t) = -\int_\Omega J(x,y)dy\, u(x,t)$ is the rate at which the individuals are leaving from location x to all other locations $y\in\Omega$. Then, we can write the equation (32) with initial condition u_0, as

$$\begin{cases} u_t(x,t) &= \displaystyle\int_\Omega J(x,y)u(y,t)dy - u(x,t), \quad x\in\Omega, \\ u(x,0) &= u_0(x), \qquad\qquad\qquad\qquad\quad x\in\Omega. \end{cases} \quad (33)$$

This problem and variations of it have been previously used to model diffusion processes, in [2], [18], [27], and [35], for example. This model allows to take into account short-range and long-range interactions, and it is possible to generalize the problem (33), for $\Omega\subset\mathbb{R}^N$, or even more general type of measurable set Ω, (see Chapter 1). The model (33) is called nonlocal diffusion model since the diffusion of the density u at point x and time t does not only depend on $u(x,t)$, but on all the values of u in a neighbourhood of x through the "convolution" term $\int_\Omega J(x,y)u(y,t)dy$.

Now, let us fix an open set $\Omega\subset\mathbb{R}^N$. For local problems as (27) the two most usual boundary conditions are Neumann's and Dirichlet's. The local heat equation with Neumann boundary condition is given by

$$\begin{cases} u_t(x,t) &= \Delta u(x,t), \quad x\in\Omega,\ t>0, \\ \dfrac{\partial u}{\partial\nu}(x,t) &= 0, \qquad\qquad x\in\partial\Omega,\ t>0, \\ u(x,0) &= u_0(x), \qquad x\in\Omega, \end{cases} \quad (34)$$

where ν denotes the (typically exterior) normal to the boundary $\partial\Omega$, and $\frac{\partial u}{\partial\nu}=0$ models that the individuals do not enter or leave Ω. An analogous nonlocal problem defined in $\Omega\subset\mathbb{R}^N$

open, proposed in [18], is given by

$$\begin{cases} u_t(x,t) = \displaystyle\int_\Omega J(x,y)\left(u(y,t)-u(x,t)\right)dy = \int_\Omega J(x,y)u(y,t)dy - h_0(x)u(x,t) \\ u(x,0) = u_0(x), \end{cases} \tag{35}$$

where $J : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$, with $\int_{\mathbb{R}^N} J(x,y)dy = 1$, and we denote

$$h_0(x) = \int_\Omega J(x,y)dy, \quad \forall x \in \Omega.$$

In (35), the integral is over Ω, then this model assumes that individuals may not enter or leave Ω, and the diffusion takes place only in Ω. Moreover, (35) shares with the local problem (34), that the constants are equilibrium solutions.

On the other hand, the local heat equation with homogeneous Dirichlet boundary conditions is given by

$$\begin{cases} u_t(x,t) &= \Delta u(x,t), \quad x \in \Omega,\ t > 0, \\ u(x,t) &= 0, \quad\quad\quad\ \ x \in \partial\Omega,\ t > 0, \\ u(x,0) &= u_0(x), \quad\quad x \in \Omega. \end{cases}$$

In this case, u is zero in the boundary of the habitat. An analogous nonlocal problem proposed in [18] with $\Omega \subset \mathbb{R}^N$ open , is given by

$$\begin{cases} u_t(x,t) = \displaystyle\int_{\mathbb{R}^N} J(x,y)u(y,t)dy - u(x,t), & x \in \Omega,\ t > 0, \\ u(x,t) = 0, & x \notin \Omega, t > 0, \\ u(x,0) = u_0(x), & x \in \Omega. \end{cases} \tag{36}$$

In this model the diffusion takes place in the whole $\mathbb{R}^N$, and $u = 0$ outside Ω. Hence, this problem models the case in which the individuals extinguish when they leave the habitat Ω, and $\int_{\mathbb{R}^N} J(x,y)u(y,t)dy = \int_\Omega J(x,y)u(y,t)dy$. Therefore, the equation in (36), is given by

$$u_t(x,t) = \int_\Omega J(x,y)u(y,t)dy - u(x,t).$$

Problems (32), (33), (35) and (36) can be unified considering the nonlocal problem

$$\begin{cases} u_t(x,t) = \displaystyle\int_\Omega J(x,y)u(y,t)dy - h(x)u(x,t), & x \in \Omega,\ t > 0, \\ u(x,0) = u_0(x), & x \in \Omega, \end{cases} \tag{37}$$

with h defined in Ω. This is the kind of linear nonlocal problems we are going to study.

Concerning nonlinear problems, we introduce the nonlocal reaction-diffusion model, by adding a local reaction term $f(x, u(x,t))$ to the diffusion population model (37),

$$\begin{cases} u_t(x,t) = \displaystyle\int_\Omega J(x,y)u(y,t)dy - h(x)u(x,t) + f(x,u(x,t)), & x \in \Omega,\ t > 0, \\ u(x,0) = u_0(x), & x \in \Omega, \end{cases} \tag{38}$$

where $f : \Omega \times \mathbb{R} \to \mathbb{R}$. This model was also considered in [35], where (38) models the population dynamics of species, and f denotes the per capita net reproduction rate at x at the given population density $u(x,t)$, to take into account the number of new individual at x at time t.

We also consider nonlocal reaction-diffusion models, with nonlocal diffusion and nonlocal reaction. The problem is given by

$$\begin{cases} u_t(x,t) = \displaystyle\int_\Omega J(x,y)u(y,t)dy - h(x)u(x,t) + f(x,u)(\cdot,t), & x \in \Omega,\ t > 0, \\ u(x,0) = u_0(x), & x \in \Omega, \end{cases} \tag{39}$$

but now $f : \Omega \times L^1(\Omega) \to \mathbb{R}$ is a nonlocal term. This model has been previously considered in [30].

The problem with local diffusion, $(-\Delta)$, and nonlocal reaction has been considered in [11], where the nonlocal reaction term takes into account a nonlocal saturation, or nonlocal competition effects.

Another kind of nonlocal diffusion model, is the one that appears in [12, 19], given by

$$u_t(x,t) = \int_{\mathbb{R}^N} J(x,y)(\Gamma(u(y,t)) - \Gamma(u(x,t)))dy, \quad x \in \mathbb{R}^N,\ t > 0, \tag{40}$$

where $\Gamma(u) = \mathrm{sign}(u)\big(|u| - 1\big)_+$. This problem is called the nonlocal Stefan problem, which models the temperature and enthalpy distribution in a phase transition between several states, for example the phase change from ice to water.

Recently, there has been a big interest in studying diffusion in spaces which are non smooth. There are many attempts to try to generalize the laplacian to nonsmooth spaces. There are Dirichlet forms, that help describing jump processes which can be defined in spaces that are nonsmooth. Hence, it is possible to define differential equation on nonsmooth spaces, like some fractal sets. With this theory, called Analysis at fractals, it is possible to extend concepts like the laplacian, Green's functions and heat kernels, (see [9, 37, 50]).

Nonlocal diffusion models like (37), (38), (39) can be naturally defined in metric measure spaces (see Chapter 1), since we just need to consider the density of probability of jumping from a location x in Ω to a location y in Ω, given by the function $J(x,y)$. And this kind of density can be defined in a general metric measure space, since, we just need the space Ω to have a measure and a metric. This allows us studying the diffusion in very different type of spaces, like: graphs, (which are used to model complicated structures in chemistry, molecular biology or electronics, or they can also represent basic electric circuits into digital computers), compact manifolds, multi-structures composed by several compact sets with different dimensions, (for example a dumbbell domain, where it is necessary to consider a perturbed domain to study local diffusion problems, as we can see in [3], whereas in the nonlocal diffusion problems we will be able study the problem directly in the domain), or even fractal sets as the Sierpinski gasket.

Let us focus in what will be done throughout this work. As we said above, in this thesis we study general nonlocal diffusion problems. Let μ be a measure and d a metric defined in Ω, we consider (Ω, μ, d) a metric measure space, which is introduced in Chapter 1.

First of all, we consider the linear nonlocal diffusion problem which is given by

$$\begin{cases} u_t(x,t) = (K - hI)(u)(x,t), & x \in \Omega,\ t > 0 \\ u(x,0) = u_0(x), & x \in \Omega, \end{cases} \tag{41}$$

where

$$K(u)(x,t) = \int_\Omega J(x,y)u(y,t)dy$$

is the integral operator, and

$$hI(u)(x,t) = h(x)u(x,t)$$

is the multiplication operator with $h \in L^\infty(\Omega)$ or in $\mathcal{C}_b(\Omega)$, where $\mathcal{C}_b(\Omega)$ are the continuous and bounded functions defined on Ω. We will not assume, unless otherwise made explicit, that $\int_\Omega J(x,y)dy = 1$. A function that will be important throughout this work is

$$h_0(x) = \int_\Omega J(x,y)dy,$$

which is not necessarily equal to the identity.

To study the linear problem (41), in Chapter 2, we first derive a complete study of the linear operator $K - hI$, studying the spaces where the operators are defined, the compactness and the spectrum of K and hI separately.

Then in Chapter 3 we concentrate on the study of existence and uniqueness of the solution of (41), as well as the monotonicity properties of the solution in $X = L^p(\Omega)$, with $1 \leq p \leq \infty$ or $X = \mathcal{C}_b(\Omega)$. We recover and generalize the study of existence and uniqueness of solution of (41), with $h = h_0$ or $h = Id$, and $\Omega \subset \mathbb{R}^N$ a domain, has been previously done in [2, 18] in $L^1(\Omega)$.

After this, we study in detail the asymptotic behaviour of the solution as time goes to infinity. We prove that if $\sigma_X(K - hI)$ is a disjoint union of two closed subsets σ_1 and σ_2 with $\mathrm{Re}(\sigma_1) \leq \delta_1$, $\mathrm{Re}(\sigma_2) \leq \delta_2$, with $\delta_2 < \delta_1$, then the asymptotic behavior of the solution of (3.1) in X is described by the Riesz Projection of $K - hI$ corresponding to σ_1. We prove also that the Riesz projection and the Hilbert projection are equal. Furthermore, we apply this result to the particular cases of the nonlocal diffusion problem (3.1) with h constant or $h = h_0$. In particular, we recover and generalize the result in [18], for $X = L^p(\Omega)$, with $1 \leq p \leq \infty$ or $X = \mathcal{C}_b(\Omega)$, whereas in [18], the authors obtain the result with $\Omega \subset \mathbb{R}^N$ an open set, in $L^2(\Omega)$ if the initial data is in $L^2(\Omega)$, and in $L^\infty(\Omega)$ if the initial data is in $\mathcal{C}(\overline{\Omega})$.

The study of the problem (41) leads us to the conclusion that equation (41) shares some properties with the classical heat equation, in particular, they both have weak and strong maximum principles, when J satisfies hypotheses of positivity, but they do not share the regularizing effect, as was pointed in [27], in the case $\Omega = \mathbb{R}^N$. This happens because the solution of (41) carries the singularities of the initial data. However, we have been able to prove that the semigroup $S(t)$ of (41) satisfies that $S(t) = S_1(t) + S_2(t)$, with $S_1(t)$ that converges to 0 as t goes to infinity in norm X, and $S_2(t)$ is compact, hence $S(t)$ is asymptotically smooth, according to the definition in [32, p. 4].

In Chapter 4, we consider a nonlocal reaction-diffusion equation, with a nonlinear reaction term, and we work with the problem

$$\begin{cases} u_t(x,t) = (K - hI)(u)(x,t) + f(x,u(x,t)), & x \in \Omega,\, t > 0 \\ u(x,t_0) = u_0(x), & x \in \Omega, \end{cases} \tag{42}$$

with $f : \Omega \times \mathbb{R} \to \mathbb{R}$, and initial data $u_0 \in L^p(\Omega)$. The function $f(x,s)$ will be assumed to be locally Lipschitz in the variable $s \in \mathbb{R}$, uniformly with respect to $x \in \Omega$.

There exists a large literature in the study of the local nonlinear reaction- diffusion equation

$$\begin{cases} u_t(x,t) = \Delta u(x,t) + f(x,u(x,t)), & x \in \Omega,\, t > 0, \\ u(x,t_0) = u_0(x), & x \in \Omega. \end{cases} \tag{43}$$

Existence, uniqueness and comparison results of the solutions of (43) with nonlinear locally Lipschitz term, f, as in (42) satisfying sign conditions are well-known, see for example [47, 4]. The arguments used for the problem (43) are essentially fixed-point arguments, but we can not use these arguments for the nonlocal problem (42), because the linear semigroup $S(t)$ associated to (41) does not regularize.

We prove first the existence for the equation (42) with f globally Lipschitz, and secondly, we prove the existence for f locally Lipschitz satisfying sign conditions with sub-supersolution arguments, in $X = L^p(\Omega)$, with $1 \le p \le \infty$ or $X = \mathcal{C}_b(\Omega)$. Hence, we recover and generalize the results of existence and uniqueness of solutions of (42), with $\Omega \subset \mathbb{R}^N$ and initial data in $\mathcal{C}(\overline{\Omega})$, in [8]. Observe that in [30], the authors study Fujita exponents for (42), which coincides with the classical one, (43).

We will also study the asymptotic behaviour of the solution of (42). In [44], under sign conditions on the nonlinear term, the authors prove the existence of two extremal equilibria of (43), with $\Omega \subset \mathbb{R}^N$ a bounded domain and different type of boundary conditions. The authors also prove that the asymptotic dynamics of the solutions enter between these extremal equilibria, uniformly in space, for bounded sets of initial data. As a consequence, they obtained a bound for the global attractor for the local reaction-diffusion equations.

On the other hand, we prove that there exist also two ordered extremal equilibria φ_m and φ_M (one minimal and another maximal), for the problem (42), and all the asymptotic dynamics of the solutions of (42) with bounded initial data, enter between the two extremal equilibria φ_m and φ_M, when time goes to infinity in $X = L^p(\Omega)$, with $1 \le p < \infty$. Moreover, the same extremal equilibria, φ_m and φ_M, are bounds of any weak limit in $L^p(\Omega)$, with $1 \le p < \infty$, of the solution of (42) with initial data u_0 in $L^p(\Omega)$. Observe that for the nonlocal problem (42), we obtain a weaker result than for the local problem (43) again by the lack of smoothing effects.

After studying the asymptotic behaviour, we discuss the existence and stability of equilibrium solutions for the problem

$$\begin{cases} u_t(x,t) &= \displaystyle\int_\Omega J(x,y)u(y,t)dy - h_0(x)\, u(x,t) + f(u(x,t)), \quad x\in\Omega,\ t>0 \\ u(x,0) &= u_0(x), \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad x\in\Omega, \end{cases} \tag{44}$$

Let F be the Nemitcky operator associated to the function f, such that $F(u)(x,t) = f(u(x,t))$. Since $F : L^p(\Omega) \to L^p(\Omega)$ is not differentiable (see Appendix B), and the semigroup associated to the linear problem (41) does not regularize then the principle of linearized stability fails. However, under hypotheses on the convexity of the function f, we prove that the stability/instability with respect to the linearization, implies the stability/instability of the equilibria of the nonlinear problem (44).

We will also prove that any continuous nonconstant equilibrium solution of (44) is, if it exists, unstable when f is convex. In [16], [14] and [40], the authors prove similar results for the local reaction-diffusion problem (43) with Neumann boundary conditions. In [14] and [40], the authors also prove that if Ω is a convex domain, then any nonconstant equilibrium, is, if it exists, unstable for any dimension. Up to our knowledge this result has not been proved for the nonlocal problem (44), and the techniques used for the local problem do not seem to be useful to prove the instability of nonconstant equilibria if the domain Ω is convex.

There exists a big interest in the study of the existence and stability of equilibria of the problem (42). In [8], the authors study the stability of the positive steady solutions, with initial data in $\mathcal{C}(\overline{\Omega})$. In particular they prove, under hypothesis on the spectrum of the linear operator K_J, that there exists a unique nonnegative equilibrium asymptotically stable in $\mathcal{C}(\overline{\Omega})_+$.

In Chapter 5, we study the nonlocal reaction-diffusion problem with both terms nonlocal, i.e., we consider the problem

$$\begin{cases} u_t(x,t) &= (K-hI)(u)(x,t) + f(x,u)(\cdot,t), \qquad x\in\Omega,\ t>0 \\ u(x,0) &= u_0(x), \quad x\in\Omega, \end{cases} \tag{45}$$

where $(K-hI)(u)$ is the nonlocal diffusion term and $f : \Omega\times L^1(\Omega)\to\mathbb{R}$ is the nonlocal reaction term, and it is defined as

$$f = g\circ m,$$

where $g:\mathbb{R}\to\mathbb{R}$ is a nonlinear function, and $m : \Omega\times L^1(\Omega)\to\mathbb{R}$ is the average of u in a ball of radius $\delta>0$ and center x, defined as

$$m(x,u(\cdot,t)) = \frac{1}{\mu(B_\delta(x))}\int_{B_\delta(x)} u(y,t)dy.$$

We first derive a complete theory of existence and uniqueness for the problem (45), in $X = L^p(\Omega)$, with $1\le p\le\infty$ or $X = \mathcal{C}_b(\Omega)$, with g globally Lipschitz.

The problem (45) with g linear, may fail to have comparison properties. Hence, we give comparison results for (45), with g globally Lipschitz, with Lipschitz constant small enough, using fixed-point arguments.

If g is locally Lipschitz, and satisfies sign conditions then we will prove the existence and uniqueness of solution of the problem (45), with nonlinear term g, such that the Lipschitz constant of g_{k_0} is small enough, where g_{k_0} is a truncated function associated to g. In fact, the existence and uniqueness, will be proved for initial data in $L^\infty(\Omega)$, such that $\|u_0\|_{L^\infty(\Omega)} \leq k_0$. Furthermore, we will prove some monotonocity properties for the solution of (45) with g and u_0 satisfying the conditions above.

We prove also that all the asymptotic dynamics of the solutions of (45) with g globally Lipschitz, enters between two extremal equilibria φ_m and φ_M, as we do for the nonlocal reaction-diffusion problem (42). In fact, the asymptotic dynamics of the solutions of (45) enters between φ_m and φ_M uniformly in compact sets of Ω.

Another advantage of this model (45), with nonlocal reaction with respect to the nonlocal reaction-diffusion problem (42), is that the nonlocal reaction term $f : L^p(\Omega) \to L^1(\Omega)$ is compact, and we prove that the semigroup associated to (45) is asymptotically smooth, and then we use [32, Theorem 3.4.6.] to prove the existence of a global attractor for the semigroup of (45).

In Chapter 6, we study the nonlocal two-phase Stefan problem in $\mathbb{R}^N$

$$\begin{cases} u_t = \displaystyle\int_{\mathbb{R}^N} J(x-y)v(y)dy - v, \quad \text{where } v = \Gamma(u), \\ u(\cdot, 0) = f, \end{cases} \tag{46}$$

where J is a smooth nonnegative convolution kernel, u is the enthalpy and

$$\Gamma(u) = \text{sign}(u)\big(|u| - 1\big)_+.$$

The Stefan problem is a non-linear and moving boundary problem which aims to describe the temperature and enthalpy distribution in a phase transition between several states. The history of the problem goes back to Lamé and Clapeyron [39], and afterwards [49]. For the local model can be seen e.g. the monographs [17] and [54] for the phenomenology and modeling; [23], [41], [45] and [53] for the mathematical aspects of the model.

The main model uses a local equation under the form $u_t = \Delta v$, $v = \Gamma(u)$, but recently, a nonlocal version of the one-phase Stefan problem was introduced in [12], which is equivalent to (46) in the case of nonnegative solutions, and $\Gamma(u)$ is given by $\Gamma(u) = \big(u-1\big)_+$.

This new mathematical model turns out to be rather interesting from the physical point of view at an intermediate (mesoscopic) scale, since it explains for instance the formation and evolution of *mushy regions* (regions which are in an intermediate state between water and ice).

We study the existence, uniqueness and comparison results along the lines of the previous Chapters, and we study the asymptotic behaviour in the spirit of [12], but for sign-changing solutions, which presents very challenging difficulties concerning the asymptotic behavior. Though we do not give a complete study of the question which appears to be rather difficult, we give some sufficient conditions which guarantee the identification of the limit when time goes to infinity.

Below, we briefly summarize the organization of the work:

In CHAPTER 1 we describe the metric measure spaces (Ω, μ, d), and enumerate the nonlocal diffusion models that will be studied in the following chapters.

In CHAPTER 2 we study the linear operator

$$\big(K - hI\big)(u)(x) = \int_{\Omega} J(x,y)u(y)dy - h(x)u(x), \qquad x \in \Omega.$$

We start studying the operator $K(u)$, which has a straight dependence with the kernel J. We give results of regularity and compactness of the operator K, in terms of the regularity of J. We study the positiveness of the operator K, and we describe the spectrum of the operator K. We will give conditions to obtain that the spectrum is independent of the Lebesgue space where we are working. After that we study the multiplication operator hI, that sends $u(x)$ to $h(x)u(x)$. In the last part of this chapter we analyze the spectrum of $K - hI$, and we will also give conditions to obtain that the spectrum of $K - hI$ is independent of the Lebesgue space.

In CHAPTER 3 we give a result of existence and uniqueness of solution of (41). We write the solution in terms of the group associated to the operator $K - hI$. We give also monotonicity results. We prove that under some hypothesis on the positivity of the kernel J, the Weak and Strong Maximum Principle. In the last part of this chapter, we prove that the solutions of the homogeneous problem (36) converges asymptotically to the eigenfunction associated to the first eigenvalue of the operator $K - hI$, and the solution of the problem (35) has an exponential convergence to the mean value of the initial data $u_0 \in L^p(\Omega)$, with $1 \leq p \leq \infty$.

In CHAPTER 4 we work with the nonlinear problem (42). We give a result of existence and uniqueness of solutions for f locally Lipschitz satisfying an increasing property. We also prove the existence of two extremal equilibria solution (one maximal and another minimal). We prove that all the solutions enter between these two extremal equilibria when time goes to infinity. We study also in the particular case of the problem (44) that if the reaction term f is strictly convex, any nonconstant equilibrium solution is unstable, if it exists.

In CHAPTER 5 we are confined to the nonlocal reaction-diffusion problem (45), with both terms nonlocal. We give a result of existence and uniqueness and comparison results of solutions of the problem (45), with g globally lipschitz. We give also a result of existence and uniqueness of solution of the problem (45) with g locally Lipschitz and sublinear and some bounded initial data. We prove that the asymptotic dynamic of the solutions enter between two extremal equilibria when time goes to infinity, and finish proving the existence of a global attractor.

In CHAPTER 6, we study the nonlocal two-phase Stefan problem in $\mathbb{R}^N$. We give results of existence and uniqueness of solutions of (46). And we focus in the study of the asymptotic behaviour of sign-changing solutions. Since, we have not been able to give a general result about this asymptotic behavior, we give some sufficient conditions to guarantee the identification of the limits. This work has been done in collaboration with Professor Emmanuel Chasseigne at the Université François Rabelais in Tours, during my stay for three months in 2011.

Chapter 1

Nonlocal diffusion on metric measure spaces

First of all we introduce some Measure Theory, to define the metric measure spaces, [46]. Then we enumerate some examples of metric measure space, in which all the theory throughout this work can be applied, we consider open subsets of $\mathbb{R}^N$, which are the most usual in the literature; graphs, which have plenty of applications; compact manifolds; multi-structures, that are the union of metric measure spaces of different dimensions, etc.

We finish introducing the linear nonlocal diffusion model, and we enumerate the different problems that will be analyzed in the following chapters.

1.1 Metric measure spaces

In this section we introduce concepts of Measure Theory, for more information see [46].

First of all, let us start defining what is a metric space (X, d) that consists of a set X and a **distance** d on X, i.e., a function $d : X \times X \to [0, \infty)$ satisfying the following properties: for all x, y, $z \in X$

i. $d(x, y) \geq 0$ and $d(x, y) = 0$ if and only if $x = y$,

ii. $d(x, y) = d(y, x)$,

iii. $d(x, y) \leq d(x, z) + d(z, y)$.

We will denote the balls in X by $B(x, r) = \{y \in X : d(x, y) < r\}$ where $x \in X$ and $r > 0$.

Let us introduce now several definitions.

Definition 1.1.1.

i. A collection $\mathfrak{M}$ of subsets of X is said to be a σ**-algebra** *in X if $\mathfrak{M}$ has the following properties:*

(a) $X \in \mathfrak{M}$.

(b) If $A \in \mathfrak{M}$, then $A^c \in \mathfrak{M}$, where $A^c = X \setminus A$.

(c) If $A = \bigcup_{n=1}^{\infty} A_n$, and $A_n \in \mathfrak{M}$ for $n = 1, 2, \ldots,$ then $A \in \mathfrak{M}$.

ii. *Let X be a topological space, we denote by $\mathscr{B}$ the smallest σ-algebra in X such that every open set in X belongs to $\mathscr{B}$. The members of $\mathscr{B}$ are called the Borel sets of X.*

Definition 1.1.2.

i. *A* **positive measure** *is a function μ, defined on a σ-algebra $\mathfrak{M}$, whose range is in $[0, \infty]$ and which is countably additive. This means that if $\{A_n\}_{n\in\mathbb{N}}$ is a pair-wise disjoint collection of members of $\mathfrak{M}$, then*

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n).$$

To avoid trivialities, we shall also assume that $\mu(A) < \infty$ for at least one $A \in \mathfrak{M}$, $A \neq \emptyset$.

ii. *A* **measure space**, *$(X, \mathfrak{M}, \mu)$, has a positive measure defined on the σ-algebra, $\mathfrak{M}$, and the members of $\mathfrak{M}$ are called the* **measurable sets** *in X.*

iii. *Let X be a measure space, Y be a topological space, and $f : X \to Y$, then f is said to be measurable provided that $f^{-1}(V)$ is a measurable set in X for every open set V in Y.*

iv. *A measure μ is called* **complete measure** *if every subset of a set with measure zero is measurable.*

v. *The measure μ defined on a σ-algebra $\mathfrak{M}$ in X is* **σ-finite measure** *if X is a countable union of sets X_i with finite measure.*

vi. *We call* **total variation of** *μ to the function $|\mu|$ defined on the Borel σ-algebra $\mathcal{B}$ in X by*

$$|\mu|(E) = \sup \sum_{i=1}^{\infty} \big|\mu(E_i)\big|, \quad E \in \mathcal{B}$$

the supremum being taken over all disjoint partitions $\{E_i\}$ of E.

The following Theorem states that every measure can be completed (see [46, p. 28]).

Theorem 1.1.3. *Let $(X, \mathfrak{M}, \mu)$ be a measure space, let $\mathfrak{M}^*$ be the collection of all $E \subset X$ for which there exists sets A and B in $\mathfrak{M}$ such that $A \subset E \subset B$ and $\mu(B - A) = 0$, and define $\mu(E) = \mu(A)$ in this situation. Then $\mathfrak{M}^*$ is a σ-algebra, and μ is a measure on $\mathfrak{M}^*$.*

Thanks to this result, whenever it is convenient, we may assume that any given measure is complete.

The following result can be found in [46, p. 40].

Theorem 1.1.4. *Let X be a locally compact Hausdorff space. Then there exists a σ-algebra $\mathfrak{M}$ in X which contains all Borel sets in X, and there exists a positive measure μ on $\mathfrak{M}$ such that*

i. $\mu(K) < \infty$ *for every compact set* $K \subset X$.

ii. For every μ-measurable set $E \subset \mathfrak{M}$ we have that

$$\mu(E) = \inf\{\mu(V) : E \subset V,\ V \text{ open}\}.$$

iii. The relation

$$\mu(E) = \sup\{\mu(K) : K \subset E,\ K \text{ compact}\}.$$

holds for every open set E, and for every $E \in \mathfrak{M}$, with $\mu(E) < \infty$.

iv. If $E \in \mathfrak{M}$ with $\mu(E) = 0$ and $A \subset E$, then $A \in \mathfrak{M}$. (μ is complete).

A measure μ defined on the σ-algebra of all Borel sets in a locally compact Hausdorff space X is called a **Borel measure** on X. If μ is positive, a Borel set $E \subset X$ is **outer regular** or **inner regular**, if E has property *ii.* or *iii.*, respectively, of Theorem 1.1.4. If every Borel set in X is both outer and inner regular, μ is **regular**.

Throughout this work we will be working with metric measure spaces, and any time we mention them, we will be referring to the following definition.

Definition 1.1.5. *A* **metric measure space** (X, μ, d) *is a metric space (X, d) with a σ-finite, regular, and complete Borel measure μ in X, that associates a finite positive measure to the balls of X.*

Remark 1.1.6. *Let (X, μ, d) be a metric measure space, according to the previous definition, the measure satisfies the properties in Theorem 1.1.4. The measure μ is a complete and regular measure, which are the properties ii., iii. and iv. in Theorem 1.1.4, and moreover, since μ associates a finite positive measure to the balls of X, then for every compact set $K \subset X$, $\mu(K) < \infty$, then the property i. in Theorem 1.1.4 is satisfied.*

1.1.1 Function spaces in a metric measure space

Let (Ω, μ) be a measure space where μ is a measure as in Definition 1.1.5. For $1 \leq p < \infty$, if f is a measurable function on Ω, we define

$$\|f\|_{L^p(\Omega)} = \left(\int_\Omega |f|^p d\mu\right)^{1/p}$$

and let $L^p(\Omega)$ consist of all f for which $\|f\|_{L^p(\Omega)} < \infty$. We call $\|f\|_{L^p(\Omega)}$ the L^p-norm of f.

Let f be a measurable function on Ω. The essential supremum of $f : \Omega \to \mathbb{R}$, $\operatorname{ess\,sup}(f)$, is defined by

$$\operatorname{ess\,sup}(f) = \inf\{a \in \mathbb{R} : \mu\big(\{x : f(x) > a\}\big) = 0\}.$$

For $p = \infty$, if f is a measurable function on Ω, we define $\|f\|_{L^\infty(\Omega)}$ to be the essential supremum of $|f|$, and we let $L^\infty(\Omega)$ consist of all f for which $\|f\|_{L^\infty(\Omega)} < \infty$.
In particular, if $\mu(\Omega) < \infty$ and $q > p$ then $L^q(\Omega) \hookrightarrow L^p(\Omega)$.

Let (Ω, μ, d) be a metric measure space, if f is a measurable function on Ω, we define $\|f\|_{\mathcal{C}_b(\Omega)}$ by the supremum of $|f|$, and we let $\mathcal{C}_b(\Omega)$ consist of all continuous and bounded functions f, such that $\|f\|_{\mathcal{C}_b(\Omega)} < \infty$. Then $\mathcal{C}_b(\Omega) \subset L^\infty(\Omega)$.

The results that will be used throughout this work, and are well known properties of the L^p-space are: Hölder's inequality and Minkowski's inequality; the Monotone Convergence Theorem and the Dominated Convergence Theorem; Fubini's Theorem and Lusin's Theorem. (These results can be see in detail in Appendix A).

Furthermore, since $L^p(\Omega)$, with $1 \leq p < \infty$ is a Banach space, we can consider its dual which is given by $L^{p'}(\Omega)$, for p' satisfying $1/p + 1/p' = 1$, and the dual space of $L^\infty(\Omega)$ is $(L^\infty(\Omega))' = \mathcal{M}(\Omega)$, where $\mathcal{M}(\Omega)$ is the set of measures satisfying the properties in Theorem 1.1.4.

1.2 Some examples of metric measure spaces

In the following chapters we will consider a general measure metric space (Ω, μ, d). In this section we enumerate some examples to which we can apply the theory developed throughout this work.

- Subset of $\mathbb{R}^N$: Let Ω be a Lebesgue measurable set of $\mathbb{R}^N$ with positive measure. A particular case is the one in which Ω is an open subset of $\mathbb{R}^N$, which can be even $\Omega = \mathbb{R}^N$. We consider the metric measure space (Ω, μ, d) with:
  - $\Omega \subseteq \mathbb{R}^N$,
  - μ the Lebesgue measure on $\mathbb{R}^N$,
  - d the Euclidean metric of $\mathbb{R}^N$.

- Graphs: We consider a graph $G = (V, E)$, where $V \subset \mathbb{R}^N$ is the finite set of *vertices*, and the *edge set* E, consists of a collection of Jordan curves
$$E = \left\{ \pi_j : [0,\, 1] \to \mathbb{R}^N \middle|\, j \in \{1,\, 2,\, 3, ...,\, n\} \right\}$$
where $\pi_j \in \mathcal{C}^1\left([0,1]\right)$ is injective. We consider that each $e_j := \pi_j([0,\, 1])$ has its end points in the set of vertices V, and any two edges $e_j \neq e_h$ satisfy that the intersection $e_j \cap e_h$ is either empty, 1 vertex or 2 vertices.

  We consider a graph in $\mathbb{R}^N$, non empty, connected and finite. From now on, we identify the graph $G \subset \mathbb{R}^N$ with its associated network.
$$G = \bigcup_{j=1}^{n} e_i = \bigcup_{j=1}^{n} \pi_j\left([0,\, 1]\right).$$

We denote $v = \pi_j(t)$ for some $t \in [0,1]$. For a function $u : G \to \mathbb{R}$ we set $u_j := u \circ \pi_j : [0,\, 1] \to \mathbb{R}$, and use the abbreviation

$$u_j(v) := u_j\big(\pi_j^{-1}(v)\big)$$

We define the measure structure of this graph. The edges have associated the Lebesgue measure in dimension 1, and the length of the edge e_i is defined as the length of the curve π_i,

$$\mu(e_i) = \mu(\pi_i[0,1]) = \int_0^1 \|\pi_i'(t)\| dt. \tag{1.1}$$

A set $A \subset e_i$ is **measurable** if and only if $\pi_i^{-1}(A) \subset [0,1]$ is measurable, and for any measurable set $A \subset e_i$, we consider the measure μ_i

$$\mu_i(A) = \int_{\pi_i^{-1}(A)} \|\pi_i'(t)\| dt.$$

Hence a set $A \subset G$ is **measurable** if and only if $A \cap e_i$ is measurable for every $i \in \{1,\, 2,\, 3, ...,\, n\}$, and its measure is given by

$$\mu(A) = \sum_{i=1}^{n} \mu_i(A \cap e_i). \tag{1.2}$$

It turns out that a function $f : G \to \mathbb{R}$ is measurable if and only if $f_{|e_i} : e_i \to \mathbb{R}$ is measurable.

For $1 \leq p < \infty$, we set $f \in L^p(G) = \prod_{i=1}^{n} L^p(e_i)$, with norm

$$\|f\|_{L^p(G)} = \sum_{i=1}^{n} \|f\|_{L^p(e_i)} < \infty,$$

where,

$$\|f\|_{L^p(e_i)} = \left(\int_0^1 |f(\pi_i(t))|^p \, \|\pi_i'(t)\| dt\right)^{1/p} = \left(\int_0^1 |f(\pi_i(\cdot))|^p \, d\mu_i\right)^{1/p}.$$

For $p = \infty$, $f \in L^\infty(G) = \prod_{i=1}^{n} L^\infty(e_i)$, with norm

$$\|f\|_{L^\infty(G)} = \max_{i=1,\dots,n} \|f\|_{L^\infty(e_i)} < \infty,$$

where,

$$\|f\|_{L^\infty(e_i)} = \sup_{t\in[0,1]} |f(\pi_i(t))|\,.$$

Furthermore, a function $f : G \to \mathbb{R}$ is continuous in the graph G, if and only if $f_{|e_i} : e_i \to \mathbb{R}$ is continuous. We set $f \in \mathcal{C}(G) = \prod_{i=1}^{n} \mathcal{C}(e_i)$, with the norm associated

$$\|f\|_{\mathcal{C}(G)} = \max_{i=1,\dots,n} \|f\|_{\mathcal{C}(e_i)} < \infty,$$

where,

$$\|f\|_{\mathcal{C}(e_i)} = \sup_{t\in[0,1]} |f(\pi_i(t))|.$$

Now, let us describe the metric associated to the graph. For v, $w \in G$ the **geodesic distance** from v to w is the length of the shortest path from v to w. This distance will be the metric associated to the graph G, and we denote the **geodesic metric** as d_g. Moreover, since the graph is connected, there always exists the path from v to w, and since the graph is finite the geodesic metric d_g is equivalent to euclidean metric in $\mathbb{R}^N$. Let us see this below:

The graph G is compact in (G, d_g), the graph with the geodesic metric, and G is compact in (G, d), the graph with the euclidean metric in $\mathbb{R}^N$. We consider the identity map $I : (G, d_g) \to (G, d)$, thus, we have that I is continuous, because for any $v, w \in G$ with $d(v, w) \leq d_g(x, y)$. Thus, since I is continuous and injective in a compact set, and $Im(G) = G$, then I is an homeomorfism. Therefore, the metrics d_g and d are equivalent.

To sum up, the metric measure space (G, μ_G, d_g) is given by:

- G is a graph with a finite number of edges and vertices,
- μ_G the measure described in (1.2),
- d_g is the geodesic metric which is equivalent to the Euclidean metric of $\mathbb{R}^N$.

1.2.1 Manifolds, Multi-structures and other metric measure spaces

Let us introduce the family of **Hausdorff measures** below, for which we follow [26, chap. 2]. A d-dimensional Hausdorff measure is a type of positive outer measure, that assigns a number in $[0, \infty]$ to a set in $\mathbb{R}^N$. The zero-dimensional Hausdorff measure is the number of points in the set (if the set is finite) or ∞ if the set is infinite. The one-dimensional Hausdorff measure of a simple curve in $\mathbb{R}^N$ is equal to the length of the curve. Likewise, the two-dimensional Hausdorff measure of a measurable subset of $\mathbb{R}^2$ is proportional to the area of the set. Thus, the concept of the Hausdorff measure generalizes counting, length, area and volume. In fact, there are d-dimensional Hausdorff measures for any $d \geq 0$, which is not necessarily an integer.

Definition 1.2.1.

i. Let $(\mathbb{R}^n, d)$ be the euclidean metric space. For any subset $E \subset \Omega$, let $diam(E)$ denote its diameter,

$$diam(E) = \sup\{d(x, y) : x, y \in E\}, \qquad diam(\emptyset) = 0.$$

Let E be any subset of Ω, and $\delta > 0$ a real number. We define

$$\mathcal{H}^s_\delta(E) = \inf \left\{ \sum_{i=1}^{\infty} (diam(E_i))^s : E \subseteq \bigcup_{i=1}^{\infty} E_i,\ diam(E_i) < \delta \right\}.$$

ii. For E and s as above, we define

$$\mathcal{H}^s(E) = \lim_{\delta \to 0} \mathcal{H}^s_\delta(E) = \sup_{\delta > 0} \mathcal{H}^s_\delta(E).$$

We call $\mathcal{H}^s$ the s-dimensional Hausdorff measure on $\mathbb{R}^n$.

In the following result, we give several properties of the Hausdorff measure.

Theorem 1.2.2. (Elementary properties of Hausdorff measure)

i. $\mathcal{H}^s$ is a Borel regular measure in $\mathbb{R}^N$ for $0 \leq s < \infty$.

ii. $\mathcal{H}^0$ is a counting measure.

iii. $\mathcal{H}^N$ is the Lebesgue measure in $\mathbb{R}^N$.

iv. $\mathcal{H}^s \equiv 0$ on $\mathbb{R}^N$ for all $s > N$.

v. Let $A \subset \mathbb{R}^N$ and $0 \leq s < t < \infty$.

(a) If $\mathcal{H}^s(A) < \infty$, then $\mathcal{H}^t(A) = 0$.

(b) If $\mathcal{H}^t(A) > 0$, then $\mathcal{H}^s(A) = +\infty$.

We define below the Hausdorff dimension of a subset of $\mathbb{R}^N$.

Definition 1.2.3. *The* **Hausdorff dimension** *of a set $A \subset \mathbb{R}^N$ is defined to be*

$$\mathcal{H}_{dim}(A) \equiv \inf\{0 \leq s < \infty : \mathcal{H}^s(A) = 0\}$$

Remark 1.2.4. *Observe $\mathcal{H}_{dim}(A) \leq N$. If we denote $s = \mathcal{H}_{dim}(A)$, then $\mathcal{H}^t(A) = 0$ for all $t > s$ and $\mathcal{H}^t(A) = +\infty$ for all $t < s$; $\mathcal{H}^s(A)$ may be any number between 0 and ∞ included. Furthermore $\mathcal{H}_{dim}(A)$ need not be a integer. Even if $\mathcal{H}_{dim}(A) = k$ is an integer and $0 < \mathcal{H}^k(A) < \infty$, A need not be a k-dimensional surface in any sense.*

Let us introduce more examples of metric measure spaces.

- Compact Manifold: Let $\mathcal{M} \subset \mathbb{R}^N$ be a compact manifold that we define as follows: Let U be an open bounded set of $\mathbb{R}^d$, with $d \leq N$, and let $\varphi : U \to \mathbb{R}^N$ be an application such that it defines a diffeomorphism from $\overline{U}$ onto its image $\varphi(\overline{U})$, then we define the compact manifold as $\mathcal{M} = \varphi(\overline{U})$.

  A natural measure in $\mathcal{M}$, is the one for which, $A \subset \mathcal{M}$ is measurable if and only if $\varphi^{-1}(A) \subset \mathbb{R}^d$ is measurable. Hence for any measurable set $A \subset \mathcal{M}$, we define the measure μ as, (see [48, p. 48])

$$\mu(A) = \int_{\varphi^{-1}(A)} \sqrt{g}\, dx, \tag{1.3}$$

  where $g = det(g_{ij})$ and $g_{ij} = \langle \frac{\partial \varphi}{\partial x_i}, \frac{\partial \varphi}{\partial x_j} \rangle$.

Since the compact manifold $\mathcal{M} \subset \mathbb{R}^N$ is given by $\varphi(\overline{U})$, with $U \subset \mathbb{R}^d$, then the "ambient measure" is the d-dimensional Hausdorff measure for the manifold $\mathcal{M}$. In fact, the measure (1.3) is equal to the d-Hausdorff measure, (see [48, p. 48]).

The natural metric in $\mathcal{M}$: Let $\ell(c)$ be the length of the curve defined as in (1.1), then we define the **geodesic distance** between two points p, q in a manifold $\mathcal{M}$ as, (see [29, p. 164]):

$$d_g(p,q) := \inf\{\ell(c) \mid c : [0,1] \to \mathcal{M} \text{ smooth curve, } c(0) = p,\, c(1) = q\}. \tag{1.4}$$

On the other hand $\mathcal{M} \subset \mathbb{R}^N$ then the "ambient metric" is the euclidean metric, d. Since the manifold $\mathcal{M}$ is compact, and arguing like we did for the graph G, we obtain the the geodesic metric, d_g, and the euclidean metric, d, are equivalent.

To sum up, the metric measure space $(\mathcal{M}, \mathcal{H}^d, d)$ is given by:

– $\mathcal{M}$ the compact manifold in $\mathbb{R}^N$.
– $\mathcal{H}^d$ the d-dimensional Hausdorff measure.
– d_g is the geodesic metric equivalent to the Euclidean metric of $\mathbb{R}^N$.

- Multi-structure: Now, we consider a multi-structure, composed by several compact sets with different dimensions. For example, we can think in a piece of plane joined to a curve that is joined to a sphere in $\mathbb{R}^N$, or we can think also in a dumbbell domain. Therefore, we are going to define an appropriate measure and metric for these multi-structures.

Let (X, μ_X, d) be the direct sum of metric measure spaces composed by a collection of metric measure spaces $\big\{(X_i, \mu_i, d_i)\big\}_{i\in\{1,\dots,n\}}$, with its respective measures, μ_i, and metrics, d_i, defined as above. Moreover, we assume the measure spaces $\big\{(X_i, \mu_i)\big\}_{i\in\{1,\dots,n\}}$ satisfy

$$\mu_i(X_i \cap X_j) = \mu_j(X_i \cap X_j) = 0,$$

for $i \neq j$, and $i,\, j \in \{1, \dots, n\}$.
We define

$$X = \bigcup_{i\in\{1,\dots,n\}} X_i, \tag{1.5}$$

and we say that $E \subset X$ is measurable if and only if $E \cap X_i$ is measurable for all $i \in \{1, \dots, n\}$. Moreover we define the measure μ_X as

$$\mu_X(E) = \sum_{i=1}^{n} \mu_i(E \cap X_i). \tag{1.6}$$

Furthermore, let us define the metric that we consider in X. We assume that $X_i \subset \mathbb{R}^N$ is compact for all $i \in \{1, \dots, n\}$, and the metrics d_i associated to each X_i, are equivalent to the euclidean metric in $\mathbb{R}^N$. Therefore, the metric d that we consider for the multi-structure, is the euclidean metric in $\mathbb{R}^N$.

To sum up, the metric measure space (X, μ_X, d) is given by:

- X the multi-structure in (1.5).
- μ_X the measure given by (1.6).
- d is the Euclidean metric of $\mathbb{R}^N$.

- Space with finite Hausdorff measure and geodesic distance: We consider a compact set $F \subset \mathbb{R}^N$, with $\mathcal{H}_{dim}(F) = s$, and such that F has finite s-Hausdorff measure, i.e., $\mathcal{H}^s(F) < \infty$. The metric associated to F is the geodesic metric, which may not be equivalent to the euclidean metric in $\mathbb{R}^N$.

  Therefore, we consider the metric measure space (F, μ_F, d_g) given by:

  - F is a compact set in $\mathbb{R}^N$.
  - $\mathcal{H}^s$ the s-dimensional Hausdorff measure.
  - d_g is the geodesic metric.

  There exist some examples with the previous metric and measure associated, some are fractal sets like the Sierpinski gasket, (see Figure 1.1). The Sierpinski gasket is a fractal set that has associated a metric and measure like the ones described above, i.e., we consider the $\frac{log(3)}{log(2)}$- dimensional Hausdorff measure $\mathcal{H}^s$, and the geodesic metric. For more information see [37] and [20].

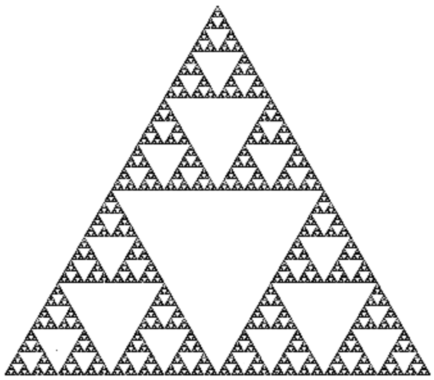

Figure 1.1: Sierpinski Gasket.

1.3 Nonlocal diffusion problems

Now, let us introduce the kind of linear nonlocal diffusion problems we are going to deal with throughout this work. We describe first the problem in a general metric measure space (Ω, μ, d), or in subsets of $\mathbb{R}^N$:

- Diffusion in a metric measure space: Let (Ω, μ, d) be a metric measure space, and let $u(x,t)$ be the density of population at the point $x \in \Omega$ at time t.

We assume J is a positive function defined in $\Omega \times \Omega$, i,e., $(x, y) \mapsto J(x, y)$ and we assume that J is the density of probability of jumping from a location y to x, and $u(x)$ is the density of population at the point $x \in \Omega$, then $\int_\Omega J(x,y)u(y)dy$ is the rate at which the individuals arrive to location x from all other locations $y \in \Omega$. Since we have assumed that J is the density of probability, and J is defined in $\Omega \times \Omega$, then $\int_\Omega J(x,y)dy = 1$, for all $x \in \Omega$. In particular, $-u(x) = -\int_\Omega J(x,y)dy\, u(x)$ is the rate at which the individuals are leaving from location x to all other locations $y \in \Omega$. Then, we consider the problem

$$\begin{cases} u_t(x,t) = \displaystyle\int_\Omega J(x,y)\big(u(y,t)-u(x,t)\big)dy = \int_\Omega J(x,y)u(y,t)dy - u(x,t), & x \in \Omega,\ t>0 \\ u(x,0) = u_0(x), & x \in \Omega. \end{cases} \tag{1.7}$$

In this problem, the integral terms only take into account the diffusion inside Ω. Thus the individuals may not enter or leave Ω. In particular, when $\Omega \subset \mathbb{R}^N$, the diffusion is forced to act only in Ω with no interchange of mass between Ω and the exterior $\mathbb{R}^N \setminus \Omega$.

- Diffusion in $\mathbb{R}^N$: Let $\Omega \subset \mathbb{R}^N$ and let us assume that $J(x,y)$ is the density of probability of jumping from x to y defined in $\mathbb{R}^N \times \mathbb{R}^N$, then we have that $\int_{\mathbb{R}^N} J(x,y)dy = 1$ for all $x \in \mathbb{R}^N$. Therefore $\|J(x,\cdot)\|_{L^1(\Omega)} = 1$ and $J(x,\cdot) \in L^1(\Omega)$ for all $x \in \Omega$. Some examples of J's in $\mathbb{R}^N$ are the following:

- $J(x,y) = e^{-\frac{|x-y|^2}{\sigma^2}}$, where $\sigma > 0$ (Normal distribution);
- $J(x,y) = \frac{1}{|x-y|^{\alpha-1}}$, where $0 < \alpha < 1$.

We are interested in two kind of nonlocal problems, which appear in [18]:

- The nonlocal problem proposed in [18] as an analogous problem to the local diffusion problem with Dirichlet boundary conditions, is the following: it is imposed $u = g$ outside Ω. Hence the nonlocal problem is given by

$$\begin{cases} u_t(x,t) = \displaystyle\int_{\mathbb{R}^N} J(x,y)u(y,t)dy - u(x,t), & x \in \Omega,\ t > 0 \\ u(x,t) = g(x), & x \notin \Omega, t > 0 \\ u(x,0) = u_0(x), & x \in \Omega. \end{cases} \tag{1.8}$$

- Let us consider a nonlocal diffusion problem where the diffusion is forced to act only in $\Omega \subset \mathbb{R}^N$, then the integrals over the whole $\mathbb{R}^N$ that appear in (1.8) are replaced by integrals only in Ω. The nonlocal diffusion problem (1.9) proposed in [18] as the nonlocal problem analogous to the classical heat equation with Neumann boundary conditions is given by

$$\begin{cases} u_t(x,t) = \displaystyle\int_\Omega J(x,y)\left(u(y,t) - u(x,t)\right)dy, & x \in \Omega,\ t > 0 \\ u(x,0) = u_0(x), & x \in \Omega. \end{cases} \tag{1.9}$$

Now, we unify the nonlocal problems (1.8) and (1.9).

The problem (1.8) can be rewritten as:

$$\begin{aligned} u_t(x,t) &= \int_\Omega J(x,y)u(y,t)dy - u(x,t) + \int_{\mathbb{R}^N\setminus\Omega} J(x,y)u(y,t)dy \\ &= (K-I)u(x,t) + G_g(x) \end{aligned}$$

where

$$K(u)(x,t) = \int_\Omega J(x,y)u(y,t)dy \tag{1.10}$$

and

$$G_g(x) = \int_{\mathbb{R}^N\setminus\Omega} J(x,y)g(y)dy.$$

The problem (1.9) can be written as:

$$\begin{aligned} u_t(x,t) &= \int_\Omega J(x,y)u(y,t)dy - \int_\Omega J(x,y)dy\, u(x,t) \\ &= (K-h_0 I)u \end{aligned}$$

with K as in (1.10), and $h_0(x) = \displaystyle\int_\Omega J(x,y)dy$. In particular, since J is nonnegative and $\displaystyle\int_{\mathbb{R}^N} J(x,y)dy = 1$, for all $x\in\Omega$, we have that $0 \le h_0(x) \le 1$.

We unify the nonlocal problems (1.8) and (1.9) as follows:

$$\begin{cases} u_t(x,t) = (K-hI)(u)(x,t) + \widetilde{G}_g(x), & x\in\Omega,\ t>0 \\ u(x,0) = u_0(x), & x\in\Omega, \end{cases} \tag{1.11}$$

with

$$h(x) = \begin{cases} 1, & \text{for the problem (1.8) ,} \\ h_0(x) = \displaystyle\int_\Omega J(x,y)dy, & \text{for the problem (1.9) ,} \end{cases}$$

and

$$\widetilde{G}_g(x) = \begin{cases} G_g(x), & \text{for the problem (1.8) ,} \\ 0, & \text{for the problem (1.9) .} \end{cases}$$

We consider now a metric measure space (Ω,μ,d), which can be even $\Omega\subset\mathbb{R}^N$, and we unify the problems (1.7), (1.8) and (1.9). Hence the problem we work with through this work is the following

$$\begin{cases} u_t(x,t) = (K-hI)(u)(x,t), & x\in\Omega,\ t>0 \\ u(x,0) = u_0(x), & x\in\Omega, \end{cases} \tag{1.12}$$

with $h\in L^\infty(\Omega)$ and $K(u) = \displaystyle\int_\Omega J(x,y)u(y)dy$, where

$$J:\Omega\times\Omega\to\mathbb{R}.$$

As we can see in (1.12), the problem is defined for $x\in\Omega$, and the integral operator $K(u)$ acts only in Ω.

Now, we enumerate the equations we are going to work with throughout this work. Let (Ω,μ,d) be a metric measure space:

i. In Chapter 3, we study the evolution linear nonlocal problem

$$u_t(x,t) = (K - hI)(u)(x,t), \quad x \in \Omega. \tag{1.13}$$

ii. In Chapter 4, we consider the the nonlocal reaction-diffusion equation. We add a local nonlinear reaction term $f : \Omega \times \mathbb{R} \to \mathbb{R}$ to the equation (1.13). Thus, we study

$$u_t(x,t) = (K - hI)(u)(x,t) + f(x, u(x,t)), \quad x \in \Omega. \tag{1.14}$$

iii. In Chapter 5, we consider a reaction-diffusion equation with a nonlocal reaction term, $f : \Omega \times L^1(\Omega) \to \mathbb{R}$, with $f = g \circ m$, where $g : \mathbb{R} \to \mathbb{R}$ is a nonlinear function, and m is a average of u in the ball centered in x of radius δ. The reaction-diffusion equation is given by

$$u_t(x,t) = (K - hI)(u)(x,t) + g\left(\frac{1}{\mu(B_\delta(x))} \int_{B_\delta(x)} u(y,t)dy\right), \quad x \in \Omega. \tag{1.15}$$

iv. In Chapter 6 we will study the two-phase Stefan problem in $\mathbb{R}^N$,

$$u_t(x,t) = \int_{\mathbb{R}^N} J(x-y)\Gamma(u)(y,t)dy - \Gamma(u)(x,t),$$

where $\Gamma(u) = \text{sign}(u)(|u| - 1)_+$.

NOTATION: Throughout this thesis we use the following notation:

- (Ω, μ, d) will always be a metric measure space , with μ as in Definition 1.1.5. Sometimes we will omit the part in which we mention μ as in Definition 1.1.5.
- $L^{p'}(\Omega)$ is used to denote the Lebesgue spaces with p' satisfying $1 = 1/p + 1/p'$, for $1 \le p \le \infty$.
- Let $L^p(\Omega)$ be a Banach space. The dual space of $L^p(\Omega)$, will be considered as:
  - for $1 \le p < \infty$, $(L^p(\Omega))' = L^{p'}(\Omega)$, where $1 = \dfrac{1}{p} + \dfrac{1}{p'}$,
  - for $p = \infty$, $(L^\infty(\Omega))' = \mathcal{M}(\Omega)$, where $\mathcal{M}(\Omega)$ is the set of Radon measures, for more information see [28, chap. 7].

Chapter 2

The linear nonlocal diffusion operator

Throughout this chapter we will work with (Ω, μ, d) a metric measure space, with the properties in Definition 1.1.5.

We consider the linear nonlocal diffusion problem:

$$\begin{cases} u_t(x,t) & = (K - hI)(u)(x,t), \quad x \in \Omega, \ t > 0, \\ u(x,0) & = u_0(x), \qquad\qquad\qquad x \in \Omega \end{cases} \tag{2.1}$$

where

$$(K - hI)(u) = \int_\Omega J(\cdot, y)u(y)dy - h(\cdot)u,$$

with J a function such that $J : \Omega \times \Omega \to \mathbb{R}$, and $h \in L^\infty(\Omega)$. In this chapter we give a comprehensive survey of the linear operator $K - hI$, and in the next chapter we apply this theory to study the existence, uniqueness, positivity, regularizing effects and the asymptotic behavior of the solution of (2.1).

We will start studying the linear nonlocal diffusive integral operator

$$K(u) = \int_\Omega J(\cdot, y)u(y)dy, \tag{2.2}$$

where J is the kernel of the operator. We will prove that under hypotheses on the integrability or continuity of J, K is a bounded linear operator in $X = L^p(\Omega)$ or $X = \mathcal{C}_b(\Omega)$. Moreover, under these same hypotheses on J, we will prove the compactness of the operator K. To prove the compactness, we will show that K can be approximated by operators with finite rank, and we will also use Ascoli-Arzela Theorem.

We will denote the operator K by K_J, to remark the dependence between J and K.

We will also study the particular case of the convolution operator

$$K_{J_0}(u) = \int_\Omega J_0(\cdot - y)u(y)dy,$$

where $J_0 : \mathbb{R}^N \to \mathbb{R}$.

We give a result of positiveness of the diffusive operator K_J: given a nonnegative function z, not identically zero, we will describe the set of points in Ω where $K_J(z)$ is strictly positive. For this, we will assume that the kernel J satisfies

$$J(x,y)>0 \text{ for all } x,\ y\in\Omega, \text{ such that } d(x,y)<R, \tag{2.3}$$

for some $R>0$ and Ω R-connected (see Definition 2.1.14). This positiveness will be used later on to prove that the solution of the problem (2.1) has a strong maximum principle.

We are also interested in the adjoint operator associated to K_J, which will be proved to be given by $(K_J)^* = K_{J^*}$, where, $J^*(x,y) = J(y,x)$. Moreover, if J satisfies that $J(x,y) = J(y,x)$, then $K_J \in \mathcal{L}(L^2(\Omega), L^2(\Omega))$ is selfadjoint. In this case we have that the spectrum is real and it is bounded above and below by

$$m = \inf_{u\in L^2, \|u\|_{L^2}=1} \langle K_J(u), u\rangle_{L^2(\Omega),L^2(\Omega)} \quad \text{and} \quad M = \sup_{u\in L^2, \|u\|_{L^2}=1} \langle K_J(u), u\rangle_{L^2(\Omega),L^2(\Omega)}.$$

Moreover, thanks to Kreĭn-Rutman Theorem, (see [38]), we will obtain that if the function J satisfies (2.3), then the spectral radius in $\mathcal{C}_b(\Omega)$ of the operator K_J is a positive simple eigenvalue, with a strictly positive eigenfunction associated. A similar result was proved by Bates and Zhao [8], for $\Omega \subset \mathbb{R}^N$ open, but their hypothesis on the positivity of J is stronger, because they assume that $J(x,y) > 0$ for all $x,\ y \in \Omega$.

Let $X = L^p(\Omega)$, with $1 \le p \le \infty$ or $X = \mathcal{C}_b(\Omega)$. If $K_J \in \mathcal{L}(L^1(\Omega), \mathcal{C}_b(\Omega))$ is compact, then we obtain that the spectrum $\sigma_X(K_J)$ is independent of X. Hence, the previous results will be also satisfied for the spectrum of K_J in X. Therefore, $\sigma_X(K_J) \subset [m, M]$ and the spectral radius of the operator K_J in X will be proved to have a strictly positive associated eigenfunction.

Finally, in the last part of this chapter we study the linear nonlocal operator $K_J - hI$, with $h \in L^\infty(\Omega)$ or $h \in \mathcal{C}_b(\Omega)$. We will give Green's formulas for the operator $K_J - hI$ when $J(x,y) = J(y,x)$. A similar result can be found in [2], for $\Omega \subset \mathbb{R}^N$ open. Moreover, we will make a general spectral study of the operator $K_J - hI$, and we will prove that $\sigma_X(K_J - hI)$ is composed by $\overline{Im(h)}$, and eigenvalues of finite multiplicity. Furthermore, we will prove that if $J(x,y) = J(y,x)$ and $h \ge h_0 = \int_\Omega J(\cdot,y)dy$, then $\sigma_X(K_J - hI)$ is nonpositive.

2.1 Properties of the operator K

We consider the function J, defined in Ω as

$$\Omega \ni x \mapsto J(x,\cdot) \ge 0$$

and define $K_J(u)(x) = \displaystyle\int_\Omega J(x,y)u(y)dy$, with $x \in \Omega$ for u defined in Ω. We call J the kernel of the operator K_J. We will not assume, unless otherwise made explicit, that Ω has a finite measure.

2.1.1 Regularity of K_J

In this section we are going to study spaces between which the linear operator is defined, depending on the integrability or continuity of the function J. Moreover, we will prove that the operator is bounded.

The following proposition states that under appropriate regularity of a general kernel J, we have that $K_J \in \mathcal{L}(L^p(\Omega), X)$, where $X = L^q(\Omega)$, $C_b(\Omega)$ or $X = W^{1,q}(\Omega)$, if $\Omega \subset \mathbb{R}^N$ is open.

Proposition 2.1.1.

i. *For* $1 \leq p,\, q \leq \infty$, *if* $J \in L^q(\Omega, L^{p'}(\Omega))$, *then* $K_J \in \mathcal{L}(L^p(\Omega), L^q(\Omega))$ *and the mapping* $J \longmapsto K_J$ *is linear and continuous, and*

$$\|K_J\|_{\mathcal{L}(L^p(\Omega),L^q(\Omega))} \leq \|J\|_{L^q(\Omega,L^{p'}(\Omega))}. \tag{2.4}$$

ii. *For* $1 \leq p \leq \infty$, *if* $J \in L^\infty(\Omega, L^{p'}(\Omega))$ *and for any measurable set* $D \subset \Omega$ *satisfying* $\mu(D) < \infty$,

$$\lim_{x \to x_0} \int_D J(x,y)dy = \int_D J(x_0,y)dy, \qquad \forall x_0 \in \Omega, \tag{2.5}$$

then $K_J \in \mathcal{L}(L^p(\Omega), \mathcal{C}_b(\Omega))$ *and the mapping* $J \longmapsto K_J$ *is linear and continuous, and*

$$\|K_J\|_{\mathcal{L}(L^p(\Omega),\mathcal{C}_b(\Omega))} \leq \|J\|_{L^\infty(\Omega,\, L^{p'}(\Omega))}. \tag{2.6}$$

In particular, if $J \in \mathcal{C}_b(\Omega, L^{p'}(\Omega))$, *then* $K_J \in \mathcal{L}(L^p(\Omega), \mathcal{C}_b(\Omega))$, *and*

$$\|K_J\|_{\mathcal{L}(L^p(\Omega),\mathcal{C}_b(\Omega))} \leq \|J\|_{\mathcal{C}_b(\Omega,\, L^{p'}(\Omega))}.$$

iii. *If* $\Omega \subset \mathbb{R}^N$ *is* **open**, *for* $1 \leq p, q \leq \infty$, *if* $J \in W^{1,q}(\Omega, L^{p'}(\Omega))$, *then* $K_J \in \mathcal{L}(L^p(\Omega), W^{1,q}(\Omega))$ *and the mapping* $J \longmapsto K_J$ *is linear and continuous, and*

$$\|K_J\|_{\mathcal{L}(L^p(\Omega),W^{1,q}(\Omega))} \leq \|J\|_{W^{1,q}(\Omega,L^{p'}(\Omega))}. \tag{2.7}$$

Proof.
i. Thanks to Hölder's inequality, we have for $1 \leq q < \infty$ and $1 \leq p \leq \infty$,

$$\begin{aligned}
\|K_J(u)\|^q_{L^q(\Omega)} &= \int_\Omega |K_J(u)(x)|^q dx \\
&= \int_\Omega \left| \int_\Omega J(x,y)u(y)\, dy \right|^q dx \\
&\leq \|u\|^q_{L^p(\Omega)} \int_\Omega \|J(x,\cdot)\|^q_{L^{p'}(\Omega)} dx \quad = \|u\|^q_{L^p(\Omega)} \|J\|^q_{L^q\left(\Omega,L^{p'}(\Omega)\right)}.
\end{aligned}$$

For $q = \infty$ and $1 \leq p \leq \infty$, for each $x \in \Omega$,

$$|K_J(u)(x)| = \left| \int_\Omega J(x,y)u(y)dy \right| \leq \|u\|_{L^p(\Omega)} \|J(x,\cdot)\|_{L^{p'}(\Omega)}. \tag{2.8}$$

Taking supremums in (2.8) in Ω, we obtain

$$\|K_J(u)\|_{L^\infty(\Omega)} = \sup_{x\in\Omega} |K_J(u)(x)| \leq \|u\|_{L^p(\Omega)} \sup_{x\in\Omega} \|J(x,\cdot)\|_{L^{p'}(\Omega)} = \|u\|_{L^p(\Omega)} \|J\|_{L^\infty\left(\Omega, L^{p'}(\Omega)\right)}.$$

Thus, the result.

ii. We have to prove, that for all $u \in L^p(\Omega)$, $K_J(u) \in \mathcal{C}_b(\Omega)$, for $1 \leq p \leq \infty$. The hypothesis (2.5) can also be written as

$$\lim_{x\to x_0} \int_\Omega J(x,y)\chi_D(y)dy = \int_\Omega J(x_0,y)\chi_D(y)dy, \qquad \forall x_0 \in \Omega, \tag{2.9}$$

what means that $K_J(\chi_D)$ is continuous in Ω, where χ_D is the characteristic function of $D \subset \Omega$, with $\mu(D) < \infty$. Moreover, since $J \in L^\infty(\Omega, L^{p'}(\Omega))$, from *i.*, we have that $K_J \in \mathcal{L}(L^p(\Omega), L^\infty(\Omega))$, since $\mu(D) < \infty$, then $\chi_D \in L^p(\Omega)$, for $1 \leq p \leq \infty$. Then $K_J(\chi_D)$ is bounded. Thus, $K_J(\chi_D) \in \mathcal{C}_b(\Omega)$ for any characteristic function χ_D. Moreover, the space

$$V = span\left[\,\chi_D\,;\, D \subset \Omega \text{ with } \mu(D) < \infty\right],$$

is dense in $L^p(\Omega)$, for $1 \leq p \leq \infty$. We prove it first for $1 \leq p < \infty$. Suppose that there exists a function $g \in (L^p(\Omega))'$, such that $g \perp V$, then,

$$\int_\Omega \chi_D(x)g(x)dx = \int_D g(x)dx = 0,\ \forall D \subset \Omega \text{ with } \mu(D) < \infty,$$

what implies that $g(x) = 0$ for a.e. $x \in D$, for all $D \subset \Omega$ with $\mu(D) < \infty$. On the other hand, μ is a σ-finite measure, then $\Omega = \bigcup_{i=1}^\infty D_i$, with $\mu(D_i) < \infty$. Thus, we have that $g(x) = 0$ for a.e. $x \in \Omega$. Hence, $V \hookrightarrow L^p(\Omega)$ densely, for all $1 \leq p < \infty$.

We prove it now for $p = \infty$. We suppose that there exists a measure $g \in (L^\infty(\Omega))' = \mathcal{M}(\Omega)$, such that $g \perp V$, where $\mathcal{M}(\Omega)$ is the set of Radon measures (see Theorem 1.1.4), then

$$\int_\Omega \chi_D dg = g(D) = 0,\ \forall D \subset \Omega \text{ with } \mu(D) < \infty,$$

what implies that the measure $g \equiv 0$ for all $D \subset \Omega$ with $\mu(D) < \infty$. Arguing like before, we obtain that $V \hookrightarrow L^\infty(\Omega)$ densely.

From *i.* , $K_J \in \mathcal{L}(L^p(\Omega), L^\infty(\Omega))$, and we have already proved that $K_J : V \to \mathcal{C}_b(\Omega)$, and $V \hookrightarrow L^p(\Omega)$ densely, then

$$K_J(L^p(\Omega)) = K_J\left(\overline{V}\right) \subset \overline{K_J(V)} \subset \mathcal{C}_b(\Omega).$$

Therefore, $\forall u \in L^p(\Omega)$, with $1 \leq p \leq \infty$, $K_J(u) \in \mathcal{C}_b(\Omega)$.
Finally, taking supremums in Ω in (2.8), we obtain the result.

In particular if $J \in C_b\big(\Omega, L^{p'}(\Omega)\big)$, then the hypothesis (2.5) is satisfied. Furthermore, $C_b\big(\Omega, L^{p'}(\Omega)\big) \hookrightarrow L^\infty\big(\Omega, L^{p'}(\Omega)\big)$. Thus, the result.

iii. As a consequence of Fubini's Theorem, and since Ω is open we have that for all $\varphi \in \mathcal{C}_c^\infty(\Omega)$ and $\forall i = 1, \dots, N$, the weak derivative of $K_J(u)$ is given for all $u \in L^p(\Omega)$ by

$$\begin{aligned} \langle \partial_{x_i} K_J(u), \varphi \rangle &= - \langle K_J(u), \partial_{x_i}\varphi \rangle \\ &= - \int_\Omega \int_\Omega J(x,y)u(y)\partial_{x_i}\varphi(x)\, dy\, dx \\ &= - \int_\Omega \int_\Omega J(x,y)\partial_{x_i}\varphi(x)u(y)\, dx\, dy \\ &= - \langle \langle J(\cdot,y), \partial_{x_i}\varphi \rangle, u \rangle \\ &= \langle \langle \partial_{x_i} J(\cdot,y), \varphi \rangle, u \rangle \\ &= \int_\Omega \int_\Omega \partial_{x_i} J(x,y)\varphi(x)u(y)\, dx\, dy \\ &= \int_\Omega \int_\Omega \partial_{x_i} J(x,y)u(y)\,\varphi(x)\, dy\, dx \qquad = \langle K_{\frac{\partial J}{\partial x_i}}(u), \varphi \rangle. \end{aligned} \tag{2.10}$$

Therefore

$$\frac{\partial}{\partial x_i} K_J(u) = K_{\frac{\partial J}{\partial x_i}}(u). \tag{2.11}$$

Since $J \in W^{1,q}(\Omega, L^{p'}(\Omega))$, and from part *i.* and (2.11), we have that for $u \in L^p(\Omega)$

$$\|K_J\|_{\mathcal{L}(L^p(\Omega),L^q(\Omega))} \leq \|J\|_{L^q(\Omega,L^{p'}(\Omega))} \tag{2.12}$$

and $\forall i = 1, \dots, N$,

$$\|\frac{\partial}{\partial x_i} K_J\|_{\mathcal{L}(L^p(\Omega),L^q(\Omega))} = \|K_{\frac{\partial}{\partial x_i}J}\|_{\mathcal{L}(L^p(\Omega),L^q(\Omega))} \leq \left\| \frac{\partial}{\partial x_i} J \right\|_{L^q(\Omega,L^{p'}(\Omega))}. \tag{2.13}$$

Hence, $K_J \in \mathcal{L}(L^p(\Omega), W^{1,q}(\Omega))$, for all $1 \leq p, q \leq \infty$ and from (2.12) and (2.13) we have (2.7). $\square$

The following result collects the cases in which $K_J \in \mathcal{L}(X, X)$, with $X = L^p(\Omega)$ or $X = \mathcal{C}_b(\Omega)$.

Corollary 2.1.2.

i. If $J \in L^p(\Omega, L^{p'}(\Omega))$ then $K_J \in \mathcal{L}(L^p(\Omega), L^p(\Omega))$, for $1 \leq p \leq \infty$ fixed.

ii. If $J \in \mathcal{C}_b(\Omega, L^1(\Omega))$ then $K_J \in \mathcal{L}(\mathcal{C}_b(\Omega), \mathcal{C}_b(\Omega))$.

iii. If $\mu(\Omega) < \infty$ and $J \in L^\infty(\Omega, L^\infty(\Omega))$ then $K_J \in \mathcal{L}(L^p(\Omega), L^p(\Omega))$, for all $1 \leq p \leq \infty$.

Proof.

i. From Proposition 2.1.1 we have the result.

ii. If $J \in \mathcal{C}_b(\Omega, L^1(\Omega))$ then, thanks to the previous Proposition 2.1.1, K_J belongs to $\mathcal{L}(L^\infty(\Omega), \mathcal{C}_b(\Omega))$. Moreover, since $\mathcal{C}_b(\Omega) \subset L^\infty(\Omega)$, we have that $K_J \in \mathcal{L}(\mathcal{C}_b(\Omega), \mathcal{C}_b(\Omega))$.

iii. From Proposition 2.1.1 we have that $K_J \in \mathcal{L}(L^1(\Omega), L^\infty(\Omega))$. Moreover, since $\mu(\Omega) < \infty$, $L^p(\Omega) \hookrightarrow L^1(\Omega)$ and $L^\infty(\Omega) \hookrightarrow L^p(\Omega)$, then

$$K_J : L^p(\Omega) \hookrightarrow L^1(\Omega) \longrightarrow L^\infty(\Omega) \hookrightarrow L^p(\Omega).$$

$\square$

2.1.2 Regularity of convolution operators

There is a large literature on nonlocal diffusion problems, where $\Omega = \mathbb{R}^N$ and the nonlocal term is the convolution with a function $J_0 : \mathbb{R}^N \to \mathbb{R}$. Hence, the convolution operator is given by

$$K_{J_0}(u) = J_0 * u.$$

Some examples in the literature with this operator are [1], [7], [18], [22], [52].

Let $\Omega \subset \mathbb{R}^N$ be a measurable set, (it can be $\Omega = \mathbb{R}^N$, or just a subset $\Omega \subset \mathbb{R}^N$). In this section, we study the regularity of the operator

$$K_{J_0}(u)(x) = \int_\Omega J_0(x-y)u(y)dy. \tag{2.14}$$

where J_0 is a function in $L^{p'}(\mathbb{R}^N)$, for $1 \leq p \leq \infty$. Hence the kernel is given by

$$J(x,y) = J_0(x-y), \ \forall x,\, y \in \Omega. \tag{2.15}$$

We want to analyze the spaces where the operator K_{J_0}, (2.14), is defined depending on the integrability of J_0, as we have done in Proposition 2.1.1 for the operator K_J. Let us see below the cases that are obtained from Proposition 2.1.1.

Corollary 2.1.3. *For $1 \leq p \leq \infty$, let $\Omega \subseteq \mathbb{R}^N$ be a measurable set, if $J_0 \in L^{p'}(\mathbb{R}^N)$, then $K_{J_0} \in \mathcal{L}(L^p(\Omega), L^\infty(\Omega))$. In particular if $\mu(\Omega) < \infty$, then $K_{J_0} \in \mathcal{L}(L^p(\Omega), L^q(\Omega))$, for $1 \leq q \leq \infty$.*

Proof. If $J_0 \in L^{p'}(\mathbb{R}^N)$, we have that J defined as in (2.15) satisfies that it belongs to $L^\infty(\Omega, L^{p'}(\Omega))$, since

$$\sup_{x\in\Omega} \|J(x,\cdot)\|_{L^{p'}(\Omega)} = \sup_{x\in\Omega} \|J_0(x-\cdot)\|_{L^{p'}(\Omega)} \leq \|J_0\|_{L^{p'}(\mathbb{R}^N)} < \infty.$$

Thus, thanks to Proposition 2.1.1, we have that $K_{J_0} \in \mathcal{L}(L^p(\Omega), L^\infty(\Omega))$. In particular, if $\mu(\Omega) < \infty$ then $K_{J_0} \in \mathcal{L}(L^p(\Omega), L^q(\Omega))$, for all $1 \leq q \leq \infty$. $\square$

On the other hand, if $\mu(\Omega) = \infty$, (like in the case of $\Omega = \mathbb{R}^N$), then K_{J_0} is not necessarily in $\mathcal{L}(L^p(\Omega), L^q(\Omega))$, for $q \neq \infty$. In the proposition below we prove the cases which **can not** be obtained as a consequence of Proposition 2.1.1.

Proposition 2.1.4. *For $1 \leq p \leq \infty$, let $\Omega \subseteq \mathbb{R}^N$ be a measurable set with $\mu(\Omega) = \infty$,*

i. if $J_0 \in L^r(\mathbb{R}^N)$ and $\frac{1}{q} = \frac{1}{p} + \frac{1}{r} - 1$ then $K_{J_0} \in \mathcal{L}(L^p(\Omega), L^q(\Omega))$, and

$$\|K_{J_0}\|_{\mathcal{L}(L^p(\Omega),L^q(\Omega))} \leq \|J_0\|_{L^r(\mathbb{R}^N)}.$$

In particular, if $r = 1$ we can take $p = q$.

ii. If $\Omega \subset \mathbb{R}^N$ is **open**, *$J_0 \in W^{1,r}(\mathbb{R}^N)$ and $\frac{1}{q} = \frac{1}{p} + \frac{1}{r} - 1$ then $K_{J_0} \in \mathcal{L}(L^p(\Omega), W^{1,q}(\Omega))$, and*

$$\|K_{J_0}\|_{\mathcal{L}(L^p(\Omega),W^{1,q}(\Omega))} \leq \|J_0\|_{W^{1,r}(\mathbb{R}^N)}.$$

Proof.

i. We use Young's inequality (see [13, p. 104]) for the convolution,

$$\|f * g\|_{L^q(\mathbb{R}^N)} \leq \|f\|_{L^r(\mathbb{R}^N)}\|g\|_{L^p(\mathbb{R}^N)}, \text{ with } \frac{1}{q} = \frac{1}{p} + \frac{1}{r} - 1, \text{ for } 1 \leq p,\, q \leq \infty.$$

Let us consider the following extension of u,

$$\tilde{u}(x) = \begin{cases} u(x), & \text{if } x \in \Omega \\ 0, & \text{if } x \notin \Omega, \end{cases}$$

thus, we have for $x \in \Omega$

$$K_{J_0}(u)(x) = \int_\Omega J_0(x-y)u(y)dy = \int_{\mathbb{R}^N} J_0(x-y)\tilde{u}(y)dy = (J_0 * \tilde{u})(x).$$

Now, we define the extension of the operator K_{J_0} as

$$\widehat{K_{J_0}}(u)(x) = (J_0 * \tilde{u})(x), \text{ for } x \in \mathbb{R}^N,$$

then $K_{J_0}(u)(x) = \left(\widehat{K_{J_0}}(u)\right)\Big|_\Omega(x)$, for $x \in \Omega$. Thanks to Young's inequality, we have

$$\|K_{J_0}(u)\|_{L^q(\Omega)} \leq \|\widehat{K_{J_0}}(u)\|_{L^q(\mathbb{R}^N)} \leq \|J_0\|_{L^r(\mathbb{R}^N)}\|\tilde{u}\|_{L^p(\mathbb{R}^N)} = \|J_0\|_{L^r(\mathbb{R}^N)}\|u\|_{L^p(\Omega)}.$$

Hence, $\|K_{J_0}(u)\|_{L^q(\Omega)} \leq \|J_0\|_{L^r(\mathbb{R}^N)}\|u\|_{L^p(\Omega)}$, for all p, q, r such that $\frac{1}{q} = \frac{1}{p} + \frac{1}{r} - 1$.

ii. Following the same arguments made in Proposition 2.1.1 in (2.10), we know that for $x \in \Omega$,

$$\frac{\partial}{\partial_{x_i}}K_{J_0}(u) = K_{\frac{\partial J_0}{\partial x_i}}(u) = \left(\widehat{K}_{\frac{\partial J_0}{\partial x_i}}(u)\right)\Big|_\Omega$$

Then, applying part *i.* to $\|K_{J_0}(u)\|_{L^q(\Omega)}$ and $\|K_{\frac{\partial J_0}{\partial x_i}}(u)\|_{L^q(\Omega)}$ we have that for p, q, r such that $\frac{1}{q} = \frac{1}{p} + \frac{1}{r} - 1$, $K_{J_0} \in \mathcal{L}(L^p(\Omega), W^{1,q}(\Omega))$. Thus, the result. □

2.1.3 Compactness of K_J

Under the hypotheses on J in Proposition 2.1.1, in this section we give a result of compactness of the operator K_J.

The following lemma is a well known characterization of compact operators (see [13, p.157]).

Lemma 2.1.5. *Let E, F be Banach spaces and $(T_n)_{n\in\mathbb{N}}$ be a sequence of operators with finite rank from E to F, and let $T \in \mathcal{L}(E,F)$ such that $\|T_n - T\|_{\mathcal{L}(E,F)} \to 0$, as n goes to ∞. Then $T \in \mathcal{K}(E,F)$, that is, T is compact.*

The lemma below will help us to apply the previous Lemma 2.1.5 to the operator K_J.

Lemma 2.1.6. *For $1 \leq q < \infty$ and $1 \leq p \leq \infty$, let (Ω,μ) be a measure space, then any function $H \in L^q(\Omega, L^{p'}(\Omega))$ can be approximated by functions of separated variables in $L^q(\Omega, L^{p'}(\Omega))$.*

Proof. Observe that if $h(x,y) = f(x)g(y)$ is a function with separated variables, with $f \in L^q(\Omega)$ and $g \in L^{p'}(\Omega)$, then $h \in L^q(\Omega, L^{p'}(\Omega))$. We consider the space

$$V = \{\text{Finite linear combinations of functions with separated variables }\} \subset L^q(\Omega, L^{p'}(\Omega)).$$

First we prove that V is densely included in $L^q(\Omega, L^{p'}(\Omega))$, for $1 \le q < \infty$ and $1 < p \le \infty$ (i.e., all the cases except $p = 1$). To prove this, we suppose that there exists a function $L \in \left(L^q(\Omega, L^{p'}(\Omega))\right)' = L^{q'}\left(\Omega, \left(L^{p'}(\Omega)\right)'\right)$, with $L \perp V$, then,

$$\begin{aligned}
\int_\Omega \int_\Omega L(x,y) \sum_{j=1}^M f_j(x) g_j(y) dx\, dy &= 0, \ \forall M \in \mathbb{N}, \ \forall f_j \in L^q(\Omega), \ g_j \in L^{p'}(\Omega) \\
\sum_{j=1}^M \int_\Omega \int_\Omega L(x,y) f_j(x) g_j(y) dx\, dy &= 0, \ \forall M \in \mathbb{N}, \ \forall f_j \in L^q(\Omega), \ g_j \in L^{p'}(\Omega) \\
\sum_{j=1}^M \int_\Omega g_j(y) \int_\Omega L(x,y) f_j(x) dx\, dy &= 0, \ \forall M \in \mathbb{N}, \ \forall f_j \in L^q(\Omega), \ g_j \in L^{p'}(\Omega).
\end{aligned}$$

In particular, if we fix $f \in L^q(\Omega)$,

$$\int_\Omega g(y) \int_\Omega L(x,y) f(x) dx\, dy = 0 \quad \forall g \in L^{p'}(\Omega),$$

then

$$\int_\Omega L(x,y) f(x) dx = 0 \quad \forall f \in L^q(\Omega), \text{ for a.e. } y \in \Omega.$$

Therefore

$$L(x,y) = 0 \ \text{ for a.e. } x, y \in \Omega.$$

With this, we have proved that $V \hookrightarrow L^q(\Omega, L^{p'}(\Omega))$ densely, for $1 \le q < \infty$ and $1 < p \le \infty$.

Now we prove that V is densely included in $L^q(\Omega, L^{p'}(\Omega))$, for $1 \le q < \infty$ and $p = 1$. To prove this, we follow the same arguments we have already used, then we suppose that there exists a function $L \in (L^q(\Omega, L^\infty(\Omega)))' = L^{q'}(\Omega, \mathcal{M}(\Omega))$, with $L \perp V$, where we denote $\mathcal{M}(\Omega)$ as the set of Radon measures (see Theorem 1.1.4). Then $L \in L^{q'}(\Omega, \mathcal{M}(\Omega))$ is defined as

$$x \mapsto L(x, \cdot) \in \mathcal{M}(\Omega).$$

Since $L \perp V$, we have that

$$\begin{aligned}
\int_\Omega \int_\Omega \sum_{j=1}^M f_j(x) g_j(y) d_y L(x,y)\, dx &= 0, \ \forall M \in \mathbb{N}, \ \forall f_j \in L^q(\Omega), \ g_j \in L^\infty(\Omega) \\
\sum_{j=1}^M \int_\Omega \int_\Omega f_j(x) g_j(y) d_y L(x,y)\, dx &= 0, \ \forall M \in \mathbb{N}, \ \forall f_j \in L^q(\Omega), \ g_j \in L^\infty(\Omega) \\
\sum_{j=1}^M \int_\Omega f_j(x) \int_\Omega g_j(y) d_y L(x,y)\, dx &= 0, \ \forall M \in \mathbb{N}, \ \forall f_j \in L^q(\Omega), \ g_j \in L^\infty(\Omega).
\end{aligned}$$

In particular, if we fix $g \in L^\infty(\Omega)$,

$$\int_\Omega f(x) \int_\Omega g(y) d_y L(x,y)\, dx = 0 \quad \forall f \in L^q(\Omega),$$

then

$$\int_\Omega g(y) d_y L(x,y) = 0 \quad \forall g \in L^\infty(\Omega), \text{ for a.e. } x \in \Omega.$$

Therefore

$$L(x,\cdot) = 0 \ \text{ for a.e. } x \in \Omega, \quad \text{thus } \ L \equiv 0$$

With this, we have proved that $V \hookrightarrow L^q(\Omega, L^\infty(\Omega))$ densely, for $1 \le q < \infty$ and $p = 1$. $\square$

In the following proposition we prove the main result of compactness. Note that we have almost the same assumptions as in Proposition 2.1.1, (see Remark 2.1.8).

Proposition 2.1.7.

i. *For $1 \le p \le \infty$ and $1 \le q < \infty$, if $J \in L^q(\Omega, L^{p'}(\Omega))$ then $K_J \in \mathcal{L}(L^p(\Omega), L^q(\Omega))$ is compact.*

ii. *For $1 \le p \le \infty$, if $J \in BUC(\Omega, L^{p'}(\Omega))$ (bounded and uniformly continuous), then $K_J \in \mathcal{L}(L^p(\Omega), \mathcal{C}_b(\Omega))$ is compact. In particular, $K_J \in \mathcal{L}(L^p(\Omega), L^\infty(\Omega))$ is compact.*

iii. *For $1 \le p \le \infty$ and $1 \le q < \infty$, if $\Omega \subset \mathbb{R}^N$ is* **open** *and $J \in W^{1,q}(\Omega, L^{p'}(\Omega))$ then $K_J \in \mathcal{L}(L^p(\Omega), W^{1,q}(\Omega))$ is compact.*

Proof.

i. Since $J \in L^q(\Omega, L^{p'}(\Omega))$, for $1 \le p \le \infty$ and $1 \le q < \infty$, we know from Lemma 2.1.6 that there exist $M(n) \in \mathbb{N}$ and $f_j^n \in L^q(\Omega)$, $g_j^n \in L^{p'}(\Omega)$ with $j = 1, ..., M(n)$ such that $J(x,y)$ can be approximated by functions that are a finite linear combination of functions with separated variables defined as,

$$J^n(x,y) = \sum_{j=1}^{M(n)} f_j^n(x) g_j^n(y)$$

and $\|J - J^n\|_{L^q(\Omega, L^{p'}(\Omega))} \to 0$, as n goes to ∞.

First of all we are going to prove that K_J can be approximated by operators with finite rank in $\mathcal{L}(L^p\Omega), L^q(\Omega))$. To do this, we first define

$$K_J^n(u)(x) = K_{J^n}(u)(x) = \sum_{j=1}^{M(n)} f_j^n(x) \int_\Omega g_j^n(y) u(y) dy,$$

and we prove that K_J^n converges strongly to the operator

$$K_J(u)(x) = \int_\Omega J(x,y) u(y) dy.$$

Since $K_J - K_J^n = K_{J-J^n}$, and thanks to Proposition 2.1.1, we have that,

$$\|K_J - K_J^n\|_{\mathcal{L}(L^p(\Omega), L^q(\Omega))} \le \|J - J^n\|_{L^q(\Omega, L^{p'}(\Omega))}.$$

Therefore, thanks to Lemma 2.1.6, we have that $\|J - J^n\|_{L^q(\Omega, L^{p'}(\Omega))} \to 0$, as n goes to ∞. Hence, we have proved that

$$\|K_J - K_J^n\|_{\mathcal{L}(L^p(\Omega), L^q(\Omega))} \to 0, \text{ as } n \text{ goes to } \infty.$$

Finally applying Lemma 2.1.5 to the operator K_J, we have that $K_J \in \mathcal{L}(L^p(\Omega), L^q(\Omega))$ is compact.

ii. If $J \in BUC(\Omega, L^{p'}(\Omega))$, then hypothesis (2.5) of Proposition 2.1.1 is satisfied and then $K_J \in \mathcal{L}(L^p(\Omega), \mathcal{C}_b(\Omega))$. Now, we consider $u \in B \subset L^p(\Omega)$, where B is the unit ball in $L^p(\Omega)$. Now, we prove using Ascoli-Arzela Theorem (see [13, p. 111]), that $K_J(B)$ is relatively compact in $\mathcal{C}_b(\Omega)$.

Let $x, z \in \Omega$, $u \in B$, thanks to Hölder's inequality, we have,

$$\begin{aligned} |K_J(u)(z) - K_J(u)(x)| &= \left| \int_\Omega J(z,y)u(y)dy - \int_\Omega J(x,y)u(y)dy \right| \\ &\le \|J(z,\cdot) - J(x,\cdot)\|_{L^{p'}(\Omega)} \|u\|_{L^p(\Omega)} \\ &\le \|J(z,\cdot) - J(x,\cdot)\|_{L^{p'}(\Omega)}. \end{aligned} \tag{2.16}$$

Since $J \in BUC(\Omega, L^{p'}(\Omega))$, then for all $\varepsilon > 0$, there exists $\delta > 0$ such that if x, $z \in \Omega$ satisfy that $d(z,x) < \delta$, then $\|J(z,\cdot) - J(x,\cdot)\|_{L^{p'}(\Omega)} < \varepsilon$. Hence, we have that $K_J(B)$ is equicontinuous.

On the other hand, thanks to Hölder's inequality, $\forall x \in \Omega$,

$$\begin{aligned} |K_J(u)(x)| &= \left| \int_\Omega J(x,y)u(y)dy \right| \\ &\le \|J(x,\cdot)\|_{L^{p'}(\Omega)} < \infty, \quad \forall u \in B. \end{aligned}$$

Thus, the hypotheses of Ascoli-Arzela Theorem are satisfied, then we have that $K_J(B)$ is precompact. Therefore we have proved that $K_J \in \mathcal{L}(L^p(\Omega), \mathcal{C}_b(\Omega))$ is compact.

The second part of the result is immediate. We have proved that $K_J \in \mathcal{L}(L^p(\Omega), \mathcal{C}_b(\Omega))$ is linear and compact. Moreover, $\mathcal{C}_b(\Omega) \subset L^\infty(\Omega)$, then we have that $K_J \in \mathcal{L}(L^p(\Omega), L^\infty(\Omega))$ is compact.

iii. Thanks to the argument (2.10) in Proposition 2.1.1, we have that $\frac{\partial}{\partial x_i} K_J(u) = K_{\frac{\partial J}{\partial x_i}}(u)$. Since $J \in W^{1,q}(\Omega, L^{p'}(\Omega))$, we have that $J \in L^q(\Omega, L^{p'}(\Omega))$ and $\frac{\partial J}{\partial x_i} \in L^q(\Omega, L^{p'}(\Omega))$, for all $i = 1, \dots, N$. Using part *i.* we obtain that $K_{\frac{\partial J}{\partial x_i}} \in \mathcal{L}(L^p(\Omega), L^q(\Omega))$ is compact. Thus, if B is the unit ball in $L^p(\Omega)$, we have that $K_J(B)$ and $K_{\frac{\partial J}{\partial x_i}}(B)$ are precompact for all $i = 1, \dots, N$.

Now we consider the mapping

$$\begin{aligned} \mathcal{T}: \ L^p(\Omega) &\longrightarrow \left(L^q(\Omega)\right)^{N+1} \\ u &\longmapsto \left(K_J(u),\, K_{\frac{\partial J}{\partial x_1}}(u), \dots,\, K_{\frac{\partial J}{\partial x_N}}(u) \right). \end{aligned}$$

Thanks to Tíkhonov's Theorem (see [42, p. 167]), we know that $\mathcal{T}(B)$ is precompact in $\left(L^q(\Omega)\right)^{N+1}$.

Moreover, we consider the mapping

$$\begin{array}{rccc} \mathcal{S}: & W^{1,q}(\Omega) & \hookrightarrow & (L^q(\Omega))^{N+1} \\ & g & \longmapsto & \left(g, \frac{\partial g}{\partial x_1}, \dots, \frac{\partial g}{\partial x_N}\right). \end{array}$$

Since $\mathcal{S}$ is an isometry, i.e., $\|g\|_{W^{1,q}(\Omega)} = \|\mathcal{S}(g)\|_{(L^q(\Omega))^{N+1}}$, then we have that $\mathcal{S}^{-1}|_{Im(\mathcal{S})}: Im(\mathcal{S}) \subset (L^q(\Omega))^{N+1} \to W^{1,q}(\Omega)$ is continuous. On the other hand, thanks to the hypotheses on J and Proposition 2.1.1, we have that $K_J \in \mathcal{L}\left(L^p(\Omega), W^{1,q}(\Omega)\right)$. Thus, $Im(\mathcal{T}) \subset Im(\mathcal{S})$. Hence, the operator $K_J : L^p(\Omega) \to W^{1,q}(\Omega)$, can be written as

$$K_J(u) = \mathcal{S}^{-1}|_{Im(\mathcal{S})} \circ \mathcal{T}(u).$$

Therefore, we have that K_J is the composition of a continuous operator $\mathcal{S}^{-1}|_{Im(\mathcal{S})}$, with a compact operator $\mathcal{T}$. Thus, the result. $\square$

Remark 2.1.8. *In general, we have proved that K_J is compact, under the same hypotheses of Proposition 2.1.1. But to prove that $K \in \mathcal{L}(L^p(\Omega), L^\infty(\Omega))$ is compact, we assume that $J \in BUC(\Omega, L^{p'}(\Omega))$, instead of $J \in L^\infty(\Omega, L^{p'}(\Omega))$, that was the assumption in Proposition 2.1.1.*

Moreover we have not proved that $K_J \in \mathcal{L}(L^p(\Omega), W^{1,\infty}(\Omega))$ is compact.

We finish this section applying interpolation theorems. The following result is valid for a general operator K, not necessarily an integral operator.

Proposition 2.1.9. *Let (Ω, μ) be a measure space, and let $\mu(\Omega) < \infty$. For $1 \leq p_0 < p_1 < \infty$, if $K \in \mathcal{L}(L^{p_0}(\Omega), L^{p_0}(\Omega))$ and $K \in \mathcal{L}(L^{p_1}(\Omega), L^{p_1}(\Omega))$ then $K \in \mathcal{L}(L^p(\Omega), L^p(\Omega))$, for all $p \in [p_0, p_1]$.*
Suppose that either:

i. $K \in \mathcal{L}(L^{p_0}(\Omega), L^{p_0}(\Omega))$ *is compact,*

ii. $K \in \mathcal{L}(L^{p_1}(\Omega), L^{p_1}(\Omega))$ *is compact,*

then $K \in \mathcal{L}(L^p(\Omega), L^p(\Omega))$ is compact for all $p \in [p_0, p_1]$.

Proof. Thanks to the hypotheses and Riesz-Thorin Theorem, (see [10, p. 196]), we have that $K \in \mathcal{L}(L^p(\Omega), L^p(\Omega))$, for all $p \in [p_0, p_1]$. The proof of the compactness can be seen in [21, p. 4]. $\square$

2.1.4 Positiveness of the operator K_J

In this section, given a nonnegative function z, which is not identically zero, we describe the set of points where $K_J(z)$ is strictly positive, under hypothesis (2.3) on the kernel J. To do this, we need first to introduce the definition of *essential support* associated to a nonnegative measurable function z.

Definition 2.1.10. *Let z be a measurable nonnegative function $z : \Omega \to \mathbb{R}$. We define the* **essential support** *associated to z as:*

$$P(z) = \big\{x \in \Omega \,:\, \forall \delta > 0,\ \mu\big(\{y \in \Omega :\ z(y) > 0\} \cap B_\delta(x)\big) > 0\big\}, \tag{2.17}$$

where μ is the measure of the set, and $B_\delta(x)$ is the ball centered in x, with radius δ.

The following lemma will be useful to understand better the essential support of a non-negative function z.

Lemma 2.1.11. *Let z be a nonnegative measurable function $z : \Omega \to \mathbb{R}$, then the following properties are equivalent:*

i. $z \geq 0$ *not identically zero.*

ii. $P(z) \neq \emptyset$.

iii. $\mu\,(P(z)) > 0$.

Proof.

i) ⇒ ii) From Definition 2.1.10 of $P(z)$, we have that $x \notin P(z)$, if and only if, there exists $\delta > 0$ such that

$$\mu\big(\{y \in \Omega :\ z(y) > 0\} \cap B_\delta(x)\big) = 0. \tag{2.18}$$

Then, we have that $B_\delta(x) \subset P(z)^c$. Indeed, since $B_\delta(x)$ is open then for any $\tilde{x} \in B_\delta(x)$, there exists $\varepsilon > 0$, such that $B_\varepsilon(\tilde{x}) \subset B_\delta(x)$. Thus, from (2.18) we obtain that

$$\mu\big(\{y \in \Omega :\ z(y) > 0\} \cap B_\varepsilon(\tilde{x})\big) = 0.$$

Hence $\tilde{x} \notin P(z)$, and we have proved that $B_\delta(x) \subset P(z)^c$.

This implies that, $P(z)^c$ is open and $z(x) = 0$ for a.e. $x \in P(z)^c$. Furthermore, we have that $P(z)^c$ is the largest open set where $z \equiv 0$ almost everywhere.

Now, we assume that $P(z) = \emptyset$, then $P(z)^c = \Omega$. Thus $z \equiv 0$ a.e. in Ω, and we arrive to contradiction. Therefore, $P(z) \neq \emptyset$.

i) ⇒ iii) If $\mu(P(z)) = 0$, then $P(z)^c = \Omega \setminus P(z)$ satisfies that $\mu(P(z)^c) = \mu(\Omega)$. Hence, $z \equiv 0$ a.e. in Ω, and we arrive to contradiction. Therefore, $\mu(P(z)) > 0$.

ii) ⇒ i) If $P(z) \neq \emptyset$ then, there exists $x \in \Omega$ and $\delta > 0$ such that

$$\mu\left(\{y \in \Omega :\ z(y) > 0\} \cap B_\delta(x)\right) > 0.$$

Thus, there exists a set with positive measure where z is strictly positive.

iii) ⇒ ii) If $\mu(P(z)) > 0$, then $P(z) \neq \emptyset$. □

Let us introduce the following definitions.

Definition 2.1.12. *Let z be a measurable nonnegative function $z : \Omega \to \mathbb{R}$. For $R > 0$, we denote*

$$P^0(z) = P(z),$$

the essential support of z, and we define the open sets:

$$P_R^1(z) = \bigcup_{x \in P^0(z)} B(x,R), \quad P_R^2(z) = \bigcup_{x \in P_R^1(z)} B(x,R), \ldots, \quad P_R^n(z) = \bigcup_{x \in P_R^{n-1}(z)} B(x,R), \ldots$$

for all $n \in \mathbb{N}$.

Remark 2.1.13. *If the metric of Ω is equivalent to the euclidean metric in $\mathbb{R}^N$, then the sets $P_R^n(z)$ in Definition 2.1.12 are equal to*

$$P_R^n(z) = (P(z) + B_{nR}) \cap \Omega,$$

where B_{nR} is the ball centered in zero with radius nR.

Definition 2.1.14. *Let (Ω, μ, d) be a metric measure space, and $R > 0$. We say that Ω is* ***R-connected*** *if $\forall x, y \in \Omega$, $\exists N \in \mathbb{N}$ and a finite set of points $\{x_0, \ldots, x_N\}$ in Ω such that $x_0 = x$, $x_N = y$ and $d(x_{i-1}, x_i) < R$, for all $i = 1, \ldots, N$.*

Lemma 2.1.15. *If Ω is compact and connected then Ω is R-connected for any $R > 0$.*

Proof. Since Ω is compact, given $R > 0$, there exists $n \in \mathbb{N}$ and $\{y_1, \ldots, y_n\} \subset \Omega$ such that $\Omega \subset \bigcup\limits_{i=1}^{n} B(y_i, R/4)$. Moreover, $\forall i = 1, \ldots, n$, $B(y_i, R/4) \cap \bigcup\limits_{j \neq i} B(y_j, R/4) \neq \emptyset$, since otherwise $\Omega = B(y_i, R/4) \cup \bigcup\limits_{j \neq i} B(y_j, R/4)$, contradicting that Ω is connected. Analogously, we have that $\forall i_1, \ldots, i_k \in \{1, \ldots, n\}$, $\bigcup\limits_{r=1}^{k} B(y_{i_r}, R/4) \ \cap \bigcup\limits_{j \notin \{i_1, \ldots, i_k\}} B(y_j, R/4) \neq \emptyset$

Now, let us prove that Ω is R-connected for any $R > 0$. Then, given any two points x, y in Ω, first of all we consider $x_0 = x$, and we choose a ball such that $x \in B(y_{i_1}, R/4)$, since Ω is connected, then there exists a ball $B(y_{i_2}, R/4)$ that intersects $B(y_{i_1}, R/4)$, and we choose $x_1 = y_{i_2}$. If $y \in B(y_{i_2}, R/4)$, we finish the proof, if not, following this constructing argument, we obtain there exists a ball $B(y_{i_3}, R/4)$ that intersects $B(y_{i_1}, R/4) \cup B(y_{i_2}, R/4)$, and we choose $x_2 = y_{i_3}$. If $y \in B(y_{i_3}, R/4)$ we finish the proof, if not, with a continuation argument, we find a finite set of points $\{x_0, \ldots, x_{n-1}\}$ such that $x_0 = x$, $x_{N-1} = y$ and $d(x_{i-1}, x_i) < R$, for $i = 1, \ldots, N$, where $N \leq n$. Thus, the result. $\square$

Lemma 2.1.16. *Let (Ω, μ, d) be a metric measure space such that Ω is R-connected. For a fixed $x_0 \in \Omega$, and for some $R > 0$, we set*

$$P_{x_0}^0 = \{x_0\}, \quad P_{R,x_0}^1 = B(x_0, R) \ \text{ and } \ P_{R,x_0}^n = \bigcup_{x \in P_{R,x_0}^{n-1}} B(x,R) \ \text{ for all } \ n \in \mathbb{N}.$$

Then, for every compact set in $\mathcal{K} \subset \Omega$, there exists $n(x_0) \in \mathbb{N}$ such that $\mathcal{K} \subset P_{R,x_0}^n$ for all $n \geq n(x_0)$.

Furthermore, if Ω is compact, there exists $n_0 \in \mathbb{N}$ such that for any $y \in \Omega$, $\Omega = P_{R,y}^n$, for all $n \geq n_0$.

Proof. Since Ω is R-connected, fixed $x_0 \in \Omega$, for any $y \in \Omega$, $\exists M = M_y \in \mathbb{N}$ and a finite set of points $\{x_0, \dots, x_M\}$ such that $x_M = y$ and $d(x_{i-1}, x_i) < R$, for all $i = 1, \dots, M$. Thus, $x_1 \in B(x_0, R) = P^1_{R,x_0}$, $x_2 \in B(x_1, R) \subset P^2_{R,x_0}$, $B(x_i, R) \subset P^{i+1}_{R,x_0}$, for all $i = 1, \dots, M$. In particular, $y \in P^M_{R,x_0}$ and

$$B(y, R) \subset P^{M+1}_{R,x_0}. \tag{2.19}$$

Arguing analogously, we obtain that $x_0 \in P^M_{R,y}$. Then we have proved that if Ω is R-connected, $\forall x, y \in \Omega$, $\exists N \in \mathbb{N}$ such that $x \in P^N_{R,y}$ and $y \in P^N_{R,x}$, i.e., there exists and R-chain of N-steps that joins x and y, and there exists and R-chain of N-steps that joins y and x.

On the other hand, since $\mathcal{K}$ is compact, $\mathcal{K} \subset \bigcup_{y\in\mathcal{K}} B(y, R)$, there exists $n \in \mathbb{N}$ such that $\mathcal{K} \subset \bigcup_{i=1}^n B(y_i, R)$. From (2.19), for every y_i there exists M_{y_i} such that $B(y_i, R) \subset P^{M_{y_i}+1}_{R,x_0}$. We choose $n(x_0) = \max\limits_{i=1,\dots,n}(M_{y_i} + 1)$, and we obtain that $\mathcal{K} \subset P^{n(x_0)}_{R,x_0}$. Therefore, $\mathcal{K} = P^n_{R,x_0}$, for all $n \geq n(x_0)$. Thus, the result.

If Ω is compact. From the previous result we know that fixed $x_0 \in \Omega$, $\exists N = N(x_0)$ such that $\Omega = P^N_{R,x_0}$. Moreover, $\Omega = P^N_{R,x_0}$ if and only if $\forall y \in \Omega$, $y \in P^N_{R,x_0}$ and $x_0 \in P^N_{R,y}$, i.e., there exists an R-chain of N-steps that joins x_0 and y. Therefore, for all $y_1, y_2 \in \Omega$ there exists an R-chain of $2N$-steps that joins y_1 and y_2. This is because there exists an R-chain of N- steps that joins y_1 with x_0, and there exists an R-chain of N-steps that joins x_0 with y_2, then joining both R-chains, we obtain that for any $y_1 \in \Omega$, $y_1 \in P^{2N}_{R,y_2}$, for all $y_2 \in \Omega$. Hence $\Omega \subset P^{2N}_{R,y_2}$, for all $y_2 \in \Omega$. Thus, we have proved the result with $n_0 = 2N$. $\square$

Now, we prove the main result.

Proposition 2.1.17. *Let (Ω, μ, d) be a metric measure space, and let J satisfy that $J \geq 0$ not identically zero, with*

$$J(x, y) > 0 \text{ for all } x,\, y \in \Omega, \text{ such that } d(x, y) < R, \tag{2.20}$$

for some $R > 0$. If z is a measurable function defined in Ω, with $z \geq 0$, not identically zero. Then,

$$P(K^n_J(z)) \supset P^n_R(z), \quad \text{for all } n \in \mathbb{N}.$$

If Ω is R-connected, then for any compact set $\mathcal{K} \subset \Omega$,

$$\exists\, n_0(z) \in \mathbb{N}, \quad \text{such that } P(K^n_J(z)) \supset \mathcal{K}, \text{ for all } n \geq n_0(z).$$

If Ω is compact and connected, then $\exists\, n_0 \in \mathbb{N}$, such that, for all $z \geq 0$ measurable and not identically zero

$$P(K^n_J(z)) = \Omega, \text{ for all } n \geq n_0.$$

Proof. First of all we prove that $P\left(K_J(z)\right) \supset P^1_R(z)$. Since $z \geq 0$, not identically zero, and as a consequence of Lemma 2.1.11, we have that $\mu\left(P(z)\right) > 0$. Then,

$$K_J(z)(x) = \int_\Omega J(x, y)z(y)dy \geq \int_{P(z)} J(x, y)z(y)dy.$$

From hypothesis (2.20) on the positivity of J, we have that

$$K_J(z)(x) > 0 \text{ for all } x \in \bigcup_{y \in P(z)} B(y, R) = P_R^1(z). \tag{2.21}$$

Since $P_R^1(z)$, is an open set in Ω, we have that, if $x \in P_R^1(z)$, then

$$\mu\left(B(x, \delta) \cap P_R^1(z)\right) > 0 \quad \text{for all } 0 < \delta \in \mathbb{R}. \tag{2.22}$$

Thus, thanks to (2.21) and (2.22), we have that

$$P\left(K_J(z)\right) \supset P_R^1(z). \tag{2.23}$$

Applying K_J to $K_J(z)$, following the previous arguments and thanks to (2.23), we obtain

$$P\left(K_J^2(z)\right) \supset P_R^1(K_J(z)) = \bigcup_{x \in P(K_J(z))} B(x, R) \supset \bigcup_{x \in P_R^1(z)} B(x, R) = P_R^2(z).$$

Therefore, iterating this process, we finally obtain that

$$P(K_J^n(z)) \supset P_R^n(z), \ \forall n \in \mathbb{N}. \tag{2.24}$$

Now consider $\mathcal{K} \subset \Omega$ a compact set in Ω, and taking $x_0 \in P(z)$, then thanks to Lemma 2.1.16 there exists $n_0(z) \in \mathbb{N}$, such that $\mathcal{K} \subset P_R^n(z)$ for all $n \geq n_0$, then thanks to (2.24), $\mathcal{K} \subset P(K_J^n(z))$ for all $n \geq n_0$.

If Ω is compact and connected, thanks to Lemma 2.1.15, Ω is R- connected. From Lemma 2.1.16 there exists $n_0 \in \mathbb{N}$ such that for any $y \in \Omega$, $\Omega = P_{R,y}^n$, for all $n \geq n_0$. Hence, from (2.24), for any $z \geq 0$ not identically zero, taking $y \in P(z)$, $P(K_J^n(z)) \supset P_{R,y}^n = \Omega$, for all $n \geq n_0$. □

Remark 2.1.18. *In Figure 2.1 can be seen which is the set where the function J is strictly positive under hypothesis* (2.20)*, in the particular case in which $\Omega \subset \mathbb{R}$.*

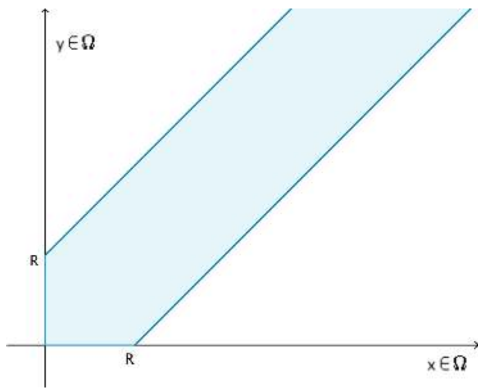

Figure 2.1: Domain where J is strictly positive if $\Omega \subset \mathbb{R}$.

Furthermore, the hypothesis (2.20) *is somehow an optimal condition. We give below a counterexample in* $\mathbb{R}$*: if the hypothesis* (2.20) *is not satisfied, we find a function* z_0*, for which the previous Proposition 2.1.17 is not satisfied.*

COUNTEREXAMPLE: *Let* $\Omega \subset \mathbb{R}$*,* $\Omega = [0, L]$*, with* $L > 0$ *and let us fix an arbitrary* $x_0 = 1/2 \in [0, 1]$*, and* $R > 0$ *small enough such that* $(1/2 - R, 1/2 + R) \subset [0, 1]$*. We consider a function* J *satisfying that* $J \geq 0$ *defined as*

$$J(x,y) = \begin{cases} 1, \, (x,\, y) \in \big([0,1] \times [0,1]\big) \setminus \big(\big(\frac{1}{2} - R, \frac{1}{2} + R\big) \times \big(\frac{1}{2} - R, \frac{1}{2} + R\big)\big), \textit{ with } d(x,y) < R, \\ 0 \quad \textit{for the rest of } (x,\, y). \end{cases} \tag{2.25}$$

We remark that, $J(x,y) = 0$, $\forall\, (x,\, y) \in \big(\frac{1}{2} - R, \frac{1}{2} + R\big) \times \big(\frac{1}{2} - R, \frac{1}{2} + R\big)$.

Now, we consider a function $z_0 : \Omega \to \mathbb{R}$ *,* $z_0 \geq 0$*, such that*

$$P(z_0) \subset [1/2, 1].$$

Since $z_0(y) = 0$ *for all* $y \notin [1/2, 1]$*, we have that*

$$K_J(z_0)(x) = \int_\Omega J(x,y) z_0(y) dy = \int_{P(z_0)} J(x,y) z_0(y) dy = \int_{1/2}^1 J(x,y) z_0(y) dy.$$

Moreover, from (2.25)*, we have that for* $\tilde{x} \in [0, 1/2)$*,* $J(\tilde{x}, y) = 0$ *for all* $y \in [1/2, 1]$*. Let us prove this below:*

- *If* $\tilde{x} \in (1/2 - R, 1/2)$*, then*
  - *if* $y \in [1/2, 1/2 + R)$*, then* $(\tilde{x}, y) \in \big(\frac{1}{2} - R, \frac{1}{2} + R\big) \times \big(\frac{1}{2} - R, \frac{1}{2} + R\big)$*, and* $J(\tilde{x}, y) = 0$*;*
  - *if* $y \in [1/2 + R, 1]$*, then* $d(\tilde{x}, y) > R$*, and* $J(\tilde{x}, y) = 0$*.*
- *If* $\tilde{x} \in [0, 1/2 - R]$*, then for* $y \in [1/2, 1]$*,* $d(\tilde{x}, y) > R$*, and* $J(\tilde{x}, y) = 0$*.*

Thus,

$$K_J(z_0)(\tilde{x}) = \int_{1/2}^1 J(\tilde{x}, y) z_0(y) dy = 0, \ \forall \tilde{x} \in [0, 1/2).$$

Hence

$$P(K_J(z_0)) \subset [1/2, 1].$$

If we apply K_J *to* $K_J(z_0)$*, we obtain that*

$$K_J^2(z_0)(x) = \int_\Omega J(x,y) K_J(z_0)(y) dy = \int_{P(K_J(z_0))} J(x,y) K_J(z_0)(y) dy = \int_{1/2}^1 J(x,y) K_J(z_0)(y) dy.$$

Arguing as above, we have that for any $\tilde{x} \in [0, 1/2)$*,* $J(\tilde{x}, y) = 0$*, for all* $y \in [1/2, 1]$*. Thus,*

$$P\big(K_J^2(z_0)\big) \subset [1/2, 1].$$

Therefore, iterating this process, we obtain that

$$P(K_J^n(z_0)) \subset [1/2, 1] \quad \textit{for all } \ n \in \mathbb{N}.$$

Hence $P(K_J^n(z_0)) \neq [0, 1]$ *for all* $n \in \mathbb{N}$*, and the hypothesis* (2.20) *is essentially optimal.*

2.1.5 The adjoint operator of K_J

In this section we describe the adjoint operator associated to K_J, and we prove that if $J \in L^2(\Omega \times \Omega)$ and $J(x,y) = J(y,x)$ then the operator K_J is selfadjoint in $L^2(\Omega)$.

Proposition 2.1.19. *For $1 \leq p < \infty$, $1 \leq q < \infty$. Let (Ω, μ) be a measure space. We assume that the mapping*

$$x \mapsto J(x,\cdot) \ \textit{ satisfies that } \ J \in L^q(\Omega, L^{p'}(\Omega)),$$

and the mapping

$$y \mapsto J(\cdot,y) \ \textit{ satisfies that } \ J \in L^{p'}(\Omega, L^q(\Omega)).$$

Then the adjoint operator associated to $K_J \in \mathcal{L}(L^p(\Omega), L^q(\Omega))$, is

$$K_J^* : L^{q'}(\Omega) \to L^{p'}(\Omega), \quad \textit{with } K_J^* = K_{J^*},$$

where $J^*(x,y) = J(y,x)$.
If J satisfies that

$$J(x,y) = J(y,x), \tag{2.26}$$

then for $u \in L^p(\Omega)$ and $v \in L^{q'}(\Omega)$,

$$\langle K_J(u),\, v\rangle_{L^q, L^{q'}} = \langle u, K_J(v)\rangle_{L^p, L^{p'}}. \tag{2.27}$$

In the particular case in which $p = q = 2$ and $J \in L^2(\Omega \times \Omega)$, the operator K_J is selfadjoint in $L^2(\Omega)$.

Proof. We consider $u \in L^p(\Omega)$ and $v \in L^{q'}(\Omega)$. Thanks to Fubini's Theorem and the hypothesis on J

$$\langle K_J(u), v\rangle_{L^q(\Omega), L^{q'}(\Omega)} = \int_\Omega \int_\Omega J(x,y)u(y)dy\, v(x)dx = \int_\Omega \int_\Omega J(x,y)v(x)dx\, u(y)dy,$$

and $\displaystyle\int_\Omega \int_\Omega J(x,y)v(x)dx\, u(y)dy = \langle u,\, K_J^*(v)\rangle_{L^p(\Omega), L^{p'}(\Omega)}$, with

$$K_J^*(v)(y) = \int_\Omega J(x,y)v(x)dx = \int_\Omega J^*(y,x)v(x)dx = K_{J^*}(v)(y),$$

and $J^*(y,x) = J(x,y)$.

In particular if $u \in L^p(\Omega)$ and $v \in L^{q'}(\Omega)$ and J satisfies that $J(x,y) = J(y,x)$, we obtain

$$\langle K_J(u),\, v\rangle_{L^q, L^{q'}} = \langle u, K_J(v)\rangle_{L^p, L^{p'}}. \tag{2.28}$$

An immediate consequence of (2.28) is the case in which $p = q = 2$, that we have that K_J is selfadjoint in $L^2(\Omega)$. $\square$

2.1.6 Spectrum of K_J

In this section, we are going to prove that under certain hypotheses on K_J, $\sigma_X(K_J)$ is independent of X, with $X = L^p(\Omega)$, where $1 \leq p \leq \infty$ or $X = \mathcal{C}_b(\Omega)$. We also characterize the spectrum of K_J when K_J is selfadjoint in $L^2(\Omega)$, and we finish this section proving that under the same hypothesis on the positivity of J in Proposition 2.1.17, the spectral radius of K_J in $\mathcal{C}_b(\Omega)$ is a simple eigenvalue that has a strictly positive eigenfunction associated.

The proposition below is for a general compact operator K, not only for the integral operator K_J (see Propositions 2.1.7 to check compactness for operators with kernel, K_J).

Proposition 2.1.20. *Let (Ω, μ, d) be a metric measure space with μ as in Definition 1.1.5 and $\mu(\Omega) < \infty$.*

i. For $1 \leq p_0 < p_1 < \infty$, if $K \in \mathcal{L}(L^{p_0}(\Omega), L^{p_1}(\Omega))$ and additionally $K \in \mathcal{L}(L^{p_0}(\Omega), L^{p_0}(\Omega))$ is compact then $K \in \mathcal{L}(L^p(\Omega), L^p(\Omega))$, $\forall p \in [p_0, p_1]$, and $\sigma_{L^p}(K)$ is independent of p.

ii. For $1 \leq p_0 < p_1 \leq \infty$, if $K \in \mathcal{L}(L^{p_0}(\Omega), L^{p_1}(\Omega))$ is compact, then $K \in \mathcal{L}(L^p(\Omega), L^p(\Omega))$, $\forall p \in [p_0, p_1]$, and $\sigma_{L^p}(K)$ is independent of p.

iii. For $1 \leq p_0 \leq \infty$, if $K \in \mathcal{L}(L^{p_0}(\Omega), \mathcal{C}_b(\Omega))$ is compact and $X = \mathcal{C}_b(\Omega)$ or $X = L^r(\Omega)$ with $r \in [p_0, \infty]$, then $K \in \mathcal{L}(X, X)$, and $\sigma_X(K)$ is independent of X.

Proof.
i. Thanks to Proposition 2.1.9, we have that $K \in \mathcal{L}(L^p(\Omega), L^p(\Omega))$ is compact for all $p \in [p_0, p_1]$. Thus the spectrum of K is composed by zero and a discrete set of eigenvalues of finite multiplicity, (see [13, chap. 6]). Let us prove now that the eigenvalues of the spectrum $\sigma_{L^p(\Omega)}(K)$ are independent of p.

We prove first that $\sigma_{L^{p_1}(\Omega)} \subset \sigma_{L^p(\Omega)}$: if $\lambda \in \sigma_{L^{p_1}}(K)$ is an eigenvalue, then the associated eigenfunction $\Phi \in L^{p_1}(\Omega)$. Since $\mu(\Omega) < \infty$ we have that $L^{p_1}(\Omega) \hookrightarrow L^p(\Omega)$ continuously, for all $p \leq p_0$, then $\Phi \in L^p(\Omega)$. Thus we obtain that $\lambda \in \sigma_{L^p}(K)$ for all $p \in [p_0, p_1]$.

Now, we prove that $\sigma_{L^p(\Omega)} \subset \sigma_{L^{p_1}(\Omega)}$: if $\lambda \in \sigma_{L^p(\Omega)}(K)$ is an eigenvalue, with $p \in [p_0, p_1)$, then the associated eigenfunction $\Phi \in L^p(\Omega)$ satisfies that

$$K(\Phi) = \lambda\Phi. \tag{2.29}$$

Since $L^p(\Omega) \hookrightarrow L^{p_0}(\Omega)$ continuously and $K : L^{p_0}(\Omega) \to L^{p_1}(\Omega)$, then $K(\Phi) \in L^{p_1}(\Omega)$. From (2.29), we obtain that $\Phi \in L^{p_1}(\Omega)$. Hence, $\Phi \in L^p(\Omega)$ for $p \in [p_0, p_1]$. Thus, the result.

ii. We know that $K \in \mathcal{L}(L^{p_0}(\Omega), L^{p_1}(\Omega))$ is compact, and we have that

$$K : L^{p_1}(\Omega) \hookrightarrow L^{p_0}(\Omega) \longrightarrow L^{p_1}(\Omega)$$

and

$$K : L^{p_0}(\Omega) \longrightarrow L^{p_1}(\Omega) \hookrightarrow L^{p_0}(\Omega).$$

Therefore $K \in \mathcal{L}(L^{p_1}(\Omega), L^{p_1}(\Omega))$ is compact, and the hypotheses of Proposition 2.1.9 are satisfied. Therefore $K \in \mathcal{L}(L^p(\Omega), L^p(\Omega))$ is compact for all $p \in [p_0, p_1]$. From part *i.*, we have the result.

iii. We know that $K \in \mathcal{L}(L^{p_0}(\Omega), \mathcal{C}_b(\Omega))$ is compact. Since $\mu(\Omega) < \infty$, we have that

$$K : \mathcal{C}_b(\Omega) \hookrightarrow L^{p_0}(\Omega) \longrightarrow \mathcal{C}_b(\Omega)$$

and

$$K : L^{p_0}(\Omega) \longrightarrow \mathcal{C}_b(\Omega) \hookrightarrow L^{p_0}(\Omega)$$

and for $r \in [p_0, \infty]$

$$K : L^r(\Omega) \hookrightarrow L^{p_0}(\Omega) \longrightarrow \mathcal{C}_b(\Omega) \hookrightarrow L^r(\Omega)$$

Therefore, $K \in \mathcal{L}(X, X)$ is compact for $X = \mathcal{C}_b(\Omega)$ or $X = L^r(\Omega)$ with $r \in [p_0, \infty]$. Hence, following the arguments in *i.* we have that $\sigma_X(K)$ is independent of X. $\square$

The following result holds for a general selfadjoint operator in a Hilbert space, and the proof can be found in [13, p. 165].

Proposition 2.1.21. *Let H be a Hilbert space and $T \in \mathcal{L}(H)$ a selfadjoint operator. Take*

$$m = \inf_{\substack{u \in H \\ \|u\|_H = 1}} \langle Tu, u \rangle_H \qquad \text{and} \qquad M = \sup_{\substack{u \in H \\ \|u\|_H = 1}} \langle Tu, u \rangle_H.$$

Then $\sigma(T) \subset [m, M] \subset \mathbb{R}$, $m \in \sigma(T)$ and $M \in \sigma(T)$.

We can apply this Proposition to the operator K_J, obtaining more details about its spectrum.

Proposition 2.1.22. *Let (Ω, μ, d) be a metric measure space with $\mu(\Omega) < \infty$. We assume $K_J \in \mathcal{L}(L^{p_0}(\Omega), \mathcal{C}_b(\Omega))$ is compact, and $p_0 \leq 2$. Let $X = L^p(\Omega)$, with $p \in [p_0, \infty]$, or $X = \mathcal{C}_b(\Omega)$, and J satisfies that*

$$J(x, y) = J(y, x).$$

Then $K_J \in \mathcal{L}(X, X)$ and $\sigma_X(K_J) \setminus \{0\}$ is a real sequence of eigenvalues of finite multiplicity, independent of X, that converges to 0.
Moreover, if we consider

$$m = \inf_{\substack{u \in L^2(\Omega) \\ \|u\|_{L^2(\Omega)} = 1}} \langle K_J(u), u \rangle_{L^2(\Omega)} \qquad \text{and} \qquad M = \sup_{\substack{u \in L^2(\Omega) \\ \|u\|_{L^2(\Omega)} = 1}} \langle K_J(u), u \rangle_{L^2(\Omega)}, \tag{2.30}$$

then $\sigma_X(K_J) \subset [m, M] \subset \mathbb{R}$, $m \in \sigma_X(K_J)$ and $M \in \sigma_X(K_J)$.
In particular, $L^2(\Omega)$ admits an orthonormal basis consisting of eigenfunctions of K_J.

Proof. Thanks to Proposition 2.1.19, K_J is selfadjoint in $L^2(\Omega)$, then $\sigma_{L^2}(K_J) \setminus \{0\}$ is a real sequence of eigenvalues of finite multiplicity that converges to 0, (see [13, chap.6]). Furthermore, from Proposition 2.1.20 we have that $\sigma_X(K_J)$ is independent of X. Thus, the result.

On the other hand, as a consequence of Proposition 2.1.21, we have that $\sigma_X(K_J) \subset [m, M] \subset \mathbb{R}$, with $m \in \sigma_X(K_J)$ and $M \in \sigma_X(K_J)$, where m and M are given by (2.30).

Thanks to the Spectral Theorem (see [13, chap.6]), we know that $L^2(\Omega)$ admits an orthonormal basis consisting of eigenfunctions of K_J. $\square$

The following Corollary states that under the hypotheses of Proposition 2.1.17, any non-negative eigenfunction associated to the operator K_J is in fact strictly positive positive as well as its associated eigenvalue.

Corollary 2.1.23. *Let (Ω, μ, d) be a metric measure space, let J satisfy the hypotheses of Proposition 2.1.17 and assume Ω is R-connected. If $\Phi \geq 0$, not identically zero, is an eigenfunction associated to an eigenvalue λ of the operator K_J, then $\Phi > 0$, and the eigenvalue, λ, is also strictly positive.*

Proof. Thanks to Proposition 2.1.17, we know that, for every function $\Phi \geq 0$, not identically zero defined in Ω, it happens that $P(K_J^n(\Phi)) \supset P_R^n(\Phi)$, $\forall n \in \mathbb{N}$.

On the other hand, since Φ is an eigenfunction associated to an eigenvalue λ of the operator K_J, we have that $K_J^n(\Phi) = \lambda^n \Phi$, $\forall n \in \mathbb{N}$. Moreover, from Proposition 2.1.17, we know that for any compact set $\mathcal{K} \subset \Omega$, there exists $n_0 \in \mathbb{N}$ such that $P(K_J^n(\Phi)) \supset \mathcal{K}$ for all $n \geq n_0$. Thus, $K_J^n(\Phi) = \lambda^n \Phi$ is strictly positive in $\mathcal{K}$ for all $n \geq n_0$. Therefore Φ must be strictly positive in any compact set $\mathcal{K}$ of Ω. Hence, $\lambda > 0$ and Φ must be strictly positive in Ω. $\square$

Now, let us give some results about the spectral radius of the operator K, where the spectral radius is

$$r(K) = \sup |\sigma(K)|.$$

To give these properties about the spectral radius we will use Kreĭn- Rutman Theorem.

The definitions below will be helpful to introduce the following results, (see [51], [38]).

Definition 2.1.24.

i. *A real Banach Space X is called ordered if there exists a given closed convex cone C in X (with the vertex at the origin) satisfying $C \cap (-C) = \{0\}$, i.e. $C \subset X$ is closed, and*

$$\begin{aligned} \alpha,\, \beta \in [0, \infty) \text{ and } x, y \in C \quad &\Longrightarrow \quad \alpha x + \beta y \in C; \\ x \in C, \;\; -x \in C \quad &\Longrightarrow \quad x = 0 \in C. \end{aligned}$$

Then C is called the positive cone of X. This is equivalent to say that $x \in C$ if and only if $x \geq 0$; and $x \geq y$ if and only if $x - y \geq 0$.

ii. *If C has no empty interior, Int(C), in X, then X is called strongly ordered.*

iii. *In a strongly ordered space, an everywhere defined linear operator $T : X \to X$ is called strongly positive if there exists $n_0 \in \mathbb{N}$ such that $T^n(C \setminus \{0\}) \subset$ Int(C), for all $n \geq n_0$.*

Theorem 2.1.25. *(Kreĭn-Rutman Theorem) Let X be a strongly ordered Banach space with positive cone C. Assume that $T : X \to X$ is a strongly positive compact linear operator on X. Then*

i. *the spectral radius of T, $r(T) = \sup |\sigma(T)|$, is a positive, simple eigenvalue of T;*

ii. *the eigenfunction u in $X \setminus \{0\}$ associated with the eigenvalue $r(T)$ can be taken in Int(C);*

iii. if μ is in the spectrum of T, $0 \neq \mu \neq r(T)$, then μ is an eigenvalue of T satisfying $|\mu| < r(T)$;

iv. if μ is an eigenvalue of T associated with an eigenfunction v in $C \setminus \{0\}$ then $\mu = r(T)$.

To apply the Kreĭn-Rutman Theorem to the operator K_J, we work in the space $\mathcal{C}_b(\Omega)$, with Ω compact, and we consider the positive cone $C = \{f \geq 0;\ f \in \mathcal{C}_b(\Omega)\}$, with $\mathrm{Int}(C) = \{f \in \mathcal{C}_b(\Omega);\ f(x) > 0,\ \forall x \in \Omega\}$. Thus, in the proposition below, we prove that the spectral radius of the operator K is a simple eigenfunction that has an associated eigenfunction that is strictly positive.

Proposition 2.1.26. *Let (Ω, μ, d) be a metric measure space, with Ω compact and connected. We assume that J satisfies*

$$J(x, y) = J(y, x)$$

and

$$J(x, y) > 0,\ \forall x,\ y \in \Omega \ \text{ such that } \ d(x, y) < R, \ \text{for some } R > 0,$$

and $K_J \in \mathcal{L}(L^p(\Omega), \mathcal{C}_b(\Omega))$, with $1 \leq p \leq \infty$, is compact, (see Proposition 2.1.7 ii.).

Then $K_J \in \mathcal{L}(\mathcal{C}_b(\Omega), \mathcal{C}_b(\Omega))$ is compact, the spectral radius $r_{\mathcal{C}_b(\Omega)}(K_J)$ is a positive simple eigenvalue, and its associated eigenfunction is strictly positive.

Proof. Since Ω is compact and connected then from Proposition 2.1.17 we obtain that, there exists $n_0 \in \mathbb{N}$ such that, for any nonnegative $u \in \mathcal{C}_b(\Omega)$, $\Omega = P_R^n(u)$, for all $n \geq n_0$, (see Definition 2.1.12), and we know that for every nonnegative $u \in \mathcal{C}_b(\Omega)$ and $\forall n \in \mathbb{N}$, $P(K^n(u)) \supset P_R^n(u)$. Therefore $\Omega = P_R^n(u) \subset P(K^n(u))$ for all $n \geq n_0$, i.e., for any nonnegative $u \in \mathcal{C}_b(\Omega)$, $K_J^n(u) > 0$ in Ω for all $n \geq n_0$. Hence, K_J is strongly positive in $\mathcal{C}_b(\Omega)$. Moreover $K_J : \mathcal{C}_b(\Omega) \hookrightarrow L^p(\Omega) \longrightarrow \mathcal{C}_b(\Omega)$ is compact. Thus, we have that all hypotheses of Kreĭn-Rutman Theorem 2.1.25 are satisfied in the space $\mathcal{C}_b(\Omega)$ for the operator K_J, then the spectral radius $r_{\mathcal{C}_b(\Omega)}(K_J)$ is a positive simple eigenvalue with an eigenfunction Φ associated to it that is strictly positive. $\square$

2.2 The multiplication operator hI

Let h be a function defined in Ω, $h : \Omega \to \mathbb{R}$. In this section be will focus in the study of the linear multiplication operator hI, that maps

$$u(x) \mapsto h(x)u(x).$$

We will start studying the spaces where the operator is defined depending on the integrability or continuity of the function h.

In particular, we are interested in the multiplication operator hI with $h \in L^\infty(\Omega)$ or $h \in \mathcal{C}_b(\Omega)$. We will describe which is its adjoint operator, and we will also prove that if $h \in L^\infty(\Omega)$, then the operator hI is selfadjoint in $L^2(\Omega)$, and we finish describing the spectrum and resolvent set of hI.

The following proposition studies the regularity of hI.

Proposition 2.2.1. *Let (Ω, μ, d) be a metric measure space.*

i. For $1 \leq p,\, q \leq \infty$, if $h \in L^r(\Omega)$, and if $\dfrac{1}{p} + \dfrac{1}{r} = \dfrac{1}{q}$, then $hI \in \mathcal{L}(L^p(\Omega), L^q(\Omega))$, and

$$\|hI\|_{\mathcal{L}(L^p(\Omega),L^q(\Omega))} \leq \|h\|_{L^r(\Omega)}.$$

ii. If $h \in L^\infty(\Omega)$ then $hI \in \mathcal{L}\left(L^p(\Omega), L^p(\Omega)\right)$, for all $1 \leq p \leq \infty$, and

$$\|hI\|_{\mathcal{L}(L^p(\Omega),L^p(\Omega))} \leq \|h\|_{L^\infty(\Omega)}.$$

iii. If $h \in \mathcal{C}_b(\Omega)$, let $X = L^p(\Omega)$, with $1 \leq p \leq \infty$ or $X = \mathcal{C}_b(\Omega)$, then $hI \in \mathcal{L}\left(X, X\right)$, and

$$\|hI\|_{\mathcal{L}(X,X)} \leq \|h\|_{\mathcal{C}_b(\Omega)}.$$

Proof.

i. Thanks to Hölder's inequality, and the fact that $h \in L^r(\Omega)$ and $u \in L^p(\Omega)$

$$\begin{aligned} \|h\, u\|^q_{L^q(\Omega)} &= \int_\Omega |h(x)u(x)|^q dx \\ &\leq \left(\int_\Omega |h(x)|^{q\alpha}\, dx\right)^{1/\alpha} \left(\int_\Omega |u(y)|^{q\beta}\, dy\right)^{1/\beta} \\ &= \|h\|^q_{L^r(\Omega)} \|u\|^q_{L^p(\Omega)}, \end{aligned}$$

with $\frac{1}{\alpha} + \frac{1}{\beta} = 1$, $q\alpha = r$ and $q\beta = p$, then p, r and q have to satisfy that $\frac{1}{p} + \frac{1}{r} = \frac{1}{q}$.

ii. For $1 \leq p < \infty$, we consider $u \in L^p(\Omega)$. Since $h \in L^\infty(\Omega)$ we have that

$$||h\, u||^p_{L^p(\Omega)} = \int_\Omega |h(x)u(x)|^p dx \leq ||h||^p_{L^\infty(\Omega)} ||u||^p_{L^p(\Omega)}.$$

For $p = \infty$, we consider $u \in L^\infty(\Omega)$. Since $h \in L^\infty(\Omega)$ we have that

$$||h\, u||_{L^\infty(\Omega)} = \sup_{x\in\Omega} |h(x)u(x)| \leq ||h||_{L^\infty(\Omega)} ||u||_{L^\infty(\Omega)}.$$

Thus, the result.

iii. Since $\mathcal{C}_b(\Omega) \subset L^\infty(\Omega)$, then from *ii.*, we have the result for $X = L^p(\Omega)$. Now, if $u \in \mathcal{C}_b(\Omega)$, then hu is continuous. Furthermore, we have that

$$||h\, u||_{\mathcal{C}_b(\Omega)} = \sup_{x\in\Omega} |h(x)u(x)| \leq ||h||_{\mathcal{C}_b(\Omega)} ||u||_{\mathcal{C}_b(\Omega)}.$$

Thus, the result. □

Lemma 2.2.2. *Let (Ω, μ, d) be a metric measure space, then*

i. $hI \in \mathcal{L}(L^p(\Omega), L^p(\Omega))$, for $1 \leq p \leq \infty$ if and only if $h \in L^\infty(\Omega)$.

ii. $hI \in \mathcal{L}(\mathcal{C}_b(\Omega), \mathcal{C}_b(\Omega))$ if and only if $h \in \mathcal{C}_b(\Omega)$.

Proof.
i. Thanks to Proposition 2.2.1, we know that if $h \in L^\infty(\Omega)$ then hI belongs to $\mathcal{L}(L^p(\Omega),\ L^p(\Omega))$. Let us see the converse implication. Since $hI \in \mathcal{L}(L^p(\Omega), L^p(\Omega))$, there exists $0 < C \in \mathbb{R}$ such that

$$\|hu\|_{L^p(\Omega)} \leq C\|u\|_{L^p(\Omega)}, \quad \forall\, u \in L^p(\Omega). \tag{2.31}$$

Now, we argue by contradiction. Assume $h \notin L^\infty(\Omega)$, then for all $k \in \mathbb{R}$, the exists a set $A_k \subset \Omega$, such that $\mu(A_k) > 0$, where $h(x) > k$, $\forall x \in A_k$. Then for any $0 < k \in \mathbb{R}$ we can choose $u_k \in L^p(\Omega)$ such that $\|u_k\|_{L^p(\Omega)} = \|u_k\|_{L^p(A_k)}$. From (2.31), we have

$$C\|u_k\|_{L^p(\Omega)} \geq \left(\int_\Omega |h\, u_k|^p dx\right)^{1/p} > k\|u_k\|_{L^p(A_k)}.$$

Thus $C > k$, for any $k > 0$. Hence, we arrive to contradiction, and $h \in L^\infty(\Omega)$.

ii. Thanks to Proposition 2.2.1, we know that if $h \in \mathcal{C}_b(\Omega)$ then $hI \in \mathcal{L}(\mathcal{C}_b(\Omega),\ \mathcal{C}_b(\Omega))$. Let us see the converse implication. The boundedness is obtained from part *i.*. Moreover, since $hI \in \mathcal{L}(\mathcal{C}_b(\Omega), \mathcal{C}_b(\Omega))$, if we choose $u \equiv 1$, then $hu = h \in \mathcal{C}_b(\Omega)$. Thus, the result. □

Remark 2.2.3. *Let $X = L^p(\Omega)$, with $1 \leq p \leq \infty$ or $X = \mathcal{C}_b(\Omega)$. In general, if $h \in L^\infty(\Omega)$ is not identically zero, then the operator $h\,I : X \to X$ is not compact. For instance, in the particular case in which the function $h(x) = 1$, $\forall x \in \Omega$, we have that hI is the identity operator, and the identity is not compact. This is because the unit ball in X is not compact, since the dimension of X is infinity.*

The following result describes the adjoint operator of the multiplication operator hI, and we prove that it is selfadjoint in $L^2(\Omega)$.

Proposition 2.2.4. *Let (Ω, μ) be a measure space and let $h \in L^\infty(\Omega)$ then the adjoint operator associated to $hI \in \mathcal{L}(L^p(\Omega),\ L^p\Omega))$, with $1 \leq p < \infty$, is*

$$(hI)^* : L^{p'}(\Omega) \to L^{p'}(\Omega).$$

where $(hI)^* = hI$.

In particular if $p = 2$, hI is selfadjoint in $L^2(\Omega)$.

Proof. For $1 \leq p < \infty$, we have that for $h \in L^\infty(\Omega)$, $u \in L^p(\Omega)$, and $v \in L^{p'}(\Omega)$ then,

$$\langle hI(u), v\rangle_{L^p,\, L^{p'}} = \int_\Omega h(x)u(x)v(x)dx = \int_\Omega u(x)h(x)v(x)dx,$$

and $\displaystyle\int_\Omega u(x)h(x)v(x)dx = \langle u, (hI)^*(v)\rangle_{L^p,\, L^{p'}}$ for

$$(hI)^*(v)(x) = h(x)v(x).$$

Thus $(hI)^* = hI$, but in this case $hI : L^{p'}(\Omega) \to L^{p'}(\Omega)$.

It is immediate that if $p = 2$, then hI is selfadjoint. □

Now, we give a description of the spectrum and the resolvent set of the multiplication operator $hI \in \mathcal{L}(X, X)$, with $X = L^p(\Omega)$, for $1 \leq p \leq \infty$ or $X = \mathcal{C}_b(\Omega)$. We denote as $EV(hI)$ the eigenvalues of the multiplication operator hI, and $\mathrm{Im}(h) \subset \mathbb{R}$ the range of h.

Proposition 2.2.5. *Let (Ω, μ, d) be a metric measure space.*

i. If $X = L^p(\Omega)$, with $1 \leq p \leq \infty$, we assume $h \in L^\infty$.

ii. If $X = \mathcal{C}_b(\Omega)$, we assume $h \in \mathcal{C}_b(\Omega)$.

The resolvent set of the multiplication operator is given by

$$\rho_X(hI) = \mathbb{C} \setminus \overline{Im(h)},$$

and its spectrum is

$$\sigma_X(hI) = \overline{Im(h)},$$

and they are independent of X. Moreover, for $X = L^p(\Omega)$, the eigenvalues associated to hI exist only when the function h is constant in subsets of Ω with positive measure, i.e.,

$$EV(hI) = \{\alpha \ ; \ \mu\left(\{x \in \Omega \ ; \ h(x) = \alpha\}\right) > 0\right).$$

The eigenvalues of the multiplication operator hI have **infinite** *multiplicity.*

For $X = \mathcal{C}_b(\Omega)$,

$$EV(hI) \supset \{\alpha \ ; \ \exists A \ \text{ open with } \ \mu(A) > 0 \ \text{ such that } \ A \subset \{x \in \Omega \ ; \ h(x) = \alpha\}\} = \mathcal{F}$$

and the eigenvalues of hI in $\mathcal{F}$ have **infinite** *multiplicity.*

Proof.

i. Thanks to Lemma 2.2.2, we know that $h \in L^\infty(\Omega)$ if and only if $hI \in \mathcal{L}(L^p(\Omega), L^p(\Omega))$, for all $1 \leq p \leq \infty$. We consider $f \in L^p(\Omega)$ and $u \in L^p(\Omega)$, then

$$\begin{aligned} h(x)u(x) - \lambda u(x) &= f(x) \\ (h(x) - \lambda)u(x) &= f(x) \\ u(x) &= \frac{f(x)}{h(x) - \lambda} = \frac{1}{h(x) - \lambda} f(x). \end{aligned} \tag{2.32}$$

Then we have that $\lambda \in \rho_{L^p(\Omega)}(hI)$ if and only if $(hI - \lambda I)^{-1} \in \mathcal{L}(L^p(\Omega), L^p(\Omega))$, and thanks to Lemma 2.2.2, $(hI - \lambda I)^{-1} \in \mathcal{L}(L^p(\Omega), L^p(\Omega))$ if and only if $\dfrac{1}{h - \lambda} \in L^\infty(\Omega)$, and this happens when

$$\left| \frac{1}{h(x) - \lambda} \right| \leq C, \quad \forall x \in \Omega, \quad \text{then, there exists } \delta > 0 \text{, such that } \ |h(x) - \lambda| > \delta, \ \forall x \in \Omega,$$

i.e., if and only if $\lambda \notin \overline{\mathrm{Im}(h)}$. Then, we have proved that $\rho_{L^p(\Omega)}(hI) = \mathbb{C} \setminus \overline{\mathrm{Im}(h)}$ and its spectrum is by definition $\sigma(hI) = \mathbb{C} \setminus \rho(hI) = \overline{\mathrm{Im}(h)}$. Since $\overline{Im(h)}$ is independent of $L^p(\Omega)$, then the spectrum of $hI \in \mathcal{L}(L^p(\Omega), L^p(\Omega))$ is independent of p.

The eigenvalues of hI satisfy by definition that there exists $\Phi \in L^p(\Omega)$ with $\Phi \not\equiv 0$, such that

$$h(x)\Phi(x) = \lambda\Phi(x)$$

and this only happens if there exists a set $A \subset \Omega$, with $\mu(A) > 0$, such that $h(x) = \lambda$ for all $x \in A \subset \Omega$. Then, the eigenfunctions Φ associated to λ satisfy that

$$\Phi \in L^p(A), \text{ and } \Phi(x) = 0, \ \forall x \in \Omega \setminus A.$$

Hence, we have that $\mathrm{Ker}(hI - \lambda I) = L^p(A)$. Thus, the result.

ii. Thanks to Proposition 2.2.1, since $h \in \mathcal{C}_b(\Omega)$ we have that $hI \in \mathcal{L}(X, X)$. The rest of the proof follows the same arguments as in *i.*. Moreover, if there exists an open set $A \subset \Omega$, with $\mu(A) > 0$, such that $h(x) = \lambda$ for all $x \in A \subset \Omega$, then λ is an eigenvalue of hI in $\mathcal{C}_b(\Omega)$, and the space of eigenfunctions associated to λ is given by $\mathrm{Ker}(hI - \lambda I) = \{\Phi \in \mathcal{C}_b(\Omega) : \Phi(x) = 0, \ \forall x \in \Omega \setminus A\}$, which has infinite dimension. Thus, the result. ☐

2.3 Green's formulas for $K_J - h_0 I$

In this section we introduce the Green's formulas for $K_J - h_0 I$, where

$$h_0(x) = \int_\Omega J(x, y)dy.$$

We will assume that $h_0 \in L^\infty(\Omega)$, and this is satisfied if and only if $J \in L^\infty(\Omega, L^1(\Omega))$.

Green's formulas will be useful to obtain some properties of the sign of the spectrum of the operator $K_J - hI$.

Proposition 2.3.1. (Green's formulas) *Let (Ω, μ, d) be a metric measure space such that $\mu(\Omega) < \infty$. If $J \in L^p(\Omega, L^{p'}(\Omega))$, for $1 \le p < \infty$, and $h_0 \in L^\infty(\Omega)$, and if*

$$J(x, y) = J(y, x), \tag{2.33}$$

then for $u \in L^p(\Omega)$ and $v \in L^{p'}(\Omega)$,

$$\langle K_J(u) - h_0\, I(u),\, v\rangle_{L^p, L^{p'}} = -\frac{1}{2}\int_\Omega\int_\Omega J(x, y)(u(y) - u(x))(v(y) - v(x))dy\, dx. \tag{2.34}$$

In particular, if $p = 2$ we have that for $u \in L^2(\Omega)$

$$\langle K_J(u) - h_0\, I(u),\, u\rangle_{L^2, L^2} = -\frac{1}{2}\int_\Omega\int_\Omega J(x, y)(u(y) - u(x))^2 dy\, dx. \tag{2.35}$$

Proof. We denote the integral term of the right hand side of (2.34) by

$$\begin{aligned} I_1 &= \int_\Omega\int_\Omega J(x, y)(u(y) - u(x))(v(y) - v(x))dy\, dx \\ &= \int_\Omega\int_\Omega J(x, y)(u(y) - u(x))v(y)dy\, dx - \int_\Omega\int_\Omega J(x, y)(u(y) - u(x))v(x)dy\, dx. \end{aligned}$$

Relabeling variables in the first term of the sum, we obtain

$$I_1 = \int_\Omega \int_\Omega J(y,x)(u(x)-u(y))v(x)dx\,dy - \int_\Omega \int_\Omega J(x,y)(u(y)-u(x))v(x)dy\,dx.$$

Now, since $J(x,y) = J(y,x)$,

$$I_1 = \int_\Omega \int_\Omega J(x,y)(u(x)-u(y))v(x)dx\,dy - \int_\Omega \int_\Omega J(x,y)(u(y)-u(x))v(x)dy\,dx.$$

Thanks to Fubini's Theorem, we have that

$$I_1 \quad = -2\int_\Omega \int_\Omega J(x,y)(u(y)-u(x))v(x)dy\,dx.$$

Therefore, we have proved that the integral term of the right hand side of (2.34) is equal to

$$\int_\Omega \int_\Omega J(x,y)(u(y)-u(x))(v(y)-v(x))dy\,dx = -2\int_\Omega \int_\Omega J(x,y)(u(y)-u(x))v(x)dy\,dx. \tag{2.36}$$

On the other hand, thanks to the hypothesis on J, $h_0 \in L^\infty(\Omega)$ and from Propositions 2.1.1 and 2.2.1, we have that $K_J - h_0 I \in \mathcal{L}(L^p(\Omega), L^p(\Omega))$, for all $1 \le p \le \infty$. Hence, if $u \in L^p(\Omega)$ and $v \in L^{p'}(\Omega)$

$$\begin{aligned} \langle K_J(u) - h_0\, I(u),\, v\rangle_{L^p, L^{p'}} &= \int_\Omega \left(\int_\Omega J(x,y)u(y)dy - \int_\Omega J(x,y)dy\, u(x)\right) v(x)\, dx \\ &= \int_\Omega \int_\Omega J(x,y)(u(y)-u(x))v(x)dy\,dx. \end{aligned} \tag{2.37}$$

Hence, from (2.36) and (2.37), we obtain (2.34). The second part of the proposition is an immediate consequence of (2.34). □

2.4 Spectrum of the operator $K - hI$

Let (Ω, μ, d) be a metric measure space. In this section we describe the spectrum of $K - hI \in \mathcal{L}(X,X)$, and we prove that, under certain conditions on the operator K, it is independent of X. Moreover, we give conditions on J and h under which the spectrum of $K_J - hI$ is nonpositive.

We start introducing some definitions used in the following theorems, that will be useful to give a description of the spectrum of $K - hI$.

Definition 2.4.1. *If T is a linear operator in a Banach space Y, a* **normal point** *of T is any complex number which is in the resolvent set, or is an isolated eigenvalue of T of finite multiplicity. Any other complex number is in the* **essential spectrum** *of T.*

To describe the spectrum of $K - hI$, we use the following theorem that can be found in [34, p. 136].

Theorem 2.4.2. *Suppose Y is a Banach space, $T : D(T) \subset Y \to Y$ is a closed linear operator, $S : D(S) \subset Y \to Y$ is linear with $D(S) \supset D(T)$ and $S(\lambda_0 - T)^{-1}$ is compact for some $\lambda_0 \in \rho(T)$. Let U be an open connected set in $\mathbb{C}$ consisting entirely of normal points of T, which are points of the resolvent of T, or isolated eigenvalues of T of finite multiplicity. Then either U consists entirely of normal points of $T+S$, or entirely of eigenvalues of $T+S$.*

Remark 2.4.3. *If $S : Y \to Y$ is compact, Theorem 2.4.2 implies that the perturbation S can not change the essential spectrum of T.*

The next theorem describes the spectrum of the operator $K - hI$ in X.

Theorem 2.4.4. *Let (Ω, μ, d) be a metric measure space.*

- *If $X = L^p(\Omega)$, with $1 \le p \le \infty$, we assume $h \in L^\infty(\Omega)$.*
- *If $X = \mathcal{C}_b(\Omega)$, we assume $h \in \mathcal{C}_b(\Omega)$.*

If $K \in \mathcal{L}(X, X)$ is compact, (see Proposition 2.1.7), then

$$\sigma(K - hI) = \overline{Im(-h)} \cup \{\mu_n\}_{n=1}^{M}, \quad \textit{with} \;\; M \in \mathbb{N} \cup \{\infty\}.$$

If $M = \infty$, then $\{\mu_n\}_{n=1}^{\infty}$ is a sequence of eigenvalues of $K - hI$ with finite multiplicity, that accumulates in $\overline{Im(-h)}$.

Proof. With the notations of Theorem 2.4.2, we consider the operators

$$S = K \;\; \text{and} \;\; T = -hI.$$

First of all, we prove that $\mathbb{C} \setminus \overline{Im(-h)} \subset \rho(K - hI)$. We choose the set U in Theorem 2.4.2 as

$$U = \rho(-hI) = \rho(T) = \mathbb{C} \setminus \overline{Im(-h)}$$

that is an open, connected set. Since $U = \rho(T)$, every $\lambda \in U$ is a normal point of T.

On the other hand, if $\lambda_0 \in \rho(T)$, then $(T - \lambda_0)^{-1} \in \mathcal{L}(X, X)$, and $S = K$ is compact. Then, we have that $S(\lambda_0 - T)^{-1} \in \mathcal{L}(X, X)$ is compact. Thus, all the hypotheses of Theorem 2.4.2 are satisfied. Now, thanks to Theorem 2.4.2, we have that $U = \mathbb{C} \setminus \overline{Im(-h)}$ consists entirely of eigenvalues of $T + S = K - hI$ or U consists entirely of normal points of $T + S = K - hI$.

If $U = \mathbb{C} \setminus \overline{Im(-h)}$ consists entirely of eigenvalues of $T + S = K - hI$, we arrive to contradiction, because the spectrum of $K - hI$ is bounded. So $U = \mathbb{C} \setminus \overline{Im(-h)}$ has to consist entirely of normal points of $T + S$. Then, they are points of the resolvent or isolated eigenvalues of $T + S = K - hI$. Since any set of isolated points in $\mathbb{C}$ is a finite set, or a numerable set, we have that the isolated eigenvalues are

$$\{\mu_n\}_{n=1}^{M}, \;\; \text{with} \;\; M \in \mathbb{N} \;\; \text{or} \;\; M = \infty.$$

Moreover, since the spectrum of $K - hI$ is bounded, if $M = \infty$ then $\{\mu_n\}_{n=1}^{\infty}$ is a sequence of eigenvalues of $K - hI$ with finite multiplicity, that accumulates in $\overline{Im(-h)}$.

Now we prove that $\overline{Im(-h)} \subset \sigma(K - hI)$. We argue by contradiction. Suppose that there exists a $\tilde{\lambda} \in \overline{Im(-h)}$ that belongs to $\rho(K - hI)$. Since the resolvent set is open, there exists a ball $B_\varepsilon(\tilde{\lambda})$ centered in $\tilde{\lambda}$, that is into the resolvent of $K - hI$. Then $U = B_\varepsilon(\tilde{\lambda})$ is an open connected set that consists of normal points of $K - hI$. With the notation of Theorem 2.4.2, we consider the operators

$$T = K - hI \quad \text{and} \quad S = -K$$

and the open, connected set

$$U = B_\varepsilon(\tilde{\lambda}).$$

Arguing like in the previous case, if $\lambda_0 \in \rho(T)$, we have that $S(\lambda_0 - T)^{-1}$ is compact, thus the hypotheses of Theorem 2.4.2 are satisfied. Hence $U = B_\varepsilon(\tilde{\lambda})$ consists entirely of eigenvalues of $T + S = -hI$ or $U = B_\varepsilon(\tilde{\lambda})$ consists entirely of normal points of $T + S = -hI$.

If $U = B_\varepsilon(\tilde{\lambda})$ consists entirely of eigenvalues of $T + S = -hI$, we would arrive to contradiction, because the eigenvalues of $-hI$ are only inside $\overline{Im(-h)}$, and the ball $B_\varepsilon(\tilde{\lambda})$ is not inside $\overline{Im(-h)}$. So $U = B_\varepsilon(\tilde{\lambda})$ has to consist of normal points of $T + S = -hI$, so they are points of the resolvent of $-hI$ or isolated eigenvalues of finite multiplicity of $-hI$. Since $\rho(-hI) = \mathbb{C} \setminus \overline{Im(-h)}$, and $\tilde{\lambda} \in \overline{Im(-h)}$, we have that $\tilde{\lambda}$ has to be an isolated eigenvalue of $-hI$, with finite multiplicity. But from Proposition 2.2.5, we know that the eigenvalues of $-hI$ have infinity multiplicity. Thus, we arrive to contradiction. Hence, we have proved that $\overline{Im(-h)} \subset \sigma(K - hI)$. With this, we have finished the proof of the theorem. $\square$

In the following proposition we prove that the spectrum of $K - hI$ is independent of $X = L^p(\Omega)$ with $1 \leq p \leq \infty$, or $X = \mathcal{C}_b(\Omega)$.

Proposition 2.4.5. *Let (Ω, μ, d) be a metric measure space with $\mu(\Omega) < \infty$.*

i. For $1 \leq p_0 < p_1 < \infty$, if $K \in \mathcal{L}(L^{p_0}(\Omega), L^{p_1}(\Omega))$ and additionally $K \in \mathcal{L}(L^{p_0}(\Omega), L^{p_0}(\Omega))$ is compact and $h \in L^\infty(\Omega)$, then $K - hI \in \mathcal{L}(L^p(\Omega), L^p(\Omega))$, $\forall p \in [p_0, p_1]$, and $\sigma_{L^p}(K - hI)$ is independent of p.

ii. For $1 \leq p_0 < p_1 \leq \infty$, if $K \in \mathcal{L}(L^{p_0}(\Omega), L^{p_1}(\Omega))$ is compact and $h \in L^\infty(\Omega)$, then $K - hI \in \mathcal{L}(L^p(\Omega), L^p(\Omega))$, $\forall p \in [p_0, p_1]$, and $\sigma_{L^p}(K - hI)$ is independent of p.

iii. For a fixed $1 \leq p_0 \leq \infty$, if $K \in \mathcal{L}(L^{p_0}(\Omega), \mathcal{C}_b(\Omega))$ is compact and $X = \mathcal{C}_b(\Omega)$ or $X = L^r(\Omega)$ with $r \in [p_0, \infty]$, and $h \in \mathcal{C}_b(\Omega)$, then $K - hI \in \mathcal{L}(X, X)$ and $\sigma_X(K - hI)$ is independent of X.

Proof. Following the same arguments in Proposition 2.1.20, we have that in any of the cases *i.*, *ii.*, or *iii.*, $K \in \mathcal{L}(X, X)$ is compact, where $X = L^p(\Omega)$ with $p_0 \leq p \leq p_1$ for the cases *i.* and *ii.*, and $X = L^p(\Omega)$ with $p_0 \leq p \leq \infty$, or $X = \mathcal{C}_b(\Omega)$ for the case *iii.*. Then, from Theorem 2.4.4 we have that

$$\sigma_X(K - hI) = \overline{Im(-h)} \cup \{\mu_n\}_{n=1}^M, \quad \text{with } M \in \mathbb{N} \text{ with } or M = \infty,$$

where $\{\mu_n\}_n$ are eigenvalues of $K - hI$, with finite multiplicity $\forall n \in \{1, \ldots, M\}$.

Since $\overline{Im(-h)}$ is independent of X, we just have to prove that the eigenvalues $\lambda \in \sigma_X(K-hI)$ satisfying that $\lambda \notin \overline{Im(-h)}$ are independent of X. Let $\lambda \in \sigma_X(K-hI)$ be an eigenvalue such that $\lambda \notin \overline{Im(-h)}$. We denote by Φ an eigenfunction associated to $\lambda \in \sigma_X(K-hI)$, then

$$K(\Phi)(x) - h(x)\Phi(x) = \lambda\Phi(x) \tag{2.38}$$

Since $\lambda \notin \overline{Im(-h)}$, then from (2.38) we obtain

$$\Phi(x) \quad = \quad \frac{1}{h(x)+\lambda} K(\Phi)(x) \tag{2.39}$$

and $\dfrac{1}{h(\cdot)+\lambda} \in L^\infty(\Omega)$. Thanks to the hypotheses on K, we have

$$\frac{1}{h(\cdot)+\lambda} K \in \mathcal{L}(L^{p_0}(\Omega), L^{p_1}(\Omega)). \tag{2.40}$$

We prove first that $\sigma_{L^{p_1}(\Omega)} \subset \sigma_{L^p(\Omega)}$: if $\lambda \in \sigma_{L^{p_1}}(K)$ is an eigenvalue, then the associated eigenfunction $\Phi \in L^{p_1}(\Omega)$. Since $\mu(\Omega) < \infty$ we have that $L^{p_1}(\Omega) \hookrightarrow L^p(\Omega)$ continuously, for all $p \leq p_0$, then $\Phi \in L^p(\Omega)$. Thus we obtain that $\lambda \in \sigma_{L^p}(K-hI)$ for all $p \in [p_0, p_1]$.

Now, we prove that $\sigma_{L^p(\Omega)} \subset \sigma_{L^{p_1}(\Omega)}$: if $\lambda \in \sigma_{L^p(\Omega)}(K)$ is an eigenvalue, with $p \in [p_0, p_1)$, then the associated eigenfunction $\Phi \in L^p(\Omega)$ satisfies (2.39). Since $L^p(\Omega) \hookrightarrow L^{p_0}(\Omega)$ continuously, then from (2.40), we have that $\frac{1}{h(\cdot)+\lambda}K(\Phi) \in L^{p_1}(\Omega)$. Hence, from (2.39), we obtain that $\Phi \in L^{p_1}(\Omega)$. Therefore, $\Phi \in L^p(\Omega)$ for $p \in [p_0, p_1]$, and we have proved the independence of the spectrum respect the space for the cases *i.* and *ii.*.

The case *iii.* is analogous to the previous result, using that $h \in \mathcal{C}_b(\Omega)$ and $\lambda \notin \overline{Im(-h)}$, then

$$\frac{1}{h(\cdot)+\lambda} K(\Phi) \in \mathcal{L}(L^{p_0}(\Omega), \mathcal{C}_b(\Omega)). \tag{2.41}$$

Thus, the result. $\square$

The following results state hypotheses to know in which cases the spectrum of $K_J - hI$ is nonpositive.

Corollary 2.4.6. *Let (Ω, μ, d) be a metric measure space with $\mu(\Omega) < \infty$. For $1 \leq p_1 \leq \infty$, let $X = L^p(\Omega)$, with $p \in [1, p_1]$ or $X = \mathcal{C}_b(\Omega)$. We assume that K and h satisfy the hypotheses in Proposition 2.4.5 with $p_0 \leq 2$ and we assume that J is such that*

$$J(x,y) = J(y,x).$$

i. If $h \equiv c$, with $c \in \mathbb{R}$ such that $c > r(K_J)$, where $r(K_J)$ is the spectral radius of K_J then $\sigma_X(K_J - hI)$ is real and nonpositive.

ii. If $h = h_0 = \int_\Omega J(x,y)dy \in L^\infty(\Omega)$ and h_0 satisfies that $h_0(x) > \alpha > 0$ for all $x \in \Omega$, then $\sigma_X(K_J - hI)$ is nonpositive and 0 is an isolated eigenvalue with finite multiplicity. Moreover if J satisfies that

$$J(x,y) > 0, \ \forall x,\, y \in \Omega \text{ such that } d(x,y) < R$$

then $\{0\}$ is a simple eigenvalue.

iii. If $h \in L^\infty(\Omega)$ satisfies that $h \geq h_0$ in Ω, then $\sigma_X(K_J - hI)$ is nonpositive.

Proof. Under the hypotheses and thanks to the previous Proposition 2.4.5, we have that $\sigma_X(K - hI)$ is independent of X. Hence the rest of the results will be proved in $L^2(\Omega)$.

i. From Proposition 2.1.19 and Proposition 2.2.4, we have that K_J and hI are selfadjoint operators in $L^2(\Omega)$, then we have that $K_J - hI$ is a selfadjoint operator in $L^2(\Omega)$.

By using Proposition 2.1.21, we know that $\sigma_{L^2(\Omega)}(K_J)$ is composed by real values that are less or equal to $r(K_J)$.
On the other hand, $\sigma_{L^2(\Omega)}(K_J - hI) = \sigma_{L^2(\Omega)}(K_J) - c$ and $c > r(K_J)$, then we have that $\sigma_{L^2(\Omega)}(K_J - hI)$ is real and nonpositive. Finally, since the spectrum is independent of X, we obtain the result.

ii. Under the hypotheses we have that $K \in \mathcal{L}(X, X)$ is compact, then thanks to Theorem 2.4.4, we know that

$$\sigma_X(K - h_0 I) = \overline{Im(-h_0)} \cup \{\mu_n\}_{n=1}^{M}, \quad \text{with } M \in \mathbb{N} \text{ or } M = \infty.$$

Since $0 \notin \overline{Im(-h_0)}$, then 0 is an isolated eigenvalue with finite multiplicity. If $M = \infty$, then $\{\mu_n\}_{n=1}^{\infty}$ is a sequence of eigenvalues of $K - h_0 I$ with finite multiplicity, that has accumulation points in $Im(-h)$.

As in part *i.* we obtain that $K_J - h_0 I$ is a selfadjoint operator in $L^2(\Omega)$. Then, thanks to Proposition 2.3.1,

$$\begin{aligned}\langle (K_J - h_0)u, u\rangle_{L^2(\Omega), L^2(\Omega)} &= \int_\Omega \int_\Omega J(x,y)u(y)\, u(x)\, dy dx - \int_\Omega \int_\Omega J(x,y)\, u^2(x) dy dx \\ &= -\frac{1}{2}\int_\Omega \int_\Omega J(x,y)(u(x) - u(y))^2 dy\, dx \leq 0,\end{aligned} \tag{2.42}$$

Then from Proposition 2.1.21, and the equality (2.42) we know that

$$\sigma_{L^2(\Omega)}(K_J - h_0) \leq \sup_{\substack{u \in L^2(\Omega) \\ \|u\|_{L^2(\Omega)} = 1}} \langle (K_J - h_0)u, u\rangle_{L^2(\Omega), L^2(\Omega)} \leq 0. \tag{2.43}$$

Thus, the spectrum is nonnegative.

Let us prove below that $\{0\}$ is a simple eigenvalue. We consider φ an eigenfunction associated to $\{0\}$. Thanks to Proposition 2.3.1 in $L^2(\Omega)$ we have

$$0 = \langle (K - h_0\, I)(\varphi),\, \varphi\rangle_{L^2(\Omega), L^2(\Omega)} = -\frac{1}{2}\int_\Omega \int_\Omega J(x,y)(\varphi(y) - \varphi(x))^2 dy\, dx. \tag{2.44}$$

Since $J(x,y) > 0$, $\forall x,\, y \in \Omega$ such that $d(x,y) < R$, then for all $x \in \Omega$, $\varphi(x) = \varphi(y)$ for any $y \in B_R(x)$. Thus, φ is a constant function in Ω. Therefore, we have proved that $\{0\}$ is a simple eigenvalue.

iii. Let us see which is the sign of the spectrum of the operator $K_J - hI$, with $h \geq h_0$. From (2.43), we have

$$\begin{aligned}\langle (K_J - hI)u, u\rangle_{L^2(\Omega), L^2(\Omega)} &= \langle (K_J - h_0 + h_0 - h)u, u\rangle_{L^2(\Omega)} \\ &= \langle (K_J - h_0)u, u\rangle_{L^2(\Omega)} + \langle (h_0 - h)u, u\rangle_{L^2(\Omega)} \leq 0.\end{aligned}$$

$\square$

Chapter 3

The linear evolution equation

Throughout this chapter, we will assume that (Ω, μ, d) is a metric measure space, with μ as in Definition 1.1.5. Let $X = L^p(\Omega)$, with $1 \le p \le \infty$ or $X = \mathcal{C}_b(\Omega)$. The problem we are going to work with in this chapter, is the following

$$\begin{cases} u_t(x,t) & = (K - hI)(u)(x,t) = L(u)(x,t), \quad x \in \Omega,\ t > 0 \\ u(x,0) & = u_0(x), \qquad\qquad\qquad\qquad\qquad\quad\ x \in \Omega, \end{cases} \tag{3.1}$$

with $u_0 \in X$, $K = K_J \in \mathcal{L}(X,X)$ and $h \in L^\infty(\Omega)$ or $h \in \mathcal{C}_b(\Omega)$. This means that the operator $L = K - hI \in \mathcal{L}(X,X)$. We will apply the results of the linear nonlocal diffusive operator $K - hI$ developed in the previous chapter to study the existence, uniqueness, positivity, regularizing effects and the asymptotic behavior of the solution of (3.1).

In this chapter, we will prove the existence and uniqueness of solution of (3.1) using the semigroup theory. We will write the solution of the problem (3.1) in terms of the group e^{Lt}, associated to the linear and continuous operator L. In fact, in Proposition 3.1.2, we prove that if $K - hI \in \mathcal{L}(X,X)$, for an initial data $u_0 \in X$ there exists a unique strong solution of (3.1), such that $u \in \mathcal{C}^\infty(\mathbb{R}, X)$.

The comparison, positivity and monotonicity results are well-known for the classical diffusion problem with the laplacian (see for example [25]). We prove such results for the nonlocal problem (3.1). In particular, we will prove that under hypothesis (2.3) on the positivity of J, and Ω R-connected (see Definition 2.1.14), we have a strong maximum principle, i.e., if the initial data $u_0 \ge 0$, then the solution to (3.1), $u(t)$, is strictly positive for all $t > 0$.

One of the main differences between the nonlocal diffusion and the local diffusion problem is that the solution of (3.1) does not have regularizing effects in positive time, since the solution carries the singularities of the initial data. However, we will see that the semigroup $S(t)$ associated to the operator $L = K - hI$ can be written as $S(t) = S_1(t) + S_2(t)$, where $S_1(t)$ is the part that is not compact, but it decays to zero exponentially as time goes to infinity; and $S_2(t)$ is compact. Then $S(t)$ is asymptotically smooth, according to the definition in [32, p. 4].

We are also interested in the asymptotic behavior of the solution of (3.1), i.e., in describing the behaviour of the solution when time goes to infinity. We use the Riesz projection, (see [24, chap. VII]), and we prove Theorem 3.4.8 which states that if $\sigma_X(K - hI)$ is a disjoint union of two closed subsets σ_1 and σ_2 with $\mathrm{Re}(\sigma_1) \leq \delta_1$, $\mathrm{Re}(\sigma_2) \leq \delta_2$, with $\delta_2 < \delta_1$, then the asymptotic behavior of the solution of (3.1) in X is described by the Riesz Projection of $K - hI$ corresponding to σ_1. We prove also that the Riesz projection and the Hilbert projection coincide.

Furthermore, we apply this result to the particular cases of the nonlocal diffusion problem (3.1) with h constant or $h = h_0 = \int_\Omega J(\cdot, y)dy$. In particular, we recover and generalize the result in [18], for $X = L^p(\Omega)$, with $1 \leq p \leq \infty$ or $X = \mathcal{C}_b(\Omega)$, whereas in [18], the authors obtain the result with open $\Omega \subset \mathbb{R}^N$, in $X = L^2(\Omega)$ if the initial data is in $L^2(\Omega)$, and in $X = L^\infty(\Omega)$ if the initial data is in $\mathcal{C}(\overline{\Omega})$.

3.1 Existence and uniqueness of solution of (3.1)

Let Y be a Banach space. We start this section defining the group associated to a general linear bounded operator F. For more information, see [43] or [36].

If $F \in \mathcal{L}(Y, Y)$, the operator e^{-Ft} can be defined by the Taylor series

$$e^{-Ft} = \sum_{k=0}^{\infty} \frac{t^k F^k}{k!}, \quad t \in \mathbb{R} \tag{3.2}$$

which converges for any t. Thus e^{-Ft} also belongs to $\mathcal{L}(Y)$. It also has the group property

$$e^{-F(s+t)} = e^{-Fs} e^{-Ft}, \text{ for } s, t \in \mathbb{R}.$$

We call e^{-Ft} the group associated to the operator F, and it satisfies that

$$\frac{d}{dt} e^{-Ft} = -F e^{-Ft} = -e^{-Ft} F.$$

Moreover, it is a uniformly continuous group (see [43, p. 2]).

Lemma 3.1.1. *Let Y be a Banach space. If $F \in \mathcal{L}(Y, Y)$ then the unique solution of the problem*

$$\begin{cases} u_t = & F(u), \\ u(0) = & u_0 \in Y \end{cases} \tag{3.3}$$

is given by

$$u(t) = e^{Ft} u_0,$$

that is differentiable in time and the mapping

$$\mathbb{R} \ni t \mapsto u(t) = e^{Ft} u_0 \in Y$$

is analytic. Moreover the mapping

$$(t, u_0) \mapsto e^{Ft} u_0$$

is continuous.

We apply this semigroup technique to prove the existence of solution of the problem (3.1). The following proposition states the uniqueness and existence of strong solution to the problem (3.1).The hypothesis on the linear operator K, if K is an operator with kernel J, can be verified using Proposition 2.1.1, and the hypothesis on hI can be checked using Proposition 2.2.1.

Proposition 3.1.2. *Let* (Ω, μ, d) *be a metric measure space.*

- *If* $X = L^p(\Omega)$, *with* $1 \leq p \leq \infty$, *we assume* $h \in L^\infty(\Omega)$.
- *If* $X = \mathcal{C}_b(\Omega)$, *we assume* $h \in \mathcal{C}_b(\Omega)$.

If $K \in \mathcal{L}(X, X)$ *then the problem* (3.1) *has a unique strong solution* $u \in \mathcal{C}^\infty(\mathbb{R}, X)$, *given by*

$$u(t) = e^{Lt} u_0,$$

where $e^{Lt} \in \mathcal{L}(X, X)$ *is the group associated to the operator* $L = K - hI$.

Proof. Since $L \in \mathcal{L}(X, X)$, applying Lemma 3.1.1 to the problem (3.1), and we obtain the result. □

We denote the group associated to the operator $L = K - hI$ with $S_{K,h}$, to remark the dependence on K and h. Hence the solution of (3.1) is

$$u(t, u_0) = S_{K,h}(t) u_0 = e^{Lt} u_0. \tag{3.4}$$

Remark 3.1.3. *Another big difference between the nonlocal problem* (3.1) *and the local problem with the laplacian is that for the local problem, the flow is not reversible at all, and as a consequence of the previous Proposition 3.1.2, the flow of the nonlocal problem is reversible.*

3.2 The solution u of (3.1) is positive if the initial data u_0 is positive

We consider the operator $K_J(u) = \int_\Omega J(x, y) u(y) dy$, with J **nonnegative**, then we prove the Weak Maximum Principle, i.e., the solution u of the problem (3.1) with a nonnegative initial data $u_0(x)$ is nonnegative.

First of all, let us consider the problem (3.1), with $h \equiv 0$,

$$\begin{cases} \dfrac{du}{dt} = & K_J(u), \\ u(0) = & u_0 \geq 0. \end{cases} \tag{3.5}$$

Formally, if $J \geq 0$ then $u_t(x, 0) = K_J(u_0)(x) \geq 0$, thus u increases with time and then $u \geq 0$ since $u_0 \geq 0$. The rigorous proof of this is that thanks to (3.2) and Lemma 3.1.1, the solution to (3.5) is given by

$$u(x, t) = e^{K_J t} u_0(x) = \left(\sum_{k=0}^{\infty} \frac{t^k K_J^k}{k!} \right) u_0(x). \tag{3.6}$$

Since J is nonnegative, we have that $K_J^k(u_0)$ is nonnegative for any u_0 nonnegative, $\forall k \in \mathbb{N}$. Then we have that the solution $u(x,t)$ is nonnegative. In fact, for any $m \geq 0$

$$u(x,t) \geq u_0(x) \geq 0, \;\; u(x,t) \geq u_0(x) + tK_J(u_0)(x) \geq 0, \text{ and } \; u(x,t) \geq \left(\sum_{k=0}^{m} \frac{t^k K_J^k}{k!} \right) u_0(x) \geq 0.$$

Now, for $h \not\equiv 0$, let u be the solution to (3.1). We take the function

$$v(t) = e^{h(\cdot)t} u(t), \quad \text{for } \; t \geq 0.$$

This function v satisfies that

$$v(0) = u_0,$$

and

$$\begin{aligned} v_t(x,t) &= h(x)e^{h(x)t}u(x,t) + e^{h(x)t}u_t(x,t) \\ &= h(x)e^{h(x)t}u(x,t) + e^{h(x)t}\big(K(u)(x,t) - h(x)u(x,t)\big) \\ &= e^{h(x)t}K(u)(x,t). \end{aligned} \tag{3.7}$$

- If h in (3.1) is constant in Ω, $h(x) = \alpha$, $\forall x \in \Omega$, with $\alpha \in \mathbb{R}$, then

$$\begin{aligned} v_t(x,t) &= e^{\alpha t}K(u)(x,t) \\ &= e^{\alpha t} \int_\Omega J(x,y)u(y)dy \\ &= \int_\Omega J(x,y)e^{\alpha t}u(y)dy = K(v)(x,t). \end{aligned}$$

  Then $v(t) = e^{\alpha t}u(t)$ is a solution to the problem

$$\begin{cases} \dfrac{dv}{dt} &= K(v), \\ v(0) &= u_0. \end{cases} \tag{3.8}$$

  We have already proved that the solution to (3.8) with nonnegative initial data is nonnegative. Thus, the solution $u(x,t) = e^{-\alpha t}v(x,t)$ of (3.1) with h constant, is also nonnegative.

- We study now the case for $h \in L^\infty(\Omega)$ nonconstant. Thanks to (3.7), we know that v satisfies

$$v_t(x,t) = e^{h(x)t}K(u)(x,t), \quad \text{and} \;\; v(x,0) = u_0(x)$$

  then v can be written as

$$v(x,t) = u_0(x) + \int_0^t e^{h(x)s}K(u)(x,s)ds.$$

  Moreover $\;u(x,t) = e^{-h(x)t}v(x,t)$, then

$$u(x,t) = e^{-h(x)t}u_0(x) + \int_0^t e^{-h(x)(t-s)}K(u)(x,s)ds. \tag{3.9}$$

Let $X = L^p(\Omega)$, with $1 \leq p \leq \infty$, or $X = \mathcal{C}_b(\Omega)$. For every $\omega_0 \in X$ and $T > 0$, we consider the mapping

$$\mathcal{F}_{\omega_0} : \mathcal{C}([0,T];X) \to \mathcal{C}([0,T];X), \qquad \text{with}$$

$$\mathcal{F}_{\omega_0}(\omega)(x,t) = e^{-h(x)t}\omega_0(x) + \int_0^t e^{-h(x)(t-s)} K(\omega)(x,s)ds.$$

Fix $T > 0$ and consider the Banach space

$$X_T = \mathcal{C}([0,T];X)$$

with the norm

$$|||\omega||| = \max_{0 \leq t \leq T} \|\omega(\cdot,t)\|_X.$$

The proof of the following lemma is included for the sake of completeness. It gives us the inequalities to prove that the mapping $\mathcal{F}_{\omega_0}$ is a contraction in X_T, and it is valid for a general operator K, not only for the integral operator K_J.

Lemma 3.2.1. *Let (Ω, μ, d) be a metric measure space.*

- *If $X = L^p(\Omega)$, with $1 \leq p \leq \infty$, we assume $h \in L^\infty(\Omega)$.*
- *If $X = \mathcal{C}_b(\Omega)$, we assume $h \in \mathcal{C}_b(\Omega)$.*

If $K \in \mathcal{L}(X,X)$, ω_0, $z_0 \in X$, and ω, $z \in X_T = \mathcal{C}([0,T];X)$, then there exist two constants C_1 and C_2 depending on h and T, such that

$$|||\mathcal{F}_{\omega_0}(\omega) - \mathcal{F}_{z_0}(z)||| \leq C_1(T)\|\omega_0 - z_0\|_X + C_2(T)|||\omega - z|||, \tag{3.10}$$

where $C_1(T) = e^{\|h_-\|_{L^\infty(\Omega)}T}$, $C_2(T) = CTe^{\|h_-\|_{L^\infty(\Omega)}T}$, $C_2 : [0,\infty) \to \mathbb{R}$ is increasing and continuous, and $C_2(T) \to 0$, as $T \to 0$.

Proof. Since $K \in \mathcal{L}(X,X)$, and considering $h = h_+ + h_-$, ($h_+ = \max\{0,h\}$ and $h_- = \min\{0,h\}$), with $h \in L^\infty(\Omega)$ or $h \in \mathcal{C}_b(\Omega)$, then we obtain

$$\begin{aligned}
\|\mathcal{F}_{\omega_0}(\omega)(\cdot,t) - \mathcal{F}_{z_0}(z)(\cdot,t)\|_X &\leq \left\|e^{-h(\cdot)t}(\omega_0 - z_0)\right\|_X + \int_0^t e^{\|h_-\|_{L^\infty(\Omega)}(t-s)}\|K(\omega - z)(s)\|_X ds \\
&\leq e^{\|h_-\|_{L^\infty(\Omega)}T}\|\omega_0 - z_0\|_X + Ce^{\|h_-\|_{L^\infty(\Omega)}T}\, T \max_{0\leq t\leq T}\|\omega - z\|_X \\
&= C_1(T)\|\omega_0 - z_0\|_X + C_2(T)\,|||\omega - z|||.
\end{aligned}$$

Taking supremum in $[0, T]$,

$$|||\mathcal{F}_{\omega_0}(\omega) - \mathcal{F}_{z_0}(z)||| \leq C_1(T)\|\omega_0 - z_0\|_X + C_2(T)|||\omega - z|||.$$

Thus, the result. $\square$

In the following propositions we will prove that the solution u written as in (3.9) is nonnegative given any nonnegative initial data u_0 . To do this, we will prove that the mapping $\mathcal{F}_{\omega_0}$ has a unique fixed point in X_T, and we will prove that u is nonnegative using Picard iterations. The proposition is valid for a general positive operator K, (i.e., if $z \geq 0$, then $K(z) \geq 0$), in particular for $K = K_J$ with $J \geq 0$.

Proposition 3.2.2. (Weak Maximum Principle) *Let* (Ω, μ, d) *be a metric measure space.*

- *If* $X = L^p(\Omega)$, *with* $1 \leq p \leq \infty$, *we assume* $h \in L^\infty(\Omega)$.
- *If* $X = \mathcal{C}_b(\Omega)$, *we assume* $h \in \mathcal{C}_b(\Omega)$.

If $K \in \mathcal{L}(X, X)$ *is a positive operator, then for every* $u_0 \in X$ *nonnegative, the solution to the problem* (3.1) *is nonnegative for all* $t \geq 0$, *and it is nontrivial if* $u_0 \not\equiv 0$.

Proof. Thanks to (3.9), we know that the solution to (3.1) can be written as

$$u(x,t) = e^{-h(x)t}u_0(x,t) + \int_0^t e^{-h(x)(t-s)}K(u)(x,s)ds = \mathcal{F}_{u_0}(u)(x,t). \tag{3.11}$$

We choose T small enough such that $C_2(T)$ in Lemma 3.2.1 satisfies that $C_2(T) < 1$. Hence, by (3.10) we have that $\mathcal{F}_{u_0}(\cdot)$ is a contraction in $X_T = \mathcal{C}\big([0,T]; X\big)$. We consider the sequence of Picard iterations,

$$u_{n+1}(x,t) = \mathcal{F}_{u_0}(u_n)(x,t) \;\; \forall n \geq 1, \; x \in \Omega, \; 0 \leq t \leq T.$$

Then the sequence u_n converges to u in X_T. We take $u_1(x,t) = u_0(x) \geq 0$, then for $t \geq 0$

$$u_2(x,t) = \mathcal{F}_{u_0}(u_1)(x,t) = e^{-h(x)t}u_0(x) + \int_0^t e^{-h(x)(t-s)}K(u_0)(x)ds \tag{3.12}$$

is nonnegative, because K is a positive operator. Thus $u_2(x,t) \geq 0$ for all $t \geq 0$. Repeating this argument for all u_n, we get that $u_n(x,t)$ is nonnegative for every $n \geq 1$, for $t \geq 0$. As $u_n(x,t)$ converges to $u(x,t)$ in X_T, we have that the solution $u(x,t)$ is nonnegative in X_T.

We have proved that for some $T > 0$, that does not depend on u_0, the unique solution u of the problem (3.1) with initial data $u(x,0) = u_0(x) \geq 0$ is nonnegative for all $t \in [0,T]$.

If we consider again the same problem with initial data $u(\cdot,T)$, then the solution $u(\cdot,t)$ is nonnegative for all $t \in [T, 2T]$. Since (3.1) has a unique solution then we have proved that the solution of (3.1), $u(x,t) \geq 0$ for all $t \in [0,2T]$. Repeating this argument, we have that the solution of (3.1) is nonnegative $\forall t \geq 0$.

Since, we have proved that the solution $u(\cdot,t)$ to (3.1) is nonnegative, and K is a positive operator, from (3.11), we have that

$$u(\cdot,t) \geq e^{-h(\cdot)t}u_0(\cdot) \not\equiv 0, \quad \text{if} \;\; u_0 \not\equiv 0.$$

Thus, the result. □

Remark 3.2.3. *In the previous proposition we can only prove the positivity forwards on time. (see Corollary 3.2.5 to see that it is only positive forwards).*

We prove below that under the same hypotheses on the positivity of the function J, assumed in Proposition 2.1.17, if the initial data u_0 is nonnegative, not identically zero, then the solution to (3.1), is strictly positive for $t > 0$.

Theorem 3.2.4. (Strong Maximum Principle) *Let* (Ω, μ, d) *be a metric measure space.*

- *If $X = L^p(\Omega)$, with $1 \leq p \leq \infty$, we assume $h \in L^\infty(\Omega)$.*
- *If $X = \mathcal{C}_b(\Omega)$, we assume $h \in \mathcal{C}_b(\Omega)$.*

If $K_J \in \mathcal{L}(X, X)$, and $J \geq 0$ with

$$J(x, y) > 0 \ \textit{for all} \ x, \, y \in \Omega, \ \ \textit{such that} \ d(x, y) < R,$$

for some $R > 0$ and Ω is R-connected.

Then for every $u_0 \geq 0$, not identically zero, in X, the solution $u(t)$ of (3.1) is strictly positive, for all $t > 0$.

Proof. Thanks to Proposition 3.2.2, we know that $u \geq 0$, and it is not trivial, for all $x \in \Omega$, and for all $t \geq 0$. We take

$$v(t) = e^{h(\cdot)t} u(t),$$

then recalling the definition of the essential support in Definition 2.1.10, we have $P(u(t)) = P(v(t))$, for all $t \geq 0$. From (3.7), we know that v satisfies

$$v_t(t) = e^{h(\cdot)t} K(u(t)) \geq 0, \quad \forall t \geq 0. \tag{3.13}$$

Integrating (3.13) in $[s, \, t]$, we obtain

$$v(t) = v(s) + \int_s^t v_t(r) dr \geq v(s), \quad \text{for any} \ \ t \geq s \geq 0. \tag{3.14}$$

Then $P(v(t)) \supset P(v(s))$, $\forall t \geq s$. Moreover, since $v(t) = e^{h(\cdot)t} u(t)$ and thanks to (3.14), we obtain

$$e^{h(\cdot)t} u(t) \geq e^{h(\cdot)s} u(s).$$

Hence,

$$u(t) \geq e^{-h(\cdot)(t-s)} u(s).$$

This implies that $P(u(t)) \supset P(u(s))$, $\forall t \geq s$. As a consequence of (3.14), we have that for all $D \subset \Omega$,

$$v|_D (t) = v|_D (s) + \int_s^t \left(e^{h(\cdot)r} K(u(r)) \right)\Big|_D dr. \tag{3.15}$$

Since $P(v(t)) \supset P(v(s))$ for all $t \geq s$, and from (3.15), we have that

$$P(u(t)) \cap D = P(v(t)) \cap D \supset P(K(u)(r)) \cap D, \quad \text{for all} \ \ r \in [s, t]. \tag{3.16}$$

Moreover, applying Proposition 2.1.17 to $u(s)$, we have

$$P(K(u)(r)) \supset P(K(u(s))) \supset P_R^1(u(s)) = \bigcup_{x \in P(u(s))} B(x, R) \quad \text{for all} \ \ r \in [s, \, t]. \tag{3.17}$$

Hence, if we consider the set $D = P_R^1(u(s))$. From (3.16) and (3.17), we have that

$$P(u(t)) \supset P_R^1(u(s)), \quad \text{for all} \ \ t > s. \tag{3.18}$$

Hence the essential support of the solution at time t, contains the balls of radius R centered at the points in the support of the solution at time $s < t$.

We fix $t > 0$, and let $\mathcal{C} \subset \Omega$ be a compact set, then Proposition 2.1.17 implies that exists $n_0 \in \mathbb{N}$, such that $\mathcal{C} \subset P^n(u_0)$ for all $n \geq n_0$. We consider the sequence of times

$$t = t_n,\ t_{n-1} = t(n-1)/n,\ ..., t_j = t\,j/n,\ \ldots,\ t_1 = t/n,\ t_0 = 0.$$

Therefore, thanks to (3.18), we have that the essential supports at time t, contains the balls of radius R centered at the points in the essential support at time t_{n-1}, $P^1_R(u(t_{n-1}))$, which contains the balls of radius R centered at the points in the essential support at time t_{n-2}, then $P^2_R(u(t_{n-2}))$. Hence repeating this argument, we have

$$P(u(t)) = P(u(t_n)) \supset P^1_R(u(t_{n-1})) \supset P^2_R(u(t_{n-2})) \supset \ldots \supset P^n_R(u_0) \supset \mathcal{C}.$$

Thus, we have proved that $u(t)$ is strictly positive for every compact set in Ω, $\forall t > 0$. Therefore, $u(t)$ is strictly positive in Ω, for all $t > 0$. □

Corollary 3.2.5. *Under the assumptions of Theorem 3.2.4, if $u_0 \geq 0$, not identically zero, and $P(u_0) \neq \Omega$, then the solution to* (3.1) *has to be sign changing in Ω, $\forall t < 0$.*

Proof. We argue by contradiction. Let us assume first that there exists $t_0 < 0$ such that $u(\cdot, t_0) \equiv 0$. We take $u(\cdot, t_0)$ as initial data, then solving forward in time $u(\cdot, t) \equiv 0$, for all $t \geq t_0$. Hence, we arrive to contradiction, and $u(t_0)$ is not identically zero.

Secondly, let us assume that there exists $t_0 < 0$ such that $u(\cdot, t_0) \leq 0$, not identically zero. We take $-u(\cdot, t_0) \geq 0$ as the initial data, then thanks to Theorem 3.2.4, the solution to (3.1), satisfies that $u(x, 0) < 0$, $\forall x \in \Omega$. Thus, we arrive to contradiction.

Now, we assume that there exists $t_0 < 0$ such that $u(x, t_0) \geq 0$. Let $u(\cdot, t_0) \geq 0$ be the initial data, then thanks to Theorem 3.2.4, the solution to (3.1), satisfies that $u(x, 0) > 0$, $\forall x \in \Omega$. Thus, we arrive to contradiction.

Therefore, the solution has to be sign changing for all negative times. □

Remark 3.2.6. *As a consequence of Theorem 3.2.4, we deduce that the flow associated to the problem* (3.1)*, sends the boundary of the positive cone of X, (see Definition 2.1.24), to the interior of it, when time moves forward. Furthermore, from Proposition 3.1.2, we obtain that the flow is reversible, but despite of this, from Corollary 3.2.5 we have that the flow is not symmetric for time $t > 0$ and $t < 0$.*

3.3 Asymptotic regularizing effects

Let (Ω, μ, d) be a metric measure space, for $K \in \mathcal{L}(X, X)$, we consider the equation

$$u_t(x, t) = K(u)(x, t) - h(x)u(x, t), \text{ for } x \in \Omega. \tag{3.19}$$

We have seen above, in (3.9), that the group associated to this equation with initial data $u_0 \in X$ can be written as

$$S_{K,h}(t)u_0(x) = e^{-h(x)t}u_0(x) + \int_0^t e^{-h(x)(t-s)}K(u)(x, s)ds. \tag{3.20}$$

In general, the group (3.20) has no regularizing effects. In particular, the solution of (3.5) (i.e. (3.19) with $h \equiv 0$) is given by (3.6), then

$$u(x,t) - u_0(x) = \left(\sum_{k=1}^{\infty} \frac{t^k K_J^k}{k!} \right) u_0(x), \tag{3.21}$$

and even if $K : L^p(\Omega) \to \mathcal{C}_b(\Omega)$, we obtain that the right hand side of (3.21) is in $\mathcal{C}_b(\Omega)$, but on the left hand side we have the initial data that is in $L^p(\Omega)$. Hence, the regularity of u is equal to the regularity of the initial data u_0. Moreover, the solution to (3.1) with h constant is given by $u(x,t) = e^{h\,t} v(x,t)$, where v is solution of (3.8), then the regularity of u is equal to the regularity of the initial data u_0. Hence there is no regularizing effect.

However, we will prove that there exists a part of the group, that we call $S_2(t)$ that is compact, so it somehow regularizes. Moreover, there exists another part of the group that we call $S_1(t)$ which does not regularize, i.e., it carries the singularities of the initial data, but it decays to zero exponentially as t goes to ∞, if $h \geq 0$. Thus, we will have a regularizing effect when t goes to ∞. Then $S_{K,h}(t)$ is asymptotically smooth, according to the definition in [32, p. 4].

Now, we introduce Mazur's Theorem (see [24, p. 416]), which is the key to prove that $S_2(t)$ is compact.

Theorem 3.3.1. *(Mazur's Theorem)*
Let X be a Banach space, and let $B \subset X$ be compact. Then $\overline{co}(B)$ is compact, where $co(B)$ is the convex hull or the convex envelope of the set B (smallest convex set that contains B).

Lemma 3.3.2. *Let $\mathscr{C}$ be a compact set in X, let $T > 0$ and $F : [0,\, T] \to X$ be continuous. If $F(s) \in \mathscr{C}$ for all $s \in [0,\, T]$, then for a fixed $t \in (0,\, T]$,*

$$\frac{1}{t} \int_0^t F(s) ds \in \overline{co}(\mathscr{C}).$$

Proof. For a continuous function, the integral is given by

$$\int_0^t F(s) ds = \lim_{\Delta t \to 0} \sum_i F(\tilde{t_i}) \Delta t_i,$$

where $\tilde{t_i} \in [t_{i-1},\, t_i]$ belongs to the partition of the interval $[0,\, t]$, $\Delta t_i = t_i - t_{i-1}$ and Δt is the diameter of the partition.Then

$$\begin{aligned} \frac{1}{t} \sum_i F(\tilde{t_i}) \Delta t_i \ &= \tfrac{1}{\sum_j \Delta t_j} \sum_i F(\tilde{t_i}) \Delta t_i \\ &= \sum_i F(\tilde{t_i}) \tfrac{\Delta t_i}{\sum_j \Delta t_j} \\ &= \sum_i F(\tilde{t_i}) \alpha_i, \end{aligned}$$

with α_i satisfying $0 \leq \alpha_i \leq 1$, $\forall i$, and $\sum\limits_i \alpha_i = 1$. Moreover $F(\tilde{t_i}) \in \mathscr{C}$, then

$$\sum_i F(\tilde{t_i}) \alpha_i \in co(\mathscr{C}).$$

Therefore, we have that $\frac{1}{t}\int_0^t F(s)ds \in \overline{co}(\mathscr{C})$. $\square$

Proposition 3.3.3. *Let (Ω, μ, d) be a metric measure space,*

- *if $X = L^p(\Omega)$, with $1 \le p \le \infty$, we assume $h \in L^\infty(\Omega)$,*
- *if $X = \mathcal{C}_b(\Omega)$, we assume $h \in \mathcal{C}_b(\Omega)$.*

For $0 < t \in \mathbb{R}$ fixed, we have that the mapping

$$\begin{array}{rrcl} M: & [0,\, t] \times X & \longrightarrow & X \\ & (s, f) & \longmapsto & e^{-h(\cdot)\,(t-s)} f \end{array}$$

is continuous.

Proof. Assume we have proved that the mapping

$$\begin{array}{rrcl} g: & [0,\, t] & \longrightarrow & L^\infty(\Omega) \\ & s & \longmapsto & e^{-h(\cdot)\,(t-s)} \end{array}$$

is continuous, then we prove that the mapping M is continuous.
Given $(s_1,\, f_1) \in [0,\, t] \times X$, for all $\varepsilon > 0$, there exist $\delta_1,\, \delta_2 \in \mathbb{R}$ positive, such that for all $(s_2,\, f_2)$ satisfying $|s_1 - s_2| < \delta_1$ and $\|f_1 - f_2\|_X < \delta_2$, we have that

$$\begin{aligned} \|M(s_1,\, f_1)\ -&M(s_2,\, f_2)\|_X = \|e^{-h(\cdot)\,(t-s_1)} f_1 - e^{-h(\cdot)\,(t-s_2)} f_2\|_X \\ &= \|e^{-h(\cdot)\,(t-s_1)} f_1 - e^{-h(\cdot)\,(t-s_2)} f_1 + e^{-h(\cdot)\,(t-s_2)} f_1 - e^{-h(\cdot)\,(t-s_2)} f_2\|_X \\ &\le \|g(s_1) - g(s_2)\|_{L^\infty(\Omega)} \|f_1\|_X + \|e^{-h(\cdot)\,(t-s_2)}\|_{L^\infty(\Omega)} \|f_1 - f_2\|_X \\ &\le \|g(s_1) - g(s_2)\|_{L^\infty(\Omega)} \|f_1\|_X + \sup_{|s_1 - s_2| < \delta} \|e^{-h(\cdot)\,(t-s)}\|_{L^\infty(\Omega)} \|f_1 - f_2\|_X. \end{aligned}$$

Since g is continuous, we can choose δ_1 such that

$$\|g(s_1) - g(s_2)\|_{L^\infty(\Omega)} < \frac{\varepsilon}{2\|f_1\|_X}, \quad \text{if } |s_1 - s_2| < \delta_1$$

and we choose $\delta_2 < \dfrac{\varepsilon}{2 \sup\limits_{B_\delta(s_1)} \|e^{-h(\cdot)\,(t-s)}\|_{L^\infty(\Omega)}}$, then we obtain that for these δ_1 and δ_2

$$\|M(s_1,\, f_1) - M(s_2,\, f_2)\|_X < \varepsilon$$

Hence, we have proved that M is continuous.

Now, we just have to prove that the mapping g is continuous. Given $s_1 \in [0, t]$. Let us prove that for all $\varepsilon > 0$ there exists $\delta > 0$, such that if $|s_1 - s_2| < \delta$ then $\|g(s_1) - g(s_2)\|_{L^\infty(\Omega)} < \varepsilon$,

$$\begin{aligned} \|g(s_1) - g(s_2)\|_{L^\infty(\Omega)} &= \|e^{-h(\cdot)(t-s_1)} - e^{-h(\cdot)(t-s_2)}\|_{L^\infty(\Omega)} \\ &= \|e^{-h(\cdot)(t-s_1)} \left(1 - e^{h(\cdot)(s_2-s_1)}\right) \|_{L^\infty(\Omega)} \\ &\le e^{t\|h\|_{L^\infty(\Omega)}} \left|1 - e^{\|h\|_{L^\infty(\Omega)}|s_2 - s_1|}\right| \\ &= C\left|1 - e^{C_1|s_2 - s_1|}\right|. \end{aligned}$$

We know that the exponential function is continuous, then we have that

$$1 - e^{C_1|s_2 - s_1|} \to 0 \ \text{ as } \ s_2 \to s_1.$$

Therefore we have that $g : [0,\, t] \longrightarrow L^\infty(\Omega)$ is continuous. $\square$

In the following proposition, we see that in general, the solution associated to the problem (3.1), $u(t) = S_{K,h}(t) = S_1(t) + S_2(t)$, has no regularizing effects. We prove that $S_2(t)$ is compact, but $S_1(t)$ is not. However, we prove that if h is strictly positive in Ω, then $S_1(t)$ decays to zero exponentially as t goes to ∞, so we have a regularizing effect when t goes to ∞, and $S_{K,h}(t)$ is asymptotically smooth.

Theorem 3.3.4. *Let (Ω, μ, d) be a metric measure space, with $\mu(\Omega) < \infty$.*

- *If $X = L^p(\Omega)$, with $1 \leq p \leq \infty$, we assume $h \in L^\infty(\Omega)$.*
- *If $X = \mathcal{C}_b(\Omega)$, we assume $h \in \mathcal{C}_b(\Omega)$.*

For $1 \leq p \leq q \leq \infty$, let $X = L^q(\Omega)$ or $\mathcal{C}_b(\Omega)$. If $K \in \mathcal{L}(L^p(\Omega), X)$ is compact, (see Proposition 2.1.7), and h satisfies

$$h(x) \geq \alpha > 0 \ \text{ for all } \ x \in \Omega,$$

and $u_0 \in L^p(\Omega)$, then the group associated to the problem (3.1), satisfies that

$$u(t) = S_{K,h}(t)u_0 = S_1(t)u_0 + S_2(t)u_0$$

with

i. $S_1(t) \in \mathcal{L}(L^p(\Omega))$ $\forall t > 0$, *and* $\|S_1(t)\|_{\mathcal{L}(L^p(\Omega),L^p(\Omega))} \to 0$ *exponentially, as t goes to ∞.*

ii. $S_2(t) \in \mathcal{L}(L^p(\Omega), X)$ *is compact,* $\forall t > 0$.

Therefore $S_{K,h}(t)$ is asymptotically smooth.

Proof. We write the solution associated to (3.1), as in (3.9), then we have that

$$u(x,t) = S_{K,h}(t)u_0(x) = e^{-h(x)t}u_0(x) + \int_0^t e^{-h(x)(t-s)}K(u)(x,s)ds, \ \forall x \in \Omega$$

and we define

$$\begin{aligned} S_1(t)u_0 &= e^{-h(\cdot)t}u_0 \\ S_2(t)u_0 &= \int_0^t e^{-h(\cdot)(t-s)}K(u)(s)ds = \int_0^t e^{-h(\cdot)(t-s)}K(S_{K,h}(s)u_0)ds. \end{aligned}$$

i. Since $u_0 \in L^p(\Omega)$ and $h \in L^\infty(\Omega)$ with $h \geq \alpha > 0$, then $S_1(t)u_0 = e^{-h(\cdot)t}u_0 \in L^p(\Omega)$ and

$$\|S_1(t)\, u_0\|_{L^p(\Omega)} = \|e^{-h(\cdot)t}u_0(\cdot)\|_{L^p(\Omega)} \leq e^{-\alpha t}\|u_0\|_{L^p(\Omega)}.$$

Therefore $\|S_1(t)\|_{\mathcal{L}(L^p(\Omega),L^p(\Omega))} \leq e^{-\alpha t}$, with $\alpha > 0$, then it converges exponentially to 0, as t goes to ∞.

ii. Fix $t > 0$, as $h \in L^\infty(\Omega)$, $S_{K,h}(s) \in \mathcal{L}(L^p(\Omega))$ $\forall s \in [0,t]$, and $K \in \mathcal{L}(L^p(\Omega), X)$, then

$$\begin{aligned} \|S_2(t)(u_0)\|_X &\leq e^{-\alpha t}\int_0^t \|K(S_{K,h}(s)u_0)\|_X ds \\ &\leq e^{-\alpha t}t \max_{0 \leq s \leq t} \|K(S_{K,h}(s)u_0)\|_X < \infty. \end{aligned}$$

Thus $S_2(t) \in \mathcal{L}(L^p(\Omega), X)$.

Let us see now that $S_2(t) \in \mathcal{L}(L^p(\Omega), X)$ is compact $\forall t > 0$. Fix $t > 0$ and consider a bounded set $\mathcal{B}$ of initial data. We know that

$$S_2(t)u_0 = \int_0^t e^{-h(\cdot)(t-s)} K(u)(\cdot, s)ds.$$

We denote $S_2(t)u_0 = \displaystyle\int_0^t F_{u_0}(s)ds$, with

$$F_{u_0}(s) = e^{-h(\cdot)(t-s)} K(S_{K,h}(s)u_0).$$

Assume we have proved that $F_{u_0}(s) \in \mathscr{C}$, were $\mathscr{C}$ is a compact set in X, for all $s \in [0, t]$ and for all $u_0 \in \mathcal{B}$. Then applying Lemma 3.3.2 to F_{u_0} we have that $\frac{1}{t}S_2(t)(u_0) \in \overline{co}(\mathscr{C})$, $\forall u_0 \in \mathcal{B}$, and thanks to the Mazur's Theorem 3.3.1, we obtain that $\frac{1}{t}S_2(t)(\mathcal{B})$ is in a compact set of X. Therefore $S_2(t)$ is compact. Now, we have to prove that $F_{u_0}(s) = e^{-h(\cdot)(t-s)}K(S_{K,h}(s)u_0)$ belongs to a compact set, for all $(s,\, u_0) \in [0, t] \times \mathcal{B}$.
First of all, we check that $K(S_{K,h}(s)u_0)$ belongs to a compact set $\mathcal{W}$ in X, for all $(s,\, u_0) \in [0, t] \times \mathcal{B}$. Since K is compact, we just have to prove that the set

$$B = \{S_{K,h}(s)u_0 : \, (s,\, u_0) \in [0, t] \times \mathcal{B}\}$$

is bounded. In fact, we have that $K - hI \in \mathcal{L}(L^p(\Omega), L^p(\Omega))$, then $\left|\sigma_{L^p(\Omega)}(K - hI)\right| \leq \|K - hI\|_{\mathcal{L}(L^p(\Omega))} \leq \delta < \infty$, thus

$$\begin{aligned} \|S_{K,h}(s)u_0\|_{L^p(\Omega)} &= \|u(\cdot, s)\|_{L^p(\Omega)} \\ &\leq Ce^{(\delta+\varepsilon)s}\|u_0\|_{L^p(\Omega)} \\ &\leq Ce^{(\delta+\varepsilon)t}\|u_0\|_{L^p(\Omega)}, \end{aligned}$$

for all $(s,\, u_0) \in [0, t] \times \mathcal{B}$. (This inequality will be proved with more details in Proposition 3.4.2). Then, since $\mathcal{B}$ is bounded, we obtain that B is bounded in $L^p(\Omega)$.

Finally, we just need to prove that $F_{u_0}(s)$ is in a compact set for all $(s,\, u_0) \in [0, t] \times \mathcal{B}$, and this is true thanks to Proposition 3.3.3 that says that the mapping

$$\begin{aligned} M : \quad [0,\, t] \times X &\longrightarrow \quad X \\ (s, f) &\longmapsto \quad e^{-h\,(t-s)} f \end{aligned}$$

is continuous. Then M sends the compact set $[0,\, t] \times \mathcal{W}$ into a compact set $\mathscr{C}$. Thus, $F_{u_0}(s)$ belongs to a compact set, $\mathscr{C}$, $\forall (s,\, u_0) \in [0, t] \times \mathcal{B}$. Therefore we have finally proved that $S_2(t)$ is compact in X, for all $t > 0$, and $S_{K,h}(t)$ is asymptotically smooth. $\square$

3.4 The Riesz projection and asymptotic behavior

In this section we want to study the asymptotic behavior of the solution of the problem (3.1). For this, we need first to introduce the concept of Riesz projection of a linear and bounded operator. Moreover, we will prove that the Riesz projection is equivalent to the Hilbert projection in $L^2(\Omega)$. The Riesz projection is given in terms of the spectrum of the operator. Since the spectrum of the operator $L = K - hI$ has been proved in Proposition 2.4.5 to be independent of $X = L^p(\Omega)$, with $1 \leq p \leq \infty$ or $X = \mathcal{C}_b(\Omega)$, then the asymptotic behavior of the solution of (3.1) will be characterized with the Riesz projection, and it can we calculated in X with the Hilbert projection.

Consider a general operator $F \in \mathcal{L}(Y, Y)$, where Y is a Banach space. The proposition below gives a bound of the norm of the group associated to the linear and bounded operator F. We will also give a general result of asymptotic behavior of the solutions associated to the problem

$$\begin{cases} u_t(x,t) & = F(u)(x,t) \\ u(x,0) & = u_0(x), \text{ with } u_0 \in Y \end{cases} \tag{3.22}$$

The definitions below, can be found in [24, chap. VII].

Definition 3.4.1. *Let $F \in \mathcal{L}(Y)$, and f be an analytic function in some neighborhood of $\sigma(F) \subset \mathbb{C}$, and let U be an open set whose boundary Γ consists of a finite number of rectifiable Jordan curves, oriented in the positive sense. Suppose that $U \supset \sigma(F)$, and that $U \cup \Gamma$ is contained in the domain of analyticity of f. Then the operator*

$$f(F) = \frac{1}{2\pi i} \int_\Gamma f(\lambda)(\lambda I - F)^{-1} d\lambda$$

is well defined and $f(F) \in \mathcal{L}(Y, Y)$.

If F is a continuous operator the eigenvalues of the operator F are bounded, and there exists $\delta \in \mathbb{R}$ such that $\mathrm{Re}(\lambda) \leq \delta$ for $\lambda \in \sigma(F)$. We can find a closed rectifiable curve Γ that contains

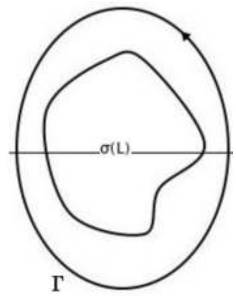

Figure 3.1: Bounded spectrum

$\sigma(F)$, without crossing any $\lambda \in \sigma(L)$, like the curve Γ in Figure 3.1.

In particular, $f(\lambda) = e^{\lambda t}$ is analytic in a neighborhood of $\sigma(F)$. Thus, we can apply Definition 3.4.1, and we obtain that

$$e^{Ft} = \sum_{k=0}^{\infty} \frac{F^k t^k}{k!} = \frac{1}{2\pi i} \int_\Gamma e^{\lambda t} (\lambda I - F)^{-1} d\lambda.$$

In the next proposition we estimate the norm of the group $e^{Ft} : Y \to Y$.

Proposition 3.4.2. *Let $F \in \mathcal{L}(Y)$ be an operator as the one described above with*

$$Re(\sigma(F)) \leq \delta,$$

then $\forall \varepsilon > 0$ there exists a constant $C_0 = C_0(\varepsilon)$ such that

$$||e^{Ft}||_{\mathcal{L}(Y)} \leq C_0 e^{(\delta+\varepsilon)t} \quad \forall t \geq 0.$$

Proof. For every curve Γ that satisfies the hypotheses of Definition 3.4.1, we have that $\mathrm{Re}(\lambda) \leq \delta + \varepsilon,\ \forall \varepsilon > 0,\ \forall \lambda \in \Gamma$, then for $t \geq 0$

$$\|e^{Ft}\|_{\mathcal{L}(Y)} = \left| \frac{1}{2\pi i} \int_\Gamma e^{\lambda t} (\lambda I - F)^{-1} d\lambda \right| \leq \frac{1}{2\pi} \int_\Gamma e^{Re(\lambda)\, t} \left|(\lambda I - F)^{-1}\right| d|\lambda| \leq C_0 e^{(\delta+\varepsilon)t}.$$

□

Corollary 3.4.3. *Let $F \in \mathcal{L}(Y)$ be an operator as the one described above with*

$$-\delta \leq Re(\sigma(F)) \leq \delta,$$

then $\forall \varepsilon > 0$ there exists a constant $C_0 = C_0(\varepsilon)$ such that

$$||e^{Ft}||_{\mathcal{L}(Y)} \leq C_0 e^{(\delta+\varepsilon)|t|} \quad \forall t \in \mathbb{R}.$$

Now, we introduce the Riesz projection, that will help us study the asymptotic behavior of the solution of (3.22). The following definitions can be found in [31, chap. 1].

Let F be a bounded and linear operator on the Banach space Y. If N is a subspace of Y invariant under F, then $F|N$ denotes the restriction of F to N, which has to be considered as an operator from N into N.

A set σ_1 is called an **isolated part** of $\sigma(F)$ if both σ_1 and $\sigma_2 = \sigma(F) \setminus \sigma_1$ are closed subsets of $\sigma(F)$. Given an isolated part σ_1 of $\sigma(F)$ we define Q_{σ_1} to be the bounded linear operator on Y given by

$$Q_{\sigma_1} = \frac{1}{2\pi i} \int_\Gamma (\lambda I - F)^{-1} d\lambda,$$

where Γ consists of a finite number of rectifiable Jordan curves, oriented in the positive sense around σ_1, separating σ_1 from σ_2. This means that σ_1 belongs to the inner region of Γ, and σ_2 belongs to the outer region of Γ. The operator Q_{σ_1} is called the **Riesz projection** of F corresponding to the isolated part σ_1, and is independent of the path Γ described as above.

The following theorem and corollary describe some properties of the Riesz projection Q_{σ_1} (see [31, p. 10]).

Theorem 3.4.4. *Let σ_1 be an isolated part of $\sigma(F)$, and put $U = Im Q_{\sigma_1}$ and $V = Ker Q_{\sigma_1}$. Then $Y = U \oplus V$, the spaces U and V are $F-$ invariant subspaces and considering $F|U = F_1$ and $F|V = F_2$*

$$\sigma(F_1) = \sigma_1 \qquad \sigma(F_2) = \sigma(F) \setminus \sigma_1.$$

Corollary 3.4.5. *Assume σ_1 is an isolated part of $\sigma(F)$, and $\sigma_2 = \sigma(F) \setminus \sigma_1$. Then,*

$$Q_{\sigma_1} + Q_{\sigma_2} = I \qquad Q_{\sigma_1} \cdot Q_{\sigma_2} = 0.$$

The following lemma will be useful to prove the next proposition, that states that the group, e^{Ft}, and the Riesz projection, Q_σ, commute, with this we will estimate the norm of the Riesz projection of the solutions to (3.22).

Lemma 3.4.6. *Let Y be a Banach space, $F : Y \to Y$ and $\lambda \in \rho(F) \subset \mathbb{C}$, then,*

$$(\lambda I - F)^{-1} F^k = F^k (\lambda I - F)^{-1}, \qquad \textit{for all } k \in \mathbb{N}.$$

Proof. Take $x \in Y$, such that there exists $y \in Y$ satisfying $(\lambda I - F)y = x$, this is $y = (\lambda I - F)^{-1} x$. Then $\lambda y - Fy = x$. Applying F, we have $\lambda F y - F^2 y = Fx$. Thus we have proved that $(\lambda I - F)Fy = Fx$, applying $(\lambda I - F)^{-1}$, we have that $Fy = (\lambda I - F)^{-1} F x$. Since $y = (\lambda I - F)^{-1} x$, we have proved that $F(\lambda I - F)^{-1} = (\lambda I - F)^{-1} F$. Following this same argument, we can prove that $(\lambda I - F)F^k y = F^k x$, for any $k \in \mathbb{N}$. Hence, the result. $\square$

Proposition 3.4.7. *Let σ be an isolated part of $\sigma(F)$, then,*

$$e^{Ft} \circ Q_\sigma = Q_\sigma \circ e^{Ft} = e^{F_1 t},$$

where, $F_1 = F | Im Q_\sigma$.

Proof. Let $e^{Ft} = \sum\limits_{k=0}^{\infty} \frac{(Ft)^k}{k!}$, then thanks to Lemma 3.4.6,

$$\begin{aligned}
(Q_\sigma \circ e^{Ft}) &= Q_\sigma \left(\sum_{k=0}^{\infty} \frac{(Ft)^k}{k!} \right) \\
&= \frac{1}{2\pi i} \int_\Gamma (\lambda - F)^{-1} \left(\sum_{k=0}^{\infty} \frac{(Ft)^k}{k!} \right) d\lambda \\
&= \frac{1}{2\pi i} \sum_{k=0}^{\infty} \int_\Gamma (\lambda - F)^{-1} \frac{F^k t^k}{k!} \, d\lambda \\
&= \frac{1}{2\pi i} \sum_{k=0}^{\infty} \int_\Gamma \frac{F^k t^k}{k!} (\lambda - F)^{-1} \, d\lambda \\
&= \sum_{k=0}^{\infty} \frac{F^k t^k}{k!} \frac{1}{2\pi i} \int_\Gamma (\lambda - F)^{-1} \, d\lambda \qquad = (e^{Ft} \circ Q_\sigma).
\end{aligned}$$

$\square$

Let Y be a Banach space, we study now the asymptotic behavior of the solution of

$$\left\{ \begin{array}{lll} u_t(t) & = F(u)(t), & \text{with } t \in \mathbb{R} \\ u(0) & = u_0, & \text{with } u_0 \in Y \end{array} \right. \tag{3.23}$$

where $F \in \mathcal{L}(Y, Y)$ and $\sigma(F)$ is a disjoint union of two closed subsets σ_1 and σ_2. Assume

$$\delta_2 < \mathrm{Re}(\sigma_1) \le \delta_1, \ \mathrm{Re}(\sigma_2) \le \delta_2\,, \ \text{ with } \ \delta_2 < \delta_1,$$

like in Figure 3.2.

Applying Corollary 3.4.5, we have that the solution to (3.23), can be written as

$$u(t) = Q_{\sigma_1}(u)(t) + Q_{\sigma_2}(u)(t).$$

On the other hand, the solution of (3.23) is equal to $u(t) = e^{Ft}u_0$. Thus, thanks to Propositions 3.4.2 and 3.4.7, and since $\mathrm{Re}(\sigma_2) \leq \delta_2$ we obtain that for $t > 0$

$$\begin{aligned} \|Q_{\sigma_2}(u(t))\|_Y &= \| \left(Q_{\sigma_2} \circ e^{F\,t}\right)(u_0)\|_Y \\ &= \|e^{F_2 t}\,\left(Q_{\sigma_2}(u_0)\right)\|_Y \\ &\leq C_2\, e^{(\delta_2+\varepsilon)t}\|Q_{\sigma_2}(u_0)\|_Y, \quad \forall \varepsilon > 0, \end{aligned} \tag{3.24}$$

where, $F_2 = F|\mathrm{Im}\, Q_{\sigma_2}$.

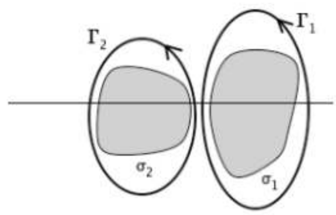

Figure 3.2: $\sigma(F) = \sigma_1 \cup \sigma_2$.

The following Theorem, which is the principal result of this section. It states which is the asymptotic behavior of the solution associated to (3.23).

Theorem 3.4.8. *Consider $F \in \mathcal{L}(Y)$ and let $\sigma(F)$ be a disjoint union of two closed subsets σ_1 and σ_2, with $\delta_2 < Re(\sigma_1) \leq \delta_1$, $Re(\sigma_2) \leq \delta_2$, with $\delta_2 < \delta_1$. Then the solution of* (3.23) *satisfies*

$$\lim_{t\to\infty} \|e^{-\mu\,t}\big(u(t) - Q_{\sigma_1}(u)(t)\big)\|_Y = 0, \; \forall \mu > \delta_2.$$

Proof. By using the definition of the Riesz projection, taking $\mu > \delta_2$ and thanks to Corollary 3.4.5

$$e^{-\mu t}(u(t) - Q_{\sigma_1}(u)(t)) \quad = \quad e^{-\mu t}\, Q_{\sigma_2}(u)(t)$$

Thanks to (3.24), we know that the right hand side of the latter equation satisfies,

$$\|e^{-\mu t}\, Q_{\sigma_2}(u)(t)\|_Y \quad \leq \quad C_2\, e^{(-\mu+\delta_2+\varepsilon)t}\|Q_{\sigma_2}(u_0)\|_Y, \qquad \forall \varepsilon > 0, \; \forall t > 0$$

Furthermore, there exists $\varepsilon_0 > 0$ such that $\forall \varepsilon$ such that $0 < \varepsilon < \varepsilon_0$, it happens that

$$(-\mu + \delta_2 + \varepsilon) < 0.$$

Hence, the result. $\square$

In the following proposition we prove that the Hilbert projection over the space generated by the eigenfunction associated to the first eigenvalue of F, is equal to the Riesz projection in X for a general operator $F : X \to X$. We denote by λ_1 the largest eigenvalue associated to F in X. We assume that λ_1 is isolated and simple, and Φ_1 is an eigenfunction associated to λ_1, with $\|\Phi\|_{L^2(\Omega)} = 1$. Taking $\sigma_1 = \{\lambda_1\}$, we know that in the Hilbert space $L^2(\Omega)$,

$$P_{\sigma_1}(u) = \langle u,\, \Phi_1\rangle \Phi_1 = \int_\Omega u(x)\, \Phi_1(x) dx\, \Phi_1, \quad \forall u \in L^2(\Omega),$$

where P_{σ_1} is the Hilbert projection over the space generated by the eigenfunction associated to σ_1.

Proposition 3.4.9. *Let (Ω, μ, d) be a metric measure space, with μ, as in Definition 1.1.5. For $1 \leq p_0 < p_1 \leq \infty$, with $2 \in [p_0, p_1]$, let $X = L^p(\Omega)$, with $p \in [p_0, p_1]$, or $X = \mathcal{C}_b(\Omega)$. We assume $F \in \mathcal{L}(X, X)$ is selfadjoint in $L^2(\Omega)$, the spectrum of F, $\sigma_X(F)$, is independent of X, and the largest eigenvalue associated to F, λ_1 is simple and isolated, with associated eigenfunction $\Phi_1 \in L^p(\Omega) \cap L^{p'}(\Omega)$, for $p \in [p_0, p_1]$, if $X = L^p(\Omega)$, or $\Phi_1 \in \mathcal{C}_b(\Omega) \cap L^1(\Omega)$, if $X = \mathcal{C}_b(\Omega)$, and $\|\Phi\|_{L^2(\Omega)} = 1$.*

If $\sigma_1 = \{\lambda_1\}$, and Γ is the curve around λ_1, such that Γ only surrounds $\{\lambda_1\}$, then for $u \in X$,

$$Q_{\sigma_1}(u) = \frac{1}{2\pi i}\int_\Gamma (\lambda I - F)^{-1} u\, d\lambda = \int_\Omega u(x)\, \Phi_1(x) dx\, \Phi_1 = \langle u,\, \Phi_1\rangle \Phi_1 = P_{\sigma_1}(u),$$

where Q_{σ_1} is the Riesz projection and P_{σ_1} is the Hilbert projection over the space generated by the eigenfunctions associated to σ_1.

Proof. We consider $L^2(\Omega) = [\Phi_1] \oplus [\Phi_1]^\perp$. Let $v(\lambda)$ be defined as,

$$(\lambda I - F) v(\lambda) = u,$$

then we can write v as follows

$$v(\lambda) = a(\lambda)\Phi_1 + W(\lambda), \tag{3.25}$$

where $W(\lambda) \in [\Phi_1]^\perp$ and $a(\lambda) = \langle v(\lambda), \Phi_1\rangle$. Now we want to describe $Q_{\sigma_1}(u)$

$$\begin{aligned} Q_{\sigma_1}(u) &= \frac{1}{2\pi i}\int_\Gamma (\lambda I - F)^{-1} u\, d\lambda \\ &= \frac{1}{2\pi i}\int_\Gamma v(\lambda) d\lambda \\ &= \frac{1}{2\pi i}\int_\Gamma a(\lambda) d\lambda\, \Phi_1 + \frac{1}{2\pi i}\int_\Gamma W(\lambda)\, d\lambda. \end{aligned} \tag{3.26}$$

Since λ_1 is a simple eigenvalue, we have that

$$Q_{\sigma_1}(u) = \alpha\, \Phi_1, \;\text{ with }\; \alpha = \langle Q_{\sigma_1}(u), \Phi_1\rangle. \tag{3.27}$$

Then, multiplying (3.26) by Φ_1 and integrating in Ω, we obtain

$$\langle Q_{\sigma_1}(u), \Phi_1\rangle = \int_\Omega \frac{1}{2\pi i}\int_\Gamma a(\lambda) d\lambda\, \Phi_1^2 dx + \int_\Omega \frac{1}{2\pi i}\int_\Gamma W(\lambda)\, d\lambda \Phi_1 dx \tag{3.28}$$

Since $W(\lambda) \in [\Phi_1]^{\perp}$ and $\|\Phi_1\|_{L^2(\Omega)} = 1$, from (3.28) we have that

$$\langle Q_{\sigma_1}(u), \Phi_1 \rangle = \frac{1}{2\pi i} \int_{\Gamma} a(\lambda) d\lambda$$

Therefore,

$$Q_{\sigma_1}(u) = \frac{1}{2\pi i} \int_{\Gamma} a(\lambda)\, d\lambda\, \Phi_1. \tag{3.29}$$

Now, we compute the Hilbert projection of u in $L^2(\Omega)$. We multiply (3.25) by Φ_1

$$\lambda \langle v(\lambda), \Phi_1 \rangle - \langle F v(\lambda), \Phi_1 \rangle = \langle u, \Phi_1 \rangle. \tag{3.30}$$

Since F is selfadjoint in $L^2(\Omega)$, (3.30) is equal to

$$\lambda \langle v(\lambda), \Phi_1 \rangle - \langle v(\lambda), F\Phi_1 \rangle = \langle u, \Phi_1 \rangle. \tag{3.31}$$

Now, since Φ_1 is an eigenfunction associated to λ_1, (3.31) becomes

$$\begin{aligned} \lambda \langle v(\lambda), \Phi_1 \rangle - \lambda_1 \langle v(\lambda), \Phi_1 \rangle &= \langle u, \Phi_1 \rangle \\ (\lambda - \lambda_1) \langle v(\lambda), \Phi_1 \rangle &= \langle u, \Phi_1 \rangle. \end{aligned} \tag{3.32}$$

Thanks to definition (3.25) and (3.32), we obtain that

$$a(\lambda) = \langle v(\lambda), \Phi_1 \rangle = \frac{\langle u, \Phi_1 \rangle}{\lambda - \lambda_1}. \tag{3.33}$$

Finally, from (3.29) and (3.33),

$$Q_{\sigma_1}(u) = \frac{1}{2\pi i} \int_{\Gamma} a(\lambda)\, d\lambda \Phi_1 = \frac{1}{2\pi i} \int_{\Gamma} \frac{\langle u, \Phi_1 \rangle}{\lambda - \lambda_1} d\lambda\, \Phi_1 = \langle u, \Phi_1 \rangle \Phi_1 = P_{\sigma_1}(u).$$

We have proved the equality in the Hilbert space $L^2(\Omega)$, but we want to prove that this is true also in $L^p(\Omega)$ for $p \in [p_0, p_1]$. Since the spectrum, $\sigma_X(F)$, is independent of X, we have that the projection $P_{\sigma_1}(u) = \langle u, \Phi_1 \rangle \Phi_1$ is well defined for $u \in X$ because by hypothesis, $\Phi_1 \in L^{p'}(\Omega) \cap L^p(\Omega)$ for all $p \in [p_0, p_1]$, if $X = L^p(\Omega)$, or $\Phi_1 \in \mathcal{C}_b(\Omega) \cap L^1(\Omega)$, if $X = \mathcal{C}_b(\Omega)$.

On the other hand, we consider the set

$$V = span\,[\, \chi_D \,;\, D \subset \Omega \text{ with } \mu(D) < \infty] \,,$$

where χ_D is the characteristic function of $D \subset \Omega$. Then, from Proposition 2.1.1, we know that $V \subset L^2(\Omega)$ is dense in $L^p(\Omega)$, with $1 \leq p \leq \infty$, and $L^2(\Omega) \cap \mathcal{C}_b(\Omega)$ is dense in $\mathcal{C}_b(\Omega)$. Since $P_{\sigma_1} \equiv Q_{\sigma_1}$ in $L^2(\Omega)$, then we have that two linear operators are equal in a dense subspace of X, then they are equal in X. Hence, the result. $\square$

3.5 Asymptotic behaviour of the solution of the nonlocal diffusion problem

Let (Ω, μ, d) be a metric measure space with Ω compact. In this section be apply the results of the previous section about the asymptotic behavior of the solution for the problem

$$\begin{cases} u_t(x,t) &= (K - hI)(u)(x,t), \quad x \in \Omega,\ t > 0, \\ u(x,0) &= u_0(x), \qquad \text{with } u_0 \in X. \end{cases} \tag{3.34}$$

We study two problems to which we apply the results of the previous sections. In particular we are going to study the asymptotic behavior of the solution of (3.34) with:

- h constant
- $h = h_0 = \displaystyle\int_\Omega J(\cdot, y)dy$, with $J \in L^\infty(\Omega, L^1(\Omega))$.

For h constant – For $h = a$ constant we have the problem

$$\begin{cases} u_t(x,t) & = (K - aI)(u)(x,t), \text{ with } a \in \mathbb{R}, \\ u(x,0) & = u_0(x), \text{ with } u_0 \in L^p(\Omega). \end{cases} \tag{3.35}$$

In the following proposition, we prove that the exponential decay in X of the asymptotic behaviour of the solution of (3.35) is given by the first eigenvalue λ_1 of $K - aI$, and the asymptotic behaviour of the solutions is described by the unique eigenfunction, Φ_1, associated to λ_1.

Proposition 3.5.1. *Let (Ω, μ, d) be a metric measure space, with Ω compact and connected. Let $X = L^p(\Omega)$, with $1 \leq p \leq \infty$, or $X = \mathcal{C}_b(\Omega)$. Let $K \in \mathcal{L}(L^1(\Omega), \mathcal{C}_b(\Omega))$ be compact, (see Proposition 2.1.7 to check compactness of integral operators K with kernel J, and assume $J(x,y) = J(y,x)$ with*

$$J(x,y) > 0, \ \forall x,\, y \in \Omega \text{ such that } d(x,y) < R, \ \text{ for some } \ R > 0. \tag{3.36}$$

Then the solution u of (3.35) satisfies that

$$\lim_{t\to\infty} \|e^{-\lambda_1 t}u(t) - C^*\Phi_1\|_X = 0, \tag{3.37}$$

where $C^* = \dfrac{\int_\Omega u_0(x)\Phi_1(x)dx}{\int_\Omega \Phi_1(x)^2 dx}$, and Φ_1 is the eigenfunction associated to λ_1. Moreover, $\Phi_1 \in L^p(\Omega) \cap L^{p'}(\Omega)$, and $\Phi_1 \in \mathcal{C}_b(\Omega) \cap L^1(\Omega)$.

Proof. From Proposition 2.1.20, we have that $\sigma_X(K)$ is independent of X. Moreover, since $J(x,y) = J(y,x)$, then from Proposition 2.1.22, we know that $\sigma(K) \setminus \{0\}$ is a real sequence of eigenvalues $\{\mu_n\}_{n\in\mathbb{N}}$ of finite multiplicity that converges to 0. Furthermore, the hypotheses of Proposition 2.1.26 are satisfied, then the largest eigenvalue, $\lambda_1 = r(K)$, is and isolated simple eigenvalue, and the eigenfunction $\Phi_1 \in \mathcal{C}_b(\Omega)$ associated to it, is positive. Since the spectrum does not depend on X, we have that, $\Phi_1 \in X$, in particular $\Phi_1 \in L^p(\Omega) \cap L^{p'}(\Omega)$, and $\Phi_1 \in \mathcal{C}_b(\Omega) \cap L^1(\Omega)$.

Thanks to Proposition 2.4.4, we know that the spectrum, $\sigma_X(K - aI) \setminus \{-a\}$, is a real sequence of eigenvalues $\{\lambda_n\}_{n\in\mathbb{N}} = \{\mu_n\}_{n\in\mathbb{N}} - a$, of finite multiplicity that converges to $-a$.

Now, we consider $\sigma_1 = \{\lambda_1\}$ and $\sigma_2 = \{\lambda_2, ..., \lambda_n, ...\} \cup \{-a\}$, and let Φ_1 be a positive eigenfunction associated to λ_1.

Since $J(x,y) = J(y,x)$, from Proposition 2.1.19, $K - aI$ is selfadjoint in $L^2(\Omega)$, then we can apply Proposition 3.4.9. Then, it holds that

$$Q_{\sigma_1}(u_0) = P_{\sigma_1}(u_0) = C^*\Phi_1, \tag{3.38}$$

where $C^* = \dfrac{\int_\Omega u_0(x)\Phi_1(x)dx}{\int_\Omega \Phi_1(x)^2 dx}$.

Furthermore, for $u_0 \in X$ thanks to Theorem 3.4.8, the solution of (3.35) satisfies

$$\lim_{t\to\infty} \|e^{-\lambda_1 t}\big(u(t) - Q_{\sigma_1}(u)(t)\big)\|_X = 0. \tag{3.39}$$

Since $u(x,t) = e^{(K-aI)t}u_0(x)$, $P_{\sigma_1} = Q_{\sigma_1}$, and thanks to Proposition 3.4.7 we have that

$$P_{\sigma_1}(u)(x,t) = P_{\sigma_1}(e^{(K-aI)t}u_0)(x,t) = e^{(K-aI)t}P_{\sigma_1}(u_0)(x). \tag{3.40}$$

On the other hand, since $P_{\sigma_1}(u_0)(x) = C^*\Phi_1$, where $C^* = \dfrac{\int_\Omega u_0(x)\Phi_1(x)dx}{\int_\Omega \Phi_1(x)^2 dx}$, then

$$e^{(K-aI)t}P_{\sigma_1}(u_0)(x) = e^{(K-aI)t}C^*\Phi_1. \tag{3.41}$$

Moreover, Φ_1 is an eigenfunction associated to λ_1 and $e^{(K-aI)t} = \sum \frac{(K-aI)^n t^n}{n!}$, then we have

$$e^{(K-aI)t}\Phi_1 = \sum \frac{(K-aI)^n t^n}{n!}\Phi_1 = \sum \frac{(K-aI)^n \Phi_1\, t^n}{n!} = \sum \frac{\lambda_1^n \Phi_1\, t^n}{n!} = e^{\lambda_1 t}\Phi_1. \tag{3.42}$$

Hence, from (3.40), (3.41) and (3.42),

$$P_{\sigma_1}(u)(x,t) = C^* e^{\lambda_1 t}\Phi_1(x). \tag{3.43}$$

Therefore, thanks to (3.39) and (3.43)

$$\lim_{t\to\infty} \|e^{-\lambda_1 t}u(t) - C^*\Phi_1\|_X = 0.$$

□

FOR $h = h_0 \in L^\infty(\Omega)$ – We consider the problem

$$\begin{cases} u_t(x,t) &= (K - h_0 I)(u)(x,t) \\ u(x,0) &= u_0(x), \text{ with } u_0 \in L^p(\Omega) \end{cases} \tag{3.44}$$

In the following proposition, we prove that the solution of (3.44) goes exponentially in norm X to the mean value in Ω of the initial data.

Proposition 3.5.2. *Let (Ω, μ, d) be a metric measure space, with $\mu(\Omega) < \infty$. Let $X = L^p(\Omega)$, with $1 \le q \le \infty$ or $X = \mathcal{C}_b(\Omega)$, let $K \in \mathcal{L}(L^1(\Omega), \mathcal{C}_b(\Omega))$ be compact, (see Proposition 2.1.7 to check compactness of integral operators with kernel J), and we assume $J \in L^\infty(\Omega, L^1(\Omega))$, $J(x,y) = J(y,x)$ and*

$$J(x,y) > 0,\ \forall x,\, y \in \Omega \text{ such that } d(x,y) < R, \text{ for some } R > 0. \tag{3.45}$$

We assume that $h_0(x) > \alpha > 0$, for all $x \in \Omega$.

Then the solution u of (3.44) satisfies that

$$\lim_{t\to\infty} \left\| e^{-(\beta_1+\varepsilon)t}\left(u(t) - \frac{1}{\mu(\Omega)}\int_\Omega u_0(x)dx\right)\right\|_X = 0, \tag{3.46}$$

where $\beta_1 < 0$, and $\varepsilon > 0$ small enough.

Proof. Since $K \in \mathcal{L}(L^1(\Omega), \mathcal{C}_b(\Omega))$ is compact, then $K \in \mathcal{L}(X,X)$ is compact. Thanks to Theorem 2.4.4, we know that

$$\sigma_X(K - h_0 I) = \overline{Im(-h_0)} \cup \{\mu_n\}_{n=1}^{M}, \quad \text{with } M \in \mathbb{N} \text{ or } M = \infty.$$

If $M = \infty$, then $\{\mu_n\}_{n=1}^{\infty}$ is a sequence of eigenvalues of $K - h_0 I$ with finite multiplicity, that has accumulation points in $Im(-h)$. Moreover, from Proposition 2.4.5, $\sigma_X(K - h_0 I)$ is independent of X.

From Corollary 2.4.6, we have that $\sigma_X(K - h_0 I) \leq 0$, and 0 is an isolated simple eigenvalue. Moreover, the constant functions v in Ω, satisfy that

$$(K - h_0 I)(v) = 0.$$

Moreover, since $J(x,y) = J(y,x)$ and thanks to Proposition 2.1.19, $K - h_0 I$ is selfadjoint in $L^2(\Omega)$, thus, from Proposition 2.1.21, $\{\mu_n\} \subset \mathbb{R}$. Hence, we consider $\sigma_1 = \{0\}$ an isolated part of $\sigma(K - h_0 I)$, with associated eigenfunction $\Phi_1 = 1/\mu(\Omega)^{1/2}$, and $\sigma_2 = \sigma(K - h_0 I) \setminus \{0\}$. Thanks to Proposition 3.4.9, if $u_0 \in X$,

$$\begin{aligned} Q_{\sigma_1}(u_0) = P_{\sigma_1}(u_0) &= \int_\Omega u_0 \Phi_1 dx\, \Phi_1 \\ &= \int_\Omega u_0(x) \frac{1}{\mu(\Omega)^{1/2}} dx \, \frac{1}{\mu(\Omega)^{1/2}} \\ &= \frac{1}{\mu(\Omega)} \int_\Omega u_0(x) dx. \end{aligned} \tag{3.47}$$

Thanks to Theorem 3.4.8, the asymptotic behavior of the solution of (3.44) is given by

$$\lim_{t\to\infty} \left\| e^{-(\beta_1+\varepsilon)t} \left(u(t) - P_{\sigma_1}(u)(t) \right) \right\|_X = 0, \tag{3.48}$$

where $\beta_1 < 0$ is upper bound of $Re\big(\sigma_X(K - h_0 I) \setminus \{0\}\big)$, and $\varepsilon > 0$ small enough, such that $\beta_1 + \varepsilon < 0$. We also know that $u(x,t) = e^{(K-h_0 I)t} u_0(x)$, and $P_{\sigma_1} = Q_{\sigma_1}$. Then, thanks to Proposition 3.4.7 we have that

$$P_{\sigma_1}(u)(x,t) = P_{\sigma_1}(e^{(K-h_0I)t} u_0)(x,t) = e^{(K-h_0I)t} P_{\sigma_1}(u_0)(x). \tag{3.49}$$

On the other hand, since $Q_{\sigma_1}(u_0)(x) = P_{\sigma_1}(u_0)(x) = \langle u_0, \Phi_1 \rangle \Phi_1 = C\Phi_1$, then

$$e^{(K-h_0I)t} P_{\sigma_1}(u_0)(x) = e^{(K-h_0I)t} C\Phi_1. \tag{3.50}$$

Furthermore, Φ_1 is an eigenfunction associated to $\{0\}$ and $e^{(K-h_0I)t} = \sum \frac{(K-h_0I)^n t^n}{n!}$, then we have

$$e^{(K-h_0I)t}\Phi_1 = \sum \frac{(K-h_0I)^n t^n}{n!} \Phi_1 = \sum \frac{(K-h_0I)^n \Phi_1\, t^n}{n!} = \sum \frac{(0)^n \Phi_1\, t^n}{n!} = e^{0t}\Phi_1 = \Phi_1. \tag{3.51}$$

Hence, from (3.49), (3.50) and (3.51)

$$P_{\sigma_1}(u)(x,t) = P_{\sigma_1}(e^{(K-h_0I)t} u_0)(x,t) = e^{(K-h_0I)t} P_{\sigma_1}(u_0)(x) = e^{0\,t} P_{\sigma_1}(u_0)(x) = P_{\sigma_1}(u_0)(x). \tag{3.52}$$

Therefore, thanks to (3.48), (3.52) and (3.47), the asymptotic behavior of the solution of (3.44) is given by

$$\lim_{t\to\infty}\left\|e^{-(\beta_1+\varepsilon)t}\left(u(t)-\frac{1}{\mu(\Omega)}\int_\Omega u_0(x)dx\right)\right\|_X=0. \tag{3.53}$$

$\square$

Remark 3.5.3. *With Propositions 3.5.1 and 3.5.2 , we recover the result of asymptotic behaviour in [18], but we obtain the results for a general metric measure space instead of an open subset of* $\mathbb{R}^N$. *Moreover we give the asymptotic behaviour in norm* $X = L^p(\Omega)$ *or* $X = \mathcal{C}_b(\Omega)$, *whereas in [18] the results are obtained in an open bounded set* $\Omega \subset \mathbb{R}^N$, *and the asymptotic behaviour is given in norm* $L^2(\Omega)$ *if* $u_0 \in L^2(\Omega)$ *and in norm* $L^\infty(\Omega)$ *if the initial data is in* $\mathcal{C}(\overline{\Omega})$.

Chapter 4

Nonlinear problem with local reaction

Throughout this chapter, we will assume that (Ω, μ, d) is a metric measure space, $X = L^p(\Omega)$, with $1 \leq p \leq \infty$, or $X = \mathcal{C}_b(\Omega)$, and the operator $K \in \mathcal{L}(X, X)$. The problem we are going to work with, is the following

$$\begin{cases} u_t(x,t) & = (K - hI)(u)(x,t) + f(x, u(x,t)) = L(u)(x,t) + f(x, u(x,t)), \ x \in \Omega, \ t > 0 \\ u(x, t_0) & = u_0(x), \quad x \in \Omega, \end{cases} \tag{4.1}$$

with $f : \Omega \times \mathbb{R} \to \mathbb{R}$ representing the local reaction term, and $u_0 \in X$.

We will write the solution of the problem (4.1), in terms of the group e^{Lt} associated to the linear operator $L = K - hI$. In fact, we will write the solution with the Variation of Constant Formula, (4.6), and we will focus in the study of the existence and uniqueness of the solution associated to (4.1), firstly for f globally Lipschitz and secondly for f locally Lipschitz and satisfying some sign-conditions.

If f is globally Lipschitz, we will prove that the solution of (4.1) with initial data $u_0 \in X$, is a global strong solution such that $u \in \mathcal{C}^1([0,T], X)$ for all $T > 0$. We will also give positivity and monotonicity results for the solution, analogous to the results of the **local** nonlinear reaction-diffusion problem with boundary conditions, (see for example [4]). In particular, we will prove the following monotonicity properties:

- Given two ordered initial data, the corresponding solutions are ordered.
- If $f(u) \geq 0$ for all $u \geq 0$. Given a nonnegative data, $u_0 \geq 0$, the corresponding solution is nonnegative.
- If $f \geq g$. If we denote by $u_f(t)$ and $u_g(t)$ the solution of (4.1) with nonlinear term f and g respectively. Then
$$u_f(t) \geq u_g(t).$$
- Let $\overline{u}(t)$ be a supersolution, and let $u(t)$ be the solution. If $\bar{u}(0) \geq u(0)$ then
$$\overline{u}(t) \geq u(t)$$
as long as the supersolution exists. The same is true for subsolutions with reversed inequality.

We will also prove the existence, uniqueness and monotonicity properties of the solution of (4.1) when the nonlinear term f, is locally Lipschitz in the variable $s \in \mathbb{R}$, uniformly with respect to $x \in \Omega$, and satisfies sign conditions: there exists $C, D \in \mathbb{R}$ with $D > 0$, such that

$$f(x,s)s \leq Cs^2 + D|s|, \quad \text{for all } \; x \in \Omega. \tag{4.2}$$

After that, we give some asymptotic estimates of the solution, and we will finish proving under hypotheses (4.2) on f, the existence of two extremal equilibria φ_m and φ_M in $L^\infty(\Omega)$. In fact, we prove that all the solutions of (4.1) with bounded initial data will enter between the two extremal equilibria when time goes to infinity for a.e. point in Ω, and if the initial data u_0 is in $L^p(\Omega)$, with $1 \leq p < \infty$, then φ_M and φ_m are bounds of any weak limit in $L^p(\Omega)$ of the solution of (4.1), when t goes to infinity. These results are weaker than the results for the local reaction-diffusion equation, where the asymptotic dynamics of the solution enter between the extremal equilibria uniformly in space, for bounded sets of initial data, (see [44]).

After studying the asymptotic behaviour we are confined to study the stability of the equilibria of the problem (4.1) with $h = h_0 = \int_\Omega J(\cdot, y)dy$. Since $F : X \to X$ globally Lipschitz is not differentiable (see Appendix B), hence we do not have that if an equilibrium is stable with respect to the linearization, then it is stable in the sense of Lyapunov. We give criterions on f to have similar results, and we prove that any nonconstant equilibria in $\mathcal{C}_b(\Omega)$ of (4.1) with $h = h_0$ is, if it exists, unstable when f is convex. Similar results are obtained in [14, 40] for the local reaction-diffusion problem.

4.1 Existence, uniqueness, positiveness and comparison of solutions with a globally Lipschitz reaction term

Let (Ω, μ, d) be a measurable metric space,

- if $X = L^p(\Omega)$, with $1 \leq p \leq \infty$, we assume $h \in L^\infty(\Omega)$,
- if $X = \mathcal{C}_b(\Omega)$, we assume $h \in \mathcal{C}_b(\Omega)$,

In this section we focus on the existence and uniqueness of solution of the problem

$$\begin{cases} u_t(x,t) & = L(u)(x,t) + f(x, u(x,t)), \; x \in \Omega, \; t > 0 \\ u(x,0) & = u_0(x), \quad x \in \Omega, \end{cases} \tag{4.3}$$

with f globally Lipschitz, whose solution will be denoted as $u(x,t,u_0)$.

Definition 4.1.1. *Let $X = L^p(\Omega)$, with $1 \leq p \leq \infty$, or $X = \mathcal{C}_b(\Omega)$, the Nemitcky operator associated to $f : \Omega \times \mathbb{R} \to \mathbb{R}$, is defined as an operator*

$$F : X \to X, \quad \textit{such that} \;\; F(u)(x) = f(x, u(x)),$$

with $u \in X$.

The following theorem gives a criterium to prove the existence of strong solutions. For more details see [43, p. 109].

Theorem 4.1.2. *Let Y be a Banach space, we assume the linear operator $H : Y \to Y$ generates a $\mathcal{C}^0$ semigroup in Y, denoted by e^{Ht}. We consider the problem*

$$\begin{cases} u_t(t) &= H(u)(t) + g(t), \quad t > t_0 \\ u(t_0) &= u_0 \in Y. \end{cases} \tag{4.4}$$

We assume $g \in \mathcal{C}([t_0, t_1], Y)$, $u_0 \in D(H)$ and u is a mild solution of (4.4) given by

$$u(t) = e^{-H(t-t_0)}u_0 + \int_{t_0}^{t} e^{-H(t-s)}g(s)ds.$$

Moreover, assume either

i. $g \in \mathcal{C}([t_0, t_1], D(H))$, i.e., $t \mapsto g(t) \in Y$ and $t \mapsto Hg(t) \in Y$ are continuous,

ii. $g \in \mathcal{C}^1([t_0, t_1], Y)$.

Then $u \in \mathcal{C}^1([t_0, t_1], Y) \cap \mathcal{C}([t_0, t_1], D(H))$, and it is a strong solution of (4.4) in Y.

Let us consider now a general globally Lipschitz operator $G : X \to X$, and we study the problem

$$\begin{cases} u_t(x,t) = (K - hI)(u)(x,t) + G(u)(x,t) = L(u)(x,t) + G(u)(x,t), & x \in \Omega,\ t \in \mathbb{R} \\ u(x,0) = u_0(x), & x \in \Omega, \end{cases} \tag{4.5}$$

In the following proposition we prove the existence and uniqueness of the solution to (4.5).

Proposition 4.1.3.

- *If $X = L^p(\Omega)$, with $1 \leq p \leq \infty$, we assume $h \in L^\infty(\Omega)$.*
- *If $X = \mathcal{C}_b(\Omega)$, we assume $h \in \mathcal{C}_b(\Omega)$.*

Let $K - hI \in \mathcal{L}(X, X)$ and let $G : X \to X$ be globally Lipschitz.

Then the problem (4.5) has a unique global solution $u \in \mathcal{C}((-\infty, \infty), X)$, for every $u_0 \in X$, with

$$u(\cdot, t) = e^{Lt}u_0 + \int_0^t e^{L(t-s)}G(u)(\cdot, s)\, ds. \tag{4.6}$$

Moreover, $u \in \mathcal{C}^1((-\infty, \infty), X)$ is a strong solution in X.

Proof. This proof is standard, however, we give it for the sake of completeness.

The solution associated to the equation (4.5) can be written as in (4.6). Denoting by $\mathcal{F}(u)$ the right hand side of (4.6), we are lead to look for fixed points of $\mathcal{F}$, in

$$V = \mathcal{C}\left([-T, T], X\right), \quad \text{for some } \ T > 0.$$

Note that V is a complete metric space for the sup norm. First we prove $\mathcal{F}$ maps V into itself.

Thus, we prove that for $u \in V$, $\mathcal{F}(u) \in \mathcal{C}([-T, T], X)$. First of all, if $u \in V$ then $G(u) \in \mathcal{C}([-T, T], X)$, because $G : X \to X$ is globally Lipschitz. Since $L = K - hI \in \mathcal{L}(X, X)$, we have that $-\|L\|_{\mathcal{L}(X)} < |\sigma(L)| < \|L\|_{\mathcal{L}(X)}$, then thanks to Corollary 3.4.3, there exists $0 < a$, $M \in \mathbb{R}$ such that

$$\|e^{Lt}\|_{\mathcal{L}(X)} \leq M e^{a\,|t|}, \quad \text{for all } \; t \in \mathbb{R}. \tag{4.7}$$

Then, since $G(u) \in \mathcal{C}([-T,T],X)$ and thanks to (4.7), we have that

$$\begin{aligned}
\|\mathcal{F}(u)(t)\|_X &\leq \|e^{Lt}u_0\|_X + \left\|\int_0^t e^{L(t-s)}G(u)(\cdot,s)\,ds\right\|_X \\
&\leq \|e^{Lt}u_0\|_X + \left|\int_0^t \|e^{L(t-s)}G(u)(s)\|_X ds\right| \\
&\leq Me^{a|t|}\|u_0\|_X + \left|\int_0^t Me^{a|t-s|}\|G(u)(s)\|_X\,ds\right| \\
&\leq Me^{a|t|}\|u_0\|_X + M|t|e^{a|t|}\sup_{s\in[-|t|,|t|]}\|G(u)(s)\|_X.
\end{aligned}$$

Thus, we have that $\mathcal{F}(u)(t) \in X$.
To prove continuity in time, we fix $t \in [-T,T]$ and $\varepsilon \in \mathbb{R}$, we have that

$$\mathcal{F}(u)(t+\varepsilon) = e^{L\varepsilon}\mathcal{F}(u)(t) + \int_t^{t+\varepsilon} e^{L(t+\varepsilon-s)}G(u)(\cdot,s)\,ds$$

and then

$$\|\mathcal{F}(u)(t+\varepsilon) - \mathcal{F}(u)(t)\|_X \leq \|(e^{L\varepsilon}-I)\mathcal{F}(u)(t)\|_X + \int_t^{t+\varepsilon}\|e^{L(t+\varepsilon-s)}\|_{\mathcal{L}(X)}\|G(u)(s)\|_X ds.$$

The first term on the right hand side above goes to zero when ε goes to zero, because e^{Lt} is a strongly continuous group and $\mathcal{F}(u)(t) \in X$. In the second term, $G(u) \in \mathcal{C}([-T,T],X)$, for $u \in V$, and $\|e^{L(t+\varepsilon-s)}\|_{\mathcal{L}(X)} \leq Me^{a|t+\varepsilon-s|}$, then the integral term is small if ε is small and continuity follows. Thus $\mathcal{F}(V) \subset V$.

Now we prove that $\mathcal{F}$ is a contraction on V if T is small enough. If $u_1, u_2 \in V$ and $t \in [-T,\, T]$, then

$$\|\mathcal{F}(u_1)(t) - \mathcal{F}(u_2)(t)\|_X \leq \left|\int_0^t \|e^{L(t-s)}\|_{\mathcal{L}(X)}\|G(u_1)(\cdot,s) - G(u_2)(\cdot,s)\|_X\,ds\right|,$$

since G is globally Lipschitz, and $\|e^{Lt}\|_{\mathcal{L}(X)} \leq Me^{a|t|}$ we get

$$\begin{aligned}
\|\mathcal{F}(u_1)(t) - \mathcal{F}(u_2)(t)\|_X &\leq L_G\left|\int_0^t \|e^{L(t-s)}\|_{\mathcal{L}(X)}\|(u_1(s)-u_2(s))\|_X ds\right| \\
&\leq ML_G\left|\int_0^t e^{a|t-s|}\|(u_1(s)-u_2(s))\|_X ds\right| \\
&\leq ML_G|t|\,e^{a|t|}\sup_{s\in[-|t|,|t|]}\|u_1(s)-u_2(s)\|_X.
\end{aligned}$$

since $t \in [-T,T]$, we have that for T small enough, $ML_G|T|e^{a|T|} < 1$. Therefore $\mathcal{F}$ is a contraction and has a unique fixed point.

Arguing by continuation. Since T does not depend on u_0, if we consider again the same problem with initial data $u(x,T)$, then we find that there exists a unique solution for all

$t \in [0, 2T]$. Also, if we consider the same problem with initial data $u(x, -T)$, then we find that there exists a unique solution for all $t \in [-2T, 0]$. Thanks to the uniqueness, we have that there exists a unique solution, u, for all $t \in [-2T, 2T]$. Repeating this argument, we prove that there exists a unique solution, $u \in \mathcal{C}^1([-T, T], X)$, of (4.5) for all $T > 0$.

We have proved that there exists a unique solution $u \in \mathcal{C}([-T, T], X)$ of (4.5) $\forall T > 0$, that satisfies the Variations of Constants Formula, (4.6). Moreover, consider $g(t) = G(u(t))$. Since $u : [-T, T] \to X$ is continuous, and $G : X \to X$ is continuous, we have that $g : [-T, T] \to X$ is continuous. Moreover, since $D(L) = X$ and $L \in \mathcal{L}(X, X)$, we can apply Theorem 4.1.2. Therefore, $u \in \mathcal{C}^1([-T, T], X)$ is a strong solution in X, $\forall T > 0$. $\square$

Now we will prove some monotonicity properties for the problem (4.5). For the linear problem the comparison results were obtained for positive time, (see Corollary 3.2.5), then for the nonlinear problem, (4.5), the results will be also proved for positive time.

In the following Proposition we prove that given two initial data ordered, the corresponding solutions remain ordered as long as they exist. Moreover, under the same hypothesis on the positivity of J in Proposition 2.1.17, the solutions are strictly ordered (i.e. $u_1 > u_2$).

Proposition 4.1.4. (Weak and Strong Maximum Principles)

- *If $X = L^p(\Omega)$, with $1 \leq p \leq \infty$, we assume $h \in L^\infty(\Omega)$.*
- *If $X = \mathcal{C}_b(\Omega)$, we assume $h \in \mathcal{C}_b(\Omega)$.*

We assume $L = K_J - hI \in \mathcal{L}(X, X)$, J nonnegative, $G : X \to X$ globally Lipschitz, and there exists a constant $\beta > 0$, such that $G + \beta I$ is increasing.

(Weak Maximum Principle): If u_0, $u_1 \in X$ satisfy that $u_0 \geq u_1$ then

$$u^0(t) \geq u^1(t), \quad \textit{for all } \ t \geq 0,$$

where $u^i(t)$ is the solution to (4.5) with initial data u_i.

(Strong Maximum Principle): In particular if J satisfies that

$$J(x, y) > 0 \textit{ for all } x,\ y \in \Omega, \textit{ such that } d(x, y) < R, \tag{4.8}$$

for some $R > 0$, and Ω is R-connected, then if $u_0 \geq u_1$, $u_0 \neq u_1$,

$$u^0(t) > u^1(t), \quad \textit{for all } \ t > 0.$$

Proof. We rewrite the equation of the problem (4.5) as

$$u_t(x, t) = L(u)(x, t) - \beta\, u(x, t) + G(u)(x, t) + \beta\, u(x, t),$$

where β is the constant in the hypotheses.

From Proposition 4.1.3 we know that $u^i(t)$ is the strong solution of (4.5), with initial data u_i, and $u^i(t)$ is the unique fixed point of

$$\mathcal{F}_i(u)(t) = e^{(L-\beta I)t} u_i + \int_0^t e^{(L-\beta I)(t-s)} \left(G(u)(s) + \beta u(s) \right) ds \tag{4.9}$$

in $V = \mathcal{C}([-T,T],X)$, because $\mathcal{F}_i$ is a contraction in V provided T small enough for $i=0$, 1. We consider the sequence of Picard iterations,

$$u^i_{n+1}(t) = \mathcal{F}_i(u^i_n)(t) \ \ \forall n \geq 1, \ \ \text{for} \ \ 0 \leq t \leq T.$$

Then the sequence $u^i_n(t)$ converges to $u^i(t)$ in V. Now, we are going to prove that the solutions are ordered for all $t \in [0,T]$.

We take the first term of the Picard iteration as $u^i_1(x,t) = u_i(x)$, then $u^0_1(t) \geq u^1_1(t)$, for all $t \geq 0$. We also have

$$u^i_2(t) = \mathcal{F}_i(u^i_1)(\cdot,t) = e^{(L-\beta I)t}u_i + \int_0^t e^{(L-\beta I)(t-s)}\left(G(u_i) + \beta u_i\right) ds.$$

Since J is nonnegative, then K_J is a positive operator. Moreover, $h+\beta$ satisfies the same hypotheses as h, and the hypotheses of Proposition 3.2.2 are satisfied for $L-\beta = K_J - (h+\beta)I$, then since $u_0 \geq u_1$, we have that

$$e^{(L-\beta I)t}u_0 \geq e^{(L-\beta I)t}u_1 \ \text{ for all } t \in [0,T]. \tag{4.10}$$

Moreover, since $G + \beta I$ is increasing and thanks to Proposition 3.2.2, we obtain that

$$e^{(L-\beta I)(t-s)}\left(G(u_0) + \beta u_0\right) \geq e^{(L-\beta I)(t-s)}\left(G(u_1) + \beta u_1\right) \ \ \text{for all } t \in [0,T] \ \text{ and } \ s \in [0,t]. \tag{4.11}$$

From (4.10) and (4.11), we have that $u^0_2(t) \geq u^1_2(t)$ for all $t \in [0,T]$. Repeating this argument, we get that

$$u^0_n(t) \geq u^1_n(t) \ \text{ for all } t \in [0,T], \ \text{ for every } \ n \geq 1.$$

Since $u^i_n(x,t)$ converges to $u^i(x,t)$ in V, we obtain that

$$u^0(t) \geq u^1(t) \ \text{ for all } t \in [0,T].$$

Now, we consider the solution of (4.5) with initial data at time T, $u^i(T)$. Then, since the initial data $u^0(T) \geq u^1(T)$, are ordered, arguing as above, we obtain that $u^0(t) \geq u^1(t)$ for all $t \in [T,2T]$. Therefore, we have that, $u^0(t) \geq u^1(t)$ for all $t \in [0,2T]$. Repeating this argument, we prove that

$$u^0(t) \geq u^1(t), \ \text{ for all } \ t \geq 0.$$

To prove the second part, we know from Proposition 4.1.3 that, $u^i(t)$, the solution of (4.5) with initial data u_i is given by (4.9). Moreover, since $h+\beta$ and J satisfy the hypotheses of Theorem 3.2.4, we have that

$$e^{(L-\beta I)t}u_0 = e^{(K-(h+\beta)I)t}u_0 > e^{(K-(h+\beta)I)t}u_1 = e^{(L-\beta I)t}u_1, \ \text{ for all } \ t > 0.$$

And, thanks to the monotonicity of $G(\cdot) + \beta I$, we obtain that

$$\int_0^t e^{(L-\beta I)(t-s)}\big(G(u^0)(x,s) + \beta u^0(x,s)\big)\, ds \geq \int_0^t e^{(L-\beta I)(t-s)}\big(G(u^1)(x,s) + \beta u^1(x,s)\big)\, ds,$$

for all $t \geq 0$. Thus, $u^0(t) > u^1(t)$ for all $t > 0$. $\square$

In the proposition below, we prove monotonicity properties with respect to the nonlinear term, for the problem (4.5).

Proposition 4.1.5.

- *If* $X = L^p(\Omega)$, *with* $1 \le p \le \infty$, *we assume* $h \in L^\infty(\Omega)$.
- *If* $X = \mathcal{C}_b(\Omega)$, *we assume* $h \in \mathcal{C}_b(\Omega)$.

If $L = K_J - hI \in \mathcal{L}(X, X)$, J *is nonnegative,* $G_i : X \to X$ *is globally Lipschitz for* $i = 1, 2$, *and there exists a constant* $\beta > 0$, *such that* $G_i + \beta I$ *is increasing for* $i = 1, 2$ *and*

$$G_1 \ge G_2$$

then

$$u^1(t) \ge u^2(t), \quad \textit{for all} \;\; t \ge 0,$$

where $u^i(t)$ *is the solution to* (4.5) *with* $G = G_i$ *and initial data* $u_0 \in X$.

In particular if Ω *is R-connected and* J *satisfies hypothesis* (4.8) *of Proposition 4.1.4, then*

$$u^1(t) > u^2(t), \quad \textit{for all} \;\; t > 0.$$

Proof. Arguing like in previous Proposition 4.1.4, we know that $u^i(t)$ is the strong solutions of (4.5) with nonlinear term G_i, and $u^i(t)$ is the unique fixed point of

$$\mathcal{F}_i(u)(t) = e^{(L-\beta I)t}u_0 + \int_0^t e^{(L-\beta I)(t-s)} \left(G_i(u)(s) + \beta u(s)\right) ds \tag{4.12}$$

in $V = \mathcal{C}([-T, T], X)$, provided T small enough, for $i = 1, 2$. We have proved in Proposition 4.1.3 that $\mathcal{F}_i$ is a contraction in V provided T small enough. We consider the sequence of Picard iterations,

$$u^i_{n+1}(t) = \mathcal{F}_i(u^i_n)(t) \;\; \forall n \ge 1.$$

Then the sequence $u^i_n(\cdot, t)$ converges to $u^i(\cdot, t)$ in V. Now, we are going to prove that the solutions are ordered for all $t \ge 0$. We take the first term of the Picard iteration as $u^i_1(x, t) = u_0(x)$, then

$$u^i_2(t) = \mathcal{F}_i(u^i_1)(\cdot, t) = e^{(L-\beta I)t}u_0 + \int_0^t e^{(L-\beta I)(t-s)} \left(G_i(u_0) + \beta u_0\right) ds.$$

In Proposition 4.1.4 we proved that under the hypotheses in this Proposition then we can use Proposition 3.2.2, and thanks to the fact that $G_1 + \beta I \ge G_2 + \beta I$, we have

$$e^{(L-\beta I)(t-s)} \left(G_1(u_0) + \beta u_0\right) \ge e^{(L-\beta I)(t-s)} \left(G_2(u_0) + \beta u_0\right), \text{ for all } t \ge 0 \text{ and } s \in [0, t].$$

Hence $u^1_2(t) \ge u^2_2(t)$ for all $t \in [0, T]$. Repeating this argument, we obtain that

$$u^1_n(x, t) \ge u^2_n(x, t) \text{ for all } t \in [0, T], \text{ for every } n \ge 1.$$

Since $u^i_n(t)$ converges to $u^i(t)$, in V, then

$$u^1(t) \ge u^2(t) \text{ for all } t \in [0, T].$$

Now, we consider the solution of (4.5) with nonlinear term G^i and with initial data $\tilde{u}_0^i(T) = u^i(x,T)$, then arguing as above and since the initial data are also ordered, we obtain that $\tilde{u}^1(t) \geq \tilde{u}^2(t)$ for all $t \in [T, 2T]$. Since the solution to (4.5) is unique, then the solutions $u^i(\cdot, t)$ are ordered for for all $t \in [0, 2T]$. Repeating this argument, we obtain that

$$u^1(t) \geq u^2(t), \quad \text{for all} \quad t \geq 0. \tag{4.13}$$

To prove the second part, we know from (4.12) that, $u^i(t)$, the solution of (4.5) with nonlinear term G^i is given by

$$u^i(t) = e^{(L-\beta I)t} u_0 + \int_0^t e^{(L-\beta I)(t-s)} \left(G_i(u^i)(s) + \beta u^i(s) \right) ds.$$

Thanks to (4.13), and the fact that $G_1 + \beta I$ is increasing, and $G_1 \geq G_2$ we have that

$$\left(G_1(u^1)(x,t) + \beta u^1(x,t)\right) \geq \left(G_1(u^2)(x,t) + \beta u^2(x,s)\right) \geq \left(G_2(u^2)(x,t) + \beta u^2(x,t)\right), \forall t \geq 0. \tag{4.14}$$

From (4.14), since $h + \beta \in L^\infty(\Omega)$, and J satisfies the hypotheses of Theorem 3.2.4, we have that

$$e^{(L-\beta I)t} \left(G_1(u^1)(x,s) + \beta u^1(x,s)\right) > e^{(L-\beta)I)t} \left(G_2(u^2)(x,s) + \beta u^2(x,s)\right), \quad \text{for all} \quad t > 0. \tag{4.15}$$

Therefore, thanks to (4.15), we obtain that

$$\int_0^t e^{(L-\beta I)(t-s)} \left(G_1(u^1)(x,s) + \beta u^1(x,s)\right) ds > \int_0^t e^{(L-\beta I)(t-s)} \left(G_2(u^2)(x,s) + \beta u^2(x,s)\right) ds,$$

for all $t > 0$. Thus, $u^1(t) > u^2(t)$, for all $t > 0$. $\square$

The following proposition states that if the initial data is nonnegative, the solution of (4.5) is also nonnegative.

Proposition 4.1.6. (Weak and Strong Positivity)

- *If $X = L^p(\Omega)$, with $1 \leq p \leq \infty$, we assume $h \in L^\infty(\Omega)$.*
- *If $X = \mathcal{C}_b(\Omega)$, we assume $h \in \mathcal{C}_b(\Omega)$.*

We assume $L = K_J - hI \in \mathcal{L}(X, X)$, J nonnegative, $G : X \to X$ globally Lipchitz, and there exists a constant $\beta > 0$, such that $G + \beta I$ is increasing, and $G(0) \geq 0$.

If $u_0 \in X$, with $u_0 \geq 0$, not identically zero, then the solution to (4.5),

$$u(t, u_0) \geq 0, \; \textit{for all}\, t \geq 0.$$

In particular if Ω is R-connected and J satisfies hypothesis (4.8) of Proposition 4.1.4, then

$$u(t, u_0) > 0, \; \textit{for all}\, t > 0.$$

Proof. Arguing like in Proposition 4.1.4 we know that $u(t)$ is the solution of (4.5), it is strong, and $u(t)$ is the unique fixed point of

$$\mathcal{F}(u)(t) = e^{(L-\beta I)t}u_0 + \int_0^t e^{(L-\beta I)(t-s)}\left(G(u)(\cdot,s) + \beta u(s)\right) ds \tag{4.16}$$

that is a contraction in $V = \mathcal{C}([-T,T],X)$, provided T small enough.

We consider the sequence of Picard iterations,

$$u_{n+1}(t) = \mathcal{F}(u_n)(t) \ \ \forall n \geq 1, \text{ for all } \ 0 \leq t \leq T.$$

Then the sequence $u_n(\cdot,t)$ converges to $u(\cdot,t)$ in V. We take $u_1(x,t) = u_0(x)$, the positive initial solution, then

$$u_2(t) = \mathcal{F}(u_1)(t) = e^{(L-\beta I)t}u_0 + \int_0^t e^{(L-\beta I)(t-s)}\left(G(u_0) + \beta u_0\right) ds.$$

In Proposition 4.1.4 we proved that under the hypotheses in this Proposition then we can use Proposition 3.2.2, then

$$e^{(L-\beta I)t}u_0 \geq 0, \quad \text{for all } \ t \in [0,T]. \tag{4.17}$$

Moreover, if $G(0) \geq 0,\ \ \beta > 0$ and $\ G(\cdot) + \beta I$ is increasing, then $G(u) + \beta u \geq 0$ for all $u \geq 0$. Hence, we obtain that

$$e^{(L-\beta I)(t-s)}\left(G(u_0) + \beta u_0\right) \geq 0, \text{ for all } \ t \in [0,T] \text{ and } s \in [0,t]. \tag{4.18}$$

Hence, from (4.17) and (4.18), $u_2(t) \geq 0$ for all $t \in [0,T]$. Repeating this argument, we get that $u_n(\cdot,t)$ is nonnegative for every $n \geq 1$. Since $u_n(t)$ converges to $u(t)$. Thus, the solution $u(t)$ is nonnegative in V, for all $t \in [0,T]$.

If we consider again the same problem with initial data $\tilde{u}_0(t) = u(x,T)$, then arguing as above we have that $\tilde{u}(t)$ is nonnegative for all $t \in [T,\, 2T]$. Thanks to the uniqueness of solution we have that $u(t) \geq 0$ for all $t \in [0,\, 2T]$. Repeating this argument, we prove that the solution of (4.5) is nonnegative $\forall t \geq 0$.

To prove that the solution $u(t)$ is strictly positive, we know that $u(t)$ is given by (4.16). Moreover, since $h + \beta$ and J satisfy the hypotheses of Theorem 3.2.4, we have that

$$e^{(L-\beta I)t}u_0 > 0, \ \text{ for all } \ t > 0. \tag{4.19}$$

Moreover, since u is nonnegative $\forall t \geq 0$, and $\big(G + \beta I\big)(u) \geq 0$ for all $u \geq 0$, thanks to Theorem 3.2.4, we have also that

$$\int_0^t e^{(L-\beta I)(t-s)}\left(G(u)(x,s) + \beta u(x,s)\right) ds \geq 0, \ \text{ for all } \ t \geq 0. \tag{4.20}$$

Thus, from (4.19) and (4.20), we have that $u(t) > 0$ for all $t > 0$. $\square$

To prove the following results, we first give the definition of supersolution and subsolution.

Definition 4.1.7. *Let $X = L^p(\Omega)$, with $1 \le p \le \infty$ or $X = \mathcal{C}_b(\Omega)$, we say that $\overline{u} \in \mathcal{C}([a,b], X)$ is a* **supersolution** *to (4.5) in $[a,b]$, if for any $t \ge s$, with $s, t \in [a,\, b]$*

$$\overline{u}(t) \ge e^{L(t-s)}\overline{u}(s) + \int_s^t e^{L(t-r)} G(\overline{u})(r)dr. \tag{4.21}$$

We say that $\underline{u}$ is a **subsolution** *if the reverse inequality holds.*

Remark 4.1.8. *We assume that e^{Lt} preserves the positivity, i.e., we assume J is nonnegative. If $\overline{u} \in \mathcal{C}([a,b], X)$ differentiable satisfies that*

$$\overline{u}_t(t) \ge L(\overline{u})(t) + G(\overline{u})(t), \quad for \;\; t \in [a,\, b] \tag{4.22}$$

then $\overline{u}$ is a supersolution that satisfies (4.21). The same happens for subsolutions if the reverse inequality holds. Let us prove this below for supersolutions.

Since (4.22) is satisfied, there exists $f : \mathbb{R} \to X$, with $f \ge 0$, such that

$$\overline{u}_t(t) = L(\overline{u})(t) + G(\overline{u})(t) + f(t), \quad for \;\; t \in [a,\, b] \tag{4.23}$$

Then

$$\overline{u}(t) = e^{Lt}\overline{u}(s) + \int_s^t e^{L(t-r)}\big(G(\overline{u})(r) + f(r)\big)dr, \quad for \;\; t, s \in [a,\, b], \;\; s \le t. \tag{4.24}$$

Since f is nonnegative and e^{Lt} preserves the positivity, then $\int_s^t e^{L(t-r)} f(r)dr \ge 0$. Hence, from (4.24) we have that (4.21) is satisfied. Thus, the result.

The following proposition states that a supersolution is greater than the solution to (4.5).

Proposition 4.1.9.

- *If $X = L^p(\Omega)$, with $1 \le p \le \infty$, we assume $h \in L^\infty(\Omega)$.*
- *If $X = \mathcal{C}_b(\Omega)$, we assume $h \in \mathcal{C}_b(\Omega)$.*

Let $L = K_J - hI \in \mathcal{L}(X, X)$, J be nonnegative, $G : X \to X$ be globally Lipchitz, and there exists a constant $\beta > 0$, such that $G + \beta I$ is increasing. Let $u(t,\, u_0)$ be the solution to (4.5) with initial data $u_0 \in X$, and let $\bar{u}(t)$ be a supersolution to (4.5) in $[0, T]$.

If $\overline{u}(0) \ge u_0$, then

$$\bar{u}(t) \ge u(t,\, u_0), \qquad for \;\; t \in [0, T].$$

The same is true for subsolutions with reversed inequality.

Proof. Arguing like in Proposition 4.1.4 we know that $u(t)$ is the solution of (4.5), it is strong, and $u(t)$ is the unique fixed point of

$$\mathcal{F}(u)(t) = e^{(L-\beta I)t}u_0 + \int_0^t e^{(L-\beta I)(t-s)}\left(G(u)(\cdot, s) + \beta u(s)\right) ds \tag{4.25}$$

in $\mathcal{C}([0, \tau], X)$, provided τ small enough. We choose $\rho \le \min\{\tau, T\}$, then the supersolution $\bar{u}(t) \in X$ exists for all $t \in [0, \rho]$. Note that $\bar{u}$ satisfies by definition that

$$\bar{u}(t) \ge \mathcal{F}(\bar{u})(t), \quad \forall t \in [0, \rho]. \tag{4.26}$$

We consider the sequence of Picard iterations in $V = \mathcal{C}([0,\rho], X)$,

$$u_{n+1}(x,t) = \mathcal{F}(u_n)(x,t) \;\; \forall n \geq 1, \tag{4.27}$$

with $u_1(t) = \overline{u}(t)$. Then the sequence $u_n(t)$ converges to $u(t)$ in V. If we show that,

$$\bar{u} \geq u_n, \quad \text{a.e. in } V, \text{ for } n = 1, 2, 3, \dots . \tag{4.28}$$

then, we have the result in V.

Since $u_1 = \bar{u}$, then $\bar{u} \geq u_1 = \bar{u}$, and (4.28) is satisfied for $n = 1$. Moreover, thanks to (4.26), we have that

$$\bar{u} \geq \mathcal{F}(\bar{u}) = u_2,$$

then (4.28) is true for $n = 2$. Assume now for induction

$$\bar{u}(t) \geq u_n(t), \;\; \text{for all } t \in [0,\rho]. \tag{4.29}$$

From Proposition 4.1.4 we have that $\mathcal{F}$ is increasing in V, and thanks to (4.26), (4.27) and (4.29), we have that

$$\overline{u} \geq \mathcal{F}(\overline{u}) \geq \mathcal{F}(u_n) = u_{n+1}, \;\; \text{for all } t \in [0,\rho].$$

Then, we have proved (4.28). Moreover, $u_n(x,t)$ converges to $u(x,t)$ in V. Then, we have that

$$\bar{u}(t) \geq u(t, u_0), \quad \text{for all } \; t \in [0,\rho].$$

Therefore, we have proved that for $\rho > 0$,

$$\bar{u}(t) \geq u(t, u_0), \; \forall t \in [0,\rho].$$

Now, we take $\widetilde{\rho} \leq T$, then $\bar{u}(t)$ exists for all $t \in [\rho, \widetilde{\rho}]$, with $\widetilde{\rho} \leq 2\tau$. If we consider again the same problem with initial data $\tilde{u}_0(\rho) = u(\cdot,\rho)$, then $\tilde{u}(t)$ is the unique fixed point of

$$\mathcal{F}(\tilde{u})(t) = e^{(L-\beta I)(t-\rho)}\tilde{u}(\cdot,\rho) + \int_\rho^t e^{(L-\beta I)(t-s)}\big(G(\tilde{u})(\cdot,s) + \beta\tilde{u}(\cdot,s)\big)ds$$

in $V = \mathcal{C}([\rho,\widetilde{\rho}], X)$, and the supersolution satisfies by definition that $\bar{u}(t) \geq \mathcal{F}(\bar{u}(t))$. Following the same argument as above, we obtain that the supersolution, $\bar{u}$, and the solution, $\tilde{u}$, are ordered for all time $t \in [\rho, \widetilde{\rho}]$. Thanks to the uniqueness of solution of (4.5) we have that $\bar{u}(t) \geq u(t, u_0)$, $\forall t \in [0, \widetilde{\rho}]$. With this continuation argument, we prove that the supersolutions are greater or equal to the solution of (4.5), in $[0,T]$. □

4.2 Existence and uniqueness of solutions, with locally Lipchitz f

Let (Ω, μ, d) be a metric measure space:

- If $X = L^p(\Omega)$, with $1 \leq q \leq \infty$, we assume $h \in L^\infty(\Omega)$.

- If $X = \mathcal{C}_b(\Omega)$, we assume $h \in \mathcal{C}_b(\Omega)$.

Let $K \in \mathcal{L}(X, X)$, in this section we prove the existence and uniqueness of solution of the problem

$$\begin{cases} u_t(x,t) & = (K - hI)(u)(x,t) + f(x, u(x,t)) = L(u)(x,t) + f(x, u(x,t)),\ x \in \Omega \\ u(x, t_0) & = u_0(x), \quad x \in \Omega, \end{cases} \tag{4.30}$$

with $u_0 \in X$, and $f : \Omega \times \mathbb{R} \to \mathbb{R}$ a function that sends (x, s) to $f(x, s)$, that is locally Lipschitz in the variable $s \in \mathbb{R}$, uniformly with respect to $x \in \Omega$, i.e., $\forall s_0 \in \mathbb{R}$, there exists a neighbourhood U of s_0 such that $\forall s_1, s_2 \in U$, $|f(x, s_1) - f(x, s_2)| < L_U|s_1 - s_2|$, $\forall x \in \Omega$, and f satisfies sign conditions.

First of all, we introduce an auxiliary problem associated to (4.30). For $k > 0$, let us introduce a globally lipschitz function, $f_k : \Omega \times \mathbb{R} \to \mathbb{R}$, associated to the locally Lipschitz function f such that

$$f_k(x, u) = f(x, u) \qquad \text{for } |u| \leq k, \text{ and } \forall x \in \Omega. \tag{4.31}$$

Hence, f_k is the truncation of the function f.

We introduce the following problem, that is equal to (4.30) substituting the locally Lipschitz function f with the associated globally Lipschitz function f_k

$$\begin{cases} u_t(x,t) = (K - hI)(u)(x,t) + f_k(x, u(x,t)) = L(u)(x,t) + F_k(u)(x,t), & x \in \Omega,\ t \in \mathbb{R} \\ u(x, 0) = u_0(x), & x \in \Omega. \end{cases} \tag{4.32}$$

where $F_k : X \to X$ is the Netmitcky operator associated to the globally Lipschitz function f_k. The solution of the problem (4.32) will be denoted as

$$u_k(t, u_0) = S_k(t)u_0.$$

Since the truncation f_k is globally Lipschitz, then the associated Nemitcky operator F_k is globally Lipschitz (see Appendix B, Lemma 6.4.14), then we can apply Proposition 4.1.3 to obtain the existence and uniqueness of solutions of the problem (4.32).

Moreover, since f_k is globally Lipschitz, there exists $\beta > 0$ such that $f_k + \beta I$ is increasing, then $F_k + \beta I$ is increasing (see Appendix B, Lemma 6.4.14). Hence, the hypotheses of Propositions 4.1.4, 4.1.5, 4.1.6 and 4.1.9 are satisfied, and we obtain those comparison results for the problem (4.32).

Now, we prove the existence and uniqueness of solutions of (4.30) with initial data $u_0 \in L^\infty(\Omega)$ or $u_0 \in \mathcal{C}_b(\Omega)$, under the sign condition (4.33) on the locally lipschitz function f.

Proposition 4.2.1. *Let $X = L^\infty(\Omega)$ or $X = \mathcal{C}_b(\Omega)$. We assume $K \in \mathcal{L}(X, X)$, J nonnegative, and $h \in X$, and we assume that the locally lipschitz function f satisfies that there exists a function $g_0 \in \mathcal{C}^1(\mathbb{R})$, and s_0, $\delta > 0$ such that*

$$(h_0(\cdot) - h(\cdot))s^2 + f(\cdot, s)s \leq g_0(s)s \leq -\delta|s|,\ \forall |s| > s_0, \tag{4.33}$$

where $h_0(x) = \int_\Omega J(x,y)dy \in L^\infty(\Omega)$, $\big(J \in L^\infty(\Omega, L^1(\Omega))\big)$.

Then there exists a unique global solution of (4.30) *with initial data* $u_0 \in X$, *such that* $u(\cdot,t)$ *is given by*

$$u(\cdot,t) = e^{Lt}u_0 + \int_0^t e^{L(t-s)} f(\cdot, u(\cdot,s))\, ds. \tag{4.34}$$

Moreover, $u \in \mathcal{C}^1([0,\infty), X)$ *is a strong solution in* X, *and*

$$\|u(t,u_0)\|_{L^\infty(\Omega)} \leq \max\left\{s_0, \|u_0\|_{L^\infty(\Omega)}\right\} \textit{ for all } t \geq 0. \tag{4.35}$$

Proof. Fix $M > s_0$. We introduce the auxiliary problem

$$\begin{cases} \dot{z}(t) & = g_0(z(t)) \\ z(0) & = M. \end{cases} \tag{4.36}$$

Since $g_0 \in \mathcal{C}^1(\mathbb{R})$, thanks to Peano's and Picard-Lindelöf Theorems, we have that there exists a unique local solution to (4.36). Thanks to second inequality in (4.33), with a continuation argument we have that z is defined for $t \geq 0$.

In fact, from (4.33), and since $\dot{z}(t) = g_0(z(t))$, then $z(t)$ decreases for every t such that $z(t) > s_0$, and $z(t) > -s_0$ for all $t \geq 0$. Since $z(0) = M$, and $M > s_0$ we have that

$$|z(t)| \leq M, \ \forall t \geq 0. \tag{4.37}$$

We consider a truncated globally Lipschitz function f_k , (4.31), associated to f. Let $u_k(\cdot, t, u_0)$ be the solution of (4.32) with initial data $u_0 \in X$, such that

$$\|u_0\|_{L^\infty(\Omega)} \leq M. \tag{4.38}$$

Thanks to Proposition 4.1.3 we know that there exists a unique strong solution $u_k(\cdot,t,u_0) \in \mathcal{C}^1(\mathbb{R}, X)$ that is given by the Variation of Constants Formula,

$$u_k(\cdot,t) = e^{Lt}u_0 + \int_0^t e^{L(t-s)} F_k(u_k)(\cdot,s)\, ds. \tag{4.39}$$

We choose

$$k = M, \tag{4.40}$$

then thanks to (4.37), we have that

$$f_k(x, z(t)) = f(x, z(t)), \forall t \geq 0, \ \ \forall x \in \Omega. \tag{4.41}$$

Moreover, since f satisfies (4.33), and from (4.41), we have that f_k satisfies

$$(h_0(\cdot) - h(\cdot))z(t)^2 + f_k(\cdot, z(t))z(t) \leq g_0(z(t))z(t) \leq -\delta|z(t)|, \ \forall t \text{ such that } \ |z(t)| > s_0. \tag{4.42}$$

Now, we are going to prove that z is a supersolution of (4.32). Since z is continuous and $z(0) > s_0$, we define

$$\tilde{t}_z := \inf\{t > 0 : \ z(t) = s_0\}.$$

We have that $z(t)$ is independent of the variable x, then $K(z(t)) = h_0 z(t)$. Thanks to (4.42) and the fact that $z(t) > s_0$ for every $t \in [0, \tilde{t}_z)$, we have that for all $\overline{t_0}$ such that $0 < \overline{t_0} < \tilde{t}_z$, we have

$$\begin{aligned} K(z)(t) - hz(t) + f_k(\cdot, z(t)) &= (h_0 - h)z(t) + f_k(\cdot, z(t)) \\ &\leq g_0(z(t)) = \dot{z}(t). \end{aligned}$$

for all $t \in [0, \overline{t_0}]$. Hence, we have proved that z is a supersolution of (4.32) in $[0, \overline{t_0}]$.

Analogously, let us consider the auxiliary problem

$$\begin{cases} \dot{w}(t) &= g_0(w(t)) \\ w(0) &= -M. \end{cases} \tag{4.43}$$

Arguing as before, we obtain that there exists $\tilde{t}_w$ such that for all $\underline{t_0} < \tilde{t}_w$, w is a subsolution of (4.32) in $\left[0, \underline{t_0}\right]$, and

$$|w(t)| \leq M, \ \forall t \geq 0. \tag{4.44}$$

We choose $T < \min\{\tilde{t}_z, \tilde{t}_w\}$, since the initial data $u_0 \in X$, satisfies that $\|u_0\|_{L^\infty(\Omega)} < M$, and $M > s_0$, then $z(t)$ and $w(t)$ are subsolution and supersolution, respectively, of (4.32) in $[0, T]$. Therefore from Proposition 4.1.9, we obtain that

$$w(t, -M) \leq u_k(t, u_0) \leq z(t, M), \quad \forall t \in [0, T]. \tag{4.45}$$

Moreover, thanks to (4.37), (4.44) and (4.45), we have that

$$|u_k(t, u_0)| \leq M = k \ \text{ for all } \ t \in [0, T]. \tag{4.46}$$

Thanks to (4.38) and since M is fixed at the beginning as $M > s_0$, we have that

$$M > \max\{s_0, \|u_0\|_{L^\infty(\Omega)}\}.$$

Thanks to the definition of f_k, (see (4.31)), and thanks to (4.46), we have that $f_k(\cdot, u_k(t)) = f(\cdot, u_k(t))$. Therefore, $u_k(\cdot, t, u_0)$ is a solution of (4.30). Hence, we denote $u_k(\cdot, t, u_0) = u(\cdot, t, u_0)$, and we have proved the existence of solution of (4.30) for all $t \in [0, T]$, moreover, u is a strong solution of (4.30) in X, given by (4.34), with $u \in \mathcal{C}^1([0, T], X)$, and thanks to (4.46), we have that

$$|u(\cdot, t, u_0)| \leq M = k \ \text{ for all } \ t \in [0, T]. \tag{4.47}$$

In fact (4.47) is satisfied for

$$M = \max\{s_0, \|u_0\|_{L^\infty(\Omega)}\}. \tag{4.48}$$

Arguing by continuation, we consider again the same problem (4.30) with initial data $\tilde{u}_0(T) = u(\cdot, T, u_0)$, then from (4.47), the initial data is bounded by M, then arguing like we have done before, considering the auxiliary problems (4.36) and (4.43), with M as in (4.48), we will have that there exists an strong solution of (4.30), $\tilde{u} \in \mathcal{C}^1([T, 2T], X)$. Since the solution constructed by truncation is unique, then we have proved that there exists an strong solution of (4.30), $u \in \mathcal{C}^1([0, 2T], X)$, given by (4.34), and

$$|u(\cdot, t, u_0)| \leq M = k \ \text{ for all } \ t \in [0, 2T]. \tag{4.49}$$

Repeating this argument, we prove that for any $T > 0$, there exists a strong solution of (4.30) $u \in \mathcal{C}^1([0,T], X)$, it is given by the Variation of Constants Formula (4.34), and it satisfies (4.35).

Now let us prove the uniqueness of solution. We consider a solution $u \in \mathcal{C}([0,T], X)$, of the problem (4.30) with initial data $u_0 \in X$ that satisfies (4.34). Since $u \in \mathcal{C}([0,T], X)$, then

$$\sup_{t\in[0,T]} \sup_{x\in\Omega} |u(x,t,u_0)| < \widetilde{C}.$$

Thus, if we choose $k > \widetilde{C}$, then $f_k(\cdot, u(\cdot,t)) = f(\cdot, u(\cdot,t))$ and then the solutions u_k of (4.32), is a solution of (4.30). Hence u and u_k coincide. Furthermore from Proposition 4.1.3, we have that the solution $u_k \in \mathcal{C}^1([0,T], X)$ is unique, it is strong and it is given by the Variation of Constant Formula. Thus, we have the uniqueness of the solution of (4.30). □

In the following proposition we prove existence and uniqueness of solution of (4.30) with initial data bounded, but now, we assume that f satisfies the sign condition (4.50).

If C is negative in hypothesis (4.50), then the sign condition (4.50), would imply the sign condition (4.33) with $h \le h_0$, in the previous Proposition 4.2.1. Hence in the proposition below, we assume that $C > 0$.

Proposition 4.2.2. *Let $X = L^\infty(\Omega)$ or $X = \mathcal{C}_b(\Omega)$. We assume $K \in \mathcal{L}(X,X)$, $h, h_0 \in X$, and the locally lipschitz function f satisfies that there exist C, $D \in \mathbb{R}$, with $C > 0$ and $D \ge 0$ such that*

$$f(\cdot,s)s \le Cs^2 + D|s|, \ \forall s. \tag{4.50}$$

Then there exists a unique solution of (4.30) *with initial data $u_0 \in X$, such that $u(\cdot,t)$ in $\mathcal{C}([0,T], X)$, for all $T > 0$, with*

$$u(\cdot,t) = e^{Lt}u_0 + \int_0^t e^{L(t-s)} f(\cdot, u(\cdot,s))\, ds. \tag{4.51}$$

Moreover, we have that u is a strong solution of (4.30) *in X.*

Proof. First of all, let us prove that $(h_0 - h)s + f(\cdot,s)$ satisfies the hypothesis (4.50). Since f satisfies (4.50) and $h, h_0 \in X$, then

$$\begin{aligned}(h_0-h)s^2 + f(\cdot,s)s \ &\le (h_0-h)s^2 + Cs^2 + D|s| \\ &\le \big(\|h_0-h\|_{L^\infty(\Omega)} + C\big)s^2 + D|s| \\ &\le C_1 s^2 + D|s|.\end{aligned} \tag{4.52}$$

We denote $C_1 = C$ to simplify the notation. Fix $0 < M \in \mathbb{R}$. We introduce the auxiliary problem

$$\begin{cases} \dot{z}(t) &= Cz(t) + D \\ z(0) &= M. \end{cases} \tag{4.53}$$

Then the solution of (4.53) is given by

$$z(t) = -\frac{D}{C} + e^{Ct}C_2, \quad \forall t \in \mathbb{R}, \text{ with } \ C_2 = M + \frac{D}{C}, \tag{4.54}$$

and $z(t) \geq 0$ increases for all $t \in \mathbb{R}$. Let $T > 0$ be an arbitrary time, then

$$0 \leq z(t) \leq z(T) \; \forall t \in [0, T]. \tag{4.55}$$

We consider a truncated globally Lipschitz function f_k associated to f. We denote by $u_k(\cdot, t, u_0)$ the solution of (4.32) with initial data $u_0 \in X$, $\|u_0\|_X \leq M$. Thanks to Proposition 4.1.3 we know that there exists a unique strong solution $u_k(\cdot, t, u_0) \in \mathcal{C}^1(\mathbb{R}, X)$ that satisfies the Variation of Constants Formula,

$$u_k(\cdot, t) = e^{Lt} u_0 + \int_0^t e^{L(t-s)} F_k(u_k)(\cdot, s)\, ds. \tag{4.56}$$

Given $T > 0$ and $M > 0$, from (4.55) we choose

$$k \geq z(T).$$

Thanks to the definition of f_k, (4.31), and (4.55) we have that

$$f_k(\cdot, z(t)) = f(\cdot, z(t)), \; \forall t \in [0, T] \tag{4.57}$$

We prove below that z is a supersolution of (4.32) for every $t \in [0, T]$. Since $z(t)$ is nonnegative for all $t \in [0, T]$, then, thanks to (4.57) and (4.52), and since $z(t)$ is independent of the variable x, we have that $K(z(t)) = h_0 z(t)$. Thus,

$$\begin{aligned} K(z)(t) - hz(t) + f_k(\cdot, z(t)) &= h_0 z(t) - hz(t) + f_k(\cdot, z(t)) \\ &\leq Cz(t) + D = \dot{z}(t), \quad \text{for all } t \in [0, T], \end{aligned}$$

hence, z is a supersolution of (4.32) for every $t \in [0, T]$.

Let us consider now the auxiliary problem

$$\begin{cases} \dot{w}(t) &= Cw(t) - D \\ w(0) &= -M, \end{cases} \tag{4.58}$$

Then $w(t) = -z(t)$, and we obtain that

$$|w(t)| < z(T) \quad \forall t \in [0, T]. \tag{4.59}$$

Arguing as before, since $w(t)$ is nonpositive for all $t \in [0, T]$. Thanks to (4.59), and since $w(t)$ is independent of the variable x, we have that $K(w(t)) = h_0 w(t)$. Thus,

$$\begin{aligned} K(w)(t) - hw(t) + f_k(\cdot, w(t)) &= h_0 w(t) - hw(t) + f_k(\cdot, w(t)) \\ &\geq Cw(t) - D = \dot{w}(t). \end{aligned}$$

Thus, w is a subsolution of (4.32) for every $t \in [0, T]$.

Since $k \geq z(T)$ and $\|u_0\|_X \leq M$, then $z(t)$ and $w(t)$ are supersolution and subsolution of (4.32) in $[0, T]$, respectively. Therefore, from Proposition 4.1.9, we obtain

$$w(t, -M) \leq u_k(t, u_0) \leq z(t, M), \quad \forall t \in [0, T]. \tag{4.60}$$

Thanks to (4.55), (4.59), and (4.60) we have that

$$|u_k(t, u_0)| \leq z(T) \leq k \ \text{ for all } \ t \in [0, T].$$

Thanks to the definition of f_k, (4.31), we obtain that $f_k(\cdot, u_k(t, u_0)) = f(\cdot, u_k(t, u_0))$. Thus, $u_k(x, t, u_0)$ is a solution to (4.30). Hence we denote $u_k(t, u_0) = u(t, u_0)$, and we have proved the existence of solution of (4.30) for all $t \in [0, T]$, moreover u is a strong solution of (4.30) in X, given by the Variation of Constants Formula (4.51).

Therefore, given any $M > 0$ and any $T > 0$, choosing $k \geq z(T)$, then we have proved the existence of solution of (4.30) with initial data $\|u_0\|_X \leq M$ for all $t \in [0, T]$.

Now, let us prove the uniqueness, arguing like in Proposition 4.2.1, let us consider a solution $u \in \mathcal{C}([0, T], X)$, of the problem 4.30 with initial data $u_0 \in X$, given by (4.51). Since $u \in \mathcal{C}([0, T], X)$, then

$$\sup_{t \in [0,T]} \sup_{x \in \Omega} |u(x, t, u_0)| < \widetilde{C}.$$

Thus, if we choose $k > \widetilde{C}$, then $f_k(\cdot, u(\cdot, t)) = f(\cdot, u(\cdot, t))$ and then the solutions u_k of (4.32), is a solution of (4.30). Hence u and u_k coincide. Furthermore from Proposition 4.1.3, we have that the solution $u_k \in \mathcal{C}^1([0, T], X)$ is unique, it is strong and it is given by the Variation of Constant Formula. Thus, we have the uniqueness of the solution of (4.30). □

Remark 4.2.3. *By using* **Kaplan's technique**, *we prove that the hypothesis* (4.50) *on f in the previous Proposition 4.2.2 is somehow optimal, in the sense that if $f(\cdot, s) = s^p$ with $p > 1$, then we do not have global existence of the solution of* (4.30). *Let us consider the nonlinear term*

$$f(s) = s^p, \quad \textit{with} \ \ p > 1.$$

Let $X = L^\infty(\Omega)$, we assume $K \in \mathcal{L}(X, X)$, $h \in L^\infty(\Omega)$, $J(x, y) = J(y, x)$, and we consider the problem

$$\begin{cases} u_t = (K - hI)u + f(u) = \displaystyle\int_\Omega J(\cdot, y)u(y)dy - h(\cdot)u + u^p \\ u(0) = u_0 \end{cases} \tag{4.61}$$

with $u_0 \in L^\infty(\Omega)$, and $u_0 \geq 0$.

Let $\Phi > 0$ be an eigenfunction associated to the first eigenvalue λ_1 of the operator $K - hI$, then $(K - hI)\Phi = \lambda_1\Phi$. We set $z(t) = \int_\Omega u(t)\Phi$. Let us see what equation does z satisfy,

$$\begin{aligned} \frac{dz}{dt}(t) \ &= \int_\Omega u_t(x, t)\Phi(x)dx \\ &= \int_\Omega \int_\Omega J(x, y)\Phi(x)dx\, u(y, t)dy - \int_\Omega h(x)\Phi(x)u(x, t)dx + \int_\Omega u^p(x)\Phi(x)dx \end{aligned} \tag{4.62}$$

Relabeling variables in the first term of the right hand side of (4.62), *since $J(x, y) = J(y, x)$ and Φ be an eigenfunction associated to the first eigenvalue λ_1 of the operator $K - hI$ we have* *that*

$$\begin{aligned}\frac{dz}{dt}(t) &= \int_\Omega\int_\Omega J(x,y)\Phi(y)dy\,u(x,t)dx - \int_\Omega h(x)\Phi(x)u(x,t)dx + \int_\Omega u^p(x,t)\Phi(x)dx \\ &= \int_\Omega \lambda_1\Phi(x)u(x,t)dx + \int_\Omega u^p(x,t)\Phi(x)dx \\ &= \lambda_1 z(t) + \int_\Omega u^p(x,t)\Phi(x)dx\end{aligned} \tag{4.63}$$

Therefore, if we consider that $\Phi(x)dx$ is a measure and we denote it by $d\mu$, then we have

$$\frac{dz}{dt}(t) \quad = \lambda_1 z(t) + \int_\Omega u^p(x,t)d\mu \tag{4.64}$$

Thanks to Jensen's Theorem (see [46, p. 62]), we know that if μ is a positive measure on a σ-algebra $\mathcal{M}$ in a set Ω such that $\mu(\Omega) = 1$, and g is a convex function, then

$$g\left(\int_\Omega f\,d\mu\right) \le \int_\Omega g\circ f\,d\mu.$$

In this case $g(s) = s^p$ with $p > 1$ is convex, and if we take an eigenfunction Φ such that $\int_\Omega \Phi(x)dx = 1$, then from (4.64) and as a consequence of Jensen's Theorem

$$\begin{aligned}\frac{dz}{dt}(t) &= \lambda_1 z(t) + \int_\Omega u^p(x,t)d\mu \\ &\ge \lambda_1 z(t) + \left(\int_\Omega u(x,t)\Phi(x)dx\right)^p \\ &= \lambda_1 z(t) + z^p(t) = F(z(t)).\end{aligned} \tag{4.65}$$

In this case, for $p > 1$, we have that if $z(0) >> 1$, $\int_0^\infty \frac{1}{F(z)} = \infty$. Thus, we do not have global existence of solution of (4.61) for all time in $t \ge 0$.

Remark 4.2.4. *In [30], the authors establish that the Fujita exponent coincides with the classical one when the diffusion is given by the Laplacian.*

In the previous Proposition 4.2.2, we have proved that the solution u of the problem (4.30), with initial data u_0 in $L^\infty(\Omega)$ or in $\mathcal{C}_b(\Omega)$ is in fact the solution of the problem (4.32), with a truncated globally Lipschitz function f_k associated to f. Then the solution u of (4.30) satisfies all the monotonicity properties that we have proved for the problem (4.32). We enumerate them in the following corollaries.

Corollary 4.2.5. (Weak and Strong Maximum Principles) *Let $X = L^\infty(\Omega)$ or $X = \mathcal{C}_b(\Omega)$. We assume J nonnegative, $K \in \mathcal{L}(X,X)$, $h \in X$, and the locally lipschitz function f satisfies that there exist C, $D \in \mathbb{R}$, with $C > 0$, $D \ge 0$ such that*

$$f(\cdot,s)s \le Cs^2 + D|s|,\ \forall s. \tag{4.66}$$

If u_0, $u_1 \in X$, satisfy that $u_0 \ge u_1$ then

$$u^0(t) \ge u^1(t),\quad \textit{for all}\ \ t > 0,$$

where $u^i(t)$ *is the solution to* (4.30) *with initial data* u_i.

In particular if J *satisfies that*

$$J(x, y) > 0 \text{ for all } x,\, y \in \Omega, \text{ such that } d(x, y) < R, \tag{4.67}$$

for some $R > 0$, *and* Ω *is* R*-connected, (see Definition 2.1.14) then*

$$u^0(t) > u^1(t), \quad \text{for all} \;\; t > 0.$$

Corollary 4.2.6. *Under the hypotheses of Corollary 4.2.5. If the initial data* $u_0 \in X$, *and the nonlinear terms* f^1, f^2 *satisfy*

$$f^1 \geq f^2$$

then

$$u^1(t) \geq u^2(t), \quad \text{for all} \;\; t \geq 0,$$

where $u^i(\cdot, t)$ *is the solution to* (4.30) *with nonlinear term* f^i.

In particular if J *satisfies hypothesis* (4.67) *of Corollary 4.2.5, and* Ω *is* R*-connected then*

$$u^1(t) > u^2(t), \quad \text{for all} \;\; t > 0.$$

Corollary 4.2.7. (Weak and Strong Positivity) *Under the hypotheses of Corollary 4.2.5. Let* $f(0) \geq 0$, *if* $u_0 \in X$, *with* $u_0 \geq 0$, *not identically zero, then the solution to* (4.30),

$$u(t,\, u_0) \geq 0, \text{ for all } t \geq 0.$$

In particular, if J *satisfies hypothesis* (4.67) *of Corollary 4.2.5, and* Ω *is* R*-connected then*

$$u(t,\, u_0) > 0, \text{ for all } t > 0.$$

Corollary 4.2.8. *Under the hypotheses of corollary 4.2.5. Let* $u(t,\, u_0)$ *be a solution to* (4.30) *with initial data* $u_0 \in X$, *and let* $\bar{u}(t)$ *be a supersolution to* (4.30) *in* $[0, T]$.

If $\bar{u}(0) \geq u_0$, *then*

$$\bar{u}(t) \geq u(t,\, u_0), \qquad \text{for all} \;\; t \in [0, T].$$

The same is true for subsolutions with reversed inequality.

In the previous Proposition 4.2.2, we have proved the existence and uniqueness of solution of the problem (4.30) with initial data in $L^\infty(\Omega)$ or in $\mathcal{C}_b(\Omega)$. Now we prove the existence and uniqueness for the problem with initial data in $L^p(\Omega)$ for all $1 < p < \infty$, and we prove also that the solution is a strong solution in $L^1(\Omega)$.

Theorem 4.2.9. *Let* $\mu(\Omega) < \infty$, *we assume* $J(x, y) = J(y, x)$, *and the locally Lipschitz function* f *satisfies that* $f(\cdot, 0) \in L^\infty(\Omega)$, *and*

$$\frac{\partial f}{\partial u}(\cdot, u) \leq \beta(\cdot) \in L^\infty(\Omega) \tag{4.68}$$

and for some $1 < p < \infty$

$$\left|\frac{\partial f}{\partial u}(\cdot, u)\right| \leq C(1 + |u|^{p-1}), \tag{4.69}$$

if $K \in \mathcal{L}(L^p(\Omega), L^p(\Omega))$ *and* h_0, $h \in L^\infty(\Omega)$, *then the equation* (4.30) *with initial data* $u_0 \in L^p(\Omega)$ *has a unique global solution given by the Variation of Constants Formula*

$$u(\cdot, t) = e^{Lt} u_0 + \int_0^t e^{L(t-s)} f(\cdot, u(\cdot, s))\, ds, \tag{4.70}$$

with

$$u \in \mathcal{C}\big([0,T], L^p(\Omega)\big) \cap \mathcal{C}^1\big([0,T], L^1(\Omega)\big), \ \forall T > 0,$$

and it is a strong solution in $L^1(\Omega)$.

Proof. We prove that f satisfies the hypothesis in Proposition 4.2.2, i.e., there exists $C, D \in \mathbb{R}$, with $C, D > 0$ such that

$$f(\cdot, s)s \leq Cs^2 + D|s|, \quad \forall s \in \mathbb{R}.$$

Let $s > 0$ be arbitrary. Integrating (4.68) in $[0, s]$, we have

$$\begin{aligned} \int_0^s \frac{\partial f}{\partial t}(\cdot, t)dt \ &\leq \int_0^s \beta(\cdot)dt \\ f(\cdot, s) - f(\cdot, 0) \ &\leq \beta(\cdot)s. \end{aligned} \tag{4.71}$$

Multiplying (4.71) by $s > 0$, and since β, $f(\cdot, 0) \in L^\infty(\Omega)$, we obtain

$$\begin{aligned} f(\cdot, s)s \ &\leq \beta(\cdot)s^2 + f(\cdot, 0)s \\ &\leq \|\beta\|_{L^\infty(\Omega)} s^2 + \|f(\cdot, 0)\|_{L^\infty(\Omega)} s \\ &\leq Cs^2 + D|s|. \end{aligned}$$

Let $s < 0$ be arbitrary. Integrating (4.68) in $[s, 0]$,

$$\begin{aligned} \int_s^0 \frac{\partial f}{\partial t}(\cdot, t)dt \ &\leq \int_s^0 \beta(\cdot)dt \\ f(\cdot, 0) - f(\cdot, s) \ &\leq -\beta(\cdot)s. \end{aligned} \tag{4.72}$$

Multiplying (4.72) by $-s$, since $s < 0$ and β, $f(\cdot, 0) \in L^\infty(\Omega)$, then

$$\begin{aligned} f(\cdot, s)s \ &\leq \beta(\cdot)s^2 + f(\cdot, 0)s \\ &\leq \|\beta\|_{L^\infty(\Omega)} s^2 + \|f(\cdot, 0)\|_{L^\infty(\Omega)} s \\ &\leq Cs^2 + D|s|. \end{aligned}$$

Thus, we have that f satisfies the hypotheses of Proposition 4.2.2, and we have the existence and uniqueness of solutions for (4.30) with initial data $u_0 \in L^\infty(\Omega)$.

Since $L^\infty(\Omega)$ is dense in $L^p(\Omega)$, we consider a sequence of initial data $\{u_0^n\}_{n\in\mathbb{N}} \subset L^\infty(\Omega)$ such that $u_0^n \to u_0$ in $L^p(\Omega)$ as n goes to ∞. Thanks to Proposition 4.2.2, we know that the solution of (4.30) associated to the initial data $u_0^n \in L^\infty(\Omega)$, satisfies

$$u_t^n(x, t) = (K - hI)(u^n)(x, t) + f(x, u^n(x, t)) = L(u^n)(x, t) + f(x, u^n(x, t)).$$

We want to see first that $\{u^n\}_{n\in\mathbb{N}} \subset \mathcal{C}([0, \infty), L^p(\Omega))$ is a Cauchy sequence in compact sets of $[0, \infty)$. Then we consider

$$u_t^k(t) - u_t^j(t) = L(u^k - u^j)(t) + f(\cdot, u^k(t)) - f(\cdot, u^j(t)). \tag{4.73}$$

Multiplying (4.73) by $|u^k - u^j|^{p-2}(u^k - u^j)(t)$, and integrating in Ω, we obtain

$$\begin{aligned}\frac{1}{p}\frac{d}{dt}\|u^k(t) - u^j(t)\|^p_{L^p(\Omega)} &= \langle L(u^k - u^j)(t),\, |u^k - u^j|^{p-2}(u^k - u^j)(t)\rangle \\ &+ \int_\Omega \Big(f(\cdot, u^k(t)) - f(\cdot, u^j(t))\Big)\, |u^k - u^j|^{p-2}(u^k - u^j)(t).\end{aligned} \tag{4.74}$$

If we denote

$$u^k(t) - u^j(t) = w(t) \text{ and } g(w) = |w|^{p-2}w \in L^{p'}(\Omega),$$

then the first term on the right hand side of (4.74) can be divided in two parts as follows. First of all, we write

$$L(w(t)) = K(w(t)) - h_0(\cdot)w(t) + h_0(\cdot)w(t) - h(\cdot)w(t). \tag{4.75}$$

Since $J(x,y) = J(y,x)$, $K \in \mathcal{L}(L^p(\Omega), L^p(\Omega))$, and thanks to Proposition 2.3.1,

$$\begin{aligned}\int_\Omega (K - h_0 I)(w)g(w)\,dx &= \int_\Omega (K - h_0 I)(w)|w|^{p-2}w\,dx \\ &= -\frac{1}{2}\int_\Omega\int_\Omega J(x,y)(w(y) - w(x))(g(w)(y) - g(w)(x))dy\,dx.\end{aligned} \tag{4.76}$$

From (4.76), since J is nonnegative and $g(w) = |w|^{p-2}w$ is increasing, then

$$\int_\Omega (K - h_0 I)(w)|w|^{p-2}w\,dx = -\frac{1}{2}\int_\Omega\int_\Omega J(x,y)(w(y) - w(x))(g(w)(y) - g(w)(x))dy\,dx \le 0. \tag{4.77}$$

Moreover, h, $h_0 \in L^\infty(\Omega)$, then the second part of (4.75) applied to (4.74) satisfies

$$\int_\Omega (h_0(x) - h(x))\,|w|^p(x)\,dx \le C\|w\|^p_{L^p(\Omega)}. \tag{4.78}$$

On the other hand, thanks to the hypothesis (4.68) and the mean value Theorem, there exists $\xi = \xi(x,t)$, such that, the second term on the right hand side of (4.74) satisfies that

$$\begin{aligned}\int_\Omega \Big(f(\cdot, u^k(t)) - f(\cdot, u^j(t))\Big)\, |u^k - u^j|^{p-2}(u^k - u^j)(t) &= \int_\Omega \frac{\partial f}{\partial u}(\cdot, \xi)|w|^p \\ &\le \|\beta\|_{L^\infty(\Omega)}\|w\|^p_{L^p(\Omega)}.\end{aligned} \tag{4.79}$$

Finally, thanks to (4.74), (4.77), (4.78), and (4.79), we obtain

$$\frac{d}{dt}\|u^k(t) - u^j(t)\|^p_{L^p(\Omega)} \le C\|u^k(t) - u^j(t)\|^p_{L^p(\Omega)}$$

Thanks to Gronwall's inequality,

$$\|u^k(t) - u^j(t)\|^p_{L^p(\Omega)} \le e^{Ct}\|u^k_0 - u^j_0\|^p_{L^p(\Omega)}, \tag{4.80}$$

and taking supremums in $[0,T]$ in (4.80), we get

$$\sup_{t\in[0,T]} \|u^k(t) - u^j(t)\|^p_{L^p(\Omega)} \le C(T)\|u^k_0 - u^j_0\|^p_{L^p(\Omega)} \tag{4.81}$$

The right hand side of (4.81) goes to zero as k and j go to ∞. Therefore we have that $\{u^n\}_n \subset \mathcal{C}([0,\infty), L^p(\Omega))$ is a Cauchy sequence in compact sets of $[0,\infty)$, and there exists the limit of the sequence $\{u^n\}_n$ in $\mathcal{C}([0,T], L^p(\Omega))$, $\forall T > 0$, denoted by

$$u(t) = \lim_{n\to\infty} u^n(t)$$

and it is independent of the sequence $\{u_0^n\}_n$. Let us see this below. We choose two different sequences $\{u_0^n\}_n$ and $\{v_0^n\}_n$ that converge to u_0, and we construct a new sequence $\{w_0^n\}_n$, that consists of $w_0^{2n+1} = u_0^n$ and $w_0^{2n} = v_0^n$, for all $n \in \mathbb{N}$. Then, w_0^n converges to u_0. Since the sequence of solutions $\{w^n(t)\}$ of (4.30) associated to the initial values w_0^n is a Cauchy sequence, then there exists a unique limit $w(t) = \lim_{n\to\infty} w^n(t)$, and this limit is the same limit of the sequences $\{u^n(t)\}_n$ and $\{v^n(t)\}_n$. Thus, the limit is independent of the sequence $\{u_0^n\}_n$.

Let us prove now that the limit u is given by the Variation of Constants Formula (4.70). We integrate (4.69) in $[0,s]$, then

$$\begin{aligned} \int_0^s \left|\frac{\partial f}{\partial s}(\cdot,t)\right| dt \quad &\leq \int_0^s C(1+|t|^{p-1})dt \\ |f(\cdot,s)| - |f(\cdot,0)| \quad &\leq C(s + \tfrac{1}{p}|s|^{p-1}s) \end{aligned}$$

Therefore, we have that

$$\begin{aligned} |f(\cdot,s)| \quad &\leq C(s + \tfrac{1}{p}|s|^p) + |f(\cdot,0)| \\ &\leq C(1 + |s| + |s|^p) \end{aligned}$$

Thus, we have proved

$$|f(\cdot,u)| \leq C(1 + |u| + |u|^p), \tag{4.82}$$

then, since $\mu(\Omega) < \infty$, and thanks to (4.82), we have that $f : L^p(\Omega) \to L^1(\Omega)$.

Now we prove that $f : L^p(\Omega) \to L^1(\Omega)$ is Lipschitz in bounded sets of $L^p(\Omega)$. Consider $u, v \in L^p(\Omega)$ with $\|u\|_{L^p(\Omega)}, \|v\|_{L^p(\Omega)} < M$, and $0 < M \in \mathbb{R}$, thanks to the Mean Value Theorem, there exists $\xi \in L^p(\Omega)$,

$$\xi(x) = \theta(x)u(x) + (1-\theta(x))v(x), \ \text{ for a.e. } \ x \in \Omega$$

with $0 \leq \theta(x) \leq 1$ for a.e. $x \in \Omega$, and $\|\xi\|_{L^p(\Omega)} < 2M$, such that $|f(u)-f(v)| = \left|\frac{\partial f}{\partial u}(\xi)\right| |u-v|$. From hypothesis (4.69) and Hölder's inequality, we have that

$$\begin{aligned} \int_\Omega |f(u)-f(v)| \quad &= \int_\Omega \left|\frac{\partial f}{\partial u}(\xi)\right| |u-v| \\ &\leq \int_\Omega C\left(1+|\xi|^{p-1}\right)|u-v| \\ &\leq \left(\int_\Omega \left|C\left(1+|\xi|^{p-1}\right)\right|^{p'}\right)^{1/p'} \|u-v\|_{L^p(\Omega)} \\ &\leq \left(C\mu(\Omega) + \left(\int_\Omega |\xi|^p dx\right)^{1/p'}\right) \|u-v\|_{L^p(\Omega)} \\ &\leq \left(C\mu(\Omega) + \left(\|u\|_{L^p(\Omega)} + \|v\|_{L^p(\Omega)}\right)^{p-1}\right) \|u-v\|_{L^p(\Omega)} \\ &\leq C(M)\|u-v\|_{L^p(\Omega)}, \end{aligned}$$

Then, since $u(t) = \lim_{n\to\infty} u^n(t)$ in $\mathcal{C}([0,T], L^p(\Omega))$, $\forall T > 0$, we have that

$$f(u^n) \to f(u) \quad \text{in } \mathcal{C}([0,T], L^1(\Omega)) \ \ \forall T > 0. \tag{4.83}$$

Since $L \in \mathcal{L}(L^1(\Omega), L^1(\Omega))$, then there exists $\delta > 0$, such that $Re(\sigma(L)) \leq \delta$. Hence thanks to Lemma 3.4.2, we know that $\|e^{Lt}\|_{\mathcal{L}(L^1(\Omega))} \leq C_0 e^{\delta t}$. Thus

$$\begin{aligned}
&\left\| \int_0^t \ e^{L(t-s)} f(\cdot, u^n(s)) ds - \int_0^t e^{L(t-s)} f(\cdot, u(s)) ds \right\|_{L^1(\Omega)} \\
&\qquad \leq \int_0^t \left\| e^{L(t-s)} \left(f(\cdot, u^n(s)) - f(\cdot, u(s)) \right) \right\|_{L^1(\Omega)} ds \\
&\qquad \leq \int_0^t \int_\Omega \left\| e^{L(t-s)} \right\|_{L^1(\Omega)} \|(f(\cdot, u^n(s)) - f(\cdot, u(s)))\|_{L^1(\Omega)}\, ds \\
&\qquad \leq \int_0^t e^{\delta s} \|f(\cdot, u^n(s)) - f(\cdot, u(s)))\|_{L^1(\Omega)}\, ds
\end{aligned} \tag{4.84}$$

Taking supremums in $t \in [0,T]$ in (4.84), and from (4.83)

$$\int_0^t e^{L(t-s)} f(\cdot, u^n(s)) ds \to \int_0^t e^{L(t-s)} f(\cdot, u(s)) ds \quad \text{in } \mathcal{C}([0,T], L^1(\Omega)), \ \ \forall T > 0.$$

Let $u_0 \in L^p(\Omega)$ be the limit of the sequence $\{u_0^n\}_{n\in\mathbb{N}}$, we have already proved that $u^n \to u$ in $\mathcal{C}\left([0,T], L^p(\Omega)\right)$, $\forall T > 0$, and since $\|e^{Lt}\|_{\mathcal{L}(L^p(\Omega))} \leq C_0 e^{\delta t}$, we obtain that

$$e^{Lt} u_0^n \to e^{Lt} u_0 \quad \text{in } \mathcal{C}\left([0,T], L^p(\Omega)\right) \ \ \forall T > 0.$$

Moreover, since

$$\int_0^t e^{L(t-s)} f(\cdot, u^n(s)) ds = u^n(t) - e^{Lt} u_0^n$$

and, $u^n(t) - e^{Lt} u_0^n$ converges to $u(t) - e^{Lt} u_0$ in $\mathcal{C}\left([0,T], L^p(\Omega)\right)$ $\forall T > 0$, as $n \to \infty$, and we have that

$$\int_0^t e^{L(t-s)} f(\cdot, u(s)) ds = u(t) - e^{Lt} u_0. \tag{4.85}$$

Then we have that

$$\int_0^t e^{L(t-s)} f(\cdot, u^n(s)) ds \ \longrightarrow \int_0^t e^{L(t-s)} f(\cdot, u(s)) ds$$

converges in $\mathcal{C}\left([0,T], L^p(\Omega)\right)$, $\forall T > 0$. Hence, we have proved the global existence of the mild solution, u in $\mathcal{C}\left([0,T], L^p(\Omega)\right)$ for all $T > 0$ of the problem (4.30), because u satisfies that

$$u(t) = e^{Lt} u_0 + \int_0^t e^{L(t-s)} f(\cdot, u(s)) ds.$$

Moreover, consider $g(t) = f(\cdot, u(t))$. Since $u : [0,T] \mapsto L^p(\Omega)$ is continuous, and $f : L^p(\Omega) \mapsto L^1(\Omega)$ is continuous, we have that $g : [0,T] \mapsto L^1(\Omega)$ is continuous. Moreover, $L \in \mathcal{L}(L^1(\Omega), L^1(\Omega))$. Then, by using Theorem 4.1.2 for the problem (4.30), we have that the initial data $u_0 \in L^p(\Omega) \hookrightarrow D(L) = L^1(\Omega)$ and $X = L^1(\Omega)$, thus $u \in$

$\mathcal{C}([0,T],L^p(\Omega))\cap\mathcal{C}^1([0,T],L^1(\Omega))$ and it is a strong solution in $L^1(\Omega)$.

Finally, let us prove the uniqueness of the solution of (4.30) with initial data $u_0\in L^p(\Omega)$, such that $u\in\mathcal{C}\left([0,T],L^p(\Omega)\right)\cap\mathcal{C}^1([0,T],L^1(\Omega))$, $\forall T>0$, is a strong solution of (4.30) and the solution is given by the Variations of Constants Formula (4.70). We consider that there exists two different strong solutions u and v. If we follow the steps of this proof from (4.73) to (4.80), replacing u^k for u and u^j for v, we obtain

$$\|u(t)-v(t)\|^p_{L^p(\Omega)}\leq e^{Ct}\|u(0)-v(0)\|^p_{L^p(\Omega)}. \tag{4.86}$$

Since $u(0)=v(0)=u_0$, then

$$0\leq\|u(t)-v(t)\|^p_{L^p(\Omega)}\leq 0\quad\forall t\geq 0. \tag{4.87}$$

Therefore $u(x,t)=v(x,t)$ for a.e. $x\in\Omega$ and $t\geq 0$. Thus, the result. $\square$

Remark 4.2.10. *In the previous Theorem 4.2.9, the sign condition on f, (4.69), we have not included the case $p=1$. This is because if $p=1$ in hypothesis (4.69), then we have that $\left|\frac{\partial f}{\partial u}(\cdot,u)\right|\leq C$, then f is globally Lipschitz, and we have proved in Proposition 4.1.3, that if f is globally Lipschitz, then we have existence and uniqueness of solution of (4.3) for any initial data in $u_0\in L^q(\Omega)$, with $1\leq q\leq\infty$.*

In the following Corollaries we enumerate the monotonicity properties that are satisfied for the solution of (4.30) with initial data $u_0\in L^p(\Omega)$ with p as in Theorem 4.2.9. We apply Corollaries 4.2.5 to 4.2.8, that state the monotonicity properties of the solution of (4.30) with initial data bounded.

Corollary 4.2.11. *Let (Ω,μ,d) be a metric measure space, with $\mu(\Omega)<\infty$, for $1\leq q\leq p$, we assume that $K\in\mathcal{L}(L^q(\Omega),L^q(\Omega))$, and $h\in L^\infty(\Omega)$. If the locally Lipschitz function f satisfies that $f(\cdot,0)\in L^\infty(\Omega)$, and*

$$\frac{\partial f}{\partial u}(\cdot,u)\leq\beta(\cdot)\in L^\infty(\Omega)$$

and, for some $1<p<\infty$

$$\left|\frac{\partial f}{\partial u}(\cdot,u)\right|\leq C(1+|u|^{p-1}).$$

If u_0, $u_1\in L^p(\Omega)$ satisfy that $u_0\geq u_1$ then

$$u^0(t)\geq u^1(t),\ \ \textit{for all}\ \ t\geq 0,$$

where $u^i(t)$ is the solution to (4.30) with initial data u_i.
In particular if J satisfies that

$$J(x,y)>0\ \textit{for all}\ x,\ y\in\Omega,\ \textit{such that}\ d(x,y)<R, \tag{4.88}$$

for some $R>0$, and Ω is R-connected, (see Definition 2.1.14), then

$$u^0(t)>u^1(t),\ \ \textit{for all}\ \ t>0.$$

Proof. Given u_0, $u_1 \in L^p(\Omega)$, with $u_0 \geq u_1$. Since $L^\infty(\Omega)$ is dense in $L^p(\Omega)$ with $1 < p < \infty$, then we choose two sequences $\{u_0^n\}_{n\in\mathbb{N}}$ and $\{u_1^n\}_{n\in\mathbb{N}}$ in $L^\infty(\Omega)$ that converge to the initial data u_0 and u_1 respectively,and such that

$$u_0^n \geq u_1^n, \forall n \in \mathbb{N}.$$

Thanks to Corollary 4.2.5, we know that the associated solutions satisfy

$$u_n^0(t) \geq u_n^1(t), \text{ for all } t \geq 0, \ \forall n \in \mathbb{N}.$$

From Theorem 4.2.9, we know that $u_n^i(t)$ converges to $u^i(t)$, for $i = 0, 1$ in $\mathcal{C}([0,T], L^p(\Omega))$. Therefore

$$u^0(t) \geq u^1(t), \text{ for all } t \geq 0.$$

Analogously we arrive to $u^0(t) > u^1(t)$, for all $t > 0$. □

Corollary 4.2.12. *Let (Ω, μ, d) be a metric measure space, with $\mu(\Omega) < \infty$, for $1 \leq q \leq p$, we assume that $K \in \mathcal{L}(L^q(\Omega), L^q(\Omega))$, $h \in L^\infty(\Omega)$, and the locally Lipschitz functions f^1 and f^2 satisfy that, $f^i(\cdot, 0) \in L^\infty(\Omega)$,*

$$\frac{\partial f^i}{\partial u}(\cdot, u) \leq \beta^i(\cdot) \in L^\infty(\Omega)$$

and for some $1 < p < \infty$

$$\left|\frac{\partial f^i}{\partial u}(\cdot, u)\right| \leq C(1 + |u|^{p-1}).$$

If the initial data $u_0 \in L^p(\Omega)$ and

$$f^1 \geq f^2,$$

then

$$u^1(t) \geq u^2(t), \text{ for all } t \geq 0,$$

where $u^i(\cdot, t)$ is the solution to (4.30) *with nonlinear term f^i and initial data u_0.*

In particular if J satisfies hypothesis (4.88) *of Corollary 4.2.11, and Ω is R-connected then*

$$u^1(t) > u^2(t), \text{ for all } t > 0.$$

Corollary 4.2.13. *Under the hypotheses of Corollary 4.2.11. Let $f(\cdot, 0) \geq 0$, if $u_0 \in L^p(\Omega)$, with $u_0 \geq 0$, not identically zero, then the solution to* (4.30),

$$u(t, u_0) \geq 0, \text{ for all } t > 0.$$

In particular if J satisfies hypothesis (4.88) *of Corollary 4.2.11, and Ω is R-connected then*

$$u(t, u_0) > 0, \text{ for all } t > 0.$$

Corollary 4.2.14. *Under the hypotheses of corollary 4.2.11, let $u_0 \in L^p(\Omega)$, and let $\bar{u}(t)$ be a supersolution to* (4.30) *in $[0, T]$, (see 4.21), and let $u(t, u_0)$ be the solution to* (4.30) *with initial data $u_0 \in L^p(\Omega)$. If $\bar{u}(0) \geq u_0$, then*

$$\bar{u}(t) \geq u(t, u_0), \quad \text{for all } t \in [0, T].$$

The same is true for subsolutions with reversed inequality.

Under the hypotheses of Propositions 4.1.3, 4.2.2 and Theorem 4.2.9 we define the non-linear semigroup associated to (4.1) written as

$$S(t)u_0 = u(t, u_0) = e^{Lt}u_0 + \int_0^t e^{L(t-s)} f(\cdot, u(s))ds.$$

4.3 Asymptotic estimates

Let (Ω, μ, d) be a metric measure space.

- If $X = L^p(\Omega)$, with $1 \le p \le \infty$, we assume $h \in L^\infty(\Omega)$.
- If $X = \mathcal{C}_b(\Omega)$, we assume $h \in \mathcal{C}_b(\Omega)$.

Let $K \in \mathcal{L}(X, X)$. Now, we study asymptotic estimates of the norm X of the solution u of the nonlinear nonlocal problem that we recall is given by

$$\begin{cases} u_t(x,t) &= (K - hI)(u)(x,t) + f(x, u(x,t)) = L(u)(x,t) + f(x, u(x,t)), \ x \in \Omega \\ u(x,0) &= u_0(x), \quad x \in \Omega, \end{cases} \tag{4.89}$$

with $u_0 \in X$ and $f : \Omega \times \mathbb{R} \to \mathbb{R}$ as in Propositions 4.1.3, 4.2.2 and Theorem 4.2.9, where the nonlinear term f satisfies that there exist $C(\cdot) \in L^\infty(\Omega)$ and $0 < D(\cdot) \in L^\infty(\Omega)$ such that

$$f(\cdot, u)u \le C(\cdot)u^2 + D(\cdot)|u| \tag{4.90}$$

This means that

$$\begin{array}{ll} f(x,u) \le C(x)u + D(x), & \text{if } u \ge 0 \\ f(x,u) \ge C(x)u - D(x), & \text{if } u \le 0. \end{array} \tag{4.91}$$

In the following proposition we give more details about C and D, and we give bounds of $|u(t)|$, where u is the solution to (4.89).

Proposition 4.3.1.

- *If $X = L^p(\Omega)$, with $1 \le p \le \infty$, we assume $h \in L^\infty(\Omega)$.*
- *If $X = \mathcal{C}_b(\Omega)$, we assume $h \in \mathcal{C}_b(\Omega)$.*

Let $K \in \mathcal{L}(X, X)$, and J be nonnegative. We assume either:

i. $u_0 \in X$, $f : \Omega \times \mathbb{R} \to \mathbb{R}$ globally Lipschitz, and $f(\cdot, 0) \in L^\infty(\Omega)$. Then there exists $C = L_f$ and $D = \|f(\cdot, 0)\|_{L^\infty(\Omega)}$, such that

$$f(\cdot, u)u \le C(\cdot)u^2 + D(\cdot)|u|, \quad \forall u. \tag{4.92}$$

ii. u_0 in $X = L^\infty(\Omega)$ or $X = \mathcal{C}_b(\Omega)$, $f(x,s)$ is locally Lipschitz in the variable $s \in \mathbb{R}$, uniformly respect to $x \in \Omega$, and there exist $C(\cdot)$, $D(\cdot) \in L^\infty(\Omega)$ with $D \ge 0$ such that

$$f(\cdot, u)u \le C(\cdot)u^2 + D(\cdot)|u|, \quad \forall u. \tag{4.93}$$

iii. $J(x,y)=J(y,x)$, *f is locally Lipschitz in the variable $s\in\mathbb{R}$, uniformly respect to $x\in\Omega$, and it satisfies that $f(\cdot,0)\in L^\infty(\Omega)$,*

$$\frac{\partial f}{\partial u}(\cdot,u)\le \beta(\cdot)\in L^\infty(\Omega) \tag{4.94}$$

and for some $1<p<\infty$,

$$\left|\frac{\partial f}{\partial u}(\cdot,u)\right|\le C_1(1+|u|^{p-1}), \tag{4.95}$$

initial data u_0 in $X=L^p(\Omega)$ and $K\in\mathcal{L}(L^q(\Omega),L^q(\Omega))$, for $1\le q\le p$. Then there exists $C=\|\beta\|_{L^\infty(\Omega)}$ and $D=\|f(\cdot,0)\|_{L^\infty(\Omega)}$, such that

$$f(\cdot,u)u\le C(\cdot)u^2+D(\cdot)|u|,\quad \forall u. \tag{4.96}$$

Let $\mathcal{U}(t)$ be the solution of

$$\begin{cases} \mathcal{U}_t(x,t)=L(\mathcal{U}(x,t))+C(x)\mathcal{U}(x,t)+D(x)=L_C(\mathcal{U}(x,t))+D(x), & x\in\Omega,\ t>0\\ \mathcal{U}(x,0)=|u_0(x)|, & x\in\Omega,\end{cases} \tag{4.97}$$

where $L_C=L+C$. Then the solution, u, of (4.89)*, satisfies that*

$$|u(t)|\le\mathcal{U}(t),\quad \textit{for all }\ t\ge 0.$$

Proof. We prove this proposition assuming hypothesis *ii.*, the rest of the cases are analogous, since hypotheses (4.94) and (4.95) imply (4.93), (see proof of Theorem 4.2.9). First of all, we prove that the solution of (4.97) is nonnegative. We know that the solution $\mathcal{U}$ can be written with the Variation of Constants Formula as

$$\mathcal{U}(t)=e^{L_Ct}|u_0|+\int_0^t e^{L_C(t-s)}D(\cdot)ds. \tag{4.98}$$

where $L_C=L+C=K-(h-C)$. Since $|u_0|\ge 0$, D is nonnegative, and J is nonnegative. If we denote $h_C=h-C$, then we can apply Proposition 3.2.2 to $L_C=K-h_CI$, and then we have that

$$e^{L_Ct}|u_0|\ge 0\ \ \forall t\ge 0\quad\text{and}\quad e^{L_C(t-s)}D\ge 0\ \forall t\ge 0 \text{ and } s\in[0,t].$$

Thus, we have that $\mathcal{U}(t)$ is nonnegative for all $t\ge 0$.

Now, we prove that $\mathcal{U}$ is a supersolution of (4.89). Since $\mathcal{U}$ is nonnegative and f satisfies (4.93), we obtain

$$L(\mathcal{U})+f(\cdot,\mathcal{U})\le L(\mathcal{U})+C(\cdot)\mathcal{U}+D(\cdot)=\mathcal{U}_t.$$

Moreover $u_0\le|u_0|=\mathcal{U}(0)$, then from Corollary 4.2.8 we have

$$u(t)\le\mathcal{U}(t),\ \forall t\ge 0. \tag{4.99}$$

Now let $\mathcal{W}=-\mathcal{U}$ be the solution to

$$\begin{cases}\mathcal{W}_t=L(\mathcal{W})+C(\cdot)\mathcal{W}-D(\cdot)=L_C(\mathcal{W})-D(\cdot)\\ \mathcal{W}(0)=-|u_0|.\end{cases}$$

Since $-|u_0| \leq 0$ then, we have that $\mathcal{W}(t) = -\mathcal{U}(t)$ is nonpositive for all $t \geq 0$.

Now, we prove that $\mathcal{W}$ is a subsolution of (4.89). Since $\mathcal{W}$ is nonpositive and f satisfies (4.93), we obtain

$$L(\mathcal{W}) + f(\cdot, \mathcal{W}) \geq L(\mathcal{W}) + C(\cdot)\mathcal{W} - D(\cdot) = \mathcal{W}_t.$$

Moreover, $u_0 \geq -|u_0| = \mathcal{W}(0)$, then from Corollary 4.2.8 we obtain that

$$u(t) \geq \mathcal{W}(t), \ \forall t \geq 0. \tag{4.100}$$

Therefore, thanks to (4.99) and (4.100) we have that

$$-\mathcal{U}(t) \leq u(t) \leq \mathcal{U}(t), \ \ \forall t \geq 0.$$

Thus, the result. $\square$

In the following proposition we give an asymptotic estimate of the norm X of the semigroup of (4.89),

$$S(t)u_0 = u(t, u_0),$$

that is given in terms of the norm of the equilibrium associated to the problem (4.97). To obtain this estimate, we assume that the operator L_C satisfies that

$$\inf \sigma_X(-L - C) \geq \delta > 0. \tag{4.101}$$

Then we have that, $\|e^{L_C t}\|_X \leq e^{-\delta t}$ for all $t \geq 0$. But first we prove a Lemma that will be useful.

Lemma 4.3.2. *Let X be a Banach space, and let $S(t) : X \to X$ be a continuous semigroup. Assume that u_0, $v \in X$ satisfy that $S(t)u_0 \to v$ in X as $t \to \infty$. Then v is an equilibrium point for $S(t)$.*

Proof. Since $v = \lim_{t\to\infty} S(t)u_0$. Then applying $S(s)$ for $s > 0$, and using the continuity of $S(t)$ for $t > 0$,

$$S(s)v = S(s) \lim_{t\to\infty} S(t)u_0 = \lim_{t\to\infty} S(s+t)u_0 = v.$$

Then v is an equilibrium point for the system. $\square$

Now, we prove the asymptotic estimate of the solution of (4.97).

Proposition 4.3.3. *Let $\mu(\Omega) < \infty$, let $X = L^p(\Omega)$, with $1 \leq p \leq \infty$, and $h \in L^\infty(\Omega)$. We assume $K \in \mathcal{L}(L^p(\Omega), L^\infty(\Omega))$ is compact (see Proposition 2.1.7), J is nonnegative, f and J satisfy the hypotheses of Proposition 4.3.1, and $C \in L^\infty(\Omega)$, $0 \leq D \in L^\infty(\Omega)$. If*

$$\inf \sigma_X(-L - C) \geq \delta > 0, \tag{4.102}$$

then there exists a unique equilibrium solution, Φ, associated to (4.97), such that

$$L(\Phi) + C(\cdot)\Phi + D(\cdot) = 0, \tag{4.103}$$

$\Phi \in L^\infty(\Omega)$ and $\Phi \geq 0$. Moreover, if $u_0 \in X$, then the solution u of (4.89) satisfies that

$$\overline{\lim_{t\to\infty}} \|u(t, u_0)\|_X \leq \|\Phi\|_X.$$

Proof. First of all, thanks to Proposition 2.4.5, we have that $\sigma_X(-L-C)$ is independent of X. Moreover, thanks to hypothesis (4.102), we have that 0 does not belong to the spectrum of L_C, then $L_C = L + C$ is invertible. Thus, the solution Φ of (4.103) is unique.

On the other hand, since Φ satisfies the equation (4.103), $D \in L^\infty(\Omega)$, and $L_C \in \mathcal{L}(L^\infty(\Omega), L^\infty(\Omega))$, then $\Phi \in L^\infty(\Omega)$.

Now, we want to prove that Φ is nonnegative. We write the solution $\mathcal{U}$ with the Variation of Constants Formula

$$\mathcal{U}(t) = e^{L_C t}|u_0| + \int_0^t e^{L_C(t-s)} D(\cdot) ds \geq 0 \quad \forall t \geq 0. \tag{4.104}$$

Thanks to hypothesis (4.102) and Proposition 3.4.2, we have that

$$\|e^{L_C t}\|_{\mathcal{L}(X,X)} \leq e^{-\delta t}, \tag{4.105}$$

then

$$\lim_{t\to\infty} \mathcal{U}(t) = \int_0^\infty e^{L_C s} D\, ds. \tag{4.106}$$

Thanks to Proposition 3.2.2, we know that $e^{L_C t}$ preserves the positivity. From (4.106), since D is nonnegative, then $\lim_{t\to\infty} \mathcal{U}(t) \geq 0$. Moreover, thanks to Lemma 4.3.2, we have that $\lim_{t\to\infty} \mathcal{U}(t)$ is an equilibrium, and the problem (4.97) has a unique equilibrium. Then, $\lim_{t\to\infty} \mathcal{U}(t) = \Phi$, and Φ is nonnegative.

Furthermore, from Proposition 4.3.1, we have that the solution u satisfies that

$$|u(t)| \leq \mathcal{U}(t) = \Phi + e^{L_C t}(|u_0| - \Phi), \tag{4.107}$$

where $\mathcal{U}(t)$ is the solution to (4.97). Let us see below that $\mathcal{U}(t) = \Phi + e^{L_C t}(|u_0| - \Phi)$ is a solution to (4.97). Since $L_C = L + C$ is a linear operator and thanks to (4.103), we have

$$\mathcal{U}_t(t) = L_C(e^{L_C t}(|u_0| - \Phi)) = L_C(\mathcal{U}(t) - \Phi) = L_C(\mathcal{U}(t)) - L_C(\Phi) = L_C(\mathcal{U}(t)) + D.$$

For $u_0 \in X$, we have that $(|u_0| - \Phi) \in X$, and thanks to (4.105) we obtain

$$\begin{aligned} \|u(t)\|_X &\leq \|\mathcal{U}(t)\|_X \\ &\leq \|\Phi\|_X + \|e^{L_C t}(|u_0| - \Phi)\|_X \\ &\leq \|\Phi\|_X + \|e^{L_C t}\|_{\mathcal{L}(X,X)} \|(|u_0| - \Phi)\|_X \\ &\leq \|\Phi\|_X + e^{-\delta t}\|(|u_0| - \Phi)\|_X \end{aligned} \tag{4.108}$$

Since $\delta > 0$, then from (4.108), we have

$$\overline{\lim_{t\to\infty}} \|u(t)\|_X \leq \|\Phi\|_X.$$

Thus, the result. □

Remark 4.3.4. *The hypotheses on the spectrum of L_C,* (4.102)*, can be obtained assuming that $J \in L^\infty(\Omega, L^1(\Omega))$, and $h \in L^\infty(\Omega)$ satisfies that*

$$h - C \geq h_0 + \delta \ \text{ in } \Omega, \quad \text{with } h_0(x) = \int_\Omega J(x,y)dy \in L^\infty(\Omega), \text{ and } \delta > 0,$$

and $J(x,y) = J(y,x)$. Then, thanks to part ii. of Corollary 2.4.6, we have

$$\sigma_X(-L-C) \geq \delta > 0. \tag{4.109}$$

Remark 4.3.5. *Another way to prove that Φ, the equilibrium solution that satisfies* (4.103), *is nonnegative assuming that $J(x,y) = J(y,x)$ is the following. Thanks to Proposition 2.1.19, we have that L_C is selfadjoint in $L^2(\Omega)$. Moreover, we know that $\Phi \in L^\infty(\Omega) \subset L^2(\Omega)$.*

We multiply (4.103) *by $\Phi^- := \min\{\Phi, 0\} \leq 0$ and we integrate in Ω, then we have that*

$$\begin{aligned}\int_\Omega D(x)\ \Phi^-(x)dx &= -\int_\Omega\int_\Omega J(x,y)\Phi(y)dy\Phi^-(x)dx + \int_\Omega h(x)\Phi(x)\Phi^-(x)dx - \int_\Omega C(x)\Phi(x)\Phi^-(x)dx \\ &= -\int_\Omega\int_\Omega J(x,y)\Phi(y)dy\Phi^-(x)dx + \int_\Omega h(x)(\Phi^-)^2(x)dx - \int_\Omega C(x)(\Phi^-)^2(x)dx\end{aligned} \tag{4.110}$$

In (4.110), *we write $\Phi(y) = \Phi^+(y) + \Phi^-(y)$, since $-\Phi^+\Phi^- = 0$, thanks to Proposition 2.1.21 and the fact that the spectrum of $-L_C$ satisfies* (4.102), *we obtain*

$$\begin{aligned}&\int_\Omega D(x)\ \Phi^-(x)dx = \\ &= -\int_\Omega\int_\Omega J(x,y)(\Phi^+(y) + \Phi^-(y))dy\Phi^-(x)dx + \int_\Omega h(x)(\Phi^-)^2(x)dx - \int_\Omega C(x)(\Phi^-)^2(x)dx \\ &\geq -\int_\Omega\int_\Omega J(x,y)\Phi^-(y)dy\Phi^-(x)dx + \int_\Omega h(x)(\Phi^-)^2(x)dx - \int_\Omega C(x)(\Phi^-)^2(x)dx \\ &= \langle -L_C(\Phi^-), \Phi^- \rangle_{L^2(\Omega),L^2(\Omega)} \\ &\geq \inf_{u \in L^2(\Omega)} \frac{\langle -L_C(u), u \rangle_{L^2(\Omega),L^2(\Omega)}}{\|u\|^2_{L^2(\Omega)}} \|\Phi^-\|^2_{L^2(\Omega)} \\ &= \inf \sigma_{L^2(\Omega)}(-L - C)\|\Phi^-\|^2_{L^2(\Omega)} \geq \delta\|\Phi^-\|^2_{L^2(\Omega)},\end{aligned} \tag{4.111}$$

We have that $D \geq 0$ and $\Phi^- \leq 0$, then thanks to (4.111), *we have that*

$$0 \geq \int_\Omega D(x)\Phi^-(x)dx \geq \delta\|\Phi^-\|^2_{L^2(\Omega)}$$

then $\Phi^- = 0$. Hence, we have that the solution Φ is nonnegative.

4.4 Extremal equilibria

In this section we prove the existence of two ordered extremal equilibria, which give some information about the set that attracts the dynamics of the semigroup $S(t)$ associated to the problem (4.89),

$$S(t)u_0 = u(\cdot, t, u_0),$$

where $u(\cdot, t, u_0)$ is the solution of (4.89).

A function $\varphi = \varphi(x)$ is said to be an *equilibrium* solution, or *steady-state* solution, of (4.89) if it satisfies the following

$$(K - hI)(\varphi)(x) + f(x, \varphi(x)) = L(\varphi)(x) + f(x, \varphi(x)) = 0. \tag{4.112}$$

First of all, we prove the existence of the extremal equilibria for the problem (4.89) with initial data $u_0 \in L^\infty(\Omega)$, i.e., we prove that there exists φ_m and φ_M in $L^\infty(\Omega)$, such that the solution of (4.89) enter between the extremal equilibria φ_m and φ_M for a.e. $x \in \Omega$, when time goes to infinity. Secondly, we will prove that the same extremal equilibria φ_m and φ_M are the bounds of any weak limit in $L^p(\Omega)$ for $1 \leq p < \infty$ of the solution of (4.89) with initial data $u_0 \in L^p(\Omega)$. This is another difference with the nonlinear local problem, with the laplacian, where the asymptotic dynamics of the solution enter between two extremal equilibria, uniformly in space, for bounded sets of initial data (see [44]). This difference is due to the lack of smoothness of the linear group e^{Lt}.

We prove now the existence of two ordered extremal equilibria for the problem (4.89) with initial data $u_0 \in L^\infty(\Omega)$.

Theorem 4.4.1. *Let (Ω, μ, d) be a metric measure space with $\mu(\Omega) < \infty$, let $X = L^p(\Omega)$, with $1 \leq p \leq \infty$, and $h \in L^\infty(\Omega)$. We assume $K \in \mathcal{L}(L^p(\Omega), L^\infty(\Omega))$ is compact, J is nonnegative, f and J satisfy the hypotheses of Proposition 4.3.1, $C \in L^\infty(\Omega)$, $0 \leq D \in L^\infty(\Omega)$, and*

$$\inf \sigma_X(-L - C) \geq \delta > 0. \tag{4.113}$$

Then there exist two ordered extremal equilibria, $\varphi_m \leq \varphi_M$, in $L^\infty(\Omega)$ of the problem (4.89), with initial data $u_0 \in L^\infty(\Omega)$, such that any other equilibria $\psi \in L^\infty(\Omega)$ of (4.89) satisfies $\varphi_m \leq \psi \leq \varphi_M$. Furthermore, the set $\{v \in L^\infty(\Omega) : \varphi_m \leq v \leq \varphi_M\}$ attracts the dynamics of the solutions $S(t)u_0$ of the problem (4.89), in the sense that, $\forall u_0 \in L^\infty(\Omega)$, there exist $\underline{u}(t)$ and $\overline{u}(t)$ in $L^\infty(\Omega)$ such that $\underline{u}(t) \leq u(t, u_0) \leq \overline{u}(t)$, and

$$\lim_{t\to\infty} \underline{u}(t) = \varphi_m$$
$$\lim_{t\to\infty} \overline{u}(t) = \varphi_M$$

in $L^p(\Omega)$ for all $1 \leq p < \infty$.

Proof. From (4.107) we have that the solution of (4.89) satisfies that

$$|u(t)| \quad \leq \Phi + e^{L_C t}(|u_0| - \Phi) \tag{4.114}$$

Since $\|e^{L_C t}\|_{\mathcal{L}(L^\infty(\Omega))} \leq e^{-\delta t}$, with $\delta > 0$, then for every initial data $u_0 \in L^\infty(\Omega)$ and for all $\varepsilon > 0$, $\exists\, T(u_0) > 0$ such that

$$\|e^{L_C t}(|u_0| - \Phi)\|_{L^\infty(\Omega)} < \varepsilon, \quad \forall t \geq T(u_0). \tag{4.115}$$

From (4.114) and (4.115) we have that

$$-\Phi - \varepsilon \leq u(\cdot, t, u_0) \leq \Phi + \varepsilon, \quad \forall t \geq T(u_0). \tag{4.116}$$

We recall that the solution u of (4.89) is written in terms of the semigroup $S(t)$ as

$$u(\cdot, t, u_0) = S(t)u_0.$$

Now, we denote $T(u_0) = T$, to simplify the notation, and we rewrite (4.116) as

$$-\Phi - \varepsilon \leq S(t + T)(u_0) \leq \Phi + \varepsilon, \quad \forall t \geq 0. \tag{4.117}$$

In the first part of the proof, we consider the initial data $u_0 = \Phi + \varepsilon$, then we have that there exists $T = T(\Phi + \varepsilon)$ such that

$$-\Phi - \varepsilon \leq S(t+T)(\Phi + \varepsilon) \leq \Phi + \varepsilon, \quad \forall t \geq 0. \tag{4.118}$$

Now, thanks to the order preserving properties, Corollary 4.2.5 and Corollary 4.2.11, and applying $S(T)$ to (4.118) with $t = 0$, we obtain that

$$-\Phi - \varepsilon \leq S(2T)(\Phi + \varepsilon) \leq S(T)(\Phi + \varepsilon) \leq \Phi + \varepsilon. \tag{4.119}$$

Iterating this process, we obtain that

$$-\Phi-\varepsilon \leq S(nT)(\Phi+\varepsilon) \leq S((n-1)T)(\Phi+\varepsilon) \leq \cdots \leq S(T)(\Phi+\varepsilon) \leq \Phi+\varepsilon, \ \forall n \in \mathbb{N}. \tag{4.120}$$

Thus, $\{S(nT)(\Phi + \varepsilon)\}_{n\in\mathbb{N}}$ is a monotonically decreasing sequence bounded from below. Then thanks to the Monotone convergence Theorem, the sequence converges in $L^p(\Omega)$, for $1 \leq p < \infty$, to some function φ_M, i.e.

$$S(nT)(\Phi + \varepsilon) \to \varphi_M \ \text{ as } \ n \to \infty \ \text{ in } \ L^p(\Omega). \tag{4.121}$$

Moreover, since $|S(nT)(\Phi+\varepsilon)| \leq \Phi+\varepsilon$, for all $n \in \mathbb{N}$ and $\Phi+\varepsilon \in L^\infty(\Omega)$, then $\varphi_M \in L^\infty(\Omega)$. Now we prove that, in fact, the whole solution $S(t)(\Phi + \varepsilon)$ converges in $L^p(\Omega)$ to φ_M as $t \to \infty$. From (4.118) we obtain that

$$S(T+t)(\Phi + \varepsilon) \leq \Phi + \varepsilon, \qquad \text{for all } \ 0 \leq t < T. \tag{4.122}$$

Let $\{t_n\}_{n\in\mathbb{N}}$ be a time sequence tending to infinity. We can assume that $t_n > T$. Then

- If $(n+1)T + t_n \geq T$, $(t_n \leq nT)$ then

  $$S((n+1)T + t_n)(\Phi + \varepsilon) \leq \Phi + \varepsilon. \tag{4.123}$$

  Applying the semigroup at time t_n on (4.123), we have that

  $$S((n+1)T)(\Phi + \varepsilon) \leq S(t_n)(\Phi + \varepsilon). \tag{4.124}$$

- If $t_n - (n-1)T \geq T$, $(t_n \geq nT)$ then

  $$S(t_n - (n-1)T)(\Phi + \varepsilon) \leq \Phi + \varepsilon \tag{4.125}$$

  Applying the semigroup at time $(n-1)T$ on (4.125), we have that

  $$S(t_n)(\Phi + \varepsilon) \leq S((n-1)T)(\Phi + \varepsilon) \tag{4.126}$$

If $t_n = nT$, we have already proved that $S(t_n)(\Phi)$ converges in $L^p(\Omega)$ to φ_M as n goes to infinity. Now, let $\{t_n\}_{n\in\mathbb{N}}$ be a general sequence, then taking limits as n goes to infinity in (4.124), we obtain that

$$\varphi_M \leq \liminf_{t_n\to\infty} S(t_n)(\Phi + \varepsilon) \quad \text{in} \quad L^p(\Omega).$$

And taking limits as n goes to infinity in (4.126), we obtain that

$$\limsup_{t_n \to \infty} S(t_n)(\Phi + \varepsilon) \leq \varphi_M \quad \text{in} \;\; L^p(\Omega).$$

Therefore,

$$\lim_{n \to \infty} S(t_n)(\Phi + \varepsilon) = \varphi_M \quad \text{in} \;\; L^p(\Omega).$$

Since the previous argument is valid for any time sequence $\{t_n\}_{n \in \mathbb{N}}$ we actually have

$$\lim_{t \to \infty} S(t)(\Phi + \varepsilon) = \varphi_M \quad \text{in} \;\; L^p(\Omega). \tag{4.127}$$

Now, we prove the result for a general initial data $u_0 \in L^\infty(\Omega)$. Thanks to (4.117), for $T = T(u_0)$

$$-\Phi - \varepsilon \leq S(t+T)(u_0) \leq \Phi + \varepsilon, \quad \forall t \geq 0. \tag{4.128}$$

Letting the semigroup act at time t in (4.128), we have

$$S(T+2t)u_0 \leq S(t)(\Phi + \varepsilon) = \overline{u}(t), \qquad \forall t \geq 0. \tag{4.129}$$

Thanks to (4.127) and (4.129), we get that

$$\lim_{t \to \infty} \overline{u}(t) = \varphi_M \quad \text{in} \;\; L^p(\Omega). \tag{4.130}$$

Finally, let ψ be another equilibrium. From (4.130), with $u_0 = \psi$, we get $\psi \leq \varphi_M$. Thus φ_M is maximal in the set of equilibrium points, i.e., for any equilibrium, ψ, we have $\psi \leq \varphi_M$. The results for φ_m can be obtained in an analogous way. □

Corollary 4.4.2. *Under the hypotheses of the previous Theorem 4.4.1. If $u_0 \in L^\infty(\Omega)$, and*

$$u_0 \geq \varphi_M,$$

then

$$\lim_{t \to \infty} S(t)(u_0) = \varphi_M,$$

in $L^p(\Omega)$, with $1 \leq p < \infty$, i.e., φ_M is "stable from above". In particular this holds for $u_0 = \Phi$.

If $u_0 \in L^\infty(\Omega)$, and $u_0 \leq \varphi_m$, then

$$\lim_{t \to \infty} S(t)(u_0) = \varphi_m, \quad \text{in} \;\; L^p(\Omega),$$

and φ_m is "stable from below".

Proof. As a consequence of Corollary 4.2.11, the associated solutions of (4.89) satisfy

$$S(t)u_0 \geq S(t)\varphi_M = \varphi_M, \quad \forall t > 0, \tag{4.131}$$

and from (4.129) and (4.131), there exists $T = T(u_0)$, such that

$$\varphi_M \leq S(T+t)u_0 \leq S(t)(\Phi + \varepsilon), \quad \forall t > 0. \tag{4.132}$$

Taking limits as $t \to \infty$ in (4.132), we obtain

$$\lim_{t \to \infty} S(t)(u_0) = \varphi_M \quad \text{in} \;\; L^p(\Omega). \tag{4.133}$$

Therefore, φ_M is "stable from above". The proof for φ_m is analogous. □

Remark 4.4.3. *If the extremal equilibria $\varphi_M \in \mathcal{C}_b(\Omega)$, then the result of the previous Theorem 4.4.1 could be improved because we would obtain the asymptotic dynamics of the solution of* (4.89) *enter between the extremal equilibria uniformly on compact sets of Ω. Thanks to Dini's Criterium (see [6, p. 194]), we have that $S(nT)(\Phi+\varepsilon)$ in* (4.121)*, converges uniformly in compact subsets of Ω to φ_M as n goes to infinity. Thus we have that*

$$\lim_{t\to\infty} S(t)(\Phi+\varepsilon) = \varphi_M \quad \text{in} \;\; L^\infty_{loc}(\Omega). \tag{4.134}$$

Since there is no regularization for the semigroup $S(t)$ associated to (4.89)*, we can not assure that $\varphi_M \in \mathcal{C}_b(\Omega)$, as happens for the local reaction diffusion equations. In fact, we give later an example of* $\mathrm{L}^\infty(\Omega)$ *discontinuous equilibria for the problem* (4.89)*, (see example 4.5.7).*

Now, we want to prove that the previous two ordered extremal equilibria, $\varphi_m, \varphi_M \in L^\infty(\Omega)$ in Theorem 4.4.1, are the bounds of any weak limit as t goes to infinity in $L^p(\Omega)$, of the solution of the problem (4.89) with initial data $u_0 \in L^p(\Omega)$.

Proposition 4.4.4. *Let (Ω, μ, d) be a metric measure space with $\mu(\Omega) < \infty$, let $X = L^p(\Omega)$, with $1 \le p < \infty$, and $h \in L^\infty(\Omega)$. We assume $K \in \mathcal{L}(L^p(\Omega), L^\infty(\Omega))$ is compact, J is nonnegative, f and J satisfy the hypotheses of Proposition 4.3.1, $C \in L^\infty(\Omega)$, $0 \le D \in L^\infty(\Omega)$, and*

$$\inf \sigma_X(-L-C) \ge \delta > 0. \tag{4.135}$$

Then there exist two ordered extremal equilibria, $\varphi_m \le \varphi_M$, in $L^\infty(\Omega)$ of (4.89)*, with initial data $u_0 \in L^p(\Omega)$, with $1 \le p < \infty$. Moreover any other equilibria ψ of* (4.89) *satisfies $\varphi_m \le \psi \le \varphi_M$, and the set*

$$\{v \in L^\infty(\Omega) : \varphi_m \le v \le \varphi_M\}$$

attracts the dynamics of the system, in the sense that for any $u_0 \in L^p(\Omega)$, if $\tilde{u}(\cdot, u_0)$ is a weak limit of $S(t)u_0$ in $L^p(\Omega)$ for $1 \le p < \infty$, when time t goes to infinity, then

$$\varphi_m(x) \le \tilde{u}(x, u_0) \le \varphi_M(x) \quad \text{for a.e. } x \in \Omega.$$

Proof. We consider as initial data, Φ, the equilibrium solution of (4.97). From Corollary 4.4.2 we have that the solution to (4.89) with initial data $\Phi \in L^\infty(\Omega)$, converges in $L^p(\Omega)$ to the maximum equilibrium $\varphi_M \in L^\infty(\Omega)$,

$$\lim_{t\to\infty} S(t)\Phi = \varphi_M \quad \text{ in } \; L^p(\Omega). \tag{4.136}$$

On the other hand, thanks to Proposition 4.3.3, we know that given an initial data $u_0 \in L^p(\Omega)$

$$S(t)u_0 = u(t, u_0) \le \Phi + e^{L_C t}(|u_0| - \Phi) \tag{4.137}$$

Applying the nonlinear semigroup $S(s)$ to (4.137), and thanks to Proposition 4.1.4, we have that

$$S(s)u(t, u_0) = u(t+s, u_0) \le \;\; S(s)(\Phi + e^{L_C t}(|u_0| - \Phi)). \tag{4.138}$$

Since the semigroup is continuous in $L^p(\Omega)$ with respect to the initial data, thanks to Proposition 4.3.3, we have the following convergence in $L^p(\Omega)$

$$\lim_{t\to\infty} S(s)(\Phi + e^{L_C t}(|u_0| - \Phi)) = S(s) \lim_{t\to\infty} (\Phi + e^{L_C t}(|u_0| - \Phi)) = S(s)\Phi. \tag{4.139}$$

Therefore, thanks to (4.138) and (4.139), and since $\{u(t+s,u_0)\}_{\{t\geq 0\}}$ is bounded in $L^p(\Omega)$, then let $\{t_n\}_{n\in\mathbb{N}}$ be a sequence that converges to infinity, such that there exists the weak limit in $L^p(\Omega)$ of $\{u(t_n+s,u_0)\}_{n\in\mathbb{N}}$ when n goes to infinity, denoted by $\tilde{u}(\cdot,u_0)$.

In (4.138) we consider $t=t_n$, we multiply (4.138) by $0\leq\psi\in L^{p'}(\Omega)$ and integrate in Ω, then

$$\int_\Omega u(x,t_n+s,u_0)\psi(x)dx \leq \int_\Omega S(s)\big(\Phi(x)+e^{L_C t_n}(|u_0(x)|-\Phi(x))\big)\psi(x)dx \tag{4.140}$$

We take limits in (4.140) when n goes to infinity, and thanks to (4.139), we have

$$\int_\Omega \tilde{u}(x,u_0)\psi(x)dx \quad \leq \int_\Omega S(s)\Phi(x)\psi(x)dx. \tag{4.141}$$

Then, we have that

$$\tilde{u}(x,u_0)\leq S(s)\Phi(x), \quad \text{for a.e.} x\in\Omega,\ \forall s>0. \tag{4.142}$$

Taking limits now, in (4.142) when s goes to infinity, thanks to (4.136)

$$\tilde{u}(x,u_0)\leq \lim_{s\to\infty} S(s)\Phi(x)=\varphi_M(x), \quad \text{for a.e.} x\in\Omega.$$

for all $u_0\in L^p(\Omega)$. Thus, the result. The reverse inequality can be proved analogously for the minimal equilibrium φ_m. □

The following proposition proves that under the hypotheses of Proposition 4.3.3, and if $f(\cdot,0)\geq 0$, then the maximal equilibria φ_M, in Theorem 4.4.1 is nonnegative. In fact, if J satisfies the hypotheses of Proposition 2.1.17, then any nontrivial nonnegative equilibria, ψ, of the problem (4.89), is strictly positive.

Proposition 4.4.5. *Let (Ω,μ,d) be a metric measure space with $\mu(\Omega)<\infty$, let $X=L^p(\Omega)$, with $1\leq p<\infty$, and $h\in L^\infty(\Omega)$. We assume $K\in\mathcal{L}(L^p(\Omega),L^\infty(\Omega))$ is compact, J is nonnegative, $C\in L^\infty(\Omega)$, $0\leq D\in L^\infty(\Omega)$, and*

$$\inf\sigma_X(-L-C)\geq\delta>0. \tag{4.143}$$

If f and J satisfy the hypotheses of Proposition 4.3.1, and f satisfies also that

$$f(\cdot,0)\geq 0,$$

then the extremal equilibria of (4.89), $\varphi_M\geq 0$.

Furthermore, if J satisfies that

$$J(x,y)>0,\ \forall x,\ y\in\Omega \text{ such that } d(x,y)<R, \tag{4.144}$$

for some $R>0$, and Ω is R-connected, (see Definition 2.1.14), then any nontrivial nonnegative equilibria ψ of (4.89) is in fact strictly positive.

Proof. Since $f(\cdot,0) \geq 0$, then 0 is a subsolution of (4.89). Under any of the hypotheses on f in Proposition 4.3.1, thanks to Corollary 4.1.9, Corollary 4.2.8 and Corollary 4.2.14, respectively, we know that the subsolutions of the problem (4.89) are below the solution, $u(\cdot,t,u_0)$, of the problem (4.89), as long as the subsolution exists. Thus, we have that if $u_0 \geq 0$, then

$$0 \leq u(x,t;u_0), \quad \forall x \in \Omega,\ t \geq 0. \tag{4.145}$$

From Proposition 4.3.3, we know that the solution of (4.103), $\Phi \in L^\infty(\Omega)$, satisfies that $\Phi \geq 0$.

Thanks to (4.145), since $\Phi \geq 0$, then the solution associated to the initial datum Φ satisfies that

$$u(\cdot,t,\Phi) \geq 0,\ \forall t \geq 0. \tag{4.146}$$

From Corollary 4.4.2

$$\lim_{t\to\infty} u(\cdot,t,\Phi) = \varphi_M, \quad \text{in} \quad L^p(\Omega). \tag{4.147}$$

Taking limits as t goes to infinity in (4.146), and from (4.147), we have

$$\varphi_M \geq 0.$$

Hence, φ_M is nonnegative.

Moreover, if ψ is a nonnegative equilibria of (4.89) and if J satisfies (4.144), then thanks to Corollary 4.2.13 then

$$\psi = u(\cdot,t,\psi) > 0 \quad \forall t > 0.$$

Thus, the result. $\square$

Under the hypotheses of Proposition 4.4.5, the following proposition states that if f satisfies (4.149) and J satisfies (4.144), then there exists a unique nontrivial nonnegative equilibria.

Proposition 4.4.6. *Let (Ω,μ,d) be a metric measure space with $\mu(\Omega) < \infty$, let $X = L^p(\Omega)$, with $1 \leq p < \infty$, and $h \in L^\infty(\Omega)$. We assume $K \in \mathcal{L}(L^p(\Omega), L^\infty(\Omega))$ is compact, J is nonnegative, $C \in L^\infty(\Omega)$, $0 \leq D \in L^\infty(\Omega)$, and*

$$\inf \sigma_X(-L - C) \geq \delta > 0. \tag{4.148}$$

If f and J satisfy the hypotheses of Proposition 4.3.1, and f satisfies also that

$$f(\cdot,0) \geq 0,$$

and

$$\frac{f(x,s)}{s} \ \text{is monotone in the variable } s,\ \forall x \in \Omega, \tag{4.149}$$

and $J(x,y) = J(y,x)$ satisfies (4.144).

Then there exists a unique nontrivial nonnegative equilibrium, φ_M, of (4.89), and the equilibrium is strictly positive.

Proof. From Theorem 4.4.1, let $\varphi_M \in L^\infty(\Omega)$ be the maximal equilibria of (4.89). Now, assume that ψ is another nontrivial nonnegative equilibria, then $\psi \leq \varphi_M$. Thus, $\psi \in L^\infty(\Omega)$. Since $f(\cdot, 0) \geq 0$, and thanks to Proposition 4.4.5, $\psi > 0$ and $\varphi_M > 0$.

On the other hand, ψ and φ_M are equilibria, then they satisfy

$$\int_\Omega J(x,y)\varphi_M(y)dy - h(x)\varphi_M(x) + f(x,\varphi_M) = 0 \tag{4.150}$$

$$\int_\Omega J(x,y)\psi(y)dy - h(x)\psi(x) + f(x,\psi) = 0 \tag{4.151}$$

We have that φ_M and ψ belong to $L^\infty(\Omega) \subset L^p(\Omega)$, for $1 \leq p \leq \infty$, then multiplying (4.150) by ψ, and (4.151) by φ_M, and integrating in Ω,

$$\int_\Omega\int_\Omega J(x,y)\varphi_M(y)dy\psi(x)dx - \int_\Omega h(x)\varphi_M(x)\psi(x)dx + \int_\Omega f(x,\varphi_M)\psi(x)dx = 0, \tag{4.152}$$

$$\int_\Omega\int_\Omega J(x,y)\psi(y)dy\varphi_M(x)dx - \int_\Omega h(x)\varphi_M(x)\psi(x)dx + \int_\Omega f(x,\psi)\varphi_M(x)dx = 0, \tag{4.153}$$

We substract (4.153) from (4.152). Then, we obtain

$$\begin{aligned}&\int_\Omega\int_\Omega J(x,y)\varphi_M(y)dy\psi(x)dx - \int_\Omega\int_\Omega J(x,y)\psi(y)dy\varphi_M(x)dx + \\ &+ \int_\Omega \frac{f(x,\varphi_M)}{\varphi_M(x)}\varphi_M(x)\psi(x)dx - \int_\Omega \frac{f(x,\psi)}{\psi(x)}\varphi_M(x)\psi(x)dx = 0.\end{aligned}$$

Relabeling variables in the first term, we have

$$\begin{aligned}&\int_\Omega\int_\Omega J(y,x)\varphi_M(x)dx\psi(y)dy - \int_\Omega\int_\Omega J(x,y)\psi(y)dy\varphi_M(x)dx + \\ &+ \int_\Omega \frac{f(x,\varphi_M)}{\varphi_M(x)}\varphi_M(x)\psi(x)dx - \int_\Omega \frac{f(x,\psi)}{\psi(x)}\varphi_M(x)\psi(x)dx = 0.\end{aligned}$$

Since $J(x,y) = J(y,x)$, we obtain

$$\int_\Omega \left(\frac{f(x,\varphi_M)}{\varphi_M(x)} - \frac{f(x,\psi)}{\psi(x)}\right)\varphi_M(x)\psi(x)dx \quad = 0$$

Moreover, $\dfrac{f(x,s)}{s}$ is monotone in the variable s, then, we have that $\dfrac{f(x,\varphi_M)}{\varphi_M(x)} - \dfrac{f(x,\psi)}{\psi(x)} \leq 0$ on sets with positive measure. Moreover φ_M and ψ are strictly positive. Hence, $\psi = \varphi_M$. Thus, the result. □

Remark 4.4.7. *Let e^{Lt} be the linear semigroup. We know from Theorem 3.3.4, that e^{Lt} is asymptotically smooth. The nonlinear semigroup associated to* (4.89) *is denoted by $S(t)$, and given by*

$$S(t)u_0(x) = e^{Lt}u_0(x) + \int_0^t e^{L(t-s)}f(x,u(x,s))\,ds. \tag{4.154}$$

If $f : L^p(\Omega) \to L^1(\Omega)$, was compact, (which is not), since $e^{L(t-s)}$ is a continuous operator, then $\int_0^t e^{L(t-s)}f(\cdot,u(s))ds$ would be compact, and we would be able to apply [32, Lemma 3.2.3.], to prove that $S(t)$ is asymptotically smooth. But, due to the lack of smoothness of the semigroup e^{Lt}, the semigroup $S(t)$ is not asymptotically smooth in general.

4.5 Instability results for nonlocal reaction diffusion problem

Let (Ω, μ, d) be a metric measure space with $\mu(\Omega) < \infty$. Let $X = L^p(\Omega)$, with $1 \le p \le \infty$. We shall deal with the following nonlocal reaction-diffusion problem, that is the nonlinear problem 4.1 with $h = h_0$, and reaction term f, depending only on u,

$$\begin{cases} u_t(x,t) = \displaystyle\int_\Omega J(x,y)\,(u(y,t) - u(x,t))\,dy + f(u(x,t)), & x \in \Omega,\ t > 0 \\ u(x,0) = u_0(x), & x \in \Omega, \end{cases} \tag{4.155}$$

where we assume $J(x,y) = J(y,x)$, $J \ge 0$, $h_0 = \int_\Omega J(\cdot,y)dy \in L^\infty(\Omega)$, and $f : \mathbb{R} \to \mathbb{R}$, $f \in \mathcal{C}^1(\mathbb{R})$ is globally Lipschitz.

The equation (4.155) can be rewritten as

$$u_t = (K - h_0 I)u + f(u).$$

It shall be shown that if f is convex or concave then any continuous nonconstant solution of (4.155) is, if it exists, unstable, in some sense to be made precise below.

We first introduce a concept of Lyapunov stability with respect to the norm in X.

Definition 4.5.1. *Let $u(x,t,u_0)$ be the solution to* (4.155) *with initial data $u_0 \in X$. An equilibrium solution $\overline{u}$ is* **Lyapunov stable** *is for each $\varepsilon > 0$, there exists $\delta > 0$ such that, if $u_0 \in X$ and $\|u_0 - \overline{u}\|_X < \delta$, then $\|u(\cdot,t,u_0) - \overline{u}\|_X < \varepsilon$, $\forall t > 0$. An equilibrium is unstable if it is not stable.*

Let us define the concept of instability defined with respect to linearization of the problem (4.155). The linearization of (4.155) around the equilibrium $\bar{u}$ is given by

$$\begin{cases} \varphi_t(x,t) & = \displaystyle\int_\Omega J(x,y)\,(\varphi(y,t) - \varphi(x,t))\,dy + f'(\overline{u}(x))\varphi(x,t), & x \in \Omega,\ t > 0 \\ \varphi(x,0) & = \varphi_0(x), & x \in \Omega. \end{cases} \tag{4.156}$$

Since f is globally Lipchitz, then $f'(\bar{u}) \in L^\infty(\Omega)$.

Definition 4.5.2. *The equilibrium $\overline{u}$ is* **stable with respect to linearization** *if for each initial data $\varphi_0 \in X$, the solution of* (4.156) *satisfies*

$$\sup_{t>0} \|\varphi(t,\varphi_0)\|_X < \infty.$$

The equilibrium $\overline{u}$ is **asymptotically stable with respect to linearization** *if it is stable and if for any initial data $\varphi_0 \in X$, the solution of* (4.156) *satisfies*

$$\lim_{t\to\infty} \varphi(x,t,\varphi_0) = 0 \quad \text{in } X.$$

The equilibrium $\overline{u}$ is **unstable with respect to linearization** *if there exists an initial data $\varphi_0 \in X$, such that the solution of* (4.156) *satisfies*

$$\sup_{t>0} \|\varphi(t,\varphi_0)\|_X = +\infty.$$

If $F : X \to X$ is $\mathcal{C}^1$, then the stability from linearization implies the stability in the sense of Lyapunov, (see [33, p. 266]). In fact, let $A \in \mathcal{L}(X, X)$, and let $F : X \to X$ be $\mathcal{C}^1$. We consider the problem

$$u_t = Au + F(u) \quad u_0 \in X. \tag{4.157}$$

Let $\bar{u}$ be the equilibrium of (4.157), then

$$A\bar{u} + F(\bar{u}) = 0 \tag{4.158}$$

We rewrite (4.157) as follows

$$u_t = Au + F(u) = A(u - \bar{u}) + A\bar{u} + F(\bar{u}) + DF(\bar{u})(u - \bar{u}) + g(u - \bar{u}), \tag{4.159}$$

with $g(u - \bar{u}) = \|F(u) - F(\bar{u}) - DF(\bar{u})(u - \bar{u})$.

If we consider $v = u - \bar{u}$, then v satisfies

$$v_t = Av + DF(\bar{u})v + g(v). \tag{4.160}$$

Since $F : X \to X$ is $\mathcal{C}^1$, then $\|g(v)\|_X = o(\|v\|_X)$. Let us consider the linearization of the problem (4.157) around the equilibrium $\bar{u}$.

$$\varphi_t = A\varphi + DF(\bar{u})\varphi. \tag{4.161}$$

In fact, if $\|u - \bar{u}\|_X << 1$ then $\|g(u - \bar{u})\|_X << 1$, and the problems (4.160) and (4.161) are almost equal. Therefore, if $F : X \to X$ is $\mathcal{C}^1$, the stability of the equilibrium of (4.157) can be studied in terms of the stability of the linearized problem (4.161).

Hence, we have that the stability from linearization implies the stability in the sense of Lyapunov.

On the other hand, we know that the Nemitcky operator associated to $f \in \mathcal{C}^1(\mathbb{R})$, $F : L^p(\Omega) \to L^p(\Omega)$ is Lipschitz, but it is not $\mathcal{C}^1$, unless it is linear, (see Appendix B). Hence $F : L^p(\Omega) \to L^p(\Omega)$ Lipschitz is not differentiable. The following result gives conditions on f under which the stability/instability respect to the linearization (4.156) implies the stability/instability in the sense of Lyapunov, even if $F : L^p(\Omega) \to L^p(\Omega)$ is not differentiable.

Proposition 4.5.3. *Let (Ω, μ, d) be a metric measure space, with $\mu(\Omega) < \infty$, $X = L^p(\Omega)$, with $1 \leq p \leq \infty$, and $K_J \in \mathcal{L}(X, X)$, $h_0 \in L^\infty(\Omega)$, and let $a < c < d < b$. We assume J nonnegative, $f \in C^2(\mathbb{R})$, nonlinear and globally Lipschitz, and $\bar{u} \in L^\infty(\Omega)$ is an equilibrium solution of* (4.155) *with values in $[c, d]$.*

i. If $f'' > 0$ in $[a, b]$, and the equilibrium $\bar{u}$ is unstable with respect to the linearization then $\bar{u}$ is unstable in the sense of Lyapunov in X.

ii. Let $X = L^p(\Omega)$, with $1 \leq p \leq \infty$, let $K \in \mathcal{L}(L^p(\Omega), L^\infty(\Omega))$ be compact and $h_0 \in L^\infty(\Omega)$. If $f'' < 0$ in $[a, b]$, $\sigma_X\big(K - (h_0 - f'(\bar{u}))I\big) \leq -\delta < 0$, and the equilibrium $\bar{u}$ is asymptotically stable respect to the linearization then $\bar{u}$ is stable from above, in the sense that, if an initial datum u_0 takes values in $[a, b]$ and satisfies that $u_0 \geq \bar{u}$, then the solution of (4.155) *with initial datum u_0 converges to $\bar{u}$ in X when time goes to infinity.*

Proof. Let z be the solution of the nonlinear problem

$$\begin{cases} z_t(x,t) &= \displaystyle\int_\Omega J(x,y)(z(y,t)-z(x,t))dy + f(z(x,t)), \quad x\in\Omega,\ t>0. \\ z(x,0) &= u_0(x), \quad x\in\Omega. \end{cases} \tag{4.162}$$

with $u_0 \in L^\infty(\Omega)$, and let $\bar{u}$ be the equilibrium solution of (4.162), then

$$\int_\Omega J(x,y)(\bar{u}(x)-\bar{u}(y))dy + f(\bar{u}(x)) = 0. \tag{4.163}$$

Let us consider the linearization of (4.162) around $\bar{u}$,

$$\begin{cases} \varphi_t(x,t) &= \displaystyle\int_\Omega J(x,y)\,(\varphi(y,t)-\varphi(x,t))\,dy + f'(\bar{u}(x))\varphi(x,t) = \widetilde{L}(\varphi)(x,t), \quad x\in\Omega,\ t>0, \\ \varphi(x,0) &= \varphi_0(x), \quad x\in\Omega, \end{cases} \tag{4.164}$$

then we consider

$$z(x,t) = \bar{u}(x) + v(x,t). \tag{4.165}$$

From (4.165) and (4.162), v satisfies

$$\begin{cases} v_t(x,t) &= \displaystyle\int_\Omega J(x,y)\,(v(y,t)-v(x,t))\,dy + \int_\Omega J(x,y)\,(\bar{u}(y)-\bar{u}(x))\,dy + f\big(\bar{u}(x)+v(x,t)\big). \\ v(x,0) &= u_0(x)-\bar{u}(x). \end{cases} \tag{4.166}$$

i. Since $f'' > 0$ in $[a,b]$, we have that f satisfies that

$$f(\bar{u}+v) \geq f(\bar{u}) + f'(\bar{u})v, \tag{4.167}$$

for v small enough such that $\bar{u}+v$ takes values in $[a,b]$.

Applying inequality (4.167) to (4.166), we obtain that

$$v_t(x,t) \geq \int_\Omega J(x,y)(v(y,t)-v(x,t))\,dy + \int_\Omega J(x,y)(\bar{u}(y)-\bar{u}(x))\,dy + f(\bar{u}(x)) + f'(\bar{u}(x))v(x,t), \tag{4.168}$$

for all t such that $\bar{u}+v(t)$ takes values in $[a,b]$.

Since $\bar{u}$ is an equilibrium, it satisfies equality (4.163), then

$$v_t(x,t) \geq \int_\Omega J(x,y)\,(v(y,t)-v(x,t))\,dy + f'(\bar{u}(x))v(x,t). \tag{4.169}$$

for all t such that $\bar{u}+v(t)$ takes values in $[a,b]$.

If $v(0) \geq \varphi_0$, then v is a supersolution of (4.164), and from Proposition 4.1.9

$$v(x,t) \geq \varphi(x,t), \quad x\in\Omega, \quad \text{for } \ t>0 \text{ such that } \ \bar{u}+v(t) \ \text{ takes values in } \ [a,b]. \tag{4.170}$$

Since $\bar{u}$ is unstable with respect to the linearization, then we prove below that there exists $\varphi_0 \in L^\infty(\Omega)$, with $\varphi_0 > 0$ such that

$$\sup_{t>0} \|\varphi(t,\varphi_0)\|_X = +\infty. \tag{4.171}$$

Let us prove that there exists $\varphi_0 > 0$ that satifies (4.171). First, we argue by contradiction in $X = L^\infty(\Omega)$, then for all $\varphi_0 \in L^\infty(\Omega)$ with $\varphi_0 \geq 0$, we have that $\sup_{t>0} \|\varphi(t,\varphi_0)\|_{L^\infty(\Omega)} < \infty$, and since (4.164) is a linear problem, we have that for all $\varphi_0 \leq 0$, $\sup_{t>0} \|\varphi(t,\varphi_0)\|_{L^\infty(\Omega)} < \infty$. Hence, for any initial data $\varphi_0 = \varphi_0^+ - \varphi_0^-$, it happens that $\varphi(t,\varphi_0) = \varphi(t,\varphi_0^+) - \varphi(t,\varphi_0^-)$, and $\|\varphi(t,\varphi_0)\|_{L^\infty(\Omega)} < \infty$. Arguing by density we obtain that $\|\varphi(t,\varphi_0)\|_X < \infty$. Thus, we arrive to contradiction with the fact that $\bar{u}$ is unstable with respect to the linearization.

Thanks to Proposition 4.1.6 we have that if $\varphi_0 \geq 0$, then $\varphi(x,t,\varphi_0) \geq 0$, for all $x \in \Omega$ and $t > 0$. Since φ is nonnegative and from (4.170) and (4.171) we have that

$$C\|v(t)\|_{L^\infty(\Omega)} \geq \|v(t)\|_X \geq \|\varphi(t,\varphi_0)\|_X, \tag{4.172}$$

for all t such that $\bar{u} + v(t)$ takes values in $[a,b]$.

On the other hand from (4.171), for all $\delta > 0$, there exists $\mu > 0$ such that $\|\mu\varphi_0\| < \delta$, and there exists t_0 such that

$$\|\varphi(t_0,\mu\varphi_0)\|_X \geq \max\{|a|,|b|\}. \tag{4.173}$$

Hence, thanks to (4.172) and (4.173), for all $\delta > 0$, if we choose $v(0) = \mu\varphi_0$ as above, then $\|v(0)\|_X = \|u_0 - \bar{u}\|_X < \delta$, and there exists $t_0 > 0$ such that

$$\|z(t_0) - \bar{u}\|_X = \|v(t_0)\|_X \geq \|\varphi(t_0,\varphi_0)\|_X \geq \max\{|a|,|b|\}.$$

Hence, the equilibrium $\bar{u}$ is Lyapunov unstable.

ii. Since $f'' < 0$ in $[a,b]$, we have that f satisfies that

$$f(\bar{u} + v) \leq f(\bar{u}) + f'(\bar{u})v, \tag{4.174}$$

for v small enough such that $\bar{u}+v$ takes values in $[a,b]$. Applying inequality (4.174) to (4.166), we obtain that

$$v_t(x,t) \leq \int_\Omega J(x,y)(v(y,t) - v(x,t))\,dy + \int_\Omega J(x,y)(\bar{u}(y) - \bar{u}(x))\,dy + f(\bar{u}(x)) + f'(\bar{u}(x))v(x,t), \tag{4.175}$$

for all t such that $\bar{u} + v(t)$ takes values in $[a,b]$.

Since $\bar{u}$ is an equilibrium, it satisfies equality (4.163), then from (4.175) we have

$$v_t(x,t) \leq \int_\Omega J(x,y)\,(v(y,t) - v(x,t))\,dy + f'(\bar{u}(x))v(x,t), \tag{4.176}$$

for all t such that $\bar{u} + v(t)$ takes values in $[a,b]$. Thus, if $v(0) \leq \varphi_0$, then v is a subsolution of (4.164), and from Proposition 4.1.9

$$v(x,t) \leq \varphi(x,t), \quad \text{for} \quad t > 0 \text{ such that } \ \bar{u} + v(t) \ \text{ takes values in } \ [a,b]. \tag{4.177}$$

Since we want to prove the stability from above, we consider an initial datum $u_0 \geq \bar{u}$, then thanks to Proposition 4.2.11, we know that $z(x,t,u_0) \geq z(x,t,\bar{u}) = \bar{u}(x)$ for all $x \in \Omega$ for all $t > 0$, then $v(x,t,u_0 - \bar{u}) = z(x,t,u_0) - \bar{u}(x) \geq 0$ for all $x \in \Omega$ for all $t > 0$.

Let us prove that under the hypotheses in the statement, $\bar{u}+v(t)$ takes values in $[a,b]$ for all $t \geq 0$. If $\varphi_0 \geq 0$, thanks to Proposition 3.2.2, we have that $\varphi(t,\varphi_0) \geq 0$, for all $t \geq 0$. Moreover, from (4.177) and since $v(t, u_0-\bar{u}) \geq 0$ for all $t \geq 0$, we have

$$\bar{u} \leq \bar{u}+v(t) \leq \bar{u}+\varphi(t), \quad \text{for } \ t>0 \text{ such that } \ \bar{u}+v(t) \ \text{ takes values in } \ [a,b] \tag{4.178}$$

Moreover, from (4.178), we have that $a \leq \bar{u}+v(t) \leq b$, for all $t \geq 0$, if $\bar{u}+\varphi(t) \leq b$, for all $t \geq 0$, i.e., if $\|\varphi(t)\|_{L^\infty(\Omega)} \leq b - \inf \bar{u}(x) = b-d$, for all $t \geq 0$. Thanks to Proposition 2.4.5, $\sigma_X(K-(h_0-f'(\bar{u}))I)$ is independent of X. Moreover, since $\sigma_X(K-(h_0-f'(\bar{u}))I) \leq -\delta < 0$ and thanks to Proposition 3.4.2, then

$$\|\varphi(t,\varphi_0)\|_{L^\infty(\Omega)} \leq C_0 e^{-\delta t}\|\varphi_0\|_{L^\infty(\Omega)} \leq C_0\|\varphi_0\|_{L^\infty(\Omega)}, \qquad \text{for all } \ t \geq 0.$$

Hence if we choose an initial datum $\varphi_0 \in L^\infty(\Omega)$, such that $C_0\|\varphi_0\|_{L^\infty(\Omega)} \leq b-d$, then $\|\varphi(t,\varphi_0)\|_{L^\infty(\Omega)} \leq b-d$ for all $t \geq 0$. Thus, $a \leq \bar{u}+v(t) \leq b$, for all $t \geq 0$, and thanks to (4.177), we obtain that

$$v(x,t) \leq \varphi(x,t), \quad \text{for all } \ t \geq 0. \tag{4.179}$$

Furthermore, since $\bar{u}$ is asymptotically stable with respect to the linearization, then for any initial data $\varphi_0 \in L^\infty(\Omega)$

$$\lim_{t\to\infty} \|\varphi(\cdot,t,\varphi_0)\|_{L^\infty(\Omega)} = 0. \tag{4.180}$$

If we choose an initial data small enough such that $v(0) \leq \varphi_0$, with φ_0 satisfying $C_0\|\varphi_0\|_{L^\infty(\Omega)} \leq b-d$, then from (4.179), (4.180), and since $v(t) \geq 0$, we have

$$\lim_{t\to\infty} \|v(\cdot,t,v(0))\|_{L^\infty(\Omega)} = 0 \tag{4.181}$$

Furthermore, since $z(x,t) = \bar{u}(x)+v(x,t)$ then z converges to $\bar{u}$ in $L^\infty(\Omega)$ when t goes to ∞.

Since we have the convergence in $L^\infty(\Omega)$, and $\mu(\Omega) < \infty$, we have also the convergence in X. Thus, the result. □

In the following result we give a criterium to prove that an equilibrium $\bar{u}$ is unstable with respect to the linearization.

Theorem 4.5.4. *Let (Ω,μ,d) be a metric measure space, with $\mu(\Omega)<\infty$. For $1 \leq p_0 \leq 2$, let $X = L^p(\Omega)$, with $p_0 \leq p \leq \infty$, and we assume $K \in \mathcal{L}(L^{p_0}(\Omega), L^\infty(\Omega))$ is compact, J nonnegative, $f \in \mathcal{C}^2(\mathbb{R})$ nonlinear and globally Lipschitz, and $\overline{u} \in L^\infty(\Omega)$ is an equilibrium of (4.155). For $\varphi \in L^2(\Omega)$, we define*

$$I(\varphi) = -\frac{1}{2}\int_\Omega\int_\Omega J(x,y)(\varphi(y)-\varphi(x))^2 dy\, dx + \int_\Omega f'(\overline{u}(x))\varphi^2(x)dx.$$

If there exists $\overline{\varphi} \in L^2(\Omega)$ such that $I(\overline{\varphi}) > 0$, then $\overline{u}$ is unstable with respect to linearization in X.

Proof. Multiplying (4.156) by φ and integrating in Ω, we obtain

$$\int_\Omega \varphi_t(x,t)\varphi(x,t)dx \ = \int_\Omega\int_\Omega J(x,y)\left(\varphi(y,t)-\varphi(x,t)\right)dy\varphi(x,t)dx + \int_\Omega f'(\overline{u}(x))\varphi^2(x,t)dx.$$

Thanks to Proposition 2.3.1 (Green's formula),

$$\frac{\partial}{\partial t}\int_\Omega \frac{\varphi^2(x,t)}{2}dx \quad = -\frac{1}{2}\int_\Omega\int_\Omega J(x,y)(\varphi(y,t)-\varphi(x,t))^2 dy\, dx + \int_\Omega f'(\overline{u}(x))\varphi^2(x,t)dx$$

Thus, we denote

$$I(\varphi) = -\frac{1}{2}\int_\Omega\int_\Omega J(x,y)(\varphi(y)-\varphi(x))^2 dy\, dx + \int_\Omega f'(\overline{u}(x))\varphi^2(x)dx.$$

Now we assume that there exists $\overline{\varphi}$ such that $I(\overline{\varphi}) > 0$. We define

$$\lambda^* = \sup_{\varphi\in L^2,\, \|\varphi\|=1} I(\varphi) \geq I(\overline{\varphi}) > 0.$$

Thanks to Proposition 2.1.21, $\lambda^* > 0$ belongs to the spectrum of $\widetilde{L} = K - (h_0 - f'(\bar{u}))I$ in $L^2(\Omega)$. Moreover, thanks to the hypotheses, and Proposition 2.4.5, $\lambda^* \in \sigma_X(\widetilde{L})$.

Now we prove that $\bar{u}$ is unstable with respect to the linearization. We argue by contradiction, we assume that $\bar{u}$ is stable with respect to the linearization, then for any $\varphi_0 \in X$, $\|\varphi(t,\varphi_0)\|_X < \infty$, for all $t \geq 0$, i.e. for any $\varphi_0 \in X$, $\|e^{\widetilde{L}t}\varphi_0\|_X \leq M(\varphi_0)$, for all $t \geq 0$, then applying Banach-Steinhaus Theorem to the family of operators $\{e^{\widetilde{L}t}\}_{t\geq 0}$, we have that there exists $M \geq 0$ such that $\|e^{\widetilde{L}t}\|_{\mathcal{L}(X,X)} \leq M$ for all $t \geq 0$. Hence, for all $\varepsilon > 0$,

$$\|e^{(\widetilde{L}-\varepsilon)t}\|_{\mathcal{L}(X,X)} \leq e^{-\varepsilon t}M \tag{4.182}$$

Furthermore for all $\lambda > 0$, the resolvent can be written as follows, (see [24, p. 614]),

$$\left((\lambda+\varepsilon)I - \widetilde{L}\right)^{-1} = \int_0^\infty e^{(\widetilde{L}-\varepsilon)t}e^{-\lambda t}dt. \tag{4.183}$$

Therefore, from (4.182) and (4.183) we have that

$$\|(\lambda I + \varepsilon I - \widetilde{L})^{-1}\|_{\mathcal{L}(X,X)} \leq \int_0^\infty \|e^{(\widetilde{L}-\varepsilon)t}e^{-\lambda t}\|_{\mathcal{L}(X,X)}dt \leq M\int_0^\infty e^{(-\lambda-\varepsilon)t}dt = M\frac{1}{\lambda+\varepsilon}.$$

Then $(\lambda I + \varepsilon I - \widetilde{L})^{-1} \in \mathcal{L}(X,X)$ for all $\lambda > 0$. Then $\{\lambda \in \mathbb{R}^+ : \ \lambda > \varepsilon\} \subset \rho_X(\widetilde{L})$ for all $\varepsilon > 0$. Hence, $\mathbb{R}^+ \subset \rho_X(\widetilde{L})$, and we arrive to contradiction with the fact that $\lambda^* \in \sigma_X(\widetilde{L})$ and $\lambda^* > 0$. Therefore, $\bar{u}$ is unstable with respect to the linearization. $\square$

The zeros of f are the constant equilibriums of (4.155), and thanks to the criterium of the previous Theorem 4.5.4, we have that a constant equilibrium $\bar{u}$ is unstable if there exists φ such that

$$I(\varphi) = -\frac{1}{2}\int_\Omega\int_\Omega J(x,y)(\varphi(y)-\varphi(x))^2 dy\, dx + \int_\Omega f'(\overline{u}(x))\varphi^2(x)dx > 0.$$

Thus, if we take φ constant, we will have that

$$I(\varphi) = \int_\Omega f'(\overline{u})\varphi^2 dx.$$

Therefore, a constant equilibrium $\bar{u}$ is unstable with respect to the linearization if

$$f'(\overline{u}) > 0.$$

In the Theorem below, we find conditions guaranteeing that for a nonconstant equilibrium $\overline{u}$, there exists $\overline{\varphi}$ such that $I(\overline{\varphi}) > 0$. The instability results depend on the function f.

First we make the observation that if u is a nonconstant equilibrium such that

$$\int_\Omega f'(u(x))dx > 0$$

then u is unstable with respect to the linearization (4.156). This follows from the fact that $I(\varphi) > 0$ for $\varphi \equiv 1$.

The following result states that if the function f is strictly convex or strictly concave, then any continuous and bounded nonconstant solution is unstable with respect to the linearization.

Theorem 4.5.5. *Let (Ω, μ, d) be a metric measure space, with $\mu(\Omega) < \infty$. For $1 \le p_0 \le 2$, let $X = L^p(\Omega)$, with $p_0 \le p \le \infty$, and let $a < c < d < b$. We assume $K \in \mathcal{L}(L^{p_0}(\Omega), L^\infty(\Omega))$ is compact, J nonnegative, $f \in \mathcal{C}^2(\mathbb{R})$ nonlinear and globally Lipschitz. Let $\bar{u} \in \mathcal{C}_b(\Omega)$ be a nonconstant equilibrium solution of* (4.155) *with values in $[c, d]$. If either $f'' > 0$ on $[a, b]$ or $f'' < 0$ on $[a, b]$, then u is unstable with respect to the linearization in X.*

Proof. Consider first the case $f'' > 0$. Let $c = \inf\limits_{x\in\Omega} \bar{u}(x)$, then we establish instability by showing that $I(\bar{u} - c) > 0$, and applying Theorem 4.5.4. Now

$$I(\bar{u} - c) = -\frac{1}{2}\int_\Omega\int_\Omega J(x, y)(\bar{u}(y) - \bar{u}(x))^2 dy\, dx + \int_\Omega f'(\bar{u}(x))(\bar{u}(x) - c)^2 dx. \tag{4.184}$$

Since $\bar{u}$ is an equilibrium solution of (4.155),

$$\int_\Omega J(x, y)\bar{u}(y)dy - \int_\Omega J(x, y)dy\, \bar{u}(x) + f(\bar{u}(x)) = 0. \tag{4.185}$$

Integrating (4.185) in Ω

$$\begin{aligned} 0 &= \int_\Omega\int_\Omega J(x, y)\bar{u}(y)dy - \int_\Omega J(x, y)dy\, \bar{u}(x)dx + \int_\Omega f(\bar{u}(x))dx \\ -\int_\Omega f(\bar{u}(x))dx &= \int_\Omega\int_\Omega J(x, y)\bar{u}(y)dy\, dx - \int_\Omega\int_\Omega J(x, y)\bar{u}(x)dy\, dx. \end{aligned}$$

Since $J(x, y) = J(y, x)$, and relabeling variables, we get

$$\int_\Omega f(\bar{u}(x))dx = \int_\Omega\int_\Omega J(x, y)\bar{u}(y)dy\, dx - \int_\Omega\int_\Omega J(y, x)\bar{u}(y)dy\, dx = 0. \tag{4.186}$$

Now, multiplying (4.185) by $\bar{u}$, integrating in Ω and thanks to Proposition 2.3.1, (Green's formula), we obtain

$$\int_\Omega f(\bar{u}(x))\bar{u}(x)\, dx = \frac{1}{2}\int_\Omega\int_\Omega J(x, y)(\bar{u}(y) - \bar{u}(x))^2 dy\, dx. \tag{4.187}$$

From (4.184), (4.186) and (4.187),

$$I(\bar{u}-c) = -\int_\Omega (\bar{u}(x)-c)\left[f(\bar{u}(x)) - f'(\bar{u}(x))(\bar{u}(x)-c)\right] dx. \tag{4.188}$$

Now, we prove that $f(c) \leq 0$. Since $\bar{u} \in \mathcal{C}_b(\Omega)$ is an equilibrium solution, $\bar{u}$ satisfies the equality (4.185), and considering that $\bar{x} \in \{x : \bar{u}(x) = c\}$, then

$$f(c) = \int_\Omega J(\bar{x}, y)(c - \bar{u}(y))dy \leq 0.$$

From the condition on f'' we have that if $\bar{u}(x) \neq c$, then

$$f(c) > f(\bar{u}(x)) + f'(\bar{u}(x))(c - \bar{u}(x)).$$

Since $f(c) \leq 0$, then $0 > f(\bar{u}(x)) - f'(\bar{u}(x))(\bar{u}(x) - c)$. Moreover, since $\bar{u}$ is nonconstant, if $\bar{u}(x) > c = \inf\limits_{x\in\Omega} \bar{u}(x)$, then $I(\bar{u}-c)$, given by (4.188), satisfies that $I(\bar{u}-c) > 0$.

The proof of the case when $f'' < 0$ follows in a similar argument except now we take $c = \max\limits_{x\in\overline{\Omega}} \bar{u}(x)$ and note that when $f'' < 0$, we will have $f(c) \geq 0$. □

Corollary 4.5.6. *Under the hypotheses of Theorem 4.5.5. Let $\bar{u} \in \mathcal{C}_b(\Omega)$ be a nonconstant equilibrium solution of* (4.155) *with values in* $[c, d]$*. If f satisfies that $f'' > 0$ on $[a, b] \supset [c, d]$, then $\bar{u}$ is unstable in the sense of Lyapunov.*

Proof. From Theorem 4.5.5, we know that if $f'' > 0$, then the nonconstant equilibrium $\bar{u}$ is unstable with respect to linearization. And thanks to Proposition 4.5.3, if $f'' > 0$, and $\bar{u}$ is unstable with respect to linearization, then it is unstable in the sense of Lyapunov. Thus, the result. □

Remark 4.5.7. (Example of non-isolated and discontinuous equilibria) *We construct a particular example of the problem* (4.155)*, in which we give an explicit expression for non-isolated and discontinuous equilibria. This is different from the local problem, since for the local reaction-diffusion problem the equilibria are continuous, thanks to the regularization of the semigroup associated to $-\Delta$.*

If we choose $J(x, y) = 1$, for all x, $y \in \Omega$, and $f(u) = \lambda u(u^2 - 1)$, then the equilibria of (4.155) *satisfy*

$$\int_\Omega J(x, y)(u(y) - u(x))dy + f(u(x)) = 0,$$

then

$$\int_\Omega u(y)dy = \mu(\Omega)u - \lambda u(u^2 - 1). \tag{4.189}$$

The left-hand side of (4.189) *is*

$$\int_\Omega u(y)dy = A, \text{ with } A \in \mathbb{R}$$

We denote the right-hand side of (4.189) *by*

$$g(u) = \mu(\Omega)u - \lambda u(u^2 - 1).$$

Hence, given A, we take the solutions u of $g(u) = A$.
In figure 4.1, we can see a particular example, in which there are three different roots, that satisfy $g(u) = A$, and we denote them by u_1, u_2, u_3. If the divide the set Ω in three arbitrary subsets Ω_1, Ω_2, Ω_3, then we can construct the equilibria

$$u(x) = u_1\,\chi_{\Omega_1}(x) + u_2\,\chi_{\Omega_2}(x) + u_3\,\chi_{\Omega_3}(x).$$

This family of equilibria is not isolated, because we can build a new partition of Ω, denoted by $\widetilde{\Omega}_1$, $\widetilde{\Omega}_2$, $\widetilde{\Omega}_3$, and we consider the equilibrium $\tilde{u}(x) = u_1\,\chi_{\widetilde{\Omega}_1}(x) + u_2\,\chi_{\widetilde{\Omega}_2}(x) + u_3\,\chi_{\widetilde{\Omega}_3}(x)$, such that $\tilde{u}$ is as close as we want, in $L^p(\Omega)$, to the equilibrium u.

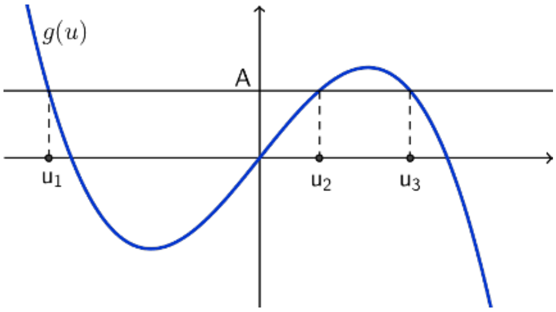

Figure 4.1: Roots of $A = g(u)$

Chapter 5

Nonlocal reaction-diffusion equation

Let (Ω, μ, d) be a metric measure space.

- If $X = L^p(\Omega)$, with $1 \leq p \leq \infty$, we assume $h \in L^\infty(\Omega)$.
- If $X = \mathcal{C}_b(\Omega)$, we assume $h \in \mathcal{C}_b(\Omega)$.

Let $K_J \in \mathcal{L}(X, X)$. In this chapter, we study a nonlinear nonlocal problem with nonlocal diffusion and nonlocal reaction, given by

$$\begin{cases} u_t(x,t) & = (K - hI)(u)(x,t) + f(x,u)(\cdot,t), \qquad x \in \Omega,\ t > 0 \\ u(x,0) & = u_0(x), \quad x \in \Omega, \end{cases} \tag{5.1}$$

where $f : \Omega \times L^1(\Omega) \to \mathbb{R}$ that maps (x, u) into $f(x, u)$ is the nonlocal reaction term. We will consider the nonlocal term given by

$$f = g \circ m, \tag{5.2}$$

where $g : \mathbb{R} \to \mathbb{R}$ is a nonlinear function and $m : \Omega \times L^1(\Omega) \to \mathbb{R}$, that sends (x, u) into $m(x, u)$, is the average of u in the ball of radius δ and center x in Ω,

$$m(x,u) = \frac{1}{\mu(B_\delta(x))} \int_{B_\delta(x)} u(y)dy. \tag{5.3}$$

In the previous chapter we have studied the problem with nonlocal diffusion and local reaction. In this chapter, first of all, we analyze the Nemitcky operator associated to the nonlocal reaction term f, (5.2). Then we will focus in the study of the existence and uniqueness of the solution associated to (5.1), firstly with g globally Lipschitz and secondly with g locally Lipschitz satisfying sign conditions. To prove all these results we will follow arguments similar to ones in chapter 4.

The existence and uniqueness of (5.1) with g globally Lipschitz, is obtained from the result of existence and uniqueness in Chapter 4, since the Nemitcky operator associated to the nonlinear term is globally Lipschitz. But we do not have comparison results for the problem (5.1) in general. To obtain them, we will ask the Lipschitz constant of the nonlinear term g to be small enough compared with the kernel J.

If $g : \mathbb{R} \to \mathbb{R}$ is locally Lipschitz, and satisfies sign conditions then we will prove the existence and uniqueness of solution of the problem (45), for g, such that the Lipschitz constant of g_{k_0} is small enough compared with J, where g_{k_0} is a truncated function associated to g. In fact, the existence and uniqueness, will be proved for initial data in $L^\infty(\Omega)$, such that $\|u_0\|_{L^\infty(\Omega)} \leq k_0$. Furthermore, we will prove some monotonocity properties for the solution of (5.1) with g and u_0 satisfying the conditions above.

We also give some asymptotic estimates of the solution of (5.1) with nonlinear term, g, globally Lipschitz. We prove the existence of two extremal equilibria φ_m and φ_M in $L^\infty(\Omega)$. If the initial data $u_0 \in L^\infty(\Omega)$, then the asymptotic dynamics of the solution enters between φ_m and φ_M . Moreover, φ_m and φ_M are bounds of the weak limits in $L^p(\Omega)$, with $1 \leq p < \infty$, of the solutions of (5.1) with initial data in $L^p(\Omega)$, with $1 \leq p < \infty$. Furthermore, we prove the existence of two extremal equilibria φ_m and φ_M in $\mathcal{C}_b(\Omega)$, and in this case, the asymptotic dynamics of the solution of (5.1) enters between φ_m and φ_M uniformly in compact sets of Ω.

In Chapter 4, the semigroup of (4.1), was not asymptotically smooth, but now, since the operator F associated to the nonlinear term f is compact, then we prove that the semigroup of (5.1) is asymptotically smooth, and with this property, we prove the existence of a global attractor for the semigroup of (5.1), by using [32, Theorem 3.4.6.].

5.1 The nonlocal reaction term

Let (Ω, μ, d) be a metric measure space, we consider $m(x, u)$, the average of u in a ball of radius δ and center x, given by

$$m(x,u) = \frac{1}{\mu(B_\delta(x))} \int_{B_\delta(x)} u(y)dy. \tag{5.4}$$

We denote

$$a(x) = \frac{1}{\mu(B_\delta(x))}.$$

Observe that, if s is a constant, then $m(x, s) = s$.

Furthermore, let $g : \mathbb{R} \to \mathbb{R}$ be a nonlinear function. We can think for example in $g(s) = \varepsilon_0 s - \varepsilon_1 |s|^{m-1} s$, with $\varepsilon_0, \varepsilon_1 > 0$ small positive constants. Hence, the nonlinear term $f : \Omega \times L^1(\Omega) \to \mathbb{R}$ is defined as $f = g \circ m$, and given by

$$f(x, u(\cdot)) = g\left(a(x) \int_{B_\delta(x)} u(y)dy\right). \tag{5.5}$$

To have everything well-defined throughout this chapter, we assume that Ω satisfies that

$$\exists\, C_0 > 0 \text{ and } C_1 > 0 \text{ constants, such that } C_1 \geq \mu\left(B_\delta(x) \cap \Omega\right) \geq C_0, \quad \text{for all } x \in \Omega, \tag{5.6}$$

with $C_0 = C_0(\delta)$ and $C_1 = C_1(\delta)$. Thanks to this assumption we have that $a(x)$ is bounded,

$$0 < \frac{1}{C_1} \leq a(x) \leq \frac{1}{C_0}, \quad \forall x \in \Omega. \tag{5.7}$$

Hence, $a \in L^\infty(\Omega)$. Sometimes, we will assume that $a \in \mathcal{C}_b(\Omega)$.

Now, we are interested in the Nemitcky operator associated to f,

$$F : X \to X, \text{ such that } F(u)(x) = f(x, u) = g(m(x, u)).$$

To study the properties of F, we study first $M(u)(x) = m(x, u)$.

In the following lemma we prove that the Nemitcky operator M associated to m is continuous, globally Lipschitz and compact.

Lemma 5.1.1. *Let (Ω, μ, d) be a metric measure space, such that $\mu(\Omega) < \infty$, the operator*

$$M(u)(x) = a(x) \int_{B_\delta(x)} u(y)dy, \tag{5.8}$$

satisfies that:

i. since $a \in L^\infty(\Omega)$, (5.7), then $M \in \mathcal{L}(L^1(\Omega), L^\infty(\Omega))$, and $M : L^p(\Omega) \to L^q(\Omega)$ is compact for $1 \leq p \leq \infty$ and $1 \leq q < \infty$;

ii. if $a \in \mathcal{C}_b(\Omega)$ and for any measurable set $D \subset \Omega$ with $\mu(D) < \infty$

$$\lim_{x \to x_0} \mu(B_\delta(x) \cap D) = \mu(B_\delta(x_0) \cap D) \quad \textit{for all } \; x_0 \in \Omega, \tag{5.9}$$

then $M \in \mathcal{L}(L^1(\Omega), \mathcal{C}_b(\Omega))$.

Proof.

i. Let $u \in L^1(\Omega)$,

$$\begin{aligned} \|M(u)\|_{L^\infty(\Omega)} &= \sup_{x \in \Omega} \left| a(x) \int_{B_\delta(x)} u(y)dy \right| \\ &\leq \frac{1}{C_0} \|u\|_{L^1(\Omega)}. \end{aligned}$$

Then $M \in \mathcal{L}(L^1(\Omega), L^\infty(\Omega))$.

Now, we prove that $M : L^p(\Omega) \to L^q(\Omega)$ is compact. Since $\mu(\Omega) < \infty$, $L^p(\Omega) \hookrightarrow L^1(\Omega)$ and $L^\infty(\Omega) \hookrightarrow L^q(\Omega)$ are continuously embedded. Hence, $M \in \mathcal{L}(L^p(\Omega), L^q(\Omega))$.

We rewrite the operator as follows,

$$M(u)(x) = \int_\Omega a(x) \chi_{B_\delta(x)}(y)\, u(y)dy \tag{5.10}$$

We consider $\widetilde{J}(x, y) = a(x)\chi_{B_\delta(x)}(y) \in L^\infty(\Omega \times \Omega)$. Since $L^\infty(\Omega \times \Omega) \hookrightarrow L^q(\Omega, L^{p'}(\Omega))$, for every $1 \leq p \leq \infty$ and $1 \leq q < \infty$. Thus $\widetilde{J}$ satisfies the hypothesis of Proposition 2.1.7, then $M : L^p(\Omega) \to L^q(\Omega)$ is compact.

ii. We consider M rewritten as in (5.10). Since $a \in \mathcal{C}_b(\Omega)$, then $\widetilde{J}(x, y) = a(x)\chi_{B_\delta(x)}(y)$ satisfies that $\widetilde{J} \in L^\infty(\Omega \times \Omega) \hookrightarrow L^\infty(\Omega, L^{p'}(\Omega))$ for $1 \leq p \leq \infty$. Moreover thanks to (5.9) we have that for any measurable $D \subset \Omega$ with $\mu(D) < \infty$,

$$\begin{aligned} \lim_{x \to x_0} \int_D \widetilde{J}(x, y)dy &= \lim_{x \to x_0} \int_D a(x)\chi_{B_\delta(x)}(y)dy = \lim_{x \to x_0} a(x)\mu(B_\delta(x) \cap D) \\ &= a(x_0)\mu(B_\delta(x_0) \cap D) = \int_D a(x_0)\chi_{B_\delta(x_0)}(y)dy = \int_D \widetilde{J}(x_0, y)dy, \quad \forall x_0 \in \Omega. \end{aligned}$$

Hence, the hypotheses of Proposition 2.1.1 are satisfied, and then for any $u \in L^1(\Omega)$, we have that $K_{\widetilde{J}} = M \in \mathcal{L}(L^1(\Omega), \mathcal{C}_b(\Omega))$. Thus, the result. $\square$

Let us consider a general globally Lipschitz operator, denoted with G. In the Lemma below, we analyze the properties of the operator given by $F = G \circ M$, where M is given by (5.8).

Lemma 5.1.2. *Let (Ω, μ, d) be a metric measure space, with $\mu(\Omega) < \infty$:*

i. Let $X = L^p(\Omega)$ with $1 \leq p \leq \infty$, or $X = \mathcal{C}_b(\Omega)$, if $G : X \to X$ is globally Lipschitz, then the operator $F = G \circ M$ satisfies that, $F : X \to X$ is globally Lipschitz.

ii. For $1 \leq p < \infty$, if $G : L^p(\Omega) \to L^p(\Omega)$ is globally Lipschitz, then

$$F = G \circ M : L^p(\Omega) \to L^p(\Omega)$$

is compact.

Proof.
i. Thanks to Lemma 5.1.1, we have that the Nemitcky operator associated to m satisfies that $M \in \mathcal{L}(X, X)$. Thanks to the hypotheses we have that $G : X \to X$ is Lipschitz, then $F = G \circ M : X \to X$ is globally Lipschitz.

ii. From Lemma 5.1.1, we have that $M : L^p(\Omega) \to L^p(\Omega)$ is compact for $1 \leq p < \infty$. Since $G : L^p(\Omega) \to L^p(\Omega)$ is Lipschitz, then we have that $F = G \circ M : L^p(\Omega) \to L^p(\Omega)$ is the composition of a compact operator, M, with a continuous operator, G. Therefore, $F : L^p(\Omega) \to L^p(\Omega)$ is compact. $\square$

5.2 Existence, uniqueness, positiveness and comparison of solutions with a nonlinear globally Lipchitz term

Let (Ω, μ, d) be a metric measure space:

- If $X = L^p(\Omega)$, with $1 \leq p \leq \infty$, we assume $h \in L^\infty(\Omega)$.
- If $X = \mathcal{C}_b(\Omega)$, we assume $h \in \mathcal{C}_b(\Omega)$.

Let $L = K - hI \in \mathcal{L}(X, X)$, then we study the general problem

$$\begin{cases} u_t(x,t) = (K - hI)(u)(x,t) + f(x,u)(x,t) = L(u)(x,t) + g(x, m(u))(x,t), & x \in \Omega,\ t \in \mathbb{R} \\ u(x,0) = u_0(x), & x \in \Omega, \end{cases} \tag{5.11}$$

where $g : \mathbb{R} \to \mathbb{R}$ is globally Lipschitz. Then $G : X \to X$ the Nemitcky operator associated to g is globally Lipschitz, and from Lemma 5.1.2, $F = G \circ M : X \to X$ is globally Lipschitz.

The results in this section are similar to the ones that appear in section 4.1 for the problem with nonlocal diffusion and local reaction. Hence, some of the results written in this section, are an immediate consequence of the results in the previous chapter, but we write the results for the problem (5.11) for the sake of completeness. Therefore, some of the proofs will not be given and we will refer the corresponding result in chapter 4. In some other results, there will be a similar argument to the respective result in chapter 4, and we will write in detail the parts of the proof that is new, referred to the problem (5.11), with $F = G \circ M$.

In the following result, we apply Proposition 4.1.3 to the problem (5.11), and we obtain the existence and uniqueness of the solution to (5.11).

Proposition 5.2.1. *Let (Ω, μ, d) be a metric measure space:*

- *If $X = L^p(\Omega)$, with $1 \leq p \leq \infty$, we assume $h \in L^\infty(\Omega)$.*
- *If $X = \mathcal{C}_b(\Omega)$, we assume $h \in \mathcal{C}_b(\Omega)$.*

Let $K \in \mathcal{L}(X, X)$ and let $g : \mathbb{R} \to \mathbb{R}$ be globally Lipchitz, then $F = G \circ M : X \to X$ is globally Lipschitz, and the problem (5.11) *has a unique global solution for every $u_0 \in X$, with*

$$u(\cdot, t) = e^{Lt}u_0 + \int_0^t e^{L(t-s)}F(u)(\cdot, s)\, ds. \tag{5.12}$$

Moreover, $u \in \mathcal{C}^1((-\infty, \infty), X)$ is a strong solution in X.

Remark 5.2.2. *For the problem* (5.11)*, the comparison results are not always satisfied, and we will need to add some conditions on the nonlinear term g. Below we give an example in which the solutions of a linear problem associated to* (5.11) *do not have comparison results.*

Let us consider the linear problem

$$\begin{cases} u_t(x,t) &= \displaystyle\int_\Omega J(x,y)u(y,t)dy - a(x)\int_{B_\delta(x)} u(y,t)dy, \quad x \in \Omega,\ t > 0 \\ u(x,0) &= u_0(x), \qquad\qquad\qquad\qquad\qquad\qquad\qquad x \in \Omega. \end{cases} \tag{5.13}$$

If $J \geq 0$, and there exist $x_0 \in \Omega$ and $\delta > 0$ such that

$$J(x,y) - a(x)\chi_{B_\delta(x)}(y) < 0, \quad \text{for all } x, y \in B_{\delta/2}(x_0), \tag{5.14}$$

then there exist nonnegative initial data such that the the solution of (5.13) *is negative for some points in Ω at some time $t_0 > 0$.*

We can rewrite the problem (5.13) *as follows*

$$u_t(x,t) = \int_\Omega \big(J(x,y) - a(x)\chi_{B_\delta(x)}(y)\big)\, u(y,t)dy.$$

If we denote $\widetilde{J}(x,y) = J(x,y) - a(x)\chi_{B_\delta(x)}(y)$, then the problem (5.13) *is given by*

$$u_t(x,t) = K_{\widetilde{J}}(u)(x,t) \tag{5.15}$$

If we choose a continuous nonnegative initial data $u_0 \geq 0$ with

$$\mathrm{supp}(u_0) = B_{\delta/4}(x_0),$$

then

$$
\begin{aligned}
u_t(x,0) &= K_{\widetilde{J}}(u_0)(x) \\
&= \int_\Omega \widetilde{J}(x,y)u_0(y)dy \\
&= \int_{B_{\delta/4}(x_0)} \widetilde{J}(x,y)u_0(y)dy.
\end{aligned}
\tag{5.16}
$$

Then, thanks to (5.14) *and* (5.16), *we know that* $u_t(x,0) < 0$ *for all* $x \in B_{\delta/2}(x_0)$. *Hence,* $u(x,t)$ *decreases in time* $t = 0$ *for all* $x \in B_{\delta/2}(x_0)$. *Moreover,* $K_{\widetilde{J}}$ *is a linear and bounded operator, then thanks to Lemma 3.1.1, the solution of* (5.13) *is continuous in time and space. Thus, there exists* $t_0 > 0$ *such that* $u(x,t) < 0$ *for all* $x \in B_{\delta/2}(x_0)\setminus B_{\delta/4}(x_0)$ *for all* $t \in (0, t_0)$. *Therefore, in this particular case, we do not have a comparison result for the linear nonlocal problem with nonlocal reaction.*

Now, we will give some monotonicity properties for the problem (5.11), with g globally Lipschitz, with Lipschitz constant small enough. In particular we will give some results with respect to the initial data and the nonlinear term.

The following Proposition proves that if two initial data in X are ordered, the corresponding solutions remain ordered.

Proposition 5.2.3. (Weak and Strong Maximum Principle) *Let* (Ω, μ, d) *be a metric measure space.*

- *If* $X = L^p(\Omega)$, *with* $1 \leq p \leq \infty$, *we assume* $h \in L^\infty(\Omega)$.
- *If* $X = \mathcal{C}_b(\Omega)$, *we assume* $h \in \mathcal{C}_b(\Omega)$.

Let $K \in \mathcal{L}(X,X)$, J *be nonnegative, and* β *be a constant defined as*

$$
\beta := \sup\left\{\lambda : J(x,y) - \lambda a(x)\chi_{B_\delta(x)}(y) \geq 0,\ \forall x,\, y \in \Omega\right\} > 0. \tag{5.17}
$$

If $g : \mathbb{R} \to \mathbb{R}$ *is globally Lipschitz and the Lipschitz constant of* g, L_g, *satisfies that*

$$
L_g < \beta,
$$

then, if $u_0, u_1 \in X$ *satisfy that* $u_0 \geq u_1$, *then*

$$
u^0(t) \geq u^1(t), \text{ for all } t \geq 0,
$$

where $u^i(t)$ *is the solution to* (5.11) *with initial data* u_i.

In particular if J *satisfies that for* $\lambda \leq \beta$ *such that* $L_g < \lambda$

$$
J(x,y) - \lambda a(x)\chi_{B_\delta(x)}(y) > 0, \quad \text{for all } x,\, y \in \Omega \quad \text{such that } d(x,y) < R, \tag{5.18}
$$

for some $R > \delta$, *and* Ω *is* R*-connected, (see Definition 2.1.14), then if* $u_0 \geq u_1$, *not identical, we have that*

$$
u^0(t) > u^1(t), \text{ for all } t > 0.
$$

Proof. We consider the Nemitcky operator $F = G \circ M$, and we rewrite the equation of the problem (5.11) as follows for some $\lambda \leq \beta$ such that $L_g < \lambda$,

$$u_t(x,t) = L(u)(x,t) - \lambda M(u)(x,t) + F(u)(x,t) + \lambda M(u)(x,t). \tag{5.19}$$

We denote $L - \lambda M = L_{\lambda M} \in \mathcal{L}(X, X)$. We rewrite $L_{\lambda M}$ as

$$\begin{aligned} L_{\lambda M}(u) &= (L - \lambda M)(u) \\ &= \int_\Omega \big(J(\cdot, y) - \lambda a(\cdot)\chi_{B_\delta(x)}(y)\big)u(y)dy - h(\cdot)u \end{aligned}$$

If we denote by $\widetilde{J}(x,y) = J(x,y) - \lambda a(x)\chi_{B_\delta(x)}(y)$, then $\widetilde{J}$ satisfies the hypotheses of Proposition 4.1.4. On the other hand, since $\lambda > L_g$, we have that $G + \lambda I$ is increasing, (see Appendix B, Lemma 6.4.14). Moreover, M is increasing in X, then $F + \lambda M = (G + \lambda I) \circ M$ is increasing. Hence, following the same arguments of Proposition 4.1.4, we obtain the result. □

In the proposition below, we prove monotonicity properties respect to the nonlinear term, for the solutions of (5.11).

Proposition 5.2.4. *Let (Ω, μ, d) be a metric measure space.*

- *If $X = L^p(\Omega)$, with $1 \leq p \leq \infty$, we assume $h \in L^\infty(\Omega)$.*
- *If $X = \mathcal{C}_b(\Omega)$, we assume $h \in \mathcal{C}_b(\Omega)$.*

Let $K - hI \in \mathcal{L}(X, X)$, J be nonnegative, and β be a constant defined as

$$\beta := \sup\left\{\lambda : J(x,y) - \lambda a(x)\chi_{B_\delta(x)}(y) \geq 0,\ \forall x,\ y \in \Omega\right\} > 0. \tag{5.20}$$

We assume g^i is globally Lipschitz for $i = 1, 2$, and L_{g^i} is the Lipschitz constant of g^i, that satisfies

$$L_{g^i} < \beta.$$

If $g^1 \geq g^2$, then

$$u^1(t) \geq u^1(t), \text{ for all } t \geq 0,$$

where $u^i(t)$ is the solution to (5.11) with initial data $u_0 \in X$, and nonlinear term g^i.

In particular if J satisfies the hypothesis (5.18) of Proposition 5.2.3, and Ω is R-connected, then, if $g^1 \geq g^2$, not equal, we have that

$$u^1(t) > u^2(t), \text{ for all } t > 0.$$

Proof. Arguing like in Proposition 5.2.3, and from Proposition 4.1.5. Thus, the result. □

The following proposition states that if the initial data is nonnegative, the solution of (5.11) is also nonnegative.

Proposition 5.2.5. (Weak and strong positivity)

- *if $X = L^p(\Omega)$, with $1 \leq p \leq \infty$, we assume $h \in L^\infty(\Omega)$.*

- *If $X = \mathcal{C}_b(\Omega)$, we assume $h \in \mathcal{C}_b(\Omega)$.*

Let $K - hI \in \mathcal{L}(X, X)$, J be nonnegative, and β be a constant defined as

$$\beta := \sup \left\{ \lambda : J(x, y) - \lambda a(x) \chi_{B_\delta(x)}(y) \geq 0, \ \forall x, \, y \in \Omega \right\} > 0. \tag{5.21}$$

We assume g is globally Lipschitz and L_g is the Lipschitz constant of g, such that

$$L_g < \beta,$$

and we assume $g(0) \geq 0$. If $u_0 \in X$, with $u_0 \geq 0$, not identically zero, then the solution to (5.11)*,*

$$u(t, u_0) \geq 0, \ \textit{for all } t \geq 0.$$

In particular if J satisfies the hypothesis (5.18) *of Proposition 5.2.3, and Ω is R-connected then, if $u_0 \geq 0$, not identically zero, we have that*

$$u(t, u_0) > 0, \ \textit{for all } t > 0.$$

Proof. Arguing like in Proposition 5.2.3, and following the same proof as in Proposition 4.1.6, we obtain the result. □

Let us recall the definition of supersolution to (5.11)

Definition 5.2.6. *We say that $\overline{u} \in \mathcal{C}([a, b], X)$ is a* **supersolution** *to* (5.11) *in $[a, b]$, if for $t \geq s$, with $s, t \in [a,\, b]$*

$$\overline{u}(\cdot, t, u_0) \geq e^{L(t-s)} \overline{u}(s) + \int_s^t e^{L(t-r)} F(\overline{u})(\cdot, r) dr.$$

We say that $\underline{u}$ is a **subsolution** *if the reverse inequality holds.*

The following proposition states that the supersolutions and the solutions of (5.11) with same initial data, are ordered as long as both exist.

Proposition 5.2.7.

- *If $X = L^p(\Omega)$, with $1 \leq p \leq \infty$, we assume $h \in L^\infty(\Omega)$.*
- *If $X = \mathcal{C}_b(\Omega)$, we assume $h \in \mathcal{C}_b(\Omega)$.*

Let $K - hI \in \mathcal{L}(X, X)$, J be nonnegative, and β be a constant defined as

$$\beta := \sup \left\{ \lambda : J(x, y) - \lambda a(x) \chi_{B_\delta(x)}(y) \geq 0, \ \forall x, \, y \in \Omega \right\} > 0. \tag{5.22}$$

We assume g is globally Lipschitz and L_g is the Lipschitz constant of g, such that

$$L_g < \beta.$$

Let $u(t,\, u_0)$ be the solution to (5.11) *with initial data $u_0 \in X$, and let $\bar{u}(t)$ be a supersolution to* (5.11) *in $[0, T]$.*

If $\bar{u}(0) \geq u_0$, then

$$\bar{u}(t) \geq u(t,\, u_0), \qquad \textit{for all } \ t \in [0, T].$$

The same is true for subsolutions if the reverse inequality holds.

Proof. Arguing like in Proposition 5.2.3, and from Proposition 4.1.9, we have the result. □

5.3 Existence and uniqueness of solutions, with a nonlinear locally Lipschitz term

Our aim in this section is to prove the existence and uniqueness of solution of the problem

$$\begin{cases} u_t(x,t) &= (K-hI)(u)(x,t) + f(x,u)(\cdot,t) = L(u)(x,t) + (g\circ m)(u)(x,t), \quad x\in\Omega,\ t\in\mathbb{R} \\ u(x,0) &= u_0(x), \qquad x\in\Omega, \end{cases} \tag{5.23}$$

with $u_0 \in L^\infty(\Omega)$ or $\mathcal{C}_b(\Omega)$ and $g:\mathbb{R}\to\mathbb{R}$ locally Lipschitz. To prove this, the nonlinear term must satisfy sign conditions and the Lipschitz constant has to be small enough compared with J. We also need to introduce an auxiliary problem associated to (5.23).

Let us introduce the globally lipschitz function, f_k, associated to the locally Lipschitz function f, that appears in the nonlinear problem (5.23) . The truncation of the function f, denoted as f_k, with $k\in\mathbb{R}$ is defined as follows. First of all, we consider g_k, a truncated globally Lipschitz function, associated to g, satisfying that

$$g_k(s) = g(s), \quad \text{for all} \quad |s|\le k, \tag{5.24}$$

and we define

$$f_k = g_k \circ m,$$

with $m:\Omega\times L^1(\Omega)\to\mathbb{R}$, $m(x,u) = a(x)\displaystyle\int_{B_\delta(x)} u(x)\,dx$.

If $|u|\le k$, then $|m(\cdot,u)|\le k$. Thus, f_k satisfies that

$$f_k(x,u) = f(x,u) \quad \text{for all } u \text{ such that } |u|\le k.$$

We introduce the following problem, that is equal to (5.23) substituting the locally Lipschitz function f with a truncated function f_k,

$$\begin{cases} \dfrac{\partial u_k}{\partial t}(x,t) = (K-hI)(u_k)(x,t) + f_k(x,u_k)(\cdot,t) = L(u_k)(x,t) + F_k(u_k)(x,t), \ x\in\Omega,\ t\in\mathbb{R} \\ u_k(x,0) \ = u_0(x), \quad x\in\Omega, \end{cases} \tag{5.25}$$

where $F_k : X\to X$ is the Nemitcky operator associated to the nonlinear term $f_k = g_k\circ m$. The solution of the problem (5.25) will be denoted as $u_k(t,u_0)$.

Since the truncated operator F_k is globally Lipschitz, then all the results of the previous section are satisfied for the problem (5.25). Thus, we can apply Proposition 5.2.1 to obtain the existence and uniqueness of the solution to the problem (5.25), and if L_{g_k} is small enough, we can apply also the Propositions 5.2.3, 5.2.4, 5.2.5 and 5.2.7 to obtain those comparison results for the problem (5.25).

In the following propositions we give a result of existence and uniqueness of solutions to the problem (5.23) with the nonlinear term g such that the associated truncated globally Lipschitz function g_k has a Lipschitz constant small enough, and the initial data u_0 is bounded by a constant that depends on the Lipschitz constant of g_k.

Proposition 5.3.1. *Let $X = L^\infty(\Omega)$ or $X = \mathcal{C}_b(\Omega)$, we assume $K \in \mathcal{L}(X, X)$, and $h \in X$, $h_0 \in L^\infty(\Omega)$. Given the nonnegative kernel J, let β be a constant defined as*

$$\beta := \sup\left\{\lambda : J(x,y) - \lambda a(x)\chi_{B_\delta(x)}(y) \geq 0,\ \forall x,\, y \in \Omega\right\} > 0. \tag{5.26}$$

Let g be locally Lipschitz such that there exists $C, D \in \mathbb{R}$ with $C < 0$ and $D \geq 0$ such that

$$(h_0(x) - h(x))s^2 + g(s)s \leq Cs^2 + D|s|, \quad \forall s,\ \forall x \in \Omega \tag{5.27}$$

with $h_0(x) = \int_\Omega J(x,y)dy$.

Let $L_{g_{k_0}}$ be the Lipschitz constant of g_{k_0}, where g_{k_0} is a truncation of g, and

$$L_{g_{k_0}} < \beta.$$

We assume that $\frac{-D}{C} \leq k_0$ and $u_0 \in X$ satisfies that $\|u_0\|_X \leq k_0$.

Then the problem (5.23) with initial data as above, has a global solution, and the solution is given by the variation of Constants Formula

$$u(\cdot,t) = e^{Lt}u_0 + \int_0^t e^{L(t-s)}F(\cdot, u(\cdot,s))ds. \tag{5.28}$$

Moreover, $u \in \mathcal{C}^1([0,T], X)$ is a strong solution of (5.23) in X, for all $T > 0$, and the solution $u \in \mathcal{C}^1([0,T], X)$ is unique in $\{u \in \mathcal{C}^1([0,T], X) :\ \|u\|_{\mathcal{C}([0,T],X)} \leq k_0\}$.

Proof. The proof is similar to the one in Proposition 4.2.2. We introduce the auxiliary problem

$$\begin{cases} \dot{z}(t) &= Cz(t) + D \\ z(0) &= k_0. \end{cases} \tag{5.29}$$

Then the solution of (5.29) is given by

$$z(t) = -\frac{D}{C} + e^{Ct}C_2, \quad \text{with } C_2 = M + \frac{D}{C},\ \forall t \in [0,T].$$

Since $C < 0$ and $k_0 \geq \frac{-D}{C}$ then

$$0 \leq z(t) \leq k_0 \quad \forall t \geq 0. \tag{5.30}$$

Thanks to Proposition 5.2.1 we know that there exists a unique strong solution $u_{k_0}(t, u_0) \in \mathcal{C}^1(\mathbb{R}, L^\infty(\Omega))$ that satisfies the Variation of Constants Formula.

Thanks to the definition of $f_k = g_k \circ m$, (5.30) we have that

$$f_k(\cdot, z(t)) = (g_k \circ m)(\cdot, z(t)) = (g \circ m)(\cdot, z(t)) = f(\cdot, z(t)),\ \forall t \geq 0. \tag{5.31}$$

Moreover, since g satisfies (5.27), $z(t) \geq 0$, $m(\cdot, z(t)) = z(t)$, and from (5.31), we have that f_k satisfies

$$\begin{aligned} (h_0 - h)(z(t)) + f_k(\cdot, z(t)) &= (h_0 - h)(z(t)) + g(m(\cdot, z(t))) \\ &\leq Cz(t) + D, \qquad\qquad \forall t \geq 0. \end{aligned} \tag{5.32}$$

Hence, thanks to (5.32), and since $z(t)$ is independent of the variable x, we have that $K(z(t)) = h_0 z(t)$. Thus,

$$\begin{aligned} K(z)(t) - hz(t) + f_k(\cdot, z(t)) &= h_0 z(t) - hz(t) + f_k(\cdot, z(t)) \\ &\leq Cz(t) + D = \dot{z}(t), \qquad \text{for all } t \geq 0. \end{aligned}$$

Hence, z is a supersolution of (5.25) for every $t \geq 0$.

Now, let $w(t, -M) = -z(t, M)$ be the solution of

$$\begin{cases} \dot{w}(t) &= Cw(t) - D \\ w(0) &= -k_0. \end{cases} \tag{5.33}$$

Then w satisfies that

$$0 \geq w(t) \geq -k_0 \quad \forall t \geq 0, \tag{5.34}$$

Arguing as above, we have that, w is a subsolution of (5.25) for every $t \geq 0$.

We have also that $\|u_0\|_{L^\infty(\Omega)} \leq k_0$. Therefore from Proposition 5.2.7, we obtain

$$w(t, -k_0) \leq u_{k_0}(t, u_0) \leq z(t, k_0), \quad \forall t \geq 0. \tag{5.35}$$

Moreover, thanks to (5.30), (5.34) and (5.35), we have that

$$|u_k(\cdot, t, u_0)| \leq k_0 \ \text{ for all } \ t \geq 0. \tag{5.36}$$

Since $f_{k_0}(u) = f(u)$ for all u such that $|u| < k_0$, and thanks to (5.36), we have that $f_{k_0}(\cdot, u_{k_0}(t)) = f(\cdot, u_{k_0}(t))$. Therefore, $u_{k_0}(\cdot, t, u_0)$ is a solution associated to (5.23) and we denote it as $u(\cdot, t, u_0)$. Moreover from (5.36) we have that

$$\|u(t, u_0)\|_X \leq k_0, \quad \forall t \geq 0. \tag{5.37}$$

Therefore, the solution $u(t, u_0)$ exists, is given by the Variation of constants Formula (5.28), and $u \in \mathcal{C}^1([0, \infty), X)$. Thus, u is a strong solution in X. Moreover, thanks to (5.37), we have that $\|u\|_{\mathcal{C}(\mathbb{R}, X)} \leq k_0$.

Now, let us prove the uniqueness. We consider a solution $u \in \mathcal{C}^1(\mathbb{R}, X)$ of (5.23), such that $\|u\|_{\mathcal{C}(\mathbb{R}, X)} \leq k_0$. Then, if we choose k_0, then $f_k(\cdot, u) = f(\cdot, u)$, and then the solutions u_k of (5.25) and u of (5.23) are the same. Furthermore, from Proposition 5.2.1 we know that the solution $u_k \in \mathcal{C}^1([0, T], X)$ is unique, strong and it is given by the Variation of Constants Formula. Thus, we have proved the uniqueness for the solutions that satisfy $\|u\|_{\mathcal{C}(\mathbb{R}, X)} \leq k_0$. □

Remark 5.3.2. *All the comparison results that are satisfied for the solutions of the truncated problem* (5.25) *are obtained also for the solution of the problem* (5.23)*, if the hypotheses of the previous Proposition 5.3.1 are satisfied.*

Remark 5.3.3. *If $h \leq h_0$, the hypotheses on g in the previous Proposition 5.3.1 are satisfied for the function*

$$g(s) = \varepsilon_0 s - \varepsilon_1 |s|^{m-1} s,$$

with $\varepsilon_0, \varepsilon_1 > 0$. Let us see this below.

First, we consider $s \geq 0$, then by using Young's inequality, we have

$$\begin{aligned} g(s) &= \varepsilon_0 s - \varepsilon_1 s^m \\ &= \varepsilon_0 s - \lambda\varepsilon_0 s + \lambda\varepsilon_0 \varepsilon \tfrac{1}{\varepsilon} s - \varepsilon_1 s^m \\ &\leq (1-\lambda)\varepsilon_0 s + \frac{1}{m'}\left(\frac{\lambda\varepsilon_0}{\varepsilon}\right)^{m'} + \left(\frac{\varepsilon}{m} - \varepsilon_1\right) s^m. \end{aligned}$$

Choosing $0 < \lambda < 1$, then $(1-\lambda)\varepsilon_0 < 0$ and choosing ε small enough such that $\frac{\varepsilon}{m} - \varepsilon_1 \leq 0$, we obtain that

$$g(s) \leq Cs + D,$$

where $C = (1-\lambda)\varepsilon_0 < 0$ and $D = \frac{1}{m'}\left(\frac{\lambda\varepsilon_0}{\varepsilon}\right)^{m'} \geq 0$.

For $s < 0$, we argue analogously and we obtain the result.

Moreover, we can choose the constants ε_0 and ε_1 of g small enough, such that the Lipschitz constant

$$L_{g_k} = (\varepsilon_0 + \varepsilon_1 N k^{m-1})$$

is as small as needed in Proposition 5.3.1.

5.4 Asymptotic estimates and extremal equilibria

Let (Ω, μ, d) be a metric measure space and let $X = L^p(\Omega)$, with $1 \leq p \leq \infty$, or $X = \mathcal{C}_b(\Omega)$. In this section, we study asymptotic estimates of the norm X of the solution u of the nonlocal reaction-diffusion problem with reaction term g globally Lipschitz, that we recall is given by

$$\begin{cases} u_t(x,t) = (K - hI)(u)(x,t) + f(x,u)(\cdot,t) = L(u)(x,t) + (g \circ m)(u)(x,t), & x \in \Omega,\ t \in \mathbb{R} \\ u(x,0) = u_0(x), \quad x \in \Omega, \end{cases} \tag{5.38}$$

with $u_0 \in X$, $g : \mathbb{R} \to \mathbb{R}$ globally Lipschitz and $m(x,u) = a(x) \int_{B_\delta(x)} u(y)dy$. We assume that g satisfies that there exist $C, D \in \mathbb{R}$, with $D \geq 0$ such that

$$g(s)s \leq Cs^2 + D|s|. \tag{5.39}$$

This means that

$$\begin{aligned} f(x,u) = g(m(x,u)) \leq Cm(x,u) + D, &\quad \text{if } m(x,u) \geq 0 \\ f(x,u) = g(m(x,u)) \geq Cm(x,u) - D, &\quad \text{if } m(x,u) \leq 0. \end{aligned} \tag{5.40}$$

The results in this section are similar to ones obtained in chapter 4 for the problem with nonlocal diffusion and local reaction, (4.89). Under the hypotheses above, we prove the existence of two ordered extremal equilibria, φ_m, φ_M, which give some information about the set that attracts the dynamics of the semigroup $S(t)u_0$, associated to (5.38), with $u_0 \in X$, where

$$S(t)u_0 = u(t, u_0).$$

In particular, if $u_0 \in L^\infty(\Omega)$ then the solutions of (5.38) associated to these initial data enter between φ_m and φ_M for a.e. $x \in \Omega$; and if $u_0 \in L^p(\Omega)$, with $1 \le p < \infty$, then φ_m and φ_M are bounds of the weak limits when time goes to infinity in $L^p(\Omega)$ with $1 \le p < \infty$ of the solutions of (5.38) when time goes to infinity. Moreover, if $a = 1/\mu(B_\delta(\cdot)) \in \mathcal{C}_b(\Omega)$, then we prove that the solutions of (5.38) enter between φ_m and φ_M uniformly on compact sets of Ω when time goes to infinity.

In the following proposition we give bounds of $|u(t)|$, where u is the solution to (5.38).

Proposition 5.4.1. *Let $\mu(\Omega) < \infty$,*

- *if $X = L^p(\Omega)$, with $1 \le p \le \infty$, we assume $h \in L^\infty(\Omega)$,*
- *if $X = \mathcal{C}_b(\Omega)$, we assume $h \in \mathcal{C}_b(\Omega)$.*

Let $L = K - hI \in \mathcal{L}(X, X)$, J be nonnegative, and let β be a constant defined as

$$\beta := \sup\left\{\lambda : J(x,y) - \lambda a(x)\chi_{B_\delta(x)}(y) \ge 0,\ \forall x,\, y \in \Omega.\right\} \tag{5.41}$$

We assume g is globally Lipscgitz, L_g is the Lipschitz constant of g, with $L_g < \beta$ and we assume there exists $C, D \in \mathbb{R}$ with $C > -\beta$ and $D \ge 0$ such that

$$g(s)s \le Cs^2 + D|s|, \quad \forall s. \tag{5.42}$$

Let $\mathcal{U}(t)$ be the solution of

$$\begin{cases} \mathcal{U}_t(x,t) = L(\mathcal{U}(x,t)) + Cm(x,\mathcal{U}(x,t)) + D = L_{J_C}(\mathcal{U}(x,t)) + D, & x \in \Omega,\ t > 0 \\ \mathcal{U}(x,0) = |u_0(x)|, & x \in \Omega, \end{cases} \tag{5.43}$$

where $L_{J_C} = K_{J_C} - hI$, with

$$J_C(x,y) = J(x,y) + Ca(x)\chi_{B_\delta(x)}(y). \tag{5.44}$$

Then the solution, u, of (5.38)*, satisfies that*

$$|u(t)| \le \mathcal{U}(t), \quad \textit{for all} \;\; t \ge 0.$$

Proof. First of all, we prove that the solution of (5.43) is nonnegative. We know that the solution $\mathcal{U}$ can be written with the Variation of Constants Formula as

$$\mathcal{U}(t) = e^{L_{J_C}t}|u_0| + \int_0^t e^{L_{J_C}(t-s)}D\, ds. \tag{5.45}$$

where $L_{J_C} = L + Cm = K_{J_C} - hI$. Since J_C is given by (5.44) and $C > -\beta$, we have that $J_C(x,y) \ge 0$ for all $x, y \in \Omega$.

Moreover, $|u_0| \ge 0$, $D \ge 0$, and J_C is nonnegative, then we can apply Proposition 3.2.2 to $L_{J_C} = K_{J_C} - hI$. Thus, we have that

$$e^{L_{J_C}t}|u_0| \ge 0 \;\; \forall t \ge 0 \quad \text{and} \quad e^{L_{J_C}(t-s)}D \ge 0\, \forall t \ge 0 \text{ and } s \in [0,t].$$

Hence, we have that $\mathcal{U}(t)$ is nonnegative for all $t \geq 0$.

Now, we prove that $\mathcal{U}$ is a supersolution of (5.38). Since $\mathcal{U}$ is nonnegative and g satisfies (5.42), we obtain

$$L(\mathcal{U}) + f(\cdot,\mathcal{U}) = L(\mathcal{U}) + g(m(\cdot,\mathcal{U})) \leq L(\mathcal{U}) + Cm(\cdot,\mathcal{U}) + D = \mathcal{U}_t.$$

Moreover $u_0 \leq |u_0|$, then from Proposition 5.2.7 we have

$$u(t) \leq \mathcal{U}(t), \ \forall t \geq 0. \tag{5.46}$$

Analogously, considering $\mathcal{W} = -\mathcal{U}$, the solution to

$$\begin{cases} \mathcal{W}_t = L(\mathcal{W}) + Cm(\cdot,\mathcal{W}) - D = L_{J_C}(\mathcal{W}) - D \\ \mathcal{W}(0) = -|u_0|. \end{cases}$$

We obtain that $\mathcal{W}$ is a subsolution of (5.38), i.e.

$$u(t) \geq \mathcal{W}(t), \ \forall t \geq 0. \tag{5.47}$$

Therefore, thanks to (5.46) and (5.47) we have that

$$-\mathcal{U}(t) \leq u(t) \leq \mathcal{U}(t), \ \ \forall t \geq 0.$$

Thus, the result. □

In following proposition we give an asymptotic estimate of the norm X of the solution of (5.38), that is given in terms of the norm of the equilibrium associated to the problem (5.43). To obtain this estimate, we assume that the operator $L_{J_C} = L + Cm$ satisfies that

$$\inf \sigma_X(L_{J_C}) \geq \delta > 0. \tag{5.48}$$

Then we have that, $\|e^{L_{J_C} t}\|_X \leq e^{-\delta\, t}$ for all $t \geq 0$.

Proposition 5.4.2. *Let (Ω, μ, d) be a metric measure space with $\mu(\Omega) < \infty$.*

- *If $X = L^p(\Omega)$, with $1 \leq p \leq \infty$, we assume $h \in L^\infty(\Omega)$.*
- *If $X = \mathcal{C}_b(\Omega)$, we assume $h \in \mathcal{C}_b(\Omega)$.*

Let $K \in \mathcal{L}(L^{p_0}(\Omega), \mathcal{C}_b(\Omega))$ be compact. We assume J is nonnegative, g and J satisfy the hypotheses of Proposition 5.4.1.

If $C, D \in \mathbb{R}$, $C > -\beta$ and $D \geq 0$, and

$$\inf \sigma_X(-L_{J_C}) \geq \delta > 0, \tag{5.49}$$

then there exists a unique equilibrium solution, Φ, associated to (5.43), such that

$$L(\Phi) + Cm(\cdot, \Phi) + D = 0, \tag{5.50}$$

$\Phi \in L^\infty(\Omega)$ and $\Phi \geq 0$. Moreover, if $u_0 \in L^p(\Omega)$, then the solution u of (5.38) satisfies that

$$\overline{\lim_{t\to\infty}} \|u(t, u_0)\|_{L^p(\Omega)} \leq \|\Phi\|_{L^p(\Omega)}.$$

In particular if $h \in \mathcal{C}_b(\Omega)$, $a \in \mathcal{C}_b(\Omega)$ and for any measurable set $D \subset \Omega$ with $\mu(D) < \infty$

$$\lim_{x \to x_0} \mu(B_\delta(x) \cap D) = \mu(B_\delta(x_0) \cap D) \quad \textit{for all} \;\; x_0 \in \Omega, \tag{5.51}$$

then $\Phi \in \mathcal{C}_b(\Omega)$ and $\Phi \geq 0$. Moreover, if $u_0 \in X$, then the solution of (5.38) *satisfies that*

$$\overline{\lim_{t \to \infty}} \|u(t, u_0)\|_X \leq \|\Phi\|_X.$$

Proof. First of all, thanks to Proposition 2.4.5, we have that $\sigma_X(-L_{J_C})$ is independent of X. Moreover, thanks to hypothesis (5.49), we have that 0 does not belong to the spectrum of $L_{J_C} = K_{J_C} - hI$, then L_{J_C} is invertible. Thus, the solution Φ of (5.50) is unique.

On the other hand, since Φ satisfies the equation (5.50), $D \in L^\infty(\Omega)$, and $L_{J_C} \in \mathcal{L}(L^\infty(\Omega))$ is invertible, then $\Phi \in L^\infty(\Omega)$. Following the proof in Proposition 4.3.3, we obtain that $\Phi \geq 0$.

From Proposition 5.4.1, the solution u satisfies that

$$|u(t, u_0)| \leq \mathcal{U}(t) = \Phi + e^{L_{J_C} t}(|u_0| - \Phi), \tag{5.52}$$

where $\mathcal{U}(t)$ is the solution to (5.43). For $u_0 \in L^p(\Omega)$, we have that $(|u_0| - \Phi) \in L^p(\Omega)$, and we obtain

$$\begin{aligned} \|u(t, u_0)\|_{L^p(\Omega)} &\leq \|\mathcal{U}(t)\|_{L^p(\Omega)} \\ &\leq \|\Phi\|_{L^p(\Omega)} + \|e^{L_{J_C} t}(|u_0| - \Phi)\|_{L^p(\Omega)} \\ &\leq \|\Phi\|_{L^p(\Omega)} + \|e^{L_{J_C} t}\|_{\mathcal{L}(L^p(\Omega))} \|(|u_0| - \Phi)\|_{L^p(\Omega)} \\ &\leq \|\Phi\|_{L^p(\Omega)} + e^{-\delta t} \|(|u_0| - \Phi)\|_{L^p(\Omega)} \end{aligned} \tag{5.53}$$

Since $\delta > 0$, then

$$\overline{\lim_{t \to \infty}} \|u\|_{L^p(\Omega)} \leq \|\Phi\|_{L^p(\Omega)}.$$

Thus, the result.

Let us prove the second part of the Proposition. Since the hypotheses of Lemma 5.1.1 are satisfied, then the Nemitcky operator associated to m satisfies that $M \in \mathcal{L}(L^1(\Omega), \mathcal{C}_b(\Omega))$, then $L_{J_C} \in \mathcal{L}(\mathcal{C}_b(\Omega))$. Since Φ satisfies (5.50), $D \in \mathcal{C}_b(\Omega)$, and and $L_{J_C} \in \mathcal{L}(\mathcal{C}_b(\Omega))$ is invertible, we obtain that $\Phi \in \mathcal{C}_b(\Omega)$. The rest of the proof is analogous to the previous one with $\Phi \in L^\infty(\Omega)$. $\square$

Now we give the results which state the existence of two ordered extremal equilibria, which give some information about the set that uniformly attracts the dynamics of the semigroup

$$S(t)u_0 = u(t, u_0)$$

associated to (5.38), as was proved in section 4.4.

The following results proves the existence of extremal equilibria for the problem (5.38) with initial data in $L^\infty(\Omega)$.

Theorem 5.4.3. *Let (Ω, μ, d) be a metric measure space with $\mu(\Omega) < \infty$.*

- *If $X = L^p(\Omega)$, with $1 \leq p \leq \infty$, we assume $h \in L^\infty(\Omega)$.*
- *If $X = \mathcal{C}_b(\Omega)$, we assume $h \in \mathcal{C}_b(\Omega)$.*

Let $K \in \mathcal{L}(L^{p_0}(\Omega), \mathcal{C}_b(\Omega))$ be compact. We assume J is nonnegative, g and J satisfy the hypotheses of Proposition 5.4.1.

If $C, D \in \mathbb{R}$, $C > -\beta$ and $D \geq 0$, and

$$\inf \sigma_X(-L_{J_C}) \geq \delta > 0, \tag{5.54}$$

then there exist two ordered extremal equilibria, $\varphi_m \leq \varphi_M$, in $L^\infty(\Omega)$ of the problem (5.38), with initial data $u_0 \in L^\infty(\Omega)$, such that any other equilibria ψ of (5.38) satisfies $\varphi_m \leq \psi \leq \varphi_M$. Furthermore, the set $\{v \in L^\infty(\Omega) : \varphi_m \leq v \leq \varphi_M\}$ attracts the dynamics of the solutions $S(t)u_0$ of (5.38), i.e., $\forall u_0 \in L^\infty(\Omega)$, there exist $\underline{u}(t)$ and $\overline{u}(t)$ in $L^\infty(\Omega)$ such that $\underline{u}(t) \leq S(t)u_0 \leq \overline{u}(t)$, and

$$\lim_{t\to\infty} \underline{u}(t) = \varphi_m, \quad \lim_{t\to\infty} \overline{u}(t) = \varphi_M \quad \text{ in } L^p(\Omega), \text{ with } 1 \leq p < \infty.$$

Moreover, if $h \in \mathcal{C}_b(\Omega)$, $a \in \mathcal{C}_b(\Omega)$ and for any measurable set $D \subset \Omega$ with $\mu(D) < \infty$

$$\lim_{x\to x_0} \mu(B_\delta(x) \cap D) = \mu(B_\delta(x_0) \cap D) \quad \text{for all } x_0 \in \Omega, \tag{5.55}$$

then φ_m and φ_M in $\mathcal{C}_b(\Omega)$, and

$$\lim_{t\to\infty} \underline{u}(t) = \varphi_m, \quad \lim_{t\to\infty} \overline{u}(t) = \varphi_M \quad \text{ in } L^\infty_{loc}(\Omega),$$

that is, uniformly in compact sets of Ω.

Proof. To see the details of this proof, go to Theorem 4.4.1, where the proof is analogous.

From Proposition 5.4.2 we know that $\Phi \in L^\infty(\Omega)$, and thanks to Proposition 5.4.2 with $X = L^\infty(\Omega)$, we have that the solution u of (5.38) satisfies that

$$\|u(t)\|_{L^\infty(\Omega)} \leq \|\Phi\|_{L^\infty(\Omega)} + \|e^{L_{J_C}t}\|_{\mathcal{L}(L^\infty(\Omega))} \||u_0| - \Phi\|_{L^\infty(\Omega)} \quad \forall t \geq 0. \tag{5.56}$$

Since $\|e^{L_{J_C}t}\|_{\mathcal{L}(L^\infty(\Omega))} \leq e^{-\delta t}$, with $\delta > 0$. Then if we fix $\varepsilon > 0$, then for every initial data $u_0 \in L^\infty(\Omega)$, there exists $T(u_0) > 0$ such that

$$-\Phi - \varepsilon \leq u(\cdot, t, u_0) \leq \Phi + \varepsilon, \quad \forall t \geq T(u_0). \tag{5.57}$$

We write the solution u of (5.38) in terms of the semigroup $S(t)$ associated to the problem. Then

$$u(\cdot, t, u_0) = S(t)u_0.$$

Now, we denote $T(u_0) = T$, to simplify the notation. Furthermore, thanks to (5.57), we obtain that

$$-\Phi - \varepsilon \leq S(t+T)(u_0) \leq \Phi + \varepsilon, \quad \forall t \geq 0. \tag{5.58}$$

First of all, we consider the case in which the initial data is $u_0 = \Phi + \varepsilon$, and we prove by using the Monotone convergence Theorem that

$$\lim_{n\to\infty} S(nT)(\Phi + \varepsilon) = \varphi_M, \quad \text{ in } L^p(\Omega) \text{ with } 1 \leq p < \infty. \tag{5.59}$$

Arguing as in Theorem 4.4.1, we obtain the convergence as t goes to infinity.

Now, we consider a general initial data $u_0 \in L^\infty(\Omega)$. Thanks to (5.58), for $T = T(u_0)$

$$-\Phi - \varepsilon \leq S(t+T)(u_0) \leq \Phi + \varepsilon, \quad \forall t \geq 0. \tag{5.60}$$

thus, letting the semigroup act at time t, we have

$$S(T+2t)u_0 \leq S(t)(\Phi + \varepsilon) = \overline{u}(t), \qquad \forall t \geq 0. \tag{5.61}$$

Thanks to (5.59) and (5.61), we get that

$$\lim_{t \to \infty} \overline{u}(t) \leq \varphi_M, \quad \text{ in } \ L^p(\Omega). \tag{5.62}$$

Finally, let ψ be another equilibrium. From (5.62), with $u_0 = \psi$, we get $\psi \leq \varphi_M$. Thus φ_M is maximal in the set of equilibrium points, i.e., for any equilibrium, ψ, we have $\psi \leq \varphi_M$. The results for φ_m can be obtained in an analogous way.

Now, let us prove the second part of the Theorem. Since $a \in \mathcal{C}_b(\Omega)$ and thanks to (5.55) the hypotheses of Lemma 5.1.1 are satisfied, and we obtain that the operator associated to m satisfies that $M \in \mathcal{L}(L^1(\Omega), \mathcal{C}_b(\Omega))$, and since $h, C \in \mathcal{C}_b(\Omega)$, then $L_{J_C} \in \mathcal{L}(\mathcal{C}_b(\Omega))$. On the other hand, thanks to Proposition 2.4.5, we have that $\sigma_X(-L_{J_C})$ is independent of X. Moreover, thanks to hypothesis (5.54), we have that 0 does not belong to the spectrum of $L_{J_C} = K_{J_C} - hI$, then L_{J_C} is invertible. Let φ be an equilibrium solution of (5.38), then φ satisfies that

$$L_{J_C}\varphi = L\varphi + Cm(\cdot, \varphi) = -g(\cdot, m(\cdot, \varphi)) + Cm(\cdot, \varphi).$$

Since $L_{J_C} \in \mathcal{L}(\mathcal{C}_b(\Omega))$ is invertible and $M \in \mathcal{L}(L^1(\Omega), \mathcal{C}_b(\Omega))$, then $\varphi \in \mathcal{C}_b(\Omega)$. Hence the extremal equilibria φ_m and φ_M belong to $\mathcal{C}_b(\Omega)$. Therefore, thanks to Dini's criterium, we have that the limit (5.59) satisfies in this case that

$$\lim_{t \to \infty} S(nT)(\Phi + \varepsilon) = \varphi_M, \quad \text{in } L^\infty_{loc}(\Omega)$$

converges uniformly in compact subsets of Ω. To obtain the convergence for any initial data in $L^\infty(\Omega)$, we follow the arguments above. Thus, the result. $\square$

Now, we prove the previous extremal equilibria, are bounds of the weak limit in $L^p(\Omega)$, with $1 \leq p < \infty$ of the solution to (5.38) with initial data $u_0 \in L^p(\Omega)$, with $1 \leq p < \infty$.

Proposition 5.4.4. *Let (Ω, μ, d) be a metric measure space with $\mu(\Omega) < \infty$.*

- *For $1 \leq p_0 \leq 2$, if $X = L^p(\Omega)$, with $p_0 \leq p \leq \infty$, we assume $h \in L^\infty(\Omega)$.*
- *If $X = \mathcal{C}_b(\Omega)$, we assume $h \in \mathcal{C}_b(\Omega)$.*

Let $K \in \mathcal{L}(L^{p_0}(\Omega), \mathcal{C}_b(\Omega))$ be compact. We assume J is nonnegative, g and J satisfy the hypotheses of Proposition 5.4.1.

If $C, D \in \mathbb{R}$, $C > -\beta$ and $D \geq 0$, and

$$\inf \sigma_X(-L_{J_C}) \geq \delta > 0, \tag{5.63}$$

then there exist two ordered extremal equilibria, $\varphi_m \leq \varphi_M$ in $L^\infty(\Omega)$, of (5.38)*, with initial data $u_0 \in L^p(\Omega)$, with $1 \leq p < \infty$ such that any other equilibria ψ of* (5.38) *satisfies $\varphi_m \leq \psi \leq \varphi_M$. Furthermore, the set*

$$\{v \in L^\infty(\Omega) : \varphi_m \leq v \leq \varphi_M\}$$

attracts the dynamics of the system.

Let $\tilde{u}(\cdot, u_0)$ be a weak limit in $L^p(\Omega)$ of $S(t)u_0$, when time t goes to infinity, then

$$\varphi_m(x) \leq \tilde{u}(x, u_0) \leq \varphi_M(x) \quad \text{for a.e. } x \in \Omega$$

for all $u_0 \in L^p(\Omega)$.

Proof. The proof is the same as in Proposition 4.4.4, but now, we apply the results in Proposition 5.4.2. □

Like in chapter 4, we write the result in which we give a criterium to know when the extremal equilibria is nonnegative, and we give sufficient hypotheses to obtain that any nonnegative equilibria is strictly positive for the problem (5.38).

Proposition 5.4.5. *If the hypotheses of Proposition 5.4.2 are satisfied, and g satisfies also that*

$$g(0) \geq 0,$$

then the extremal equilibria of (5.38)*, $\varphi_M \geq 0$.*

Furthermore, if J satisfies that

$$J(x, y) + Ca(x)\chi_{B_\delta(x)}(y) > 0, \ \forall x,\ y \in \Omega \text{ such that } d(x, y) < R, \tag{5.64}$$

for some $R > 0$, and Ω is R-connected, then any nonnegative equilibria ψ of (5.38) *is in fact strictly positive.*

Proof. The proof is the same as in Proposition 4.4.5. □

5.5 Attractor

In this section we prove the existence of an attractor for the problem (5.38).

The following proposition states that the the semigroup associated to (5.1) is asymptotically smooth.

Proposition 5.5.1. *Let $\mu(\Omega) < \infty$, and let $h \in L^\infty(\Omega)$. For $1 \leq p < \infty$, we assume $K \in \mathcal{L}(L^p(\Omega), L^p(\Omega))$ is compact, h satisfies*

$$h(x) \geq \alpha > 0, \ \text{ for all } x \in \Omega,$$

$G : L^p(\Omega) \to L^p(\Omega)$ is globally Lipschitz and $u_0 \in L^p(\Omega)$. Then $S(t)$, the semigroup associated to the problem, (5.38)*, is asymptotically smooth.*

Proof. In this proof we follow arguments analogous to the proof of Proposition 3.3.4.

We write the solution of (5.38) with the Variation of Constants Formula

$$S(t)u_0 = e^{Lt}u_0 + \int_0^t e^{L(t-s)}F(u)(\cdot,s)\,ds. \tag{5.65}$$

Thanks to Proposition 3.3.4, the linear semigroup e^{Lt} can be written as

$$e^{Lt}u_0 = e^{-h(x)t}u_0 + \int_0^t e^{-h(\cdot)(t-s)}K(u)(\cdot,s)ds.$$

and $e^{Lt}u_0$ is asymptotically smooth. Hence, if we prove that $\int_0^t e^{L(t-s)}F(u)(\cdot,s)\,ds$ is compact, then $S(t)$ is asymptotically smooth.

Let us prove that $\int_0^t e^{L(t-s)}F(u)(\cdot,s)\,ds$ is compact. Thanks to Proposition 5.2.1, we know that $u(\cdot,s) \in L^p(\Omega)$, for all $s \geq 0$, and thanks to the second part of Lemma 5.1.2, we have that $F : L^p(\Omega) \to L^p(\Omega)$ is compact. Since $e^{Lt} \in \mathcal{L}(L^p(\Omega), L^p(\Omega))$, then $e^{L(t-s)}F(u)(\cdot,s) \in L^p(\Omega)$ is compact for all $s \in [0,t]$. Moreover, thanks to Lemma 3.3.2 and Mazur's Theorem 3.3.1, we obtain that $\int_0^t e^{L(t-s)}f(x,u(\cdot,s))\,ds$ is compact. Thus, the result. □

Let X be a Banach space, we give some definitions related with a semigroup $T(t) : X \to X$.

Definition 5.5.2.

- *The semigroup $T(t) : X \to X$ is said to be* **point dissipative** *if there is a bounded set $B \subset X$ that attracts each point of X.*
- *An invariant set A is said to be a* **global attractor** *if A is a maximal compact invariant set which attracts each bounded set $B \subset X$*

The Theorem below states under which circumstances a semigroup $T(t) : X \to X$ has an attractor. The proof can be found in [32, Theorem 3.4.6.].

Theorem 5.5.3. *If $T(t) : X \to X$, $t \geq 0$, is asymptotically smooth, point dissipative, and orbits of bounded sets are bounded, then, there exists a global attractor A. If additionally X is a Banach space then the global attractor is connected.*

The following Theorem proves the existence of a global attractor of the problem (5.38).

Theorem 5.5.4. *Let $\mu(\Omega) < \infty$ and $1 \leq p < \infty$. Under the hypotheses of Proposition 5.4.2, and Proposition 5.5.1. Let $S(t) : L^p(\Omega) \to L^p(\Omega)$ be the semigroup associated to the problem* (5.38)*,with $u_0 \in L^p(\Omega)$, then there exists a global attractor A, and A is connected.*

Proof. From Proposition 5.5.1 we have that $S(t)$ is asymptotically smooth. Thanks to Proposition 5.4.2, we have that the orbits of bounded sets in $L^p(\Omega)$ are bounded. We just need to prove that $S(t)$ is point dissipative, and this is true because $|u(t)| \leq \mathcal{U}(t) = \Phi + e^{L_C t}(|u_0| - \Phi)$, and $\overline{\lim_{t\to\infty}} \|u\|_{L^p(\Omega)} \leq \|\Phi\|_{L^p(\Omega)}$, then for any $u_0 \in L^p(\Omega)$, there exists a time $T(u_0) > 0$ such that

$$\|u(t)\|_{L^p(\Omega)} \leq \|\Phi\|_{L^p(\Omega)} + 1, \qquad \forall t \geq T(u_0).$$

Hence, the closure of the ball of radius $\|\Phi\|_{L^p(\Omega)} + 1$ in $L^p(\Omega)$ attracts each $u_0 \in L^p(\Omega)$.

Therefore, the hypotheses of Theorem 5.5.3 are satisfied. Thus, the result. □

Chapter 6

A nonlocal two phase Stefan problem

The aim of this chapter is to study the following nonlocal version of the two-phase Stefan problem in $\mathbb{R}^N$

$$\begin{cases} u_t = \displaystyle\int_{\mathbb{R}^N} J(x-y)v(y)dy - v, & \text{in } \mathbb{R}^N \\ u(\cdot,0) = f, & \text{in } \mathbb{R}^N, \end{cases} \tag{6.1}$$

where J is a smooth nonnegative convolution kernel, then J is defined in this chapter as

$$J : \mathbb{R}^N \to \mathbb{R},$$

u is called the **enthalpy** and

$$v = \Gamma(u) = \operatorname{sign}(u)\big(|u| - 1\big)_+$$

is the **temperature**, (see below more precise assumptions and explanations). We study this nonlocal equation for sign-changing solutions, which presents very challenging difficulties concerning the asymptotic behavior.

In general, the Stefan problem is a non-linear and moving boundary problem which aims to describe the temperature and enthalpy distribution in a phase transition between several states.

The main model uses a local equation under the form $u_t = \Delta v$, $v = \Gamma(u)$, [39, 49], but recently, a nonlocal version of the one-phase Stefan problem was introduced in [12], which is equivalent to (6.1) in the case of nonnegative solutions.

Let us mention some basic facts about the one-phase Stefan problem: this problem models for instance the transition between ice and water: the "usual" heat equation (whether local or nonlocal) governs the evolution in the water phase while the temperature does not evolve in the ice phase, maintained at 0° C. The free boundary separating water from ice evolves according to how the heat contained in water is used to break the ice.

In the two-phase Stefan problem, the temperature can also evolve in the second phase, modeled by a second heat equation with different parameters. In this model, the temperature $v = \Gamma(u)$ is the quantity which identifies the different phases: the region $\{v > 0\}$ is the first phase, $\{v < 0\}$ represents the second phase and the intermediate region, $\{v = 0\}$ is where the transition occurs, containing what is called a *mushy region*.

In all the chapter, the function $J : \mathbb{R}^N \to \mathbb{R}$ in equation (6.1) is assumed to be continuous, non negative, compactly supported, radially symmetric, with

$$\int_{\mathbb{R}^N} J = 1.$$

In particular, $J \in L^\infty(\mathbb{R}^N) \cap L^1(\mathbb{R}^N)$. We denote by R_J the radius of the support of J,

$$\mathrm{supp}(J) = B_{R_J},$$

where B_{R_J} is the ball centered in zero with radius R_J. The graph $v = \Gamma(u)$, is defined generally as follows

$$\Gamma(u) = \begin{cases} c_1(u - e_1), & \text{if } u < e_1 \\ 0, & \text{if } e_1 \le u \le e_2 \\ c_2(u - e_2), & \text{if } u > e_2. \end{cases} \tag{6.2}$$

with e_1, e_2, c_1 and c_2 real variables, that satisfy that $e_1 < 0 < e_2$ and $c_1, c_2 > 0$ (see Figure 6.1 below). After a simple change of units, we arrive at the graph of equation (6.1): $\Gamma(u) = \mathrm{sign}(u)\,(|u| - 1)_+$, where we denote by s_+ the quantity $\max(s, 0)$, as is standard and $\mathrm{sign}(s)$ equals -1, $+1$ or 0 according to $s < 0$, $s > 0$, or $s = 0$.

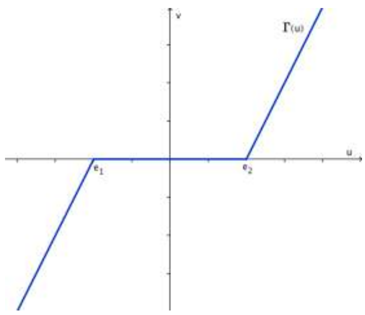

Figure 6.1: A typical graph Γ

In [12], the authors proved several qualitative properties for the nonlocal one-phase Stefan problem. Most of them are also valid in the two-phase problem, but the asymptotic behavior is far from being fully understood when solutions change sign.

Actually, up to our knowledge, there are no results for the asymptotic behavior of sign-changing solutions even in the local two-phase Stefan problem. The aim of this chapter is to try to provide at least some partial answers.

Going back to the one-phase Stefan problem, to identify the asymptotic limit for u, it is used the *Baiocchi variable*,

$$w(t) = \int_0^t v(s)ds, \tag{6.3}$$

where $v = \Gamma(u)$, (see [5]). If $\int_0^\infty \|v(t)\|_{L^1(\mathbb{R}^N)}dt < \infty$, then $w(t)$ converges monotonically, (since $v \ge 0$ in the one-phase Stefan problem), in $L^1(\mathbb{R}^N)$ as t goes to ∞ to

$$w_\infty = \int_0^\infty v(s)ds \in L^1(\mathbb{R}^N),$$

and u converges point-wise in $L^1(\mathbb{R}^N)$ to

$$\mathcal{P}f = f + J * w_\infty - w_\infty. \tag{6.4}$$

Moreover, w_∞ is a solution to the *nonlocal obstacle problem* at level one, with data f:

$$\text{(OP)} \quad \begin{cases} \text{Given a data } f \in L^1\left(\mathbb{R}^N\right), \text{ find a function } w \in L^1\left(\mathbb{R}^N\right) \text{ such that} \\ 0 \le f + J * w - w \le 1\,, \\ \big(f + J * w - w - 1\big) w = 0. \end{cases} \tag{6.5}$$

This problem is called "obstacle" since the values of the solution are cut at level 1. The asymptotic behavior of the solution u starting with f is given by $\mathcal{P}f$, (6.4).

A key argument in the one-phase Stefan problem is the *retention property*, which means that once the solution becomes positive at some point, it remains positive for greater times. In this case, the interfaces are monotone: the positivity sets (of u and v) grow. With this particular property, the Baiocchi transform gives all necessary and sufficient information to derive the asymptotic obstacle problem.

In the case of the two-phase Stefan problem, the situation is far more delicate to handle, due to the fact that sign-changing solutions do not enjoy a similar retention property in general: a solution can be positive, but later on it can become negative due to the presence of a high negative mass nearby. This implies that the Baiocchi transform is not a relevant variable anymore in general and many arguments fail. However, we shall study here some situations in which we can still apply, up to some extent, the techniques using the Baiocchi transform and get the asymptotic behavior for sign-changing solutions.

In this chapter, we first briefly derive a complete theory of existence, uniqueness and comparison results for the nonlocal two-phase Stefan problem. Then we concentrate on the asymptotic behavior of sign-changing solutions. Though we do not provide a complete picture of the question which appears to be rather difficult, we give some sufficient conditions which guarantee the identification of the limit. Namely, we first give in Section 6.2 a criterium which ensures that the positive and negative phases will never interact. This implies that the asymptotic behavior is given separately by each phase, considered as solutions of the one-phase Stefan problem.

Then we study the case when some interaction between the phases can occur, but only in the mushy zone, $\{|u| < 1\}$. We consider the same Baiocchi variable (6.3) used for the one-phase Stefan problem. And in the two-phase Stefan problem, we will prove that if $\int_0^\infty \|v(t)\|_{L^1(\mathbb{R}^N)} dt < \infty$, then $w(t)$ converges in $L^1(\mathbb{R}^N)$ as t goes to ∞ to

$$w_\infty = \int_0^\infty v(s) ds \in L^1(\mathbb{R}^N),$$

and u converges point-wise in $L^1(\mathbb{R}^N)$ to

$$\mathcal{P}f = f + J * w_\infty - w_\infty. \tag{6.6}$$

Now, w_∞ is a solution to the *nonlocal biobstacle problem* with data f:

$$\text{(BOP)}\quad \begin{cases} \text{Given a data } f \in L^1\left(\mathbb{R}^N\right), \text{ find a function } w \in L^1\left(\mathbb{R}^N\right) \text{ such that} \\ 0 \leq \operatorname{sign}(w)\,(f + J * w - w) \leq 1\,, \\ \big(f + J * w - w - \operatorname{sign}(w)\big)|w| = 0. \end{cases}$$

This problem is called "biobstacle" since the values of the solution are cut at both levels $+1$ and -1.

Hence, we prove that the asymptotic behavior of the solutions of the nonlocal two-phase Stefan problem can be described by the bi-obstacle problem, (BOP), the solution being cut at levels -1 and $+1$. We prove that this obstacle problem has a unique solution in a suitable class, and then we extend the operator which maps the initial data to the asymptotic limit to more general data by a standard approximation procedure. Notice that for the local model, such a result would be rather trivial since the mushy regions do not evolve. However, here those regions do evolve due to the nonlocal character of the equation.

Finally, we give an explicit example when the enthalpy becomes nonnegative in finite time even if the initial data is not, so that the asymptotic behavior is driven by the one-phase Stefan regime.

Throughout the chapter, we consider the spaces:

- $\mathcal{C}_c\left(\mathbb{R}^N\right) = \{\varphi \in \mathcal{C}\left(\mathbb{R}^N\right) : \varphi \text{ compactly supported }\}$;
- $\mathcal{C}_0\left(\mathbb{R}^N\right) = \{\varphi \in \mathcal{C}\left(\mathbb{R}^N\right) : \varphi \to 0 \text{ as } |x| \to \infty\}$;

Recall that throughout this chapter, J is **nonnegative, radially symmetric, compactly supported** with $\int J = 1$ and $\operatorname{supp}(J) = B_{R_J}$. Finally, we denote by $s_+ = \max(s, 0)$ and $s_- = \max(-s, 0)$.

6.1 Basic theory of the model

In this section we will develop the basic theory for the solution of the two-phase Stefan problem following arguments similar to the ones in [12]. This is due to the fact that for the one-phase Stefan model, $\Gamma(u) = (u-1)_+$, while here, we deal with a symmetric function $\Gamma(u) = \operatorname{sign}(u)(|u|-1)_+$. However, for the sake of completeness, we shall rewrite the proofs.

6.1.1 Existence, positiveness and comparison of solutions

Let $X = L^p(\mathbb{R}^N)$, with $1 \leq p \leq \infty$ or $X = \mathcal{C}_b(\mathbb{R}^N)$. We start with the theory for initial data in X. In this case the solution is regarded as a continuous curve in X. Below, we introduce the definition of solutions of (6.1).

Definition 6.1.1. *Let $f \in X$.*

- *A* **solution** *of* (6.1) *is a function $u \in \mathcal{C}([0,\infty); X)$ such for every $t > 0$, $u(t) \in X$ and*

$$u(t) = f + \int_0^t (J * \Gamma(u)(s) - \Gamma(u)(s))ds. \tag{6.7}$$

- *A function* $u \in \mathcal{C}^1([0,T],X)$ *is called* **strong solution** *of* (6.1) *if* $u(x,0) = f(x)$ *and* $u_t = J * \Gamma(u) - \Gamma(u)$ *in* $[0,T]$.

If the solution $u(t)$ is in $L^p(\mathbb{R}^N)$, with $1 \le p \le \infty$, then the equality (6.7) is satisfied a.e., and if the solution $u(t)$ is in $\mathcal{C}_b(\mathbb{R}^N)$ then the equality (6.7) is satisfied for all $x \in \mathbb{R}^N$.

Theorem 6.1.2. *Given any* $f \in X$*, there exists a unique solution of* (6.1)*,* $u \in \mathcal{C}^1([0,\infty),X)$*, that is a strong solution in* X*.*

Proof. Since $\Gamma : \mathbb{R} \to \mathbb{R}$ is globally Lipschitz, with Lipschitz constant $L_\Gamma = 1$, then the Nemitcky operator $\Gamma : X \to X$ is also globally Lipschitz, with Lipschitz constant $L_\Gamma = 1$. (see Appendix B, Lemma 6.4.14). Let X_{t_0} be the Banach space consisting of the functions $u \in \mathcal{C}([0,t_0];X)$ endowed with the norm,

$$|||u||| = \max_{0 \le t \le t_0} \|u(t)\|_X.$$

For any given $f \in X$, we define the operator $\mathcal{T}_f : X_{t_0} \to X_{t_0}$ through

$$(\mathcal{T}_f u)(t) = f + \int_0^t (J * \Gamma(u)(s) - \Gamma(u)(s))\, ds. \tag{6.8}$$

Given $u \in X_{t_0}$, since Γ is Lipschitz, J is continuous and $f \in X$, we have that $\mathcal{T}_f u \in X_{t_0}$. Moreover, since Γ is Lipschitz continuous, and thanks to Proposition 2.1.4 we have the estimate

$$\begin{aligned} |||\mathcal{T}_f u - \mathcal{T}_f w||| &\le \int_0^{t_0} \left\| J * \big(\Gamma(u)(s) - \Gamma(w)(s)\big) + \big(\Gamma(u)(s) - \Gamma(w)(s)\big) \right\|_X ds \\ &\le \int_0^{t_0} \big(\|J\|_{L^1(\mathbb{R}^N)} + 1\big) \|u(s) - w(s)\|_X ds \\ &= 2\int_0^{t_0} \|u(s) - w(s)\|_X ds \le 2t_0 |||u - w|||\,. \end{aligned}$$

Hence if $t_0 < 1/2$, the operator $\mathcal{T}_f$ turns out to be contractive.

Existence and uniqueness in the time interval $[0,t_0]$ follow by using Banach's fixed point Theorem. The length of the existence and uniqueness time interval does not depend on the initial data, so, we can iterate the argument to extend the result to all positive times by a standard procedure, and we end up with a solution in $\mathcal{C}([0,\infty);X)$. Moreover, since $\Gamma(u) \in \mathcal{C}([0,\infty);X)$, from (6.7) we also have $u \in \mathcal{C}^1([0,\infty);X)$, and the equation holds a.e. in x for all $t \ge 0$. Thus, the solution u of (6.1) is a strong solution in X. □

Notice that the solutions depend continuously on the initial data, on any finite time interval.

Lemma 6.1.3. *Let* $X = L^p(\mathbb{R}^N)$*, with* $1 \le p \le \infty$ *or* $X = \mathcal{C}_b(\mathbb{R}^N)$*. Let* u^1 *and* u^2 *be the solutions of* (6.1) *with initial data respectively* f_1*,* $f_2 \in X$*. Then, for all* $T \in (0,\infty)$ *there exists a constant* $C = C(T)$ *such that*

$$\|u^1(t) - u^2(t)\|_X \le C(T)\|f_1 - f_2\|_X, \quad t \in [0,T].$$

Proof. Since u_i is a fixed point of the operator $\mathcal{T}_{f_i}$, (6.8). Repeating the arguments in Theorem 6.1.2, we have that

$$\|u^1(t) - u^2(t)\|_X \leq \|f_1 - f_2\|_X + 2\, t_0 \sup_{t\in[0,t_0]} \|u^1(t) - u^2(t)\|, \ t \in [0, t_0].$$

Taking $t_0 = 1/4$, we get

$$\sup_{t\in[0,1/4]} \|u^1(t) - u^2(t)\|_X \leq 2\|f_1 - f_2\|_X,$$

from where the result follows by iteration, with a constant $C(T) = 2^{4T}$. □

Now, we will prove that the solution u of (6.1) with a nonnegative initial data f, is nonnegative.

Proposition 6.1.4. *Given any $f \in X$ nonnegative, the solution of* (6.1) *is also nonnegative.*

Proof. Since $-\Gamma$ is Lipschitz, then there exists a constant $\lambda_0 > L_\Gamma = 1$ such that $H(u) = \lambda_0 u - \Gamma(u)$ is monotonically increasing, (see Appendix B, Lemma 6.4.14). We rewrite the problem (6.1) as follows

$$\begin{cases} u_t = J * \Gamma(u) + \lambda_0 u - \Gamma(u) - \lambda_0 u, \\ u(\cdot, 0) = f. \end{cases} \tag{6.9}$$

We take the function

$$v(t) = e^{\lambda_0 t} u(t), \tag{6.10}$$

then

$$\begin{aligned} v_t(t) &= \lambda_0 e^{\lambda_0 t} u(t) + e^{\lambda_0 t} u_t(t) \\ &= \lambda_0 v(t) + e^{\lambda_0 t} \left(J * \Gamma(u(t)) + \lambda_0 u(t) - \Gamma(u(t))\right) - \lambda_0 v(t). \end{aligned}$$

Then, $v(t)$ is the solution of the problem

$$\begin{cases} v_t = e^{\lambda_0 t} \left(J * \Gamma(u(t)) + \lambda_0 u(t) - \Gamma(u(t))\right), \\ v(\cdot, 0) = f. \end{cases} \tag{6.11}$$

Integrating (6.11) in $[0, t]$, we have that

$$v(t) = f + \int_0^t e^{\lambda_0 s} \big(J * \Gamma(u(s)) + \lambda_0 u(s) - \Gamma(u(s))\big) ds.$$

Therefore, $u(t) = e^{-\lambda_0 t} v(t)$ is given by

$$u(t) = e^{-\lambda_0 t} f + \int_0^t \left(e^{-\lambda_0 (t-s)} J * \Gamma(u(s)) + e^{-\lambda_0 (t-s)} \big(\lambda_0 u(s) - \Gamma(u(s))\big)\right) ds. \tag{6.12}$$

Like in Theorem 6.1.2, let X_{t_0} be the Banach space consisting of the functions $u \in \mathcal{C}\left([0, t_0]; X\right)$ endowed with the norm,

$$|||u||| = \max_{0\leq t\leq t_0} \|u(t)\|_X.$$

For any given $f \in X$, we define the operator $\mathcal{T}_f : X_{t_0} \to X_{t_0}$ through

$$(\mathcal{T}_f u)(t) = e^{-\lambda_0 t} f + \int_0^t e^{-\lambda_0 (t-s)} \left(J * \Gamma(u(s)) + \lambda_0 u(s) - \Gamma(u(s)) \right) ds.$$

Since $\lambda_0 > 0$ and Γ is Lipschitz continuous, and thanks to Proposition 2.1.4 we have the estimate

$$\begin{aligned}
|||\mathcal{T}_f u - \mathcal{T}_f w||| &\leq \int_0^{t_0} \left\| \left(J * \left(\Gamma(u)(s) - \Gamma(w)(s) \right) + (\lambda_0 u(s) - \lambda_0 v(s)) + (\Gamma(u)(s) - \Gamma(w)(s)) \right) \right\|_X ds \\
&\leq \int_0^{t_0} \left(\|J\|_{L^1(\mathbb{R}^N)} + \lambda_0 + 1 \right) \|u(s) - w(s)\|_X ds \\
&= (2 + \lambda_0) \int_0^{t_0} \|u(s) - w(s)\|_X ds \leq (2 + \lambda_0) t_0 |||u - w||| .
\end{aligned}$$

Hence if $t_0 < \frac{1}{2+\lambda_0}$, the operator $\mathcal{T}_f$ turns out to be contractive.

Now, we want to prove that the solution u written as in (6.12) is nonnegative given any initial data f nonnegative. We have that the mapping $\mathcal{T}_f$ has a unique fixed point in X_{t_0}, we will prove that u is nonnegative using Picard iterations.

We consider the sequence of Picard iterations,

$$u_{n+1}(x,t) = \mathcal{T}_f(u_n)(x,t) \;\; \forall n \geq 1,$$

with $u_1 = f$. Then the sequence $u_n(x,t)$ converges to $u(x,t)$ in X_{t_0}.
Since $u_1(x,t) = f(x)$ is nonnegative, then for $t \geq 0$

$$u_2(x,t) = \mathcal{T}_f(u_1)(x,t) = e^{-\lambda_0 t} f + \int_0^t e^{-\lambda_0 (t-s)} J * \Gamma(f) + e^{-\lambda_0 (t-s)} \left(\lambda_0 f - \Gamma(f) \right) ds \quad (6.13)$$

Since $J * \Gamma(f)$ is nonnegative, $\lambda_0 I - \Gamma$ is increasing, and $(\lambda_0 I - \Gamma)(0) = 0$, then $u_2(t)$ is nonnegative.

Since $u_{n+1}(x,t) = \mathcal{T}_f(u_n)(x,t)$, if u_n is nonnegative, then following the arguments for u_2, we obtain that u_{n+1} is nonnegative for every $n \geq 1$, for $t \geq 0$. As $u_n(x,t)$ converges to $u(x,t)$, we have that the solution $u(x,t)$ is nonnegative in X_{t_0}. Hence, we have proved that for some $t_0 > 0$, that depends on λ_0, but does not depend on u_0, we find a unique solution $u \in X_{t_0}$ of the problem (6.1) with initial data $u(x,0) = f(x)$ nonnegative that is nonnegative for all $t \in [0, t_0]$. With a continuation argument, we have that the solution $u(\cdot, t)$ to (6.1) is nonnegative, not identically zero, for all $t \geq 0$. $\square$

In the following proposition we prove that given two initial data ordered, the corresponding solutions remain ordered.

Proposition 6.1.5. *If $f_1, f_2 \in X$ satisfy that $f_1 \geq f_2$, then*

$$u^1(t) \geq u^2(t), \quad \forall t \geq 0,$$

where u^i is the solution of (6.1) *with initial data f_i.*

Proof. Let $u^i(t)$ be the unique fixed point of

$$\mathcal{T}_{f_i}(u^i)(t) = e^{-\lambda_0 t} f_i + \int_0^t e^{-\lambda_0(t-s)} \left(J * \Gamma(u^i(s)) + \lambda_0 u^i(s) - \Gamma(u^i(s))\right) ds$$

in $X_{t_0} = \mathcal{C}\left([0,t_0];X\right)$. From the previous Proposition 6.1.4 we know that $\mathcal{T}_{f_i}$ is a contraction in X_{t_0}, provided t_0 small enough. We consider the sequence of Picard iterations

$$u^i_{n+1}(x,t) = \mathcal{T}_{f_i}(u^i_n)(x,t) \;\; \forall n \geq 1, \; x \in \Omega, \; 0 \leq t \leq T.$$

Then the sequence $u^i_n(x,t)$ converges to $u^i(x,t)$ in X_{t_0}. Now, we are going to prove that the solutions are ordered for all $t \geq 0$. We take the first term of the Picard iteration as $u^i_1(x,t) = f_i(x)$, then $u^1_1(t) = f_1 \geq f_2 = u^2_1(t)$, for all $t \geq 0$, and

$$u^i_2(t) = \mathcal{T}_{f_i}(u^i_1)(\cdot,t) = e^{-\lambda_0 t} f_i + \int_0^t e^{-\lambda_0(t-s)} \left(J * \Gamma(f_i) + \lambda_0 f_i - \Gamma(f_i)\right) ds.$$

Since $f_1 \geq f_2$, and Γ and $\lambda_0 I - \Gamma$ are increasing in X, we have that

$$u^1_2(t) \geq u^2_2(t), \qquad \text{for all } \; t \in [0,t_0].$$

Following this argument, we get that

$$u^1_n(t) \geq u^2_n(t), \qquad \text{for all } \; t \in [0,t_0], \; \forall n \geq 1.$$

Since $u^i_n(t)$ converges to $u^i(t)$ in X_{t_0}, we obtain that

$$u^1(t) \geq u^2(t), \quad \text{for all } t \in [0,t_0].$$

With a continuation argument, we prove that $u^1(t) \geq u^2(t)$, for all $t \geq 0$. □

To prove the following results, we need first to give the definition of supersolution and subsolution to (6.1).

Definition 6.1.6. *We say that* $\overline{u} \in \mathcal{C}([a,b],X)$ *is a* **supersolution** *to* (6.1) *in* $[a,b]$*, if for* $t \geq s$*, with* $s,t \in [a,\, b]$

$$\overline{u}(\cdot,t) \geq e^{-\lambda_0(t-s)}\overline{u}(\cdot,s) + \int_s^t e^{-\lambda_0(t-r)} \left(J * \Gamma(\overline{u}(r)) + \lambda_0 \overline{u}(r) - \Gamma(\overline{u}(r))\right) dr. \tag{6.14}$$

We say that $\underline{u}$ *is a* **subsolution** *if the reverse inequality holds.*

Remark 6.1.7. *If* $\overline{u} \in \mathcal{C}([a,b],X)$ *satisfies that*

$$\overline{u}_t \geq J * \Gamma(\overline{u}) - \Gamma(\overline{u}) \tag{6.15}$$

Adding and subtracting $\lambda_0 \overline{u}$ *to* (6.15)*. Repeating the arguments in Proposition 6.1.4 and integrating in* $[s,t]$*, we obtain that* $\overline{u}$ *is a supersolution that satisfies* (6.14)*.*

The same happens for subsolutions if the reverse inequality holds.

The following proposition states that the supersolution is greater than the solutions to (6.1).

Proposition 6.1.8. *Let $u(\cdot, t, f)$ be a solution to* (6.1) *with initial data $f \in X$ and let $\bar{u}(\cdot, t)$ be a supersolution to* (6.1) *in* $[0, T]$. *If $\bar{u}(\cdot, 0) \geq f$, then*

$$\bar{u}(t) \geq u(t, f), \quad \text{for all } t \in [0, T].$$

The same is true for subsolutions with reversed inequality.

Proof. Let $u(t)$ be the unique fixed point of

$$\mathcal{T}_f(u)(t) = e^{-\lambda_0 t} f + \int_0^t e^{-\lambda_0 (t-s)} \left(J * \Gamma(u(s)) + \lambda_0 u(s) - \Gamma(u(s)) \right) ds$$

in $\mathcal{C}([0, \tau]; X)$ provided τ small enough. We choose $t_0 \leq \tau$ such that $t_0 \leq T$, then the supersolution $\bar{u}(t) \in X$ exists for all $t \in [0, t_0]$. The supersolution $\bar{u}$ satisfies by definition that

$$\bar{u}(t) \geq \mathcal{T}_f(\bar{u})(t) \tag{6.16}$$

and $\bar{u}(0) \geq f$. We consider the sequence of Picard iterations,

$$u_{n+1}(t) = \mathcal{T}_f(u_n)(t) \;\; \forall n \geq 1. \tag{6.17}$$

Then the sequence $u_n(x, t)$ converges to $u(x, t)$ in X_{t_0}. If we show that,

$$\bar{u} \geq u_n, \quad \text{a.e. in } X_{t_0}, \text{ for } n = 1, 2, 3, \ldots, \tag{6.18}$$

then, we have the result. We take $u_1(t) = \bar{u}(t)$, then

$$\bar{u} \geq u_1 = \bar{u},$$

and (6.18) is satisfied for $n = 1$. Moreover, thanks to (6.16), we have that

$$\bar{u}(t) \geq \mathcal{T}_f(\bar{u})(t) = u_2(t), \quad t \in [0, t_0],$$

then (6.18) is true for $n = 2$. Assume now for induction

$$\bar{u}(t) \geq u_n(t), \;\; \text{for all } t \in [0, t_0]. \tag{6.19}$$

Since $\bar{u}(t)$ satisfies (6.16), $\mathcal{T}_f$ is increasing, and from (6.19), we have that

$$\bar{u}(t) \geq \mathcal{T}_f(\bar{u})(t) \geq \mathcal{T}_f(u_n)(t) = u_{n+1}(t), \;\; \text{for all } t \in [0, t_0].$$

Thus, we have that

$$\bar{u}(t) - u_{n+1}(t) \geq 0, \;\; \text{for all } t \in [0, t_0].$$

Hence, $\bar{u}(t) \geq u_{n+1}(t)$, for all $n \in \mathbb{N}$, and $u_n(t)$ converges to $u(t)$ in X_{t_0}. Therefore,

$$\bar{u}(t, u_0) \geq u(t, u_0)$$

for all $t \in [0, t_0]$. With a continuation argument, we have the result. □

We recall that given a **nonlinear** function $g : \mathbb{R} \to \mathbb{R}$ globally Lipschitz, if we consider the Nemitcky operator $G : L^p(\mathbb{R}^N) \to L^p(\mathbb{R}^N)$, then G is **not differentiable** (see Appendix B). Then, in the following results, we will consider the derivative in the sense of distributions.

Let $X = L^p(\mathbb{R}^N)$ with $1 \leq p \leq \infty$ or $X = \mathcal{C}_b(\mathbb{R}^N)$, and let us introduce the definition of derivative in the sense of distributions, (see [15, p. 10]).

Definition 6.1.9. *Let $h \in L^1_{loc}([a,b], X)$ We define the distributional derivative of h, h' by*

$$\langle h', \varphi \rangle = -\langle h, \varphi' \rangle,$$

for all $\varphi \in \mathcal{C}_c^\infty(\mathbb{R}^N)$.

This concept of distributional derivative is equivalent to the following concept of derivative:

Proposition 6.1.10. *Let $g \in L^1_{loc}([a,b], X)$, $t_0 \in [a,b] \subset \mathbb{R}$ and let $h \in \mathcal{C}([a,b], X)$ be given by $h(t) = \int_{t_0}^t g(s)ds$. Then*

i. $h' = g$ in the sense of distributions.

ii. h is differentiable a.e. and $h' = g$ a.e.

Definition 6.1.11. *We denote by $W^{1,p}\big([a,b], L^1(\mathbb{R}^N)\big)$ the space of functions $h \in L^1\big([a,b], L^p(\mathbb{R}^N)\big)$ such that $h' \in L^1\big([a,b], L^p(\mathbb{R}^N)\big)$, in the sense of distributions.*

Moreover, the derivative in the sense of distributions satisfy also the Fundamental Theorem of Calculus, see [15, Th. 1.4.35].

Theorem 6.1.12. *Let $h \in L^1\big([a,b], L^p(\mathbb{R}^N)\big)$, with $[a,b] \subset \mathbb{R}$. Then the following properties are equivalent:*

- *$h \in W^{1,p}\big([a,b], L^p(\mathbb{R}^N)\big)$,*
- *there exists $g \in L^1\big([a,b], L^p(\mathbb{R}^N)\big)$ such that the Fundamental Theorem of Calculus is satisfied, i.e.,*
$$h(t) = h(t_0) + \int_{t_0}^t g(s)ds.$$

From now on, we are interested on L^1-solutions, for which we have conservation of energy.

Theorem 6.1.13. (Conservation of energy of the L^1-solutions) *Let $f \in L^1\left(\mathbb{R}^N\right)$. The L^1-solution u to* (6.1) *satisfies*

$$\int_{\mathbb{R}^N} u(t) = \int_{\mathbb{R}^N} f, \quad \textit{for every } t > 0.$$

Proof. Since $u(t) \in L^1\left(\mathbb{R}^N\right)$ for any $t \geq 0$, we integrate equation (6.7) in space:

$$\int_{\mathbb{R}^N} u(t) = \int_{\mathbb{R}^N} f + \int_0^t \bigg(\int_{\mathbb{R}^N} J * \Gamma(u) - \int_{\mathbb{R}^N} \Gamma(u) \bigg) ds\,.$$

By Fubini's Theorem, and since $\int J = 1$, we have that $\int J * \Gamma(u) = \int J \cdot \int \Gamma(u) = \int \Gamma(u)$, (where the integrals are taken over all $\mathbb{R}^N$), which yields the result. $\square$

L^1-contraction property for L^1-solutions.

In order to obtain the contraction property, we need first to approximate the graph $\Gamma(s)$ by a sequence of strictly monotone $\Gamma_n(s)$ such that:

(i) there is a constant L independent of n such that $|\Gamma_n(s)-\Gamma_n(t)| \leq L|s-t|$, for all $n \in \mathbb{N}$;

(ii) for all $n \in \mathbb{N}$, $\Gamma_n(0) = 0$ and Γ_n is strictly increasing on $(-\infty, \infty)$;

(iii) $|\Gamma_n(s)| \leq |s|$, for all $n \in \mathbb{N}$;

(iv) $\Gamma_n \to \Gamma$ as $n \to \infty$ uniformly in $(-\infty, \infty)$.

Take for instance

$$\Gamma_n(s) = \begin{cases} (s+1), & \text{for } s < \frac{-n-1}{n} \\ \dfrac{s}{n+1}, & \text{for } \frac{-n-1}{n} \leq s \leq \frac{n+1}{n} \\ (s-1), & \text{for } s > \frac{n+1}{n}. \end{cases}$$

Since Γ_n is Lipschitz, thanks to Theorem 6.1.2, for any $f \in L^1\left(\mathbb{R}^N\right)$ and any $n \in \mathbb{N}$, there exists a unique L^1-solution $u_n \in \mathcal{C}\left([0,\infty); L^1\left(\mathbb{R}^N\right)\right)$ of the approximate problem

$$\partial_t u_n = J * \Gamma_n(u_n) - \Gamma_n(u_n) \tag{6.20}$$

with initial data $u_n(0) = f$. Moreover, $\Gamma(u_n) \in \mathcal{C}\left([0,\infty); L^1\left(\mathbb{R}^N\right)\right)$, and hence, $u_n \in \mathcal{C}^1\big([0,\infty); L^1\left(\mathbb{R}^N\right)\big)$. Thanks to Theorem 6.1.13, the conservation of energy also holds for the solutions u_n.

Now we state the L^1-contraction property for the approximate problem. This property can only be obtained if Γ_n is strictly decreasing, and this property is needed in order to obtain that $\chi_{\{r>s\}} = \chi_{\{\Gamma_n(r)>\Gamma_n(s)\}}$.

Lemma 6.1.14. *Let $u_{n,1}$ and $u_{n,2}$ be two L^1-solutions of* (6.20) *with initial data f_1, $f_2 \in L^1\left(\mathbb{R}^N\right)$. Then,*

$$\int_{\mathbb{R}^N} (u_{n,1}(t) - u_{n,2}(t))_+ dx \leq \int_{\mathbb{R}^N} (f_1 - f_2)_+ dx, \quad \forall t \geq 0, \tag{6.21}$$

$$\int_{\mathbb{R}^N} (u_{n,1}(t) - u_{n,2}(t))_- dx \leq \int_{\mathbb{R}^N} (f_1 - f_2)_- dx, \quad \forall t \geq 0, \tag{6.22}$$

and

$$\|(u_{n,1} - u_{n,2})(t)\|_{L^1(\mathbb{R}^N)} \leq \|f_1 - f_2\|_{L^1(\mathbb{R}^N)}, \quad \forall t \geq 0. \tag{6.23}$$

Proof. We begin by proving a contraction property for the positive part $(u_{n,1} - u_{n,2})_+$. To do so, we subtract the equations for $u_{n,1}$ and $u_{n,2}$ and multiply by $\chi_{\{u_{n,1}>u_{n,2}\}}$. Since $u_{n,1}-u_{n,2} \in \mathcal{C}^1\big([0,\infty); L^1(\mathbb{R}^N)\big)$, then in the sense of distributions, we have that

$$\partial_t(u_{n,1} - u_{n,2})\chi_{\{u_{n,1}>u_{n,2}\}} = \partial_t(u_{n,1} - u_{n,2})_+.$$

On the other hand, since $0 \leq \chi_{\{u_{n,1}>u_{n,2}\}} \leq 1$, we have

$$J * (\Gamma_n(u_{n,1}) - \Gamma_n(u_{n,2}))\chi_{\{u_{n,1}>u_{n,2}\}} \leq J * (\Gamma_n(u_{n,1}) - \Gamma_n(u_{n,2}))_+.$$

Finally, since Γ_n is strictly monotone, $\chi_{\{u_{n,1}>u_{n,2}\}} = \chi_{\{\Gamma_n(u_{n,1})>\Gamma_n(u_{n,2})\}}$. Thus,

$$(\Gamma_n(u_{n,1}) - \Gamma_n(u_{n,2}))\chi_{\{u_{n,1}>u_{n,2}\}} = (\Gamma_n(u_{n,1}) - \Gamma_n(u_{n,2}))_+.$$

We end up with

$$\partial_t(u_{n,1} - u_{n,2})_+ \leq J * (\Gamma_n(u_{n,1}) - \Gamma_n(u_{n,2}))_+ - (\Gamma_n(u_{n,1}) - \Gamma_n(u_{n,2}))_+.$$

Integrating in space, and using Fubini's Theorem, which can be applied, since $(\Gamma_n(u_{n,1}(t)) - \Gamma_n(u_{n,2}(t)))_+ \in L^1(\mathbb{R}^N)$, we get

$$\partial_t \int_{\mathbb{R}^N} (u_{n,1} - u_{n,2})_+(t) \leq 0\,,$$

which implies

$$\int_{\mathbb{R}^N} (u_{n,1}(t) - u_{n,2}(t))_+ dx \leq \int_{\mathbb{R}^N} (f_1 - f_2)_+ dx\,.$$

Then, a similar computation gives the contraction for the negative parts, so that the L^1-contraction holds. $\square$

Then we deduce the L^1-contraction property for the original problem after passing to the limit.

Corollary 6.1.15. *Let u_1 and u_2 be two L^1- solutions of* (6.1) *with initial data f_1, $f_2 \in L^1(\mathbb{R}^N)$. Then for every $t \geq 0$,*

$$\|(u_1 - u_2)(t)\|_{L^1(\mathbb{R}^N)} \leq \|f_1 - f_2\|_{L^1(\mathbb{R}^N)}\,, \tag{6.24}$$

and the same result holds for the positive part of $(u_1 - u_2)$,

$$\|(u_1 - u_2)_+(t)\|_{L^1(\mathbb{R}^N)} \leq \|(f_1 - f_2)_+\|_{L^1(\mathbb{R}^N)}\,, \tag{6.25}$$

and for the negative part of $(u_1 - u_2)$,

$$\|(u_1 - u_2)_-(t)\|_{L^1(\mathbb{R}^N)} \leq \|(f_1 - f_2)_-\|_{L^1(\mathbb{R}^N)}\,. \tag{6.26}$$

Proof. Passing to the limit in the approximated problems requires some compactness argument which is obtained through the Fréchet-Kolmogorov criterium.

The first step is to prove that the solutions u_n of (6.20) converge to the solution u of (6.1). Let ω be an open set whose closure is contained in $\mathbb{R}^N \times (0, \infty)$, $\omega \subset\subset \mathbb{R}^N \times (0, \infty)$. By the conservation of energy, Theorem 6.1.13, $\|u_n(t)\|_{L^1(\mathbb{R}^N)} = \|f\|_{L^1(\mathbb{R}^N)}$. Hence $\{u_n\}$ is uniformly bounded in $L^1(\omega)$. Therefore, in order to apply Fréchet-Kolmogorov's compactness criterium, we prove that

$$I = \iint_\omega |u_n(x + h,\, t + s) - u_n(x, t)|\, dx\, dt \tag{6.27}$$

goes to zero as h and s go to zero.

On one hand, thanks to the L^1-contraction property, and since the translations are continuous in $L^1(\mathbb{R}^N)$, we get

$$\begin{aligned}\int_0^T\int_{\mathbb{R}^{\mathbb{N}}} |u_n(x+h,\, t+s) - u_n(x, t+s)|\, dx\, dt \\ \leq \int_0^T\int_{\mathbb{R}^{\mathbb{N}}} |f(x+h) - f(x)|\, dx\, dt.\end{aligned} \tag{6.28}$$

Then (6.28) goes to zero as h goes to 0, uniformly in s and n. On the other hand, using the regularity in time, then Fubini's Theorem, and finally the fact that $|\Gamma_n(s)| \leq |s|$ and the L^1-contraction property, we get

$$\begin{aligned}&\int_0^T \int_{\mathbb{R}^{\mathbb{N}}} |u_n\,(x,\, t+s) - u_n(x,t)|\, dx\, dt \\ &\quad\leq \int_0^T \int_{\mathbb{R}^{\mathbb{N}}} \int_t^{t+s} |\partial_t u_n|(x,\tau)\, d\tau\, dx\, dt \\ &\quad\leq \int_0^T \int_t^{t+s} \int_{\mathbb{R}^{\mathbb{N}}} |J * \Gamma_n(u_n) - \Gamma_n(u_n)|(x,\tau)\, dx\, d\tau\, dt \\ &\quad\leq \int_0^T \int_t^{t+s} \Big(\|J\|_{L^\infty(\mathbb{R}^N)} + 1\Big)\, \|\Gamma_n(u_n)(\tau)\|_{L^1(\mathbb{R}^N)}\, d\tau\, dt \\ &\quad\leq s\,T\,\Big(\|J\|_{L^\infty(\mathbb{R}^N)} + 1\Big)\, \|f\|_{L^1(\mathbb{R}^N)}.\end{aligned} \tag{6.29}$$

Taking T such that $\omega \subset \mathbb{R}^N \times (0,T)$, and using the estimates (6.28) and (6.29) we get that (6.27) goes to 0 as h and s go to 0.

Summarizing, along a subsequence (still noted u_n), $u_n \to \eta$ in $L^1_{loc}(\mathbb{R}^N \times (0,\infty))$ for some function η. Moreover: (i) since the sequence $\{u_n(t)\}$ is uniformly bounded in $L^1(\mathbb{R}^N)$, we deduce from Fatou's lemma that for almost every $t > 0$, $\eta(t) \in L^1(\mathbb{R}^N)$; (ii) using that the nonlinearities Γ_n are uniformly Lipschitz, and their uniform convergence, we get that $\Gamma_n(u_n) \to \Gamma(\eta)$ in $L^1_{loc}(\mathbb{R}^N \times (0,\infty))$; (iii) as a consequence, since J is compactly supported, $J * \Gamma_n(u_n) \to J * \Gamma(\eta)$ in $L^1_{loc}(\mathbb{R}^N \times (0,\infty))$. All this is enough to pass to the limit in the integrated version of (6.20),

$$\eta(t) = f + \int_0^t (J * \Gamma(\eta(s)) - \Gamma(\eta(s))ds,$$

for almost every $t > 0$. In particular the integral

$$I(t) = \int_0^t \big(J * \Gamma(\eta(s)) - \Gamma(\eta(s))\big)ds \tag{6.30}$$

makes sense for every $t > 0$, and it is continue in time t with values in $L^1(\mathbb{R}^N)$. Let us prove this below. Let $0 < t < s$, then

$$\|I(t) - I(s)\|_{L^1(\mathbb{R}^N)} \leq \int_t^s \|J * \eta(r) - \eta(r)\|_{L^1(\mathbb{R}^N)}dr \leq \int_t^s 2\|\eta(r)\|_{L^1(\mathbb{R}^N)}dr. \tag{6.31}$$

Since $\eta \in L^1_{loc}(\mathbb{R}^N \times (0,\infty))$ and $\eta(t) \in L^1(\mathbb{R}^N)$ for all $t > 0$, then (6.31) is bounded and $\|I(t) - I(s)\|_{L^1(\mathbb{R}^N)}$ goes to zero as s goes to t. Hence, we can extend $\eta(t)$ to all $t > 0$ by continuity, so that it belongs to the space $\mathcal{C}^1\big([0,\infty); L^1(\mathbb{R}^N)\big)$, we get that η is the L^1-solution to (3.1) with initial data f, i.e., $\eta = u$. As a consequence, convergence is not restricted to a subsequence.

Now we turn to the contraction property. Let u_1, u_2 be the L^1-solutions with initial data f_1 and f_2 respectively. We approximate them by the above procedure, which yields sequences $u_{n,i}$, $i = 1, 2$, such that $u_{n,i} \to u_i$ in $L^1_{loc}((0,\infty), L^1(\mathbb{R}^N))$ (and hence a.e.). The approximations satisfy (6.23). Using Fatou's lemma to pass to the limit in the inequality (6.23), we get that (6.24) holds for almost every $t \geq 0$. Finally, since the solutions are in $C([0,\infty); L^1(\mathbb{R}^N))$, we deduce that this inequality holds for any $t \geq 0$.

The contractions (6.25) and (6.26) are obtained as above, taking into account that the approximations $u_{n,i}$ satisfy (6.21) and (6.22), respectively. $\square$

The following Lemma shows that the positive and negative parts of $\Gamma(u)$ are subcaloric, in the sense that (6.32) is satisfied.

Lemma 6.1.16. *Let $f \in L^1\left(\mathbb{R}^N\right)$ and u the corresponding L^1-solution. Then the functions $\big(\Gamma(u)\big)_-$, $\big(\Gamma(u)\big)_+$ and $|\Gamma(u)|$ all satisfy the inequality:*

$$\chi_t \leq J * \chi - \chi \tag{6.32}$$

in the sense of distributions a.e. in $\mathbb{R}^N \times (0,\infty)$.

Proof. We do the computation for $\chi = |\Gamma(u)| = (|u| - 1)_+$, with the proof being the same for the other functions. We take $\omega \in \mathcal{C}_c^\infty(\mathbb{R}^N \times [0,\infty))$, we consider a test function $\varphi \in \mathcal{C}_c^\infty(\mathbb{R}^N \times (0,\infty))$, then

$$\begin{aligned}
\langle |\Gamma(\omega)|, \varphi_t \rangle &= \int_0^\infty \int_{\mathbb{R}^N} (|\omega| - 1)_+(x,s)\varphi_t(x,s)\,dx\,ds \\
&= \int_{\{(x,t):\omega(x,t)>1\}} (\omega - 1)(x,s)\varphi_t(x,s)\,dx\,ds + \int_{\{(x,t):\omega(x,t)<-1\}} (-\omega + 1)(x,s)\varphi_t(x,s)\,dx\,ds.
\end{aligned} \tag{6.33}$$

Integrating by parts (6.33), and since φ has compact support in $\mathbb{R}^N \times (0,\infty)$, the terms in the boundary disappear, and we get

$$\begin{aligned}
\langle |\Gamma(\omega)|, \varphi_t \rangle &= -\int_{\{(x,t):\omega>1\}} \omega_t(x,s)\varphi(x,s)\,dx\,ds - \int_{\{(x,t):\omega<-1\}} -\omega_t(x,s)\varphi(x,s)\,dx\,ds \\
&= -\int_{\{(x,t):|\omega|>1\}} sign(\omega)\omega_t(x,s)\varphi(x,s)\,dx\,ds
\end{aligned}$$

Therefore, we have,

$$|\Gamma(\omega)|_t = \operatorname{sign}(\omega)\, \chi_{\{|\omega|>1\}}\, \omega_t \tag{6.34}$$

in the sense of distributions and for any $\omega \in \mathcal{C}_c^\infty(\mathbb{R}^N \times [0,\infty))$. Now, since $\mathcal{C}_c^\infty(\mathbb{R}^N \times [0,\infty))$ is dense in $\mathcal{C}^1([0,\infty); L^1\left(\mathbb{R}^N\right))$, given u in $\mathcal{C}^1([0,\infty); L^1\left(\mathbb{R}^N\right))$, we consider a sequence of functions $\omega^n \in \mathcal{C}_c^\infty(\mathbb{R}^N \times [0,\infty))$ such that ω^n converges to u and in $\mathcal{C}^1\big([0,\infty); L^1\left(\mathbb{R}^N\right)\big)$ as n goes to ∞. Moreover, since $\operatorname{sign}(\omega^n)\,\chi_{\{|\omega^n|>1\}}\,\omega_t$ converges to $\operatorname{sign}(u)\,\chi_{\{|u|>1\}}\,u_t$ in $\mathcal{C}^1\big([0,\infty); L^1\left(\mathbb{R}^N\right)\big)$. Then $|\Gamma(\omega^n)|_t$ converges to $|\Gamma(u)|_t = \operatorname{sign}(u)\,\chi_{\{|u|>1\}}\,u_t$ in the sense of distributions.

Now, given $u \in \mathcal{C}^1\big([0,\infty); L^1\left(\mathbb{R}^N\right)\big)$, let us see below that $|\Gamma(u)|_t \leq J * |\Gamma(u)| - |\Gamma(u)|$ in the sense of distributions:

- On the set $\{(x,t) : |u| \leq 1\}$ we have $|\Gamma(u)| = |\Gamma(u)|_t = 0$ while $0 \leq J * |\Gamma(u)|$, so that the following inequality necessarily holds:

$$|\Gamma(u)|_t \leq J * |\Gamma(u)| - |\Gamma(u)|\,. \tag{6.35}$$

- On the set $\{(x,t) : |u| > 1\}$, using that $\operatorname{sign}(u) J * \Gamma(u) \leq \big|J * \Gamma(u)\big| \leq J * |\Gamma(u)|$ and $\operatorname{sign}(u)\Gamma(u) = |\Gamma(u)|$, we obtain

$$\begin{aligned} |\Gamma(u)|_t &= \operatorname{sign}(u) J * \Gamma(u) - \operatorname{sign}(u)\Gamma(u) \\ &\leq J * |\Gamma(u)| - |\Gamma(u)|\,. \end{aligned} \tag{6.36}$$

Thus from (6.35) and (6.36) we have that

$$|\Gamma(u)|_t \leq J * |\Gamma(u)| - |\Gamma(u)|$$

in the sense of distributions. Moreover, since $|\Gamma(u)|_t$ and $J * |\Gamma(u)| - |\Gamma(u)|$ belong to $L^1(\mathbb{R}^N)$, then $|\Gamma(u)|_t \leq J * |\Gamma(u)| - |\Gamma(u)|$ a.e.. □

Let us prove below, that in fact, $\big(\Gamma(u)\big)_-$, $\big(\Gamma(u)\big)_+$ and $|\Gamma(u)|$ are subsolutions of the problem

$$V_t = J * V - V, \quad V(0) = |\Gamma(f)| \in L^1(\mathbb{R}^N). \tag{6.37}$$

But first, we need first to give the definition of supersolution and subsolution.

Definition 6.1.17. *We say that $\overline{V} \in \mathcal{C}([a,b], L^1(\mathbb{R}^N))$ is a* **supersolution** *to* (6.37) *with initial data $|\Gamma(f)|$ in $[a,b]$, if for $t \geq s$, with $s,t \in [a,\, b]$*

$$\overline{V}(\cdot,t) \geq \overline{V}(\cdot,s) + \int_s^t \big(J * \overline{V}(r) - \overline{V}(r)\big)\,dr.$$

We say that $\underline{V}$ is a **subsolution** *if the reverse inequality holds.*

We prove below that $\big(\Gamma(u)\big)_-$, $\big(\Gamma(u)\big)_+$ and $|\Gamma(u)|$ are subsolutions of (6.37).

Lemma 6.1.18. *Let $f \in L^1\left(\mathbb{R}^N\right)$ and u the corresponding L^1-solution of* (6.1). *Then the functions $\big(\Gamma(u)\big)_-$, $\big(\Gamma(u)\big)_+$ and $|\Gamma(u)|$ all are subsolutions of*

$$V_t = J * V - V, \quad V(0) = |\Gamma(f)|. \tag{6.38}$$

Proof. We do the computation for $\chi = |\Gamma(u)| = sg(u)(|u| - 1)_+$, with the proof being the same for the other functions. Since u is the L^1-solution of (6.1), then $u \in \mathcal{C}^1\big([0,\infty), L^1(\mathbb{R}^N)\big)$, and $u \in W^{1,1}\big(\mathbb{R}^N \times [0,T]\big)$, for all $T > 0$. Thanks to Definition 6.1.11, we have that $|\Gamma(u)| \in W^{1,1}\big(\mathbb{R}^N \times [0,T]\big)$, for all $T > 0$. From Lemma 6.1.16, we have that

$$|\Gamma(u)|_t \leq J * |\Gamma(u)| - |\Gamma(u)| \quad \text{a.e.} \tag{6.39}$$

We integrate (6.39) in $[s,t]$,

$$\int_s^t |\Gamma(u)|_t ds \leq \int_s^t (J * |\Gamma(u)(s)| - |\Gamma(u)(s)|)\, ds, \quad \text{a.e.},$$

and thanks to Theorem 6.1.12, we have that

$$|\Gamma(u)|(t) \leq |\Gamma(u)|(s) + \int_s^t \big(J * |\Gamma(u)(s)| - |\Gamma(u)(s)|\big) ds.$$

Thus, the result. □

The following result states that the solution of $V_t = J * V - V$, with initial data integrable, goes to zero asymptotically like $ct^{-N/2}$. (See the proof in [12, Th. A.1])

Theorem 6.1.19. *Let $g \in L^1(\mathbb{R}^N)$, let V be the solution of*

$$V_t = J * V - V, \qquad V(0) = g,$$

and let h be the solution of the local problem $h_t(t) = \Delta h(t)$ with the same initial condition $h(0) = g$. then there exists a function $\varepsilon(t) \to 0$ (depending on J and N) such that

$$t^{N/2} \max_{\mathbb{R}^N} |V(t) - e^{-t} g - h(t)| \leq \|g\|_{L^1(\mathbb{R}^N)} \varepsilon(t).$$

The property of the previous Lemma 6.1.18 allows to estimate the size of the solution in terms of the L^∞-norm of the initial data. In particular, we have that if the initial data $\|f\|_{L^\infty(\mathbb{R}^N)} \leq 1$, we have that $u(t) = f$ for any $t > 0$. Observe also that since $\int_{\mathbb{R}^N} J = 1$, the constant functions are solutions of (6.37).

Lemma 6.1.20. *Let $f \in L^1\left(\mathbb{R}^N\right) \cap L^\infty\left(\mathbb{R}^N\right)$. Then the L^1- solution u of* (6.1) *satisfies $\|u(t)\|_{L^\infty(\mathbb{R}^N)} \leq \|f\|_{L^\infty(\mathbb{R}^N)}$ for any $t > 0$. Moreover,*

$$\limsup_{t\to\infty} u(t) \leq 1 \ \textit{ and } \ \liminf_{t\to\infty} u(t) \geq -1 \textit{ a.e. in } \mathbb{R}^N.$$

Proof. First, the result is obvious if $\|f\|_{L^\infty(\mathbb{R}^N)} \leq 1$, since in this case $u(t) = f$ for any $t > 0$. So let us assume that $\|f\|_{L^\infty(\mathbb{R}^N)} > 1$. Since $\chi = |\Gamma(u)|$ is subcaloric by Lemma 6.1.16, we may compare it with the solution V of the following problem:

$$V_t = J * V - V, \quad V(0) = |\Gamma(f)| \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N). \tag{6.40}$$

The constant functions are solutions of (6.40), and $0 \leq V(t) \leq \|V(0)\|_\infty = \|\Gamma(f)\|_\infty$. Now, from Lemma 6.1.18, we know that $\chi = |\Gamma(u)|$ is a subsolution of (6.40), and thanks to Proposition 4.1.9 we obtain

$$0 \leq \|\chi(t)\|_{L^\infty(\mathbb{R}^N)} \leq \|V(t)\|_{L^\infty(\mathbb{R}^N)} \leq \|\Gamma(f)\|_{L^\infty(\mathbb{R}^N)} = \|f\|_{L^\infty(\mathbb{R}^N)} - 1.$$

Therefore, $\|u(t)\|_{L^\infty(\mathbb{R}^N)} \leq 1 + \|\chi(t)\|_{L^\infty(\mathbb{R}^N)} \leq \|f\|_{L^\infty(\mathbb{R}^N)}$.

Thanks to Theorem 6.1.19, we have that V goes to zero asymptotically like $ct^{-N/2}$, and then $\Gamma(u) \to 0$ almost everywhere, which implies the result. □

6.1.2 Free boundaries

In the sequel, unless we say explicitly something different, we will be dealing with L^1-solutions. Since the functions we are handling are in general not continuous in the space variable, their support has to be considered in the distributional sense. To be precise, for any locally integrable and nonnegative function g in $\mathbb{R}^N$, we can consider the distribution T_g associated to the function g. Then the distributional support of g, $\mathrm{supp}_{\mathcal{D}'}(g)$ is defined as the support of T_g:

$\mathrm{supp}_{\mathcal{D}'}(g) := \mathbb{R}^N \backslash \mathcal{O}$, where $\mathcal{O} \subset \mathbb{R}^N$ is the biggest open set such that $T_g|\,\mathcal{O} = 0$.
In the case of nonnegative locally integrable functions g, this means that $x \in \mathrm{supp}_{\mathcal{D}'}(g)$ if and only if

$$\forall \varphi \in \mathcal{C}_c^\infty(\mathbb{R}^N),\ \varphi \geq 0 \text{ not identically zero },\ \text{ happens that } \int_{\mathbb{R}^N} g(y)\varphi(y)dy > 0.$$

If g is continuous, then the support of g is nothing but the usual closure of the positivity set, $\mathrm{supp}_{\mathcal{D}'}(g) = \overline{\{g > 0\}}$.

We first prove that the solution does not move far away from the support of $\Gamma(u)$.

Lemma 6.1.21. *Let $f \in L^1\left(\mathbb{R}^N\right)$. Then,* $\mathrm{supp}_{\mathcal{D}'}(u_t(t)) \subset \mathrm{supp}_{\mathcal{D}'}(\Gamma(u)(t)) + B_{R_J}$ *for any $t > 0$, where B_{R_J} is the support of J.*

Proof. Recall first that the equation holds down to $t = 0$ so that we may consider here $t \geq 0$ (and not only $t > 0$). Let $\varphi \in \mathcal{C}_c^\infty(A^c)$, where $A = \mathrm{supp}_{\mathcal{D}'}(\Gamma(u)(t)) + B_{R_J}$. Notice that the support of $J * \Gamma(u)$ (which is a continuous function) lies inside A, so that

$$\int_{\mathbb{R}^N} (J * \Gamma(u))\varphi = 0.$$

Similarly, the supports of $\Gamma(u)$ and φ do not intersect, so that

$$\int_{\mathbb{R}^N} u_t\varphi = \int_{\mathbb{R}^N} (J * \Gamma(u))\varphi - \int_{\mathbb{R}^N} \Gamma(u)\varphi = 0,$$

which means that the support of u_t is contained in A. □

The following Theorem gives a control of the support of the solution $u(t)$ and the corresponding temperature $\Gamma(u)(t)$.

Theorem 6.1.22. *Let $f \in L^1(\mathbb{R}^N)$ be compactly supported. Then, for any $t > 0$, the solution $u(t)$ of* (6.1) *and the corresponding temperature $\Gamma(u)(t)$ are compactly supported.*

Proof. Estimate of the support of $\Gamma(u)$. Since $|\Gamma(u)|$ is subcaloric, we have that $\|\Gamma(u)\|_{L^1(\mathbb{R}^N)} \leq \|\Gamma(f)\|_{L^1(\mathbb{R}^N)}$, then

$$J * |\Gamma(u)| \leq \|J\|_{L^\infty(\mathbb{R}^N)}\|\Gamma(u)\|_{L^1(\mathbb{R}^N)} \leq \|J\|_{L^\infty(\mathbb{R}^N)}\|\Gamma(f)\|_{L^1(\mathbb{R}^N)}. \tag{6.41}$$

We denote $c_0 = \|J\|_{L^\infty(\mathbb{R}^N)}\|\Gamma(f)\|_{L^1(\mathbb{R}^N)}$. Now, we multiply equation (6.1) by $\mathrm{sign}(u)$, then

$$\mathrm{sign}(u)u_t = \mathrm{sign}(u)J * \Gamma(u) - \mathrm{sign}(u)\Gamma(u) \tag{6.42}$$

Since $\operatorname{sign}(u)J * \Gamma(u) \le |J * \Gamma(u)| \le J * |\Gamma(u)|$, and $\operatorname{sign}(u)\Gamma(u) = |\Gamma(u)|$, from (6.42) we have that

$$\operatorname{sign}(u)u_t \le J * |\Gamma(u)| - |\Gamma(u)|. \tag{6.43}$$

Since $\operatorname{sign}(u)u_t = |u|_t$ in the sense of distributions and integrating (6.43) in $[0,t]$,

$$\int_0^t \operatorname{sign}(u)u_t ds \le \int_0^t (J * |\Gamma(u)(s)| - |\Gamma(u)(s)|)\, ds, \quad \text{a.e.},$$

and thanks to Theorem 6.1.12, we have that

$$|u(t)| \le |f| + \int_0^t \big(J * |\Gamma(u)|(s)\big) - |\Gamma(u)|(s) ds \tag{6.44}$$

Multiplying (6.44) by a nonnegative test function $\varphi \in \mathcal{C}_c^\infty((\operatorname{supp}_{\mathcal{D}'} f)^c)$, integrating in space, and from (6.41), we have

$$\int_{\mathbb{R}^N} |u(t)|\varphi \le \int_0^t \int_{\mathbb{R}^N} (J * |\Gamma(u)|)\varphi \le c_0 t \int_{\mathbb{R}^N} \varphi.$$

Taking $t_0 = 1/c_0$, we get $\int_{\mathbb{R}^N} (|u(t)| - 1)\varphi \le 0$ for all $t \in [0, t_0]$, for any nonnegative test function $\varphi \in \mathcal{C}_c^\infty((\operatorname{supp}_{\mathcal{D}'} f)^c)$, then $|u(t)| - 1 \le 0$. Thus, $\Gamma(u(t)) = 0$, for all $t \in [0, t_0]$ in $(\operatorname{supp}_{\mathcal{D}'} f)^c$. Therefore

$$\operatorname*{supp}_{\mathcal{D}'}(\Gamma(u)(t)) \subset \operatorname*{supp}_{\mathcal{D}'}(f), \quad \text{for all } t \in [0, t_0]. \tag{6.45}$$

Estimate of the support of u. Thanks to Lemma 6.1.21 we know that $\operatorname{supp}_{\mathcal{D}'}(u_t(t)) \subset \operatorname{supp}_{\mathcal{D}'}(\Gamma(u)(t)) + B_{R_J} \subset \operatorname{supp}_{\mathcal{D}'}(f) + B_{R_J}$, for all $t \in [0, t_0]$. This means that for any $\varphi \in \mathcal{C}_c^\infty((\operatorname{supp}_{\mathcal{D}'}(f) + B_{R_J})^c)$, we have,

$$\int_{\mathbb{R}^N} u\varphi = \int_{\mathbb{R}^N} u\varphi - \int_{\mathbb{R}^N} f\varphi = \int_0^t \int_{\mathbb{R}^N} u_t\varphi = 0, \quad \text{for all } t \in [0, t_0]$$

that is,

$$\operatorname*{supp}_{\mathcal{D}'}(u(t)) \subset \operatorname*{supp}_{\mathcal{D}'}(f) + B_{R_J}, \quad \text{for all } t \in [0, t_0]. \tag{6.46}$$

Iteration. Consider now $u(t_0)$ as initial data at time t_0, whose support satisfies that,

$$\operatorname*{supp}_{\mathcal{D}'}(u(t_0)) \subset \operatorname*{supp}_{\mathcal{D}'}(f) + B_{R_J},$$

then, thanks to (6.45) and (6.46), repeating the arguments, we obtain

$$\operatorname*{supp}_{\mathcal{D}'}(u(t)) \subset \operatorname*{supp}_{\mathcal{D}'}(f) + 2\, B_{R_J}, \quad \text{for all } t \in [0, 2\, t_0].$$

Iterating this process we arrive to,

$$\operatorname*{supp}_{\mathcal{D}'}(\Gamma(u)(t)) \subset \operatorname*{supp}_{\mathcal{D}'}(f) + nB_{R_J}, \quad \text{with } n = \lfloor t/t_0 \rfloor,$$

and

$$\operatorname*{supp}_{\mathcal{D}'}(u(t)) \subset \operatorname*{supp}_{\mathcal{D}'}(f) + nB_{R_J}, \quad \text{with } n = \lfloor t/t_0 \rfloor + 1,$$

where $\lfloor x \rfloor$ is the integer part of x. Thus, we have proved that the speed of expansion of of the support of $u(t)$ is less or equal to R_J/t_0. $\square$

The last results have counterparts for $\mathcal{C}_b$-solutions:

Theorem 6.1.23. *Let $f \in \mathcal{C}_b(\mathbb{R}^N)$, and let u be the corresponding $\mathcal{C}_b$-solution. Then*

(i) $\sup(u_t(t)) \subset \sup(\Gamma(u)(t)) + B_{R_J}$ *for all* $t > 0$.

(ii) *If* $\sup_{|x|\geq R} |f(x)| < 1$ *for some* $R > 0$*, then* $\Gamma(u)(\cdot, t)$ *is compactly supported for all* $t > 0$*. If moreover* $f \in \mathcal{C}_c(\mathbb{R}^N)$*, then* $u(\cdot, t)$ *is also compactly supported for all* $t > 0$.

Proof. (i) The proof is similar (though even easier, since the supports are understood in the classical sense) to the one for L^1-solutions.
(ii) Since $\chi = |\Gamma(u)|$ is subcaloric, we get

$$\Big|\big(J * \Gamma(u)\big)(x, t)\Big| \leq \|J\|_{L^1(\mathbb{R}^N)}\|\Gamma(u)(t)\|_{L^\infty(\mathbb{R}^N)} \leq \|\Gamma(f)\|_{L^\infty(\mathbb{R}^N)}. \tag{6.47}$$

This estimate comes from comparison in L^∞ with constants, exactly as in Lemma 6.1.20. Therefore, from the integral equation, (6.7)

$$\begin{aligned} u(x, t) &= f(x) + \int_0^t \big(J * \Gamma(u)(s) - \Gamma(u)(s)\big)ds \\ &\leq f(x) + \int_0^t \|\Gamma(u)(s)\|_{L^\infty(\mathbb{R}^N)} + \|\Gamma(u)(s)\|_{L^\infty(\mathbb{R}^N)}ds \end{aligned} \tag{6.48}$$

Hence, thanks to (6.47), (6.48) and hypothesis in (ii), for $|x| \geq R$ we have

$$\begin{cases} u(x, t) \leq f(x) + t\,2\|\Gamma(f)\|_{L^\infty(\mathbb{R}^N)} \leq \sup\limits_{|x|\geq R} |f(x)| + t\,2\|\Gamma(f)\|_{L^\infty(\mathbb{R}^N)}\,, \\ u(x, t) \geq f(x) - t\,2\|\Gamma(f)\|_{L^\infty(\mathbb{R}^N)} \geq -\sup\limits_{|x|\geq R} |f(x)| - t\,2\|\Gamma(f)\|_{L^\infty(\mathbb{R}^N)}\,. \end{cases} \tag{6.49}$$

Thus, for all $|x| \geq R$ and $t \leq \big(1 - \sup_{|x|\geq R} |f(x)|\big)/\big(2\|\Gamma(f)\|_{L^\infty(\mathbb{R}^N)}\big)$ we have $-1 < u(x, t) < 1$. Hence, for such x, t, we have $\Gamma(u)(x, t) = 0$. Then, by (i), $u(x, t) = f(x)$ for all $|x| \geq R + R_J$ and $t = \big(1 - \sup_{|x|\geq R} |f(x)|\big)/\big(2\|\Gamma(f)\|_{L^\infty(\mathbb{R}^N)}\big)$. We finally proceed by iteration to get the result for all times. $\square$

6.2 First results concerning the asymptotic behavior

In the following sections we study the asymptotic behavior of the solutions of the two-phase Stefan problem, with different sign-changing initial data chosen in such a way that the solutions, $u(t)$, satisfy either:

(i) the positive and negative part do not interact, in any time $t > 0$;

(ii) the positive and negative temperature $v = \Gamma(u)$ do not interact, in any time $t > 0$;

(iii) the positive and negative part of $\Gamma(u)$ interact but the solution is driven by the one-phase Stefan regime after some time.

In order to describe the asymptotic behavior, we write the initial data as

$$f = f_+ - f_-,$$

separating the positive and negative parts where we recall the notations $f_+ = \max(f, 0)$ and $f_- = \max(-f, 0)$. Let us first introduce the following solutions of (6.1): the solution $\mathbb{U}^+$, corresponds to the initial data $\mathbb{U}^+(0) = f_+$ and the solution $\mathbb{U}^-$, corresponds to the initial data $\mathbb{U}^-(0) = f_-$.

Lemma 6.2.1. *The functions $\mathbb{U}^+$ and $\mathbb{U}^-$ are solutions of the* **one-phase Stefan problem***:*

$$\partial_t u = J * (u-1)_+ - (u-1)_+ \,.$$

Proof. By comparison in L^1 for the two-phase Stefan problem (see Proposition 6.1.4), we know that $\mathbb{U}^+$ and $\mathbb{U}^-$ are nonnegative because their respective initial data are nonnegative. Hence, for any (x,t) we have in fact $\Gamma(\mathbb{U}^+(x,t)) = (\mathbb{U}(x,t)-1)_+$. Thus, the equation for $\mathbb{U}^+$ reduces to the one-phase Stefan problem. The same happens for $\mathbb{U}^-$. □

Lemma 6.2.2. *Let $\mathbb{U}^+$ be a solution of the one-phase Stefan problem, the supports of $\mathbb{U}^+$ and $\Gamma(\mathbb{U}^+)$ are nondecreasing*

$$\begin{array}{ll} \mathrm{supp}_{\mathcal{D}'}(\mathbb{U}^+(s)) \subset \mathrm{supp}_{\mathcal{D}'}(\mathbb{U}^+(t)), & 0 \le s \le t \\ \mathrm{supp}_{\mathcal{D}'}(\Gamma(\mathbb{U}^+)(s)) \subset \mathrm{supp}_{\mathcal{D}'}(\Gamma(\mathbb{U}^+)(t)), & 0 \le s \le t. \end{array} \tag{6.50}$$

We denote this property as **retention***. It is satisfied also for $\mathbb{U}^-$ and $\Gamma(\mathbb{U}^-)$.*

Proof. We have

$$\left(\mathbb{U}^+\right)_t = J * (\mathbb{U}^+ - 1)_+ - (\mathbb{U}^+ - 1)_+,$$

since $J \ge 0$ and $(\mathbb{U}^+ - 1)_+ \ge 0$, then

$$\left(\mathbb{U}^+\right)_t \ge -(\mathbb{U}^+ - 1)_+ \ge -\mathbb{U}^+,$$

which after integration, yields

$$\mathbb{U}^+(x,t) \ge \mathbb{U}^+(x,s)e^{-(t-s)}, \quad t \ge s.$$

This implies retention for $\mathbb{U}^+$.

Concerning $\Gamma(\mathbb{U}^+) = (\mathbb{U}^+ - 1)_+$, following the same arguments for $(\mathbb{U}^+)_t$, above, we have that the time derivative of $\Gamma(\mathbb{U}^+)$ in the sense of distributions satisfies

$$\frac{\partial(\mathbb{U}^+ - 1)_+}{\partial t} = \mathbb{U}^+_t \chi_{\{\mathbb{U}^+ > 1\}} \ge -(\mathbb{U}^+ - 1)_+,$$

that is $\Gamma(\mathbb{U}^+)_t \ge -\Gamma(\mathbb{U}^+)$, from where retention follows. □

We shall use the results concerning the asymptotic behavior studied in [12, Cor 3.10, Cor 3.11], of the L^1-norm of the temperature $\Gamma(\mathbb{U}^+)$ and $\Gamma(\mathbb{U}^-)$, of the one phase Stefan problem, and we need to add new hypotheses for J.

Corollary 6.2.3. *Let us assume that $f \in L^1(\mathbb{R}^{\mathbb{N}})$, if J is non increasing in the radial variable, and if $0 \leq f \leq g$ for some $g \in L^1\left(\mathbb{R}^N\right) \cap \mathcal{C}_0(\mathbb{R}^N)$, radial and strictly decreasing in the radial variable. Then there are constants $C, k > 0$ such that*

$$\|\Gamma(\mathbb{U}^+)(t)\|_{L^1(\mathbb{R}^N)} \leq Ce^{-kt}$$

for all $t \geq 0$. It is satisfied also for $\Gamma(\mathbb{U}^-)$.

Corollary 6.2.4. *Let $f \in L^1(\mathbb{R}^{\mathbb{N}})$. Then $\|\Gamma(\mathbb{U}^+)(t)\|_{L^1(\mathbb{R}^N)} = \mathcal{O}(t^{-N/2})$. It is satisfied also for $\Gamma(\mathbb{U}^-)$*

We know that in particular if f satisfies the hypothesis of Corollaries 6.2.3 and 6.2.4, $\mathbb{U}^+$ and $\mathbb{U}^-$ have limits as $t \to \infty$ which are obtained by means of the projection operator $\mathcal{P}$, described in (6.4). We recall that this operator maps any nonnegative initial data f to $\mathcal{P}f$, which is the unique solution to a nonlocal obstacle problem at level one, (6.5). For $\mathbb{U}^+$, the limit is $\mathcal{P}f_+$ and for $\mathbb{U}^-$, the limit is $\mathcal{P}f_-$. Now the relation between u the solution of (6.1), $\mathbb{U}^+$ and $\mathbb{U}^-$ is given in the following result.

Lemma 6.2.5. *For any $t > 0$, $-\mathbb{U}^-(t) \leq -u_-(t) \leq u(t) \leq u_+(t) \leq \mathbb{U}^+(t)$.*

Proof. This result follows from a simple comparison result in $L^1(\mathbb{R}^N)$. Since initially we have $\mathbb{U}^+(0) = f_+ \geq u(0)$, from Proposition 6.1.5, for any $t > 0$, $\mathbb{U}^+(t) \geq u(t)$. On the other hand, since $\mathbb{U}^+(0) = f_+ \geq 0$, thanks to Proposition 6.1.4, we have also that for any $t > 0$, $\mathbb{U}^+(t) \geq 0$, then for any $t > 0$, $\mathbb{U}^+(t) \geq u_+(t)$.

The other inequalities are obtained the same way, since Γ is an odd function. □

This comparison allows us to prove that the asymptotic limit is well-defined:

Proposition 6.2.6. *Let us assume that $f \in L^1(\mathbb{R}^N)$ and if $N = 1, 2$, assume in addition that J is non increasing in the radial variable, and $f_+ \leq g_1$, $f_- \leq g_2$ for some $g_1, g_2 \in L^1\left(\mathbb{R}^N\right) \cap \mathcal{C}_0(\mathbb{R}^N)$, radial and strictly decreasing in the radial variable. Let u be the L^1-solution of (6.1). Then the following limit is defined in $L^1(\mathbb{R}^N)$:*

$$u_\infty(x) := \lim_{t \to \infty} u(x, t) .$$

Proof. From (6.7), we have that

$$u(t) = f + \int_0^t J * \Gamma(u)(s)ds - \int_0^t \Gamma(u)(s)ds . \tag{6.51}$$

Then we recall that under the hypotheses of this proposition, thanks to Corollaries 6.2.3 and 6.2.4 the integrals

$$\int_0^t (\mathbb{U}^+(s) - 1)_+ ds \quad \text{and} \int_0^t (\mathbb{U}^-(s) - 1)_+ ds \;\; \text{converge in} \;\; L^1(\mathbb{R}^N) \;\; \text{as } t \to \infty. \tag{6.52}$$

Using the estimate from Lemma 6.2.5

$$|\Gamma(u)| \leq \max\left\{(\mathbb{U}^+ - 1)_+ \, ; \, (\mathbb{U}^- - 1)_+\right\}, \tag{6.53}$$

then, from (6.52) and (6.53) and we have that

$$\begin{aligned}\int_0^\infty \|\Gamma(u)(s)\|_{L^1(\mathbb{R}^N)}ds &\leq \max\left\{\int_0^\infty \|(\mathbb{U}^+-1)_+\|_{L^1(\mathbb{R}^N)}ds\,;\int_0^\infty \|(\mathbb{U}^--1)_+\|_{L^1(\mathbb{R}^N)}ds\right\}\\ &\leq C.\end{aligned}$$

We obtain that the right-hand side of (6.51) has a limit as $t\to\infty$. Hence we deduce that $u(t)$ has a limit in $L^1(\mathbb{R}^N)$ which can be written as:

$$\lim_{t\to\infty} u(t) = f + \int_0^\infty J*\Gamma(u)(s)ds - \int_0^\infty \Gamma(u)(s)ds := u_\infty(x)\,.$$

□

The question is now to identify this limit u_∞ and we begin with a simple case when the positive and negative parts, $\mathbb{U}^+$ and $\mathbb{U}^-$, never interact. In particular, we assume that the limits of $\mathbb{U}^+$ and $\mathbb{U}^-$, denoted by $\mathcal{P}f_+$ and $\mathcal{P}f_-$, respectively, are at an strictly positive distance, where we consider the distance between two sets as follows: Given $A,B\subset\mathbb{R}^N$

$$\mathrm{dist}(A,B)=\inf_{x\in A,y\in B}|x-y|,$$

which is **not** the Hausdorff distance.

Lemma 6.2.7. *Let us assume that J and f satisfy the hypotheses of Proposition 6.2.6, and that*

$$\mathrm{dist}\big(\mathrm{supp}(\mathcal{P}f_+),\mathrm{supp}(\mathcal{P}f_-)\big)\geq r>0\,. \tag{6.54}$$

Then for any $t>0$, $\mathrm{dist}\big(\mathrm{supp}(u_-(t)),\mathrm{supp}(u_+(t))\big)\geq r$.

Proof. By the retention property (6.50) for $\mathbb{U}^+$ and $\mathbb{U}^-$, and hypothesis (6.54) we first know that for any $t>0$,

$$\mathrm{dist}\big(\mathrm{supp}(\mathbb{U}^+(t)),\mathrm{supp}(\mathbb{U}^-(t))\big)\geq r\,.$$

Thanks to Lemma 6.2.5, we have $0\leq u_+(t)\leq \mathbb{U}^+(t)$. Thus, the support of $u_+(t)$ is contained inside the one of $\mathbb{U}^+(t)$. The same holds for $u_-(t)$ and $\mathbb{U}^-(t)$ so that finally, the supports of $u_-(t)$ and $u_+(t)$ are necessarily at distance at least r, $\forall t\geq 0$. □

Now, we prove the main result.

Theorem 6.2.8. *Let us assume J and f satisfy the hypothesis of Proposition 6.2.6, and let* $\mathrm{supp}(J)=B_{R_J}$. *If*

$$\mathrm{dist}\big(\mathrm{supp}(\mathcal{P}f_+),\mathrm{supp}(\mathcal{P}f_-)\big)>2R_J\,, \tag{6.55}$$

then the solution of (6.1) with initial data f is given by $u(t)=\mathbb{U}^+(t)-\mathbb{U}^-(t)$, and the asymptotic behavior is given by

$$u_\infty(x)=\mathcal{P}f_+(x)-\mathcal{P}f_-(x)\,.$$

Proof. Since the supports of $\mathbb{U}^+(t)$ and $\mathbb{U}^-(t)$ are always at distance greater that $2R_J$, we have that

$$\mathbb{U}(t) = \mathbb{U}^+(t) - \mathbb{U}^-(t).$$

Moreover, the convolution $J * \Gamma(\mathbb{U}(t))$ is either equal to $J * \Gamma(\mathbb{U}^+(t))$, or to $-J * \Gamma(\mathbb{U}^-(t))$, and those last convolutions have disjoint supports. Hence we can also write

$$J * \Gamma(\mathbb{U}(t)) = J * \Gamma(\mathbb{U}^+(t)) - J * \Gamma(\mathbb{U}^-(t)) \,.$$

This implies that $\mathbb{U}$ is actually a solution of the equation:

$$\begin{aligned}\partial_t \mathbb{U} &= \partial_t \mathbb{U}^+ - \partial_t \mathbb{U}^- \\ &= J * \Gamma(\mathbb{U}^+(t)) - \Gamma(\mathbb{U}^+(t)) - J * \Gamma(\mathbb{U}^-(t)) + \Gamma(\mathbb{U}^-(t)) \\ &= J * \Gamma(\mathbb{U}(t)) - \Gamma(\mathbb{U}(t)) \,.\end{aligned}$$

But since $\mathbb{U}(0) = f_+ - f_- = f$, we conclude by uniqueness in $L^1(\mathbb{R}^N)$ that $u \equiv \mathbb{U}$ is the solution of (6.1) □

6.3 Asymptotic behavior when the positive and the negative part of the temperature do not interact

The aim of this section is to identify the limit u_∞, that is the limit of the solution u of (6.1) when time goes to infinity, in the case when the positive and negative part of the temperature, $\Gamma(u)$, never interact, this is,

$$\operatorname{dist}\left(\operatorname*{supp}_{\mathcal{D}'}\left(\Gamma(\mathcal{P}f_+)\right), \operatorname*{supp}_{\mathcal{D}'}(\Gamma\left(\mathcal{P}f_-\right))\right) \geq R_J \,, \tag{6.56}$$

but

$$\operatorname*{supp}_{\mathcal{D}'}\left(\mathcal{P}f_+\right) \cap \operatorname*{supp}_{\mathcal{D}'}\left(\mathcal{P}f_-\right) \neq \emptyset. \tag{6.57}$$

These hypotheses are less restrictive that the ones in the previous section (see hypothesis (6.55) in Theorem 6.2.8).

We know that there exists the retention property for $\mathbb{U}^+$ and $\mathbb{U}^-$, i.e., the supports of $\mathbb{U}^+$ and $\mathbb{U}^-$ are nondecreasing, which holds since these are solutions of the one-phase Stefan problem. We use the Baiocchi transform, to describe the asymptotic behavior of the solution to (6.1), as in [12].

On the other hand, we can not say that the solution is $u(t) = \mathbb{U}^+(t) - \mathbb{U}^-(t)$, like in the example we have studied in the previous section, because the supports of $\mathbb{U}^+$ and $\mathbb{U}^-$ may have a nonempty intersection.

6.3.1 Formulation in terms of the Baiocchi variable

Our next aim is to describe the large time behavior of the solutions of the two-phase Stefan problem satisfying hypotheses (6.56) and (6.57). We make a formulation of the Stefan problem as a parabolic nonlocal biobstacle problem. To identify the asymptotic limit for u, we define the Baiocchi variable, like in [12]

$$w(t) = \int_0^t \Gamma(u)(s)ds.$$

The enthalpy and the temperature can be recovered from w through the formulas

$$u = f + J * w - w, \qquad \Gamma(u) = w_t, \tag{6.58}$$

where the time derivative has to be understood in the sense of distributions.

Lemma 6.3.1. *If*

$$\operatorname{dist}\Big(\operatorname*{supp}_{\mathcal{D}'}\big(\Gamma(\mathcal{P}f_+)\big), \operatorname*{supp}_{\mathcal{D}'}(\Gamma\big(\mathcal{P}f_-)\big)\Big) \geq R_J\,, \tag{6.59}$$

then the function $\Gamma(u)$ *satisfies the following* **retention property**: *for any* $0 < s < t$,

$$\operatorname*{supp}_{\mathcal{D}'}\big(\Gamma(u(s))_+\big) \subset \operatorname*{supp}_{\mathcal{D}'}\big(\Gamma(u(t))_+\big)\,, \quad \operatorname*{supp}_{\mathcal{D}'}\big(\Gamma(u(s))_-\big) \subset \operatorname*{supp}_{\mathcal{D}'}\big(\Gamma(u(t))_-\big)\,. \tag{6.60}$$

As a consequence, we have for any $t > 0$:

$$\operatorname*{supp}_{\mathcal{D}'}\big(\Gamma(u(t))_+\big) = \operatorname*{supp}_{\mathcal{D}'}\big(w(t)_+\big)\,, \quad \operatorname*{supp}_{\mathcal{D}'}\big(\Gamma(u(t))_-\big) = \operatorname*{supp}_{\mathcal{D}'}\big(w(t)_-\big)\,.$$

Proof. We use the same ideas as in the previous section. From Lemma 6.2.5, we have that

$$\operatorname*{supp}_{\mathcal{D}'}\big(\Gamma(u(t))_+\big) \subset \operatorname*{supp}_{\mathcal{D}'}\big(\Gamma(\mathbb{U}^+(t))\big) \quad \text{and} \quad \operatorname*{supp}_{\mathcal{D}'}\big(\Gamma(u(t))_-\big) \subset \operatorname*{supp}_{\mathcal{D}'}\big(\Gamma(\mathbb{U}^-(t))\big). \tag{6.61}$$

From (6.61) and the retention property (6.50) for $\Gamma(\mathbb{U}^+)$ and $\Gamma(\mathbb{U}^-)$, we know that for any $t > 0$, there holds:

$$\operatorname{dist}\Big(\operatorname*{supp}_{\mathcal{D}'}\big(\Gamma(u(t))_+\big); \operatorname*{supp}_{\mathcal{D}'}\big(\Gamma(u(t))_-\big)\Big) \geq \operatorname{dist}\Big(\operatorname*{supp}_{\mathcal{D}'}\big(\Gamma(\mathcal{P}f_+)\big); \operatorname*{supp}_{\mathcal{D}'}(\Gamma\big(\mathcal{P}f_-)\big)\Big)\,, \tag{6.62}$$

and this distance is at least R_J under assumption (6.59). Take now a nonnegative test function $\varphi \in \mathcal{C}^\infty(\mathbb{R}^N)$ (not identically zero) with compact support in $\operatorname{supp}_{\mathcal{D}'}\big(\Gamma(u(s))_+\big)$ and consider $t > s$. Using that $\partial_t \Gamma(u)_+ = \chi_{\{u>0\}}\partial_t u$, in the sense of distributions, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big(\int_{\mathbb{R}^N} \Gamma(u(t))_+ \varphi\Big) = \int_{\mathbb{R}^N} \big(J * \Gamma(u(t))\big)\varphi\chi_{\{u>0\}} - \int_{\mathbb{R}^N} \Gamma(u(t))\varphi\chi_{\{u>0\}}\,.$$

From (6.62), for any $t > 0$, the support of $\Gamma(u(t))_+$ is at least at distance R_J from the support of $\Gamma(u(t))_-$, then we have $\big(J * \Gamma(u(t))\big)\chi_{\{u>0\}} = \big(J * \Gamma(u(t))_+\big) \geq 0$ for any $t > s$. Hence

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big(\int_{\mathbb{R}^N} \Gamma(u(t))_+ \varphi\Big) \geq -\int_{\mathbb{R}^N} \Gamma(u(t))_+ \varphi\,,$$

which can be written as $h'(t) \geq -h(t)$ where $h(t) := \int_{\mathbb{R}^N} \Gamma(u)(t)_+ \varphi$. Hence $h(t) \geq h(s)e^{-(t-s)} > 0$ which proves the retention property for $\Gamma(u)_+$. The property for $\Gamma(u)_-$ is proved analogously.

Now, take a nonnegative test function φ, not identically zero, with compact support in $\mathrm{supp}_{\mathcal{D}'}(\Gamma(u(t))_+)$. We know from the first part that for $0 < s < t$, the support of φ never intersects the support of the negative part of $\Gamma(u(s))$, hence

$$\int_{\mathbb{R}^N} w(t)\varphi\,dx = \int_0^t \int_{\mathbb{R}^N} \Gamma(u(s))\varphi\,dx\,ds = \int_0^t \int_{\mathbb{R}^N} \Gamma(u(s))_+\varphi\,dx\,ds\,.$$

Moreover, since the space integrals are continuous in time, we know that the integral $\int_{\mathbb{R}^N} \Gamma(u(s))_+\varphi\,dx$ is not only positive at time $s = t$, but also in an open time interval around t. So, we get $\int_{\mathbb{R}^N} w(t)\varphi\,dx > 0$ which proves that $\mathrm{supp}_{\mathcal{D}'}\big(\Gamma(u(t))_+\big) \subset \mathrm{supp}_{\mathcal{D}'}\big(w(t)_+\big)$.

On the other hand, if φ is a nonnegative test function such that $\int_{\mathbb{R}^N} \Gamma(u(t))_+\varphi dx = 0$, the retention property, (6.60), implies that this integral is also zero for all times $0 < s < t$, which yields $\int_{\mathbb{R}^N} w_+(t)\varphi dx = 0$. We conclude that the distributional support of $w_+(t)$ coincides with that of $\Gamma(u(t))_+$. The proof is similar for the negative part. □

The Baiocchi variable satisfies a complementary problem, that is introduced in the following result.

Lemma 6.3.2. *Under hypotheses* (6.56) *and* (6.57)*, the Baiocchi variable,* $w(t) = \displaystyle\int_0^t \Gamma(u)(s)ds$*, satisfies the complementary problem almost everywhere*

$$\begin{cases} 0 \le \mathrm{sign}(w)\,(f + J * w - w - w_t) \le 1\,, \\ \big(f + J * w - w - w_t - \mathrm{sign}(w)\big)|w| = 0\,, \\ w(0) = 0\,. \end{cases} \tag{6.63}$$

Proof. The graph condition $\Gamma(u) = \mathrm{sign}(u)(|u| - 1)_+$ can be written as

$$0 \le \mathrm{sign}(u)\big(u - \Gamma(u)\big) \le 1, \quad \big(\mathrm{sign}(u)\big(u - \Gamma(u)\big) - 1\big)\,\Gamma(u) = 0\,,$$

almost everywhere in $\mathbb{R}^N \times (0,\infty)$. In order to translate this condition in the w variable, we first notice that that if $\mathrm{sign}\,\big(\Gamma(u)\big) > 0$ then $\mathrm{sign}(u) > 0$ and similarly, $\mathrm{sign}\,\big(\Gamma(u)\big) < 0$ implies $\mathrm{sign}(u) < 0$ (only the condition $\Gamma(u) = 0$ does not imply a sign condition on u). Hence we can also write

$$0 \le \mathrm{sign}\,\big(\Gamma(u)\big)\big(u - \Gamma(u)\big) \le 1, \quad \big(\mathrm{sign}\,\big(\Gamma(u)\big)\big(u - \Gamma(u)\big) - 1\big)\,\Gamma(u) = 0\,. \tag{6.64}$$

Now we use the retention property of $\Gamma(u)$, Lemma 6.3.1, which implies that the distributional supports of $\Gamma(u)$ and w coincide for all times. Then replacing the equalities (6.58) in terms of w in (6.64), then we have

$$\begin{cases} 0 \le \mathrm{sign}(w)\,(f + J * w - w - w_t) \le 1\,, \\ (\mathrm{sign}(w)\,(f + J * w - w - w_t) - 1)\,w = 0. \end{cases}$$

Therefore, we obtain that w solves a.e. the complementary problem (6.63). □

6.3.2 A nonlocal elliptic biobstacle problem

If $\int_0^\infty \|\Gamma(u)(t)\|_{L^1(\mathbb{R}^N)}\,\mathrm{d}t < \infty$, the function $w(t)$ converges in $L^1\left(\mathbb{R}^N\right)$ as $t \to \infty$ to

$$w_\infty = \int_0^\infty \Gamma(u)(s)ds \in L^1\left(\mathbb{R}^N\right).$$

Thus, thanks to (6.58), $u(\cdot,t)$ converges point-wise and in $L^1\left(\mathbb{R}^N\right)$ to

$$\tilde{f} = f + J * w_\infty - w_\infty.$$

Passing to the limit as $t \to \infty$ in (6.63), we get that w_∞ is a solution with data f to the *nonlocal biobstacle problem*:

$$(\mathrm{BOP}) \qquad \begin{cases} \text{Given a data } f \in L^1\left(\mathbb{R}^N\right), \text{ find a function } w \in L^1\left(\mathbb{R}^N\right) \text{ such that} \\ 0 \le \operatorname{sign}(w)\left(f + J * w - w\right) \le 1\,, \\ \big(f + J * w - w - \operatorname{sign}(w)\big)|w| = 0. \end{cases}$$

This problem is called "biobstacle" since the values of the solution are cut at both levels $+1$ and -1. Under some conditions we have existence:

Lemma 6.3.3. *Let $f \in L^1(\mathbb{R}^N)$ satisfy the hypotheses* (6.56) *and* (6.57)*, and assume in addition, if $N = 1$ or $N = 2$, that J is non increasing in the radial variable, and $f_+ \le g_1$, $f_- \le g_2$ for some $g_1, g_2 \in L^1\left(\mathbb{R}^N\right) \cap \mathcal{C}_0(\mathbb{R}^N)$, radial and strictly decreasing in the radial variable. Then, problem* (BOP) *has at least a solution $w_\infty \in L^1(\mathbb{R}^N)$.*

Proof. Given the assumptions, we construct the solution u of (6.1) associated to the initial data f. Then we use the estimate

$$|\Gamma(u)| \le \max\left((\mathbb{U}^+ - 1)_+\,;\,(\mathbb{U}^- - 1)_+\right).$$

If $N \ge 3$, we use Corollary 6.2.3 to get $\|\Gamma(u(t))\|_{L^1(\mathbb{R}^N)} = O(t^{-N/2})$. For dimensions $N = 1, 2$, we use the extra assumption and Corollary 6.2.4 which states $\|\Gamma(u(t))\|_{L^1(\mathbb{R}^N)} \le Ce^{-\kappa t}$ for some $C, \kappa > 0$. In both cases, we obtain that $\int_0^\infty \Gamma(u(s))ds$ converges in $L^1(\mathbb{R}^N)$ to some function w_∞, and $\omega_t = \Gamma(u)$ converges to zero in $L^1(\mathbb{R}^N)$, then passing to the limit in (6.63) we see that w_∞ is a solution of (BOP). □

We now have a more general uniqueness result (without extra assumptions in lower dimensions). To prove the uniqueness we will need the following Lemma, from [12, Lemma 5.2.].

Lemma 6.3.4. *Let $w \in L^1(\mathbb{R}^N)$ such that $w \ge 0$, $w \le J * w$ a.e. Then $w = 0$ a.e.*

Proof. Assume first that w is continuous, and fix $\varepsilon > 0$. Since w is integrable, there is a radius R such that

$$\int_{|x| \ge R} w \le \frac{\varepsilon}{\|J\|_{L^\infty(\mathbb{R}^N)}}.$$

Hence, for $|x| \geq R + R_J$

$$w(x) \leq (J * w)(x) \leq \|J\|_{L^\infty(\mathbb{R}^N)} \int_{B_{R_J}(x)} w \leq \|J\|_{L^\infty(\mathbb{R}^N)} \int_{|x|\geq R} w \leq \varepsilon. \tag{6.65}$$

So, let us assume that for some $x \in \mathbb{R}^N$, $w(x) > \varepsilon$. Then, the maximum of w is attained at some point $\bar{x} \in B_{R+R_J}$ and

$$\max_{\mathbb{R}^N} w = w(\bar{x}) > \varepsilon. \tag{6.66}$$

Using that $w \leq J * w$, then

$$\varepsilon < w(\bar{x}) \leq J * w(\bar{x}) = \int_{B_{R_J}(\bar{x})} J(x-y)w(y)dy. \tag{6.67}$$

From (6.66) and (6.67), we obtain that $w(x) = w(\bar{x})$ in $B_{R_J}(\bar{x})$ and then, spreading this property to all the space by adding each time the support of J, i.e., arguing like above for any point $x \in B_{R_J}(\bar{x})$, we obtain that $w(x) = w(\bar{x})$ for all $x \in B_{2R_J}(\bar{x})$. Iterating this process we conclude that $w = w(\bar{x}) > \varepsilon$ in all $\mathbb{R}^N$. But this is a contradiction with (6.65). So, we deduce that $0 \leq w \leq \varepsilon$ for any $\varepsilon > 0$, hence $w \equiv 0$.

If w is not continuous, we argue by density and we take a sequence of functions $\{w_n\}_n \subset \mathcal{C}_c(\mathbb{R}^N)$, such that $w_n \to w$ as $n \to \infty$ in $L^1(\mathbb{R}^N)$. The continuous functions w_n satisfy all the hypotheses of this Lemma, and we have proved that $w_n \equiv 0$. Letting n go to ∞, we obtain $w \equiv 0$ a.e. in $L^1(\mathbb{R}^N)$. □

Below, we prove the uniqueness of solution of the problem (BOP).

Proposition 6.3.5. *Given any function $f \in L^1(\mathbb{R}^N)$, the problem* (BOP) *has at most one solution $w \in L^1(\mathbb{R}^N)$.*

Proof. The proof follows the same arguments as in [12, Thm 5.3]. For the sake of completeness we reproduce here the argument: a solutions of (BOP) satisfy,

$$\tilde{f} = f + J * w - w\,, \quad \tilde{f} \in \beta(w) \text{ a.e.}\,,$$

where $\beta(\cdot)$ is the graph of the sign function: $\beta(w) = \operatorname{sign}(w)$ if $w \neq 0$, and $\beta(\{0\}) = [-1, 1]$. We take two solutions w_i, $i = 1, 2$ of (BOP) associated with the data f and let $\tilde{f}_i$ be an associated projection, defined as

$$\tilde{f}_i = f + J * w_i - w_i, \quad \tilde{f} \in \beta(w_i) \text{ a.e.}\,.$$

Since $\tilde{f}_i \in \beta(w_i)$ we have

$$0 \leq (\tilde{f}_1 - \tilde{f}_2)\chi_{\{w_1 > w_2\}} = \big(J * (w_1 - w_2) - (w_1 - w_2)\big)\chi_{\{w_1 > w_2\}} \quad \text{a.e.}\,. \tag{6.68}$$

We then use the following inequality, that is the nonlocal version of Kato's inequality, valid for integrable functions:

$$\begin{aligned} (J * w - w)\chi_{\{w>0\}} &= \left(\int_{\{w\geq 0\}} J(x,y)w(y)dy + \int_{\{w<0\}} J(x,y)w(y)dy\right)\chi_{\{w>0\}} - w\chi_{\{w>0\}} \\ &\leq \int_{\{w\geq 0\}} J(x,y)w(y)dy\chi_{\{w>0\}} - w_+ \\ &\leq J * w_+ - w_+ \quad \text{a.e.,} \end{aligned} \tag{6.69}$$

(6.68) and (6.69) imply

$$(w_1 - w_2)_+ \leq J * (w_1 - w_2)_+ \,.$$

We end by using Lemma 6.3.4, from which we infer that $(w_1 - w_2)_+ = 0$. Reversing the roles of w_1 and w_2 we get uniqueness. $\square$

Combining the results above, we can now give our main theorem concerning the asymptotic behavior for solutions of (6.1).

Theorem 6.3.6. *Let $f \in L^1\left(\mathbb{R}^N\right)$, under the hypotheses of Lemma 6.3.3. If u is the unique solution to the problem* (6.1) *and w_∞ is the unique solution of the problem* (BOP)*, we have*

$$u(t) \to \tilde{f} := f + J * w_\infty - w_\infty \quad \text{in } L^1\left(\mathbb{R}^N\right) \quad \text{as } t \to \infty \,.$$

6.3.3 Asymptotic limit for general data

Up to now we have been able to prove the existence of a solution of (BOP) for any $f \in L^1\left(\mathbb{R}^N\right)$ satisfying (6.56) and (6.57) only if $N \geq 3$. For low dimensions, $N = 1, 2$, we have needed to add the hypotheses of Lemma 6.3.3. Hence, for lower dimensions the projection operator $\mathcal{P}$ which maps f to $\tilde{f}$

$$\mathcal{P}f = \tilde{f} = f + J * w_\infty - w_\infty$$

is in principle only defined under the extra assumptions.

However, $\mathcal{P}$ is continuous, in the L^1-norm, in the subset of $L^1\left(\mathbb{R}^N\right)$ of functions satisfying the hypotheses of Lemma 6.3.3. Since the class of functions satisfying those hypotheses is dense in set of function in $L^1\left(\mathbb{R}^N\right)$ satisfying (6.56) and (6.57), we can extend the operator to all L^1 under assumptions (6.56) and (6.57) by a standard procedure. We consider a sequence $\{f_n\}_n$ in $L^1(\mathbb{R}^N)$ satisfying the hypotheses of Lemma 6.3.3, that converges to f in $L^1\left(\mathbb{R}^N\right)$ satisfying (6.56) and (6.57). Then we define the projection $\mathcal{P}f$ as the limit $\lim\limits_{n\to\infty} \mathcal{P}f_n$ in $L^1(\mathbb{R}^N)$. Furthermore, we prove that the limit does no depend on the sequence $\{f_n\}_n$ we choose: given two sequences $\{f_n\}_n$ and $\{g_n\}_n$ such that $f_n \to f$ and $g_n \to f$, then we construct a new sequence $\{h_n\}_n$, that consists of $h_{2n+1} = f_n$ and $h_{2n} = g_n$, for all $n \in \mathbb{N}$. Then, h_n converges to f as n goes to ∞ in $L^1(\mathbb{R}^N)$, and the limit $\lim\limits_{n\to\infty} \mathcal{P}h_n$ is the same limit of the sequences $\{\mathcal{P}f_n\}_n$ and $\{\mathcal{P}g_n\}_n$. Thus, the limit is independent of the sequence $\{f_n\}_n$.

Let us prove below that $\mathcal{P}$ is continuous, in the L^1-norm.

Corollary 6.3.7. *Let $f_i,\ i = 1, 2$, satisfy the hypotheses of Theorem 6.3.6. Then*

$$\|\tilde{f}_1 - \tilde{f}_2\|_{L^1(\mathbb{R}^N)} \leq \|f_1 - f_2\|_{L^1(\mathbb{R}^N)}.$$

Proof. Since (BOP) has a unique solution, any solution with initial data satisfying the hypotheses of Theorem 6.3.6 can be obtained as the limit as $t \to \infty$ of the solution of the nolocal Stefan problem (6.1). Hence the result is obtained passing to the limit in the contraction property, (6.24), for this latter problem. $\square$

Now, we prove the main result.

Theorem 6.3.8. *Let $f \in L^1(\mathbb{R}^N)$ satisfy* (6.56) *and* (6.57)*, and let u be the corresponding solution to problem* (6.1)*. Let $\mathcal{P}f$ be the projection of f onto $\tilde{f}$. Then $u(\cdot,t) \to \mathcal{P}f$ in $L^1(\mathbb{R}^N)$ as $t \to \infty$.*

Proof. Given f, let $\{f_n\} \subset L^1(\mathbb{R}^N)$ be a sequence of functions satisfying the hypotheses of Lemma 6.3.3 which approximate f in $L^1(\mathbb{R}^N)$. Take for instance a sequence of compactly supported functions. Let u_n be the corresponding solutions to the nonlocal Stefan problem. We have,

$$\|u(t) - \mathcal{P}f\|_{L^1(\mathbb{R}^N)} \le \|u(t) - u_n(t)\|_{L^1(\mathbb{R}^N)} + \|u_n(t) - \mathcal{P}f_n\|_{L^1(\mathbb{R}^N)} + \|\mathcal{P}f_n - \mathcal{P}f\|_{L^1(\mathbb{R}^N)}.$$

Using Corollary 6.1.15, which gives the contraction property for the nonlocal Stefan problem, and Theorem 6.3.6, that states the large time behavior for bounded and compactly supported initial data, we obtain

$$\limsup_{t\to\infty} \|u(t) - \mathcal{P}f\|_{L^1(\mathbb{R}^N)} \le \|f - f_n\|_{L^1(\mathbb{R}^N)} + \|\mathcal{P}f_n - \mathcal{P}f\|_{L^1(\mathbb{R}^N)}.$$

By using Corollary 6.3.7 and letting $n \to \infty$ we get the result. □

6.4 Solutions losing one phase in finite time

In this section we we give some partial results on the asymptotic behavior of solutions for which either u or $\Gamma(u)$ becomes nonnegative (or nonpositive) in finite time. In this case, we can prove that the asymptotic behavior is driven by the one-phase Stefan regime, however we cannot identify the limit exactly.

6.4.1 A theoretical result

In the following theorem we describe the asymptotic behaviour of the solution u of (6.1) if the temperature $\Gamma(u)$ becomes nonnegative in finite time.

Theorem 6.4.1. *Let $f \in L^1(\mathbb{R}^N)$ and let u be the corresponding solution. Assume that for some $t_0 \ge 0$, there holds $f^* := u(t_0) \ge -1$ in $\mathbb{R}^N$. Then the asymptotic behavior is given by: $u(t) \to \mathcal{P}f^*$ as t goes to infinity.*

Proof. We just have to consider $u^*(t) := u(t-t_0)$ for $t \ge t_0$. Then u^* is the solution associated to the initial data f^*. That satisfies that $u^*(t) \ge -1$ for all $t \ge 0$, then f^* satisfies (6.56) and (6.57). Hence we know that $u^*(t) \to \mathcal{P}f^*$, as $t \to \infty$. Therefore, the same happens for $u(t)$. □

Of course a similar result holds if $\Gamma(u)$ becomes nonpositive in finite time. However, the problem remains open as to identify $\mathcal{P}f^*$ since we do not know what is exactly f^*.

Below, we give an example where such a phenomenon occurs, $v = \Gamma(u)$ becomes positive in finite time.

6.4.2 Sufficient conditions to lie above level -1 in finite time

In this subsection we assume for simplicity that the initial data f is continuous and compactly supported, and that J is nonincreasing in the radial variable. We assume $f_+ \leq g_1$ and $f_- \leq g_2$, for some g_1, $g_2 \in L^1\left(\mathbb{R}^N\right) \cap \mathcal{C}_0(\mathbb{R}^N)$ radial and strictly decreasing in the radial variable. The following result states that under the previous hypotheses, the support of $\Gamma(\mathbb{U}^+)$ is a subset of B_R, where $R = R(g_1)$, (see proof in [12, Lemma 3.9]).

Lemma 6.4.2. *Let J be nonincreasing in the radial variable. We assume $0 \leq f_+ \leq g$, for some $g \in L^1\left(\mathbb{R}^N\right) \cap \mathcal{C}_0(\mathbb{R}^N)$ radial and strictly decreasing in the radial variable. Then there exists some $R = R(g)$ such that*

$$\operatorname{supp}\left(\Gamma(\mathbb{U}^+)(t)\right) \subset B_R, \quad \textit{for all} \ \ t \geq 0.$$

Thanks to Lemma 6.4.2, under the previous hypotheses on f_+ and f_-, we have that there exists $R = R(g_1, g_2)$ such that

$$\operatorname{supp}\left(\Gamma(\mathbb{U}^+)(t)\right) \subset B_R, \ \text{ and } \ \operatorname{supp}\left(\Gamma(\mathbb{U}^-)(t)\right) \subset B_R, \ \text{ for all } \ t \geq 0.$$

Moreover, thanks to Lemma 6.2.5,

$$\operatorname{supp}(v(t)) \subset \operatorname{supp}(\Gamma(\mathbb{U}^+(t))) \cup \operatorname{supp}(\Gamma(\mathbb{U}^-(t))) \subset B_R, \ \text{ for any } \ t \geq 0, \tag{6.70}$$

(recall thatwe denote by $v = \Gamma(u)$). Notice that R does not depend on J, only on the L^1-norm of g_1 and g_2.

We make first the following important assumption:

$$\alpha(v_0, J) := \inf_{x \in B_R} \int_{\mathbb{R}^N} J(x-y) v_+(y,0) dy > 0 \tag{6.71}$$

(see in Remark 6.4.4 below some comments on this assumption). Let us also denote

$$\beta(J) := \sup_{x \in B_{2R}} J(x).$$

Then we shall also assume that the negative part of $v_0 := v(0) = \Gamma(f)$ is "small" compared to the positive part in the following sense:

$$\|v_-(0)\|_{L^1(\mathbb{R}^N)} < \frac{\alpha(v_0, J)}{\beta(J)}\,. \tag{6.72}$$

In such a situation, we first define

$$\bar{\eta} := \alpha(v_0, J) - \beta(J)\|v_-(0)\|_{L^1(\mathbb{R}^N)} > 0\,.$$

Then, for $\eta \in (0, \bar{\eta})$ we introduce the following function

$$\varphi(\eta) := \eta \ln\left(\frac{\alpha(v_0, J)}{\eta + \beta(J)\|v_-(0)\|_{L^1(\mathbb{R}^N)}}\right) > 0$$

and set

$$\kappa := \max\left\{\varphi(\eta) : \eta \in (0, \bar{\eta})\right\} > 0.$$

Since actually, κ depends only on J and the mass of the positive and negative parts of $v(0)$, we denote it by $\kappa(v_0, J)$. We are then ready to formulate the following result.

Proposition 6.4.3. *Let f be continuous and compactly supported, and J be nonincreasing in the radial variable. We assume $f_+ \leq g_1$ and $f_- \leq g_2$, for some g_1, $g_2 \in L^1\left(\mathbb{R}^N\right) \cap \mathcal{C}_0(\mathbb{R}^N)$ radial and strictly decreasing in the radial variable. Assume* (6.72) *and moreover that the negative part of f is controlled in the sup norm as follows*

$$\|f_-\|_\infty \leq 1 + \kappa(v_0, J)\,.$$

Then in a finite time $t_1 = t_1(f)$, the solution satisfies $u(x, t_1) \geq -1$ for all $x \in \mathbb{R}^N$.

Proof. By our assumptions, for all x we have $f(x) \geq -1 - \kappa(v_0, J)$. Then for any $x \in B_R$,

$$\begin{aligned} J * v(x,0) \;&= \int_{\{v>0\}} J(x-y)v(y,0)dy + \int_{\{v<0\}} J(x-y)v(y,0)dy \\ &\geq \alpha(v_0, J) - \beta(J)\|v_-(0)\|_{L^1(\mathbb{R}^N)} > 0. \end{aligned}$$

Thanks to (6.70), for the points $x \notin B_R$, we have $v_0(x) = 0$ and also $v(x,t) = 0$ for any time $t \geq 0$ (though we may —and will— have mushy regions, $\{|v| < 1\}$, outside B_R of course).

Thanks to the continuity of u (and v), the following time is well-defined:

$$t_0 := \sup\{t \geq 0 : J * v(x,t) > 0 \text{ for any } x \in B_R\} > 0\,.$$

This implies that

$$u_t \geq -v, \quad \text{in} \;\; B_R \times (0, t_0),$$

so that

$$\partial_t v_+ = \chi_{\{v>0\}} \partial_t u \geq -v\chi_{\{v>0\}} = -v_+ \quad \text{in} \;\; B_R \times (0, t_0).$$

Hence, in $B_R \times (0, t_0)$, v_+ enjoys the following retention property:

$$v_+(x,t) \geq e^{-t} v_+(x,0), \;\; \forall t \in [0, t_0). \tag{6.73}$$

This implies in particular that if $v(x,0)$ is positive at some point, $v(x,t)$ remains positive at this point at least until t_0.

Now, let us estimate t_0. First we use (6.73), and then Corollary 6.1.15, which gives the L^1-contraction property for v_-. Thus, for any $x \in B_R$ and $t \in (0, t_0)$, we have

$$\begin{aligned} J * v(x,t) \;&\geq \int_{\{v>0\}} J(x-y)v(y,t)dy + \int_{\{v<0\}} J(x-y)v(y,t)dy \\ &\geq e^{-t} \int_{\{v>0\}} J(x-y)v(y,0)dy - \beta(J)\|v_-(t)\|_{L^1(\mathbb{R}^N)} \\ &\geq \alpha(v_0, J)e^{-t} - \beta(J)\|v_-(0)\|_{L^1(\mathbb{R}^N)}. \end{aligned}$$

So, if we take η reaching the max of $\varphi(\eta) = \kappa$ and set

$$t_1(\eta) := \ln\left(\frac{\alpha(v_0, J)}{\eta + \beta(J)\|v_-(0)\|_{L^1(\mathbb{R}^N)}}\right),$$

then for any $t \in (0, t_1)$, we have $\alpha(v_0, J)e^{-t} - \beta(J)\|v_-(0)\|_{L^1(\mathbb{R}^N)} > \eta > 0$. This proves that $t_0 \geq t_1$.

Since v_+ has the retention property (6.73) in $(0, t_0)$, the points in

$$\mathcal{C}^+ := \{x \in \mathbb{R}^N : v(x,0) > 0\}$$

remain in this set at least until t_0. Then, for any $x \in \mathcal{C}^- := \{x \in \mathbb{R}^N : v(x,0) \leq 0\}$, we define

$$t(x) := \sup\{t > 0 : v(x,t) \leq 0\}.$$

If $t(x) = 0$, this means that $v(x,t)$ becomes positive immediately and will remain as such at least until t_1 so we do not need to consider such points. We are left with assuming $t(x) > 0$ (or infinite). Then if $t(x) > 0$, we shall prove that $t(x) \leq t_1$ by contradiction: let us assume that $t(x) > t_1$ and let us come back to the previous estimate. We then have, for any $t \in (0, t_1)$:

$$u_t(x,t) = J * v(x,t) - v(x,t) \geq J * v(x,t) > \eta > 0\,.$$

Thus, integrating the equation in time at x yields

$$u(x,t) > -1 - \kappa(v_0, J) + \eta \cdot t, \quad \forall t \in [0, t_1].$$

By our choice we have precisely $\kappa(v_0, J) = \varphi(\eta) = \eta \cdot t_1(\eta)$. Therefore, at least for $t = t_1$, we have

$$u(x, t_1) > -1 - \kappa(v_0, J) + \eta \cdot t_1 > -1\,,$$

which is a contradiction with the fact that $t(x) > t_1$. Hence $t(x) \leq t_1$, which means that at such points, the solution becomes equal to or above level -1 before t_1.

So, combining everything, we have finally obtained that for any point $x \in \mathbb{R}^N$, $u(x,t)$ becomes greater than or equal to -1 before the time t_1, which ends the proof. $\square$

Remark 6.4.4. *Hypothesis* (6.71) *expresses that for any $x \in B_R$, there is some positive contribution in the convolution with the positive part of v_0. So, this implies that at least the following condition on the intersection of the supports should hold:*

$$\forall x \in B_R\,, \quad \big(x + B_{R_J}(0)\big) \cap \operatorname{supp}\big((v_0)_+\big) \neq \emptyset\,.$$

Actually, if the radius R_J is big enough to contain all the support of v_0 this is satisfied. But even if it is not so big, and there are positive values of v_0 which spread in many directions, this condition can be satisfied.

Then, (6.72) *is a condition on the negative part, which should not be too big so that all the possible points such that $v(x,0) < 0$ will enter into the positive set for v in finite time. The exact control is a mix between the mass and the infinite norm of the various quantities.*

Appendix A: L^p-spaces

In this appendix we enumerate several well known results for the L^p-spaces: Hölder's inequality and Minkowski's inequality; the Monotone Convergence Theorem and the Dominated Convergence Theorem; Fubini's Theorem and Lusin's Theorem. These notes have been written following [46].

Let (Ω, μ) be a measure space where μ is an outer regular Borel measure in Ω that associates a finite positive measure to the balls of Ω, we give below the results that are used throughout this work.

Theorem 6.4.5. *For $1 \leq p \leq \infty$, if p and p' satisfy $1/p + 1/p' = 1$, and if $f \in L^p(\Omega)$ and $g \in L^{p'}(\Omega)$, then $fg \in L^1(\Omega)$, and*

$$\int_\Omega |fg| d\mu \leq \left(\int_\Omega |f|^p d\mu \right)^{1/p} \left(\int_\Omega |g|^{p'} d\mu \right)^{1/p'} \tag{6.74}$$

and

$$\left(\int_\Omega |f+g|^p d\mu \right)^{1/p} \leq \left(\int_\Omega |f|^p d\mu \right)^{1/p} + \left(\int_\Omega |g|^{p'} d\mu \right)^{1/p'} \tag{6.75}$$

The inequility (6.74) is Hölder's inequality and (6.75) is Minkowski's inequality.

Let μ be a positive measure then we have the following Convergence results.

Theorem 6.4.6. (Monotone Convergence Theorem): *Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of measurable functions in Ω, and assume that*

1. $0 \leq f_1 \leq f_2 \leq \cdots \leq \infty$ *in* Ω,
2. $f_n \to f$ *as* $n \to \infty$.

Then f is measurable, and

$$\int_\Omega f_n d\mu \to \int_\Omega f d\mu \qquad as \ \ n \to \infty.$$

Theorem 6.4.7. (Dominated Convergence Theorem): *Suppose $\{f_n\}_{n\in\mathbb{N}}$ is a sequence of measurable functions in Ω such that*

$$f = \lim_{n\to\infty} f_n.$$

If there exists a function $g \in L^1(\Omega)$ *such that*

$$f_n \leq g, \qquad \forall n = 1, 2, \dots,$$

then $f \in L^1(\Omega)$

$$\lim_{n\to\infty} \int_\Omega |f_n - f| d\mu = 0,$$

and

$$\lim_{n\to\infty} \int_\Omega |f_n| d\mu = \int_\Omega |f| d\mu.$$

Approximation by Continuous Functions: Let (Ω, μ, d) be a metric measure space, like in Definition 1.1.5. Under these circumstances, we have the following theorems:

Theorem 6.4.8. (Lusin's Theorem): *For* $1 \leq p < \infty$,

- $C_c(\Omega)$ *is dense in* $L^p(\Omega)$.
- $C_c(\Omega)$ *is dense in* $C_0(\Omega)$.

Theorem 6.4.9. (Fubini's Theorem): *Let* (Ω_1, μ_1) *and* (Ω_2, μ_2) *be* σ*-finite measure spaces, and let* f *be a* $\mu_1 \times \mu_2$*-measurable function on* $\Omega_1 \times \Omega_2$. *If* $0 \leq f \leq \infty$, *and if*

$$\varphi(x) = \int_{\Omega_2} f(x,y) d\mu_2(y), \quad \psi(y) = \int_{\Omega_1} f(x,y) d\mu_1(x), \quad x \in \Omega_1,\ y \in \Omega_2,$$

then φ *is* μ_1*-measurable and* ψ *is* μ_2*- measurable, and*

$$\int_{\Omega_1} \varphi(x) d\mu_1(x) = \int_{\Omega_1 \times \Omega_2} f(x,y) d\mu_1(x)\, d\mu_2(y) = \int_{\Omega_2} \psi(y) d\mu_2(y), \quad x \in \Omega_1,\ y \in \Omega_2.$$

Theorem 6.4.10. (Dual Space of $L^p(\Omega)$): *Suppose* $1 \leq p < \infty$, μ *is a* σ*-finite positive measure on* Ω, *and* Φ *is a bounded linear functional on* $L^p(\Omega)$. *For* p *and* p' *satisfying* $1/p + 1/p' = 1$ *there is a unique* $g \in L^{p'}(\Omega)$, *such that*

$$\Phi(f) = \int_\Omega f g d\mu, \quad f \in L^p(\Omega). \tag{6.76}$$

Moreover, if Φ *and* g *are related as in* (6.76), *we have*

$$\|\Phi\| = \|g\|_{L^{p'}(\Omega)}.$$

In other words, $L^{p'}(\Omega)$ *is isometrically isomorphic to the dual space of* $L^p(\Omega)$, *under the stated conditions.*

Theorem 6.4.11. (Dual Space of $C_0(\Omega)$): *If* Ω *is a locally compact Hausdorff space, then every bounded linear functional on* Φ *on* $C_0(\Omega)$ *is represented by a unique regular measure* μ, *satisfying the properties in Definition 1.1.5, in the sense that*

$$\Phi(f) = \int_\Omega f d\mu, \quad \forall f \in C_0(\Omega). \tag{6.77}$$

Moreover, the norm of Φ *is the total variation of* μ

$$\|\Phi\| = |\mu|(\Omega).$$

Since $C_c(\Omega)$ is a dense subspace of $C_0(\Omega)$, relative to the supremum norm, every bounded linear functional on $C_c(\Omega)$ has a unique extension to a bounded linear functional on $C_0(\Omega)$.

Appendix B: Nemitcky operators

In this appendix we analyze some properties of the Nemitcky operators associated to the nonlinear terms f.

Let us start introduce some definitions.

Definition 6.4.12. *Let $X = L^p(\Omega)$, with $1 \leq p \leq \infty$, or $X = \mathcal{C}_b(\Omega)$, the Nemitcky operator associated to $f : \Omega \times \mathbb{R} \to \mathbb{R}$, that sends (x, u) to $f(x, u)$ is defined as an operator*

$$F : X \to X, \quad \textit{such that} \;\; F(u)(x) = f(x, u(x)),$$

for $u : \Omega \to \mathbb{R}$.

Definition 6.4.13. *Let $X = L^p(\Omega)$, with $1 \leq p \leq \infty$, or $X = \mathcal{C}_b(\Omega)$ be an ordered Banach space. An operator $F \in \mathcal{L}(X, X)$ is increasing if given $\omega_1, \omega_2 \in Y$ such that $\omega_1 \geq \omega_2$ then $F(\omega_1) \geq F(\omega_2)$.*

In the Lemma below, we give a relation between the properties of the function f and its associated Nemitcky operator, F, and properties of globally Lipschitz functions f.

Lemma 6.4.14. *Let $X = L^p(\Omega)$, with $1 \leq p \leq \infty$ or $X = \mathcal{C}_b(\Omega)$, and let $F : X \to X$ be the Nemitcky operator associated to the function $f : \Omega \times \mathbb{R} \to \mathbb{R}$, that maps (x, s) into $f(x, s)$:*

i. if f is increasing respect the variable $s \in \mathbb{R}$, uniformly with respect to $x \in \Omega$, then F is increasing;

ii. if f is globally Lipschitz in the variable $s \in \mathbb{R}$, uniformly with respect to $x \in \Omega$, then F is globally Lipschitz;

iii. if f is globally Lipschitz in the variable $s \in \mathbb{R}$, uniformly with respect to $x \in \Omega$, then for a constant $\beta > L_f$, where L_f is the Lipchitz constant of f, $f(x, s) + \beta s$ is increasing, i.e.,

$$\textit{for all} \;\; s,\, t \in \mathbb{R} \;\textit{such that} \;\; s \geq t, \;\, f(x, s) + \beta s \geq f(x, t) + \beta t, \quad \forall x \in \Omega.$$

Proof.
i. Since f is increasing in the second variable, we have that

$$f(x, s) - f(x, t) \geq 0, \text{ for } s,\, t \in \mathbb{R} \text{ such that } \;s \geq t, \;\text{ and for all } \;x \in \Omega.$$

Consider now u, $v \in X$, with $u \geq v$, then

$$F(u)(x) - F(v)(x) = f(x, u(x)) - f(x, v(x)) \geq 0, \quad \text{for all } \ x \in \Omega.$$

Thus, F is increasing.

ii. Since f is globally Lipschitz, there exists a constant $L_f \in \mathbb{R}$ such that for s, $t \in \mathbb{R}$, it is satisfied that

$$|f(x,s) - f(x,t)| \leq L_f |s-t|, \quad \forall x \in \Omega.$$

We prove first that F is globally Lipschitz in $X = L^p(\Omega)$, for $1 \leq p < \infty$. Let u, $v \in L^p(\Omega)$ then

$$\begin{aligned}
\|F(u) - F(v)\|_{L^p(\Omega)} &= \left(\int_\Omega |F(u)(x) - F(v)(x)|^p dx \right)^{1/p} \\
&= \left(\int_\Omega |f(x,u(x)) - f(x,v(x))|^p dx \right)^{1/p} \\
&\leq \left(\int_\Omega L_f^p |u(x) - v(x)|^p dx \right)^{1/p} \\
&= L_f \|u - v\|_{L^p(\Omega)}.
\end{aligned}$$

Thus, F is globally Lipschitz in $L^p(\Omega)$, with $1 \leq p < \infty$.
Let us prove it now for $X = L^\infty(\Omega)$ or $X = \mathcal{C}_b(\Omega)$. Let u, $v \in X$ then

$$\begin{aligned}
\|F(u) - F(v)\|_X &= \sup_{x \in \Omega} |F(u)(x) - F(v)(x)| \\
&= \sup_{x \in \Omega} |f(x,u(x)) - f(x,v(x))| \\
&\leq \sup_{x \in \Omega} L_f \, |u(x) - v(x)| \\
&= L_f \|u - v\|_X.
\end{aligned}$$

Thus, F is globally Lipschitz in X.

iii. Since f is globally Lipschitz, we have that $|f(x,s) - f(x,t)| \leq L_f |s-t|$, then for $s \geq t$

$$-L_f(s-t) \leq f(x,s) - f(x,t) \leq L_f(s-t)$$

If we choose $\beta > L_f$, then we obtain the result. We prove that for $s \geq t$

$$\begin{aligned}
\big(f(x,s) + \beta s\big) - \big(f(x,t) + \beta t\big) &= f(x,s) - f(x,t) + \beta(s-t) \\
&\geq -L_f(s-t) + \beta(s-t) \\
&= (\beta - L_f)(s-t) \qquad\qquad \geq 0.
\end{aligned}$$

Hence, $f(x,s) + \beta s$ is increasing. □

Differentiability of the Nemitcky operator: We say that a Nemitcky operator $F : L^p(\Omega) \to L^q(\Omega)$ is *differentiable* at $u_0 \in L^p(\Omega)$ if there exists a continuous linear map $DF(u_0) : L^p(\Omega) \to L^q(\Omega)$ so that

$$\|F(u) - F(u_0) - DF(u_0)(u - u_0)\|_{L^q(\Omega)} = o(\, \|u - u_0\|_{L^p(\Omega)}), \quad \text{as } u \to u_0. \tag{6.78}$$

In this case $DF(u_0)$ is called the *Frechet derivative* at a. The map is *continuously differentiable* on an open set $U \subset L^p(\Omega)$ if it is differentiable at each point of U and the mapping $u \mapsto DF(u)$ defined in $U \to \mathcal{L}(L^p(\Omega), L^q(\Omega))$ is continuous.

In the following Lemma we prove that if f is not an affine function and it is Lipschitz then $F : L^q(\Omega) \to L^p(\Omega)$ is differentiable if $q > p$, but $F : L^p(\Omega) \to L^p(\Omega)$ is not differentiable.

Lemma 6.4.15. *Let (Ω, μ) be a measure space with $\mu(\Omega) < \infty$.*

(i) If $f : \mathbb{R} \to \mathbb{R}$ is Lipschitz with $f \in \mathcal{C}^2(\mathbb{R})$, and f, f' and f'' bounded then for $q > p$, $F : L^q(\Omega) \to L^p(\Omega)$ is differentiable, and in fact, $F \in \mathcal{C}^{1,\theta}(L^q(\Omega), L^p(\Omega))$, for $\theta = \min\left\{1, \frac{q}{p} - 1\right\}$ and

$$DF(u)v = f'(u)v. \tag{6.79}$$

(ii) If $f : \mathbb{R} \to \mathbb{R}$ not an affine function and it is Lipschitz, then $F : L^p(\Omega) \to L^p(\Omega)$ is not differentiable.

Proof.
i. Let us prove that $DF(u)$ defined as

$$\begin{array}{rcl} DF(u) : L^q(\Omega) & \longrightarrow & L^p(\Omega) \\ v & \longmapsto & f'(u)v \end{array}$$

with $u, h \in L^q(\Omega)$, is the Frechet derivative, i.e., $DF(u)$ satisfies (6.78).

Thanks to the Mean Value Theorem, there exists $\xi \in L^q(\Omega)$ such that

$$\|F(u+h) - F(u) - f'(u)(h)\|_{L^p(\Omega)} = \|\big(f'(\xi) - f'(u)\big)h\|_{L^p(\Omega)} \tag{6.80}$$

Since f' is bounded, then $|f'(\xi) - f'(u)| \leq 2C$, and by the Mean Value Theorem $|f'(\xi) - f'(u)| \leq C|\xi - u| \leq C|h|$. Then, for any $0 \leq \theta \leq 1$

$$\begin{aligned} |f'(\xi) - f'(u)| &= |f'(\xi) - f'(u)|^{1-\theta} \, |f'(\xi) - f'(u)|^{\theta} \\ &\leq (2C)^{1-\theta} C^{\theta} |h|^{\theta} \\ &= C|h|^{\theta}. \end{aligned} \tag{6.81}$$

Hence, from (6.80) and (6.81)

$$\|F(u+h) - F(u) - f'(u)(h)\|_{L^p(\Omega)} \leq C\|h^{1+\theta}\|_{L^p(\Omega)} = C\|h\|^{1+\theta}_{L^{p(1+\theta)}(\Omega)}.$$

Hence, if $q \geq p(1+\theta)$ and $0 < \theta \leq \min\{1, \frac{q}{p} - 1\}$, then

$$\frac{\|F(u+h) - F(u) - f'(u)(h)\|_{L^p(\Omega)}}{\|h\|_{L^q(\Omega)}} \to 0, \quad \text{as } \|h\|_{L^q(\Omega)} \to 0.$$

Therefore, $DF(u)v = f'(u)v$ is the Frechet derivative that satisfies (6.78).

Let us see now that $F \in \mathcal{C}^{1,\theta}(L^q(\Omega), L^p(\Omega))$. Thanks to Hölder's inequality

$$\begin{aligned} \|DF(u) - DF(v)\|_{\mathcal{L}(L^q(\Omega), L^p(\Omega))} &= \sup_{w \in L^q(\Omega), \|w\|_{L^q} = 1} \|f'(u)w - f'(v)w\|_{L^p(\Omega)} \\ &\leq \|f'(u) - f'(v)\|_{L^r(\Omega)}, \end{aligned}$$

where $r = \frac{pq}{q-p}$. Moreover, thanks to (6.81), we have that for any $0 \leq \theta \leq 1$

$$\|f'(u) - f'(v)\|_{L^r(\Omega)} \leq C\|(u-v)^\theta\|_{L^r(\Omega)} = C\|u-v\|^\theta_{L^{\theta r}(\Omega)} \leq C\|u-v\|^\theta_{L^q(\Omega)}.$$

for $q \geq r\theta$, then $q \geq p(1+\theta)$. Thus, the result.

ii. First, we will prove that F is not differentiable at 0. On one hand, since f is not an affine function, given $\delta > 0$ there exists $a \in \mathbb{R}$ such that

$$|f(a) - f(0) - f'(0)a| = \delta. \tag{6.82}$$

We define h as

$$h(x) = \begin{cases} a, & \text{if } x \in \mathcal{A}_\rho \\ 0, & \text{if } x \in \Omega \setminus \mathcal{A}_\rho, \end{cases} \tag{6.83}$$

where $\mathcal{A}_\rho \subset \Omega$ is a set such that $\mu(\mathcal{A}_\rho) > 0$ and $\mu(\mathcal{A}_\rho) \to 0$ as $\rho \to 0$. We have that $\|h\|_{L^p(\Omega)} = a\mu\big(\mathcal{A}_\rho\big)^{1/p}$, then $\|h\|_{L^p(\Omega)} \to 0$ as $\rho \to 0$.

On the other hand, from part *i.*, we know that if $q > p$ then $F : L^q(\Omega) \to L^p(\Omega)$ is differentiable and $DF(0)v = f'(0)v$. Moreover, $i : L^q(\Omega) \hookrightarrow L^p(\Omega)$ is continuous then $F = F \circ i : L^p(\Omega) \to L^p(\Omega)$, and for any $v \in L^p(\Omega)$, $DF(0)v = \big(DF(0) \circ i\big)v$. Hence, if F is differentiable at 0, from (6.79), we have that for any $v \in L^p(\Omega)$,

$$DF(0)v = f'(0)v.$$

Let us prove that F is not differentiability at 0. We argue by If F was differentiable at 0, then we have is satisfied

$$\|F(h) - F(0) - DF(0)h\|^p_{L^p(\Omega)} = \int_\Omega |f(h(x)) - f(0) - f'(0)h(x)|^p dx.$$

For h as in (6.83), we have that

$$\|F(h) - F(0) - DF(0)h\|^p_{L^p(\Omega)} = \int_{\mathcal{A}_\rho} |f(a) - f(0) - f'(0)a|^p dx$$

Thanks to (6.82),

$$\begin{aligned} \|F(h) - F(0) - DF(0)h\|^p_{L^p(\Omega)} &= \delta^p \mu\big(\mathcal{A}_\rho\big) \\ &= \frac{\delta^p}{a^p}\|h\|^p_{L^p(\Omega)} \end{aligned} \tag{6.84}$$

Since δ in (6.82) is strictly positive, we do not have that $\|F(h) - F(0) - DF(0)h\|^p_{L^p(\Omega)}/\|h\|^p_{L^p(\Omega)}$ goes to 0 as $\|h\|_{L^p(\Omega)}$ goes to 0. Hence, F is not differentiable at 0.

Now, we will prove that $F : L^p(\Omega) \to L^p(\Omega)$ is not differentiable at any function $u \in L^p(\Omega)$. Since f is not affine, given $\delta > 0$, for all $s \in \mathbb{R}$ there exists $t \in \mathbb{R}$ such that

$$|f(t) - f(s) - f'(s)(t-s)| > \delta,$$

then given $\delta > 0$, for every $x \in \Omega$ and $u(x) = s \in \mathbb{R}$, there exists $b(x) = t$ such that

$$\big|f(b(x)) - f(u(x)) - f'(u(x))\big(b(x) - u(x)\big)\big| > \delta. \tag{6.85}$$

Since $u \in L^p(\Omega)$, for b constructed as above, there exists a set $\mathcal{A}_\rho \subset \Omega$ such that $\mu(\mathcal{A}_\rho) > 0$ and $\mu(\mathcal{A}_\rho) \to 0$ as $\rho \to 0$, such that $\|b - u\|_{L^\infty(\mathcal{A}_\rho)} < \infty$. Let us prove this by contradiction, if there is no set $\mathcal{A}_\rho$ with positive measure where $\|b - u\|_{L^\infty(\mathcal{A}_\rho)} < \infty$, then $b - u = \infty$ for almost every $x \in \Omega$. This contradicts the fact that $u \in L^p(\Omega)$.

Now, we define

$$h(x) = \begin{cases} b(x) - u(x), & \text{if } x \in \mathcal{A}_\rho \\ 0, & \text{if } x \in \Omega \setminus \mathcal{A}_\rho, \end{cases} \tag{6.86}$$

with $\|h\|_{L^p(\Omega)} \leq \|b - u\|_{L^\infty(\mathcal{A}_\rho)} \mu\big(\mathcal{A}_\rho\big)^{1/p}$, then $\|h\|_{L^p(\Omega)} \to 0$ as $\rho \to 0$.
On the other hand, if F is differentiable at $u \in L^p(\Omega)$, arguing as above for zero, we would have that for any $v \in L^p(\Omega)$,

$$DF(u)v = f'(u)v.$$

Let us see now, if the definition of differentiability at u is satisfied for h as in (6.86),

$$\|F(h+u) - F(u) - DF(u)h\|^p_{L^p(\Omega)} \geq \int_{\mathcal{A}_\rho} \big|f(b(x)) - f(u(x)) - f'(u(x))\big(b(x) - u(x)\big)\big|^p dx,$$

thanks to (6.85),

$$\begin{aligned} \|F(h+u) - F(u) - DF(u)h\|^p_{L^p(\Omega)} &\geq \delta^p \mu\big(\mathcal{A}_\rho\big) \\ &\geq \frac{\delta^p}{\|b - u\|^p_{L^\infty(\mathcal{A}_\rho)}} \|h\|^p_{L^p(\Omega)} \end{aligned} \tag{6.87}$$

Since δ in (6.82) is strictly positive, we do not have that

$$\|F(h+u) - F(u) - DF(u)h\|^p_{L^p(\Omega)} / \|h\|^p_{L^p(\Omega)}$$

goes to 0 as $\|h\|_{L^p(\Omega)}$ goes to 0. $\square$

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