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Article

Nonlocal Probability Theory: General Fractional Calculus Approach

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Abstract: Nonlocal generalization of the standard (classical) probability theory of a continuous distribution on a positive semi-axis is proposed. An approach to the formulation of a nonlocal generalization of the standard probability theory based on the use of the general fractional calculus in the Luchko form is proposed. Some basic concepts of the nonlocal probability theory are proposed, including nonlocal (general fractional) generalizations of probability density, cumulative distribution functions, probability, average values, and characteristic functions. Nonlocality is described by the pairs of Sonin kernels that belong to the Luchko set. Properties of the general fractional probability density function and the general fractional cumulative distribution function are described. The truncated GF probability density function, truncated GF cumulative distribution function, and truncated GF average values are defined. Examples of the general fractional (GF) probability distributions, the corresponding probability density functions, and cumulative distribution functions are described. Nonlocal (general fractional) distributions are described, including generalizations of uniform, degenerate, and exponential type distributions; distributions with the Mittag-Leffler, power law, Prabhakar, Kilbas–Saigo functions; and distributions that are described as convolutions of the operator kernels and standard probability density.

Keywords: non-local probability; probability theory; nonlocal theories; general fractional calculus; fractional derivatives; fractional integrals

MSC: 60Axx; 26A33; 60A99; 60E05

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1. Introduction

Fractional calculus involving differential and integral operators of the non-integer order (see [1–7]) has been actively used in recent decades to describe nonlocal systems and processes in physics (for example, see the handbooks [8,9] and books [10–17]). Fractional calculus has a rich history, which was first described in 1868 by Letnikov in his work [18], and then continued by other authors in book [1] and papers [19–24].

Nonlocal models are also studied in modern statistical physics, including the following areas (A) fractional physical kinetics and fractional anomalous diffusion; (B) statistical physics of lattices with long-range interactions; (C) fractional statistical mechanics. Let us note some reviews, books, and works on statistical physics, in which nonlocal models were considered.

(A) Fractional kinetics has been described in many reviews and books [13,17,25–27], and [28–32]. Regarding anomalous and fractional diffusion, there are many works [13,17,33–37]. These works use fractional calculus to describe nonlocality in space and time.

(B) The lattice models of statistical physics, which take into account long-range interactions, began with the work of Dyson [38–40] in 1969–1971, and other scientists in [41–45], and then began to be actively developed. Remarkable properties of lattice models of systems with long-range interactions have been rigorously proven in works by [46,47].

These properties involve the continuous limits of lattice models of systems with power law types of long-range interactions that can be described by the equations with fractional derivatives of non-integer orders. The mapping of lattice models into continuous models is defined by a special transform operator [10,46–51]. The models with long-range interactions are used in nonlinear dynamics to describe processes with spatial nonlocality [52–59].

(C) In statistical mechanics, the power law type of the nonlocality can be described using methods of integro-differential equations with non-integer order derivatives [60–62] and [63–66]. In these works, fractional generalizations of some equations of statistical mechanics are suggested. To obtain these equations, the conservation of probability in the phase space is used [60,61]. The Liouville equations with fractional derivatives (with respect to coordinates and momenta) are derived in works [60–64]. The use of fractional calculus to construct a non-local generalization of the normalization conditions for the density of the distribution and a generalization of the definition of the average value were first proposed in 2004 [67–69].

Many models consider fractional differential equations that describe nonlocal statistical systems and processes. However, in many works, these equations are simply postulated and are not derived strictly from the basic laws and equations. This is largely due to the fact that these equations can be rigorously derived by introducing certain assumptions concerning the elements of probability theory for the nonlocal case. Because of this, it is important to understand the possibility of generalization of the standard probability theory to the nonlocal case.

As a mathematical tool for constructing basic concepts of the nonlocal probability theory, one could use fractional calculus instead of standard mathematical analysis, which uses integrals and derivatives of integer orders. Fractional differential equations of non-integer orders are powerful mathematical tools to describe nonlocal systems and processes. Unfortunately, in the theory of integrals and derivatives of non-integer orders, a narrow set of operator kernels is used, which is mainly of the power type.

The use of integral and integro-differential operators with the general form of operator kernels is very important to describe the widest possible types of nonlocalities in space. However, the use of integral and integro-differential operators of general forms has a significant drawback. The disadvantage is the lack of mutual consistency between integral and integro-differential operators, which leads to the fact that these operators do not form a general calculus.

As a generalization of FC, one can use the general fractional calculus (GFC) that is based on the concept of kernel pairs, which was proposed by Sonin [70] in an 1884 article [71] (see also [72]). Recently, the GFC and its applications have been actively developed by Kochubei [73–75], Samko and Cardoso [76,77], Luchko and Yamamoto [78,79], Kochubei and Kondratiev [80,81], Sin [82], Kinash and Janno [83,84], Hanyga [85], Giusti [86], Luchko [87–93], Tarasov [94–99], Al-Kandari, Hanna and Luchko [100], and Al-Refai and Luchko [101]. A very important form of the GFC was proposed by Luchko in 2021 [87–89]. In articles [87,88], the fundamental theorems for the general fractional integrals (GFIs) and the general fractional derivatives (GFDs) were proved. The general fractional calculus with functions of several variables (the multivariable) GFC and the general fractional vector calculus (GFVC) was proposed in 2021 [95].

The concept of the nonlocal probability theory (NPT) can be considered in a broad sense, namely, as a statistical theory of systems with nonlocality in space and time. Note that the theory of stochastic or random processes with nonlocality in time (with fading memory) is well known [102–107], and is actively used to describe economic processes.

To take into account nonlocality in the probability theory, it is necessary to generalize basic concepts of the standard (classical) probability theory. Such nonlocal generalizations must not only be correctly defined but also mutually consistent in order to form a self-consistent mathematical theory. At present, there is no mathematically correct and self-consistent "nonlocal probability theory", if we do not include the so-called "quantum nonlocality" in the concept of nonlocality.

In this article, Luchko's GFC and the GFVC are proposed to construct the nonlocal generalization of standard (classical) probability theory. The nonlocal probability theory is considered a theory in which nonlocalities in space can be described by the pairs of Sonin kernels from the Luchko sets. In the nonlocal probability theory, it is assumed that properties of the probability density and cumulative distribution function at any point P in the space depending on the properties of the probability density and cumulative distribution function in all other points in the space in addition to properties at point P. This paper proposes a generalization of integral and differential forms of the equations of the probability density and cumulative distribution functions to the nonlocal case. These equations have the form of fractional integral and differential equations with GFD and GFI with Sonin kernels from the Luchko set.

Let us note some mathematical features of the proposed approach. First of all, it should be emphasized that GF differential operators are nonlocal, since they are, in fact, integro-differential operators. In addition, the fractional and the general fractional derivatives of the Riemann–Liouville type of a constant function are not equal to zero. The standard Leibniz rule (the product rule) and the chain rule for fractional derivatives of the non-integer order and GF derivatives do not hold [108,109]. In addition, a violation of the chain rule leads to the fact that the operators defined in different orthogonal curvilinear coordinate (OCC) systems (Cartesian, cylindrical, and spherical) are not related to each other by coordinate transformations. At the same time, the equations defining these nonlocal operators in the OCC are correct for different coordinate systems [95] since mutual consistency between integral and integro-differential operators in OCC is based on the fundamental theorems of GFC. The requirement of mutual consistency of the nonlocal generalizations of integral and differential operators, which is expressed as a generalization of fundamental theorems, imposes restrictions on the properties of operator kernels and their applications [87,95,110–112]. A consistent formulation of the nonlocal probability theory should also be based on the basic theorems of some nonlocal calculus, which can be the general fractional calculus.

Let us briefly describe the content of this article.

In Section 2, nonlocal generalizations of the probability density function, the cumulative distribution function, and probability are proposed. The properties of these functions are described and proved.

In Section 3, relationships between local and nonlocal quantities are described to explain the concept of nonlocality. This section does not consider questions of constructing a nonlocal probability theory. It only describes what exactly is meant under nonlinearity in this paper.

In Section 4, nonlocal analogs of uniform and degenerate distributions, which are called general fractional uniform and degenerate distributions, are described.

In Section 5, general fractional distributions with some special functions are described. The GF distributions with the Mittag-Leffler function, the power law function, the Prabhakar function, and the Kilbas–Saigo function are suggested. Examples of GF distributions that can be represented as convolutions of the operator kernels, which describe nonlocality and standard probability density, are considered. A property of non-equivalence of equations with GFD and their solutions in different spaces is described.

In Section 6, general fractional distributions of the exponential types are suggested as a generalization of the standard exponential distributions by using the solution of linear GF differential equations.

In Section 7, truncated GF distributions and moment functions are defined. The truncated GF probability density function, truncated GF cumulative distribution function, and the truncated GF average values and moments, are considered. Two examples of the calculation of the truncated GF average value are given.

A brief conclusion is given in Section 7.

2. Toward Nonlocal Probability Theory

2.1. Remarks about the Concept of the Nonlocal Probability Theory

Let us give remarks about the concept of the nonlocal probability theory.

- *Requirements of the nonlocality Theory*
  To describe nonlocality in space, integral operators and integro-differential operators should be used. Moreover, the kernels of these operators must depend on at least two points in space. If these kernels are dependent on only one point, then they could be interpreted as densities of states that are often used in statistical physics. The density of states (DOS) describes the distribution of permitted states in space and the probability density function (PDF) describes the placement of particles by these permitted states. In this case, it is obvious that such kernels cannot describe nonlocality in space.
  For example, let functions $F(x)$ and $f(x)$ be related by the equation

$$F(x) \;=\; \int_0^x M(u)\, f(u)\, du, \tag{1}$$

  where $M(x)$ is a kernel, which can be interpreted as the density of states. Such a relationship of functions $f(x)$ and $F(x)$ cannot be interpreted as nonlocal since the equation can be represented in the form

$$\frac{dF(x)}{dx} \;=\; M(x)\, f(x). \tag{2}$$

  Equation (2) is a differential equation of an integer (first) order, in which the functions are determined by properties in an infinitesimal neighborhood of the considered point with coordinate x. Because of this, we can formulate the requirement of nonlocality in the following form. The equations that describe nonlocality cannot be represented as an equation or a system of a finite number of differential equations of an integer order. It is possible to consider the kernel in the form $M(x,u)$, instead of $M(u)$; that is,

$$F(x) \;=\; \int_0^x M(x,u)\, f(u)\, du. \tag{3}$$

  In the general case, Equation (3) cannot be represented as a differential equation of the integer order for a wide class of operator kernels $M(x,u)$.
  Therefore, to take into account nonlocality, the operator kernels must depend on at least two points in space ($M(P_1,P_2)$, $K(P_1,P_2)$). In this case, operator kernels can have physical interpretations of the nonlocal density of states in space. For simplicity, one can consider the dependence on the distance between points or the difference in the coordinates of these points. In the one-dimensional case, this leads, for example, to operator kernels of the form ($M(P,P') = M(x_1 - x_2)$, $K(P,P') = K(x_1 - x_2)$).
- *Requirement of Self-Consistency of Theory*
  For the mathematical self-consistency of the theory, the integral and integro-differential operators must be mutually consistent and must form some calculus. In order to explain this requirement, consider the following.
  Suppose that the generalized cumulative distribution function F can be obtained from the generalized probability density function f by the action of some nonlocal integral operator $\widehat{I^M}$, i.e.,

$$F \;=\; \widehat{I^M}\, f. \tag{4}$$

Suppose also that the generalized probability density function f can be obtained from the generalized cumulative distribution function F by the action of some nonlocal integro-differential operator $\widehat{D^K}$, i.e.,

$$f = \widehat{D^K} F. \tag{5}$$

If the operators are not mutually consistent, then the sequential action of these operators leads to the mathematical non-self-consistency of the theory. The non-self-consistency of the theory is expressed in the fact that the substitution of the first equation into the second, as well as the substitution of the second equation into the first, will not result in identities. This is expressed in the form of inequalities

$$f = \widehat{D^K} F = \widehat{D^K} \widehat{I^M} f \neq f, \tag{6}$$

$$F = \widehat{I^M} f = \widehat{I^M} \widehat{D^K} F \neq F. \tag{7}$$

In the standard probability theory, the requirement of self-consistency is satisfied by the virtue of the first and second fundamental theorems of the mathematical analysis.

The proposed two requirements can be satisfied by using the general fractional calculus (GFC) in the Luchko form.

- (1) The requirement of the nonlocality in this case is realized as follows. In the GFC, the integral operators and integro-differential operators are represented as Laplace convolutions, in which operator kernels are differences in the coordinates of two different points ($M(P, P') = M(x_1 - x_2)$, $K(P, P') = K(x_1 - x_2)$). The proposed approach to the nonlocal probability theory can be characterized as an approach, in which nonlocality is described by the pair of kernels $(M_x(x), K_x(x))$ from the Luchko sets, and equations with GFI and GFD. In GFC, operator kernels from the Luchko sets cannot be represented as the kernel pairs $(M_x(x) = 1, K_x(x) = \delta(x))$,
- (2) The requirement of self-consistency in this case is realized as follows. In the GFC, the integral operators and integro-differential operators are called the general fractional integrals (GFIs) and the general fractional derivatives (GFDs). These operators satisfy the first and second fundamental theorems of GFC.

In this paper, to construct a nonlocal generalization of standard probability theory (see books [113–117] and handbook [118–120]), the general fractional calculus (GFC) in the Luchko form [87,88] is proposed. Because of this, restrictions on function spaces, which can be used in nonlocal theory, are dictated by the restrictions that are used in the GFC. Therefore, one can consider a generalization of a special case of the standard probability theory, in which only continuous distributions on the positive semi-axis are considered. Obviously, in such a standard theory, a large number of probability distributions are left out of consideration, including, for example, the standard uniform distribution.

2.2. Standard Continuous Distributions on the Positive Semi-Axis

In this subsection, some equations of the standard probability theory will be written out to simplify further references. As an example, the one-dimensional continuous distributions on the positive semi-axis are used for these equations.

Let us consider a probability distribution of one random variable X on the set $\mathbb{R}_+ = (0, \infty)$. Let the function $f_X(x)$ belong to the set $C(0, \infty)$. For a function $f_X(x)$ to be a probability density function (PDF), it must satisfy [113], p. 3, the non-negativity condition $f_X(x) \geq 0$ and the normalization condition

$$f_X(x) \geq 0, \quad \int_0^\infty f_X(x)\, dx = 1. \tag{8}$$

Using the fact that continuous functions $f_X(x) \in C(0,\infty)$ are integrable, the distribution function $F_X(x)$ can be defined by the integration of the first-order. For each probability density function $f_X(x) \in C(0,\infty)$, one can put in correspondence [113], p. 3, its cumulative distribution function $F_X(x)$ defined by

$$F_X(x) \;=\; \int_0^x f_X(u)\,du. \tag{9}$$

Function (9) satisfies all standard properties of the cumulative distribution function. It is a non-decreasing continuous function that takes values from 0 to 1. Since $f_X(x)$ belongs to the set $C(0,\infty)$, Function (9) is continuously differentiable, i.e., $F_X(x) \in C^1(0,\infty)$. Then, the probability density function $f_X(x)$ can be obtained by the action of the first-order derivative

$$f_X(x) \;=\; \frac{d}{dx} F_X(x). \tag{10}$$

This statement is proved by the substitution of (9) into Equation (10), which gives the identity

$$f_X(x) \;=\; \frac{d}{dx} F_X(x) \;=\; \frac{d}{dx} \int_0^x f_X(u)\,du \;=\; f_X(x) \tag{11}$$

by the first fundamental theorem of calculus in the form

$$\frac{d}{dx} \int_0^x f(u)\,du \;=\; f(x). \tag{12}$$

Substitution of (10) into Equation (9) should also give the identity

$$F_X(x) \;=\; \int_0^x f_X(u)\,du \;=\; \int_0^x \frac{dF_X(u)}{du}\,du \;=$$

$$\int_0^x dF_X(u) \;=\; F_X(x) \;-\; F_X(0) \tag{13}$$

which is satisfied by the second fundamental theorem of calculus, if $F_X(x) \in C^1(0,\infty)$. Substitution of (10) into non-negativity and normalization conditions gives

$$\frac{dF_X(x)}{dx} \geq 0, \quad F_X(\infty) \;=\; 1. \tag{14}$$

The probability for the region $[a,b] \subset \mathbb{R}_{0,+} = [0,\infty)$ is described by the equation

$$P([a,b]) \;=\; F_X(b) \;-\; F_X(a), \tag{15}$$

where $F_X(x)$ is defined by Equation (9).

2.3. Toward Generalizations of Standard PDF and CDF for Nonlocal Cases

Despite the fact that Equation (9) uses an integral operator, the connection of functions $F_X(x)$ and $f_X(x)$ can be interpreted as local. This is due to the fact that the relationships of these functions, which are described by Equation (9), can be represented as differential equations of the first-order (10). Differential equations of integer powers for each point are given by the properties of functions in an infinitely small neighborhood of this point.

To take into account a nonlocality in the probability theory, integral and integro-differential operators with kernels, which depend on at least two points, should be used. For simplicity, one can consider the dependence on the distance between points or the difference in the coordinates of these points, $M(x,u) = M(x-u)$ and $K(x,u) = K(x-u)$.

As candidates for nonlocal analogs of Equations (9) and (10), one can consider the following equations

$$F(x) \;=\; \int_0^x M_x(x-u)\, f(u)\, du, \tag{16}$$

$$f(x) \;=\; \frac{d}{dx}\int_0^x K_x(x-u)\, F(u)\, du. \tag{17}$$

Instead of Equation (17), one can also consider the equation

$$f(x) \;=\; \int_0^x K_x(x-u)\, F^{(1)}(u)\, du, \tag{18}$$

where $F^{(1)}(x) = dF(x)/dx$.

For the kernels

$$M_x(x) \;=\; h(x), \quad K_x(x) \;=\; \delta(x), \tag{19}$$

where $h(x)$ is the Heaviside step function and $\delta(x)$ is the Dirac delta function, Equations (16) and (17) (or (18)) give standard Equations (9) and (10), respectively.

The properties of operator kernels and integrands must be such that Equations (16) and (17) exist. In this case, these equations must be mutually consistent, such that the substitution (substituting one equation into another) should give the identities. These restrictions must also be imposed on the operator kernels $M_x(x)$ and $K_x(x)$. These restrictions on operator kernels and function spaces are described by fundamental theorems of the general fractional calculus. The main restrictions, which can be used for these purposes, are proposed by Luchko [87,88] in the form of the following conditions.

The main restrictions are *Sonin's condition* and *Luchko's conditions*.

Definition 1 (Sonin's condition)**.** *Let functions $M_x(x)$ and $K_x(x)$ satisfy the condition*

$$(M_x \,*\, K_x)(x) \;:=\; \int_0^x M_x(x-u)\, K_x(u)\, du \;=\; \{1\} \tag{20}$$

for all $x \in (0,\infty)$, where $*$ denoted the Laplace convolution (see [4], p. 19, and [113], p. 6-7). In condition (20), $\{1\}$ denotes the function that is identically equal to 1 on $[0,\infty)$ [87], p. 3.

Then, the Sonin condition is satisfied.

In order for Functions (16), (17) to exist, condition (20) to be satisfied, and the nonlocal analog of the fundamental theorem of calculus be proved, one can use Luchko's conditions.

Definition 2 (Luchko's first condition)**.** *Let functions $M_x(x)$, $K_x(x)$ be represented in the form the functions $M_x(x)$, $K_x(x)$ can be represented in the form $M_x(x) = x^{\alpha}\, m(x)$, and $K_x(x) = x^{\beta}\, k(x)$ for all $x > 0$, where α, $\beta \in (-1,0)$, and $m(x)$, $k(x) \in C[0,\infty)$.*

Then, the functions $M_x(x)$, $K_x(x)$ belong to the set $C_{-1,0}(0,\infty)$, and the first Luchko condition is satisfied.

Definition 3 (Luchko's second condition)**.** *Let function $f_X(x)$ be represented as $f_X(x) = x^{a}\, h_1(x)$ for all $x > 0$, where $a > -1$, and $h_1(x) \in C[0,\infty)$. Then, the set of such functions is denoted as $C_{-1}(0,\infty)$,*

Let a function $F_X(x)$ satisfy the condition $dF_X(x)/dx \in C_{-1}(0,\infty)$. Then, the set of such functions is denoted as $C^1_{-1}(0,\infty)$.

Let functions $f_X(x)$, $F_X(x)$ satisfy the condition $f_X(x) \in C_{-1}(0,\infty)$, $F_X(x) \in C^1_{-1}(0,\infty)$.

Then, the second Luchko condition is satisfied.

In the proposed approach to nonlocal probability theory, it will be assumed that nonlocality is described by the kernel pairs that belong to the Luchko set.

Definition 4 (Luchko set). *A pair of kernels $(M_x(x), K_x(x))$ belongs to the Luchko set, if the Sonin condition and Luchko's first condition are satisfied.*

Note the following inclusions

$$C[0,\infty) \subset C_{-1}(0,\infty) \subset C(0,\infty). \tag{21}$$

It should be also noted that kernels (19) do not belong to the Luchko set.

Equations (16) and (17), in which the pair of kernels $(M_x(x), K_x(x))$ belong to the Luchko set and functions $f(x)$ and $F(x)$ satisfy Luchko's second condition, can be written by using the general fractional integral (GFI) and general fractional derivatives (GFDs) [87,88].

Definition 5 (General fractional integral). *Let $f(x) \in C_{-1}(0,\infty)$ and $M_x(x)$, $K_x(x) \in C_{-1,0}(0,\infty)$. If $M_x(x)$ and $K_x(x)$ satisfy the Sonin condition (20), then the general fractional integral $I^x_{(M)}$ is defined by the equation*

$$I^x_{(M_x)}[u]\, f(u) = (M_x * f)(x) = \int_0^x du\, M_x(x-u)\, f(u). \tag{22}$$

Definition 6 (General fractional derivatives). *Let $f(x) \in C^1_{-1}(0,\infty)$ and $M_x(x)$, $K_x(x) \in C_{-1,0}(0,\infty)$. If $M_x(x)$ and $K_X(x)$ satisfy the Sonin condition (20), then the general fractional derivative $D^x_{(K_x)}$ of the Riemann–Liouville (RL) type is defined by the equation*

$$D^x_{(K_x)}[u]\, f(u) = \frac{d}{dx}(K_x * f)(x) = \frac{d}{dx}\int_0^x du\, K_x(x-u)\, f(u), \tag{23}$$

and the general fractional derivative $D^{x,*}_{(K_x)}$ of the Caputo type is defined by the equation

$$D^{x,*}_{(K_x)}[u]\, f(u) = (K_x * f^{(1)})(x) = \int_0^x du\, K_x(x-u)\, f^{(1)}(u), \tag{24}$$

where $f^{(1)}(x) = df(x)/dx$.

Using the definitions of GFI and GFD, Equations (16)–(18) can be represented in the forms

$$F(x) = I^x_{(M_x)}[u]\, f(u), \tag{25}$$

$$f(x) = D^x_{(K_x)}[u]\, F(u), \tag{26}$$

$$f(x) = D^{x,*}_{(K_x)}[u]\, F(u). \tag{27}$$

Substitution of expression (25) into expression (26) gives the identity

$$f(x) = D^x_{(K_x)}[u]\, F(u) = D^x_{(K_x)}[u]\, I^u_{(M_x)}[w]\, f(w) = f(x), \tag{28}$$

if the pair of kernels (M_x, K_x) belongs to the Luchko set and $f(x)$ belongs to the space $C_{-1}(0,\infty)$. This identity follows from the first fundamental theorem of GFC for the GFD of the Riemann–Liouville type [87,88].

Definition 7 (Luchko's third condition). *Let functions $M_x(x)$ and $K_x(x)$ be kernels of the GFI and GFD, respectively, and let this pair of kernels belong to the Luchko set.*

Let a function $f(x)$ be represented as a GFI with the kernel $K_x(x)$, such that

$$f(x) = I^x_{(K_x)}[u]\, \varphi(u) \quad (\text{for all } x > 0, \tag{29}$$

where $\varphi(x) \in C_{-1}(0,\infty)$.

Then, the function $f(x)$ belongs to the set $C_{-1,(K)}(0,\infty)$, and the third Luchko condition is satisfied.

It should be noted that using the GFD of the Caputo type

$$D^{x,*}_{(K_x)}[u]\, F(u) \;=\; \left(K_x \,*\, \frac{dF(x)}{dx} \right) \;=\; \int_0^x du\, K_x(x-u)\, \frac{dF(u)}{du} \tag{30}$$

one can consider an action of the GFD (30) on Function (25), which gives the identity

$$f(x) \;=\; D^{x,*}_{(K_x)}[u]\, F(u) \;=\; D^{x,*}_{(K_x)}[u]\, I^{u}_{(M_x)}[w]\, f(w) \;=\; f(x), \tag{31}$$

if $f(x)$ belongs to the space $C_{-1,(K)}(0,\infty)$ and the kernel pair (M_x, K_x) belongs to the Luchko set. This identity follows from the first fundamental theorem of GFC for the GFD of the Caputo type [87,88].

A function $f(x)$ that belongs to the set $C_{-1,(K)}(0,\infty)$ is a continuous function on the positive semi-axis, for which the following inclusions are satisfied

$$C_{-1,(K)}(0,\infty) \;\subset\; C_{-1}(0,\infty) \;\subset\; C(0,\infty). \tag{32}$$

Theorem 1 (First fundamental theorem for the GFC)**.** *Let a kernel pair (M_x, K_x) belong to the Luchko set. If $f(x)$ belongs to the space $C_{-1}(0,\infty)$, then*

$$D^{x}_{(K_x)}[u]\, I^{u}_{(M_x)}[w]\, f(w) \;=\; f(x) \tag{33}$$

for all $x > 0$.

If $f(x)$ belongs to the space $C_{-1,(K)}(0,\infty)$, then

$$D^{x,*}_{(K_x)}[u]\, I^{u}_{(M_x)}[w]\, f(w) \;=\; f(x) \tag{34}$$

for all $x > 0$.

Proof. Theorem 1 is proved in [87,88] (see Theorem 3 in [87], p. 9, and Theorem 1 in [88], p. 6). □

Substitution of expression (26) into expression (25) gives the identity

$$F(u) \;=\; I^{x}_{(M_x)}[u]\, f(u) \;=\; I^{x}_{(M_x)}[u]\, D^{u}_{(K_x)}[w]\, F(w) \;=\; F^{(M)}_{X}(u)(x), \tag{35}$$

if the pair of kernels (M_x, K_x) belongs to the Luchko set and $f(x)$ belongs to the space $C_{-1}(0,\infty)$. This identity follows from the second fundamental theorem of GFC for the GFD of the RL-type [87,88].

Theorem 2 (Second fundamental theorem for the GFC)**.** *Let a kernel pair (M_x, K_x) belong to the Luchko set. If $F(x)$ belongs to the space $C^1_{-1}(0,\infty)$, then*

$$I^{u}_{(M_x)}[u]\, D^{u,*}_{(K_x)}[w]\, F(w) \;=\; F(x) \;-\; F(0) \tag{36}$$

$$I^{u}_{(M_x)}[u]\, D^{u}_{(K_x)}[w]\, F(w) \;=\; F(x) \tag{37}$$

for all $x > 0$.

Proof. Theorem 2 is proved in [87,88] (see Theorem 4 in [87], p. 11, and Theorem 2 in [88], p. 7). □

Note that the equation

$$D^{x}_{(K_x)}[u]\, F(u) \;=\; D^{x,*}_{(K_x)}[u]\, F(x) \,+\, K_x(x)\, F(0) \tag{38}$$

holds for all $x > 0$, if $F(x) \in C^1_{-1}(0,\infty)$. Using Equation (38), one can see that the functions

$$f_{RL}(x) \;=\; D^{x}_{(K_x)}[u]\, F(u), \quad f_C(x) \;=\; D^{x,*}_{(K_x)}[u]\, F(u) \tag{39}$$

coincide, if $F(0) = 0$.

2.4. General Fractional (GF) Probability Density Function

The Luchko conditions ensure the existence of GFI, GFD, and the fulfillment of the fundamental theorems of GFC for these operators, but do not guarantee the fulfillment of the probabilities density properties for $f(x)$. In order for a function $f(x)$ to be a probability density, it is necessary to impose additional conditions on the function.

Definition 8 (General fractional (GF) probability density)**.** *Let a pair of kernels $(M_x(x), K_x(x))$ belong to the Luchko set.*

Let $f(x)$ be a function that satisfies the following conditions.

(1) The function $f(x)$ is a continuous function on the positive semi-axis $(0,\infty)$, such that

$$f(x) \;\in\; C_{-1,(K)}(0,\infty). \tag{40}$$

(2) The function $f(x)$ is a non-negative function ($f(x) \geq 0$) for all $x > 0$.
(3) The function $f(x)$ satisfies the normalization condition

$$0 \;<\; N(M_x, f) \;<\; \infty, \tag{41}$$

where

$$N(M_x, f) \;:=\; \lim_{x\to\infty} I^{x}_{(M_x)}[u]\, f(u) \;=\; \lim_{x\to\infty} \int_0^x M_x(x-u)\, f(u)\, du. \tag{42}$$

Then, the function

$$f_X(x) \;=\; N^{-1}(M_x, f)\, f(x) \tag{43}$$

is called the GF probability density. The set of such functions is denoted by the symbol $C^{(M)}_{-1}(0,\infty)$.

One can define the standard probability density function in the following form. In Definition (8), one can consider that the function $f_X(x)$ belongs to the set $C_{-1}(0,\infty)$ instead of condition (40), and the function $f_X(x)$ satisfies the standard normalization condition instead of condition (41).

Definition 9 (Standard probability density function)**.** *Let $f(x)$ be a function that satisfies the following conditions.*

(1) The function $f(x)$ is a continuous function on the positive semi-axis $(0,\infty)$, such that

$$f(x) \;\in\; C_{-1}(0,\infty). \tag{44}$$

(2) The function $f(x)$ is a non-negative function ($f(x) \geq 0$) for all $x > 0$.
(3) The function $f(x)$ satisfies the condition

$$\lim_{x\to\infty} \int_0^x f_X(u)\, du \;=\; 1. \tag{45}$$

Then, such a function $f_X(x)$ is called the standard probability density, and the set of such functions is denoted as $C^{(\{1\})}_{-1}(0,\infty)$.

Note that the kernels $M_x(x)$ and $K_x(x)$, a pair of which belongs to the Luchko set, are non-negative and non-increasing functions [87,88].

Remark 1. *Note that $C_{-1}^{(\{1\})}(0,\infty)$ cannot be considered as a subset of $C_{-1}^{(M)}(0,\infty)$ since the kernel $M_x(x) = \{1\}$ for all $x > 0$ cannot be a kernel of a pair from the Luchko set.*

2.5. General Fractional (GF) Cumulative Distribution Function

Let us formulate some restrictions on the nonlocal generalization of the standard cumulative distribution function. In this case, the definitions will not be formulated in maximum generality, and will consider only a simplified case of continuity and differentiability at all points of an open interval $(0,\infty)$.

Definition 10 (General fractional (GF) cumulative distribution function)**.** *Let a pair of kernels $(M_x(x), K_x(x))$ belong to the Luchko set.*

If $f_X(x) \in C_{-1}^{(M)}(0,\infty)$, then the function $F_X^{(M)}(x)$ that is defined by the equation

$$F_X^{(M)}(x) \;=\; I^x_{(M_x)}[u]\, f_X(u) \;=\; \int_0^x M_x(x-u)\, f_X(u)\, du \tag{46}$$

is called the GF cumulative distribution function. The set of such functions is denoted as $C_{CDF}^{(M)}(0,\infty)$.

If $f_X(x) \in C_{-1}^{(\{1\})}(0,\infty)$, then the function $F_X(x)$ that is defined by the equation

$$F_X(x) \;=\; \int_0^x f_X(u)\, du \tag{47}$$

is called the standard cumulative distribution function. The set of such functions is denoted as $C_{CDF}^{(\{1\})}(0,\infty)$.

The following theorem is important for describing the properties of the GF cumulative distribution functions (46).

Theorem 3 (The Luchko theorem about set $C_{-1,(K)}(0,\infty)$)**.** *Let a pair $(M_x(x), K_x(x))$ belong to the Luchko set.*

If $f(x) \in C_{-1,(K)}(0,\infty)$, then

$$\lim_{x\to 0+} I^x_{(M_x)}[u]\, f(u) \;=\; 0, \quad I^x_{(M_x)}[u]\, f(u) \;\in\; C^1_{-1}(0,\infty). \tag{48}$$

The inverse statement is also satisfied: If conditions (48) are satisfied, then $f(x) \in C_{-1,(K)}(0,\infty)$.

Proof. The statements of this theorem are proven by Luchko in [87], (see comments on p. 9, and Remark 1 on p. 10 of [87]). □

Using the Luchko theorem (Theorem 3) and the properties of functions $f_X(x) \in C_{-1}^{(M)}(0,\infty)$, one can prove the following properties of functions (46); the following properties the GF cumulative distribution functions eqrefDEF-FM can be proved.

Theorem 4 (Property of GF cumulative distribution functions)**.** *Let a pair $(M_x(x), K_x(x))$ belong to the Luchko set and a function $f_X(x)$ belong to the set $C_{-1}^{(M)}(0,\infty)$.*

Then, the function $F_X^{(M)}(x)$, which is defined by Equation (46), satisfies the following properties.

(A) *The function* $F^{(M)}_X(x)$ *belongs to the set* $C^1_{-1}(0,\infty)$ *i.e.,*

$$\frac{dF^{(M)}_X(x)}{dx} \in C_{-1}(0,\infty). \tag{49}$$

(B) *The behavior of the function* $F^{(M)}_X(x)$ *at zero is described as*

$$\lim_{x\to 0+} F^{(M)}_X(x) = 0. \tag{50}$$

(C) *The behavior of the function* $F^{(M)}_X(x)$ *at infinity is described as*

$$\lim_{x\to +\infty} F^{(M)}_X(x) = 1. \tag{51}$$

(D) *The GF derivatives of the Caputo type of* $F^{(M)}_X(x)$ *is a non-negative function*

$$D^{x,*}_{(K_x)}[u]\, F^{(M)}_X(u) \geq 0. \tag{52}$$

(E) *The GF derivatives of the Riemann–Liouville type of* $F^{(M)}_X(x)$ *is a non-negative function*

$$D^{x}_{(K_x)}[u]\, F^{(M)}_X(u) \geq 0. \tag{53}$$

Proof. (A+B) By Definition 8 of a GF probability density function $f_X(x) \in C^{(M)}_{-1}(0,\infty)$, the function $f_X(x)$ belongs to the set $C_{-1,(K)}(0,\infty)$. This means (see Definition 7) that the function $f_X(x)$ can be represented as

$$f_X(x) = I^x_{(K_x)}[u]\, \varphi(u) \tag{54}$$

for all $x > 0$, where $\varphi(x) \in C_{-1}(0,\infty)$. According to the Luchko theorem (Theorem 3), such functions have two following important properties

$$I^x_{(M_x)}[u]\, f_X(u) \in C^1_{-1}(0,\infty), \tag{55}$$

$$\lim_{x\to 0+} I^x_{(M_x)}[u]\, f_X(u) = 0. \tag{56}$$

Using Definition 10, the GF cumulative distribution function $F^{(M)}_X(x)$ is defined by the equation

$$F^{(M)}_X(x) = I^x_{(M_x)}[u]\, f_X(u), \tag{57}$$

where $f_X(x) \in C_{-1,(K)}(0,\infty)$. Using Equation (57), the properties (55) and (56) can be rewritten as

$$F^{(M)}_X(x) \in C^1_{-1}(0,\infty), \tag{58}$$

$$\lim_{x\to 0+} F^{(M)}_X(x) = 0. \tag{59}$$

These properties coincide with properties A and B.

(C) By Definition 8, the GF probability density function $f_X(x)$ satisfies the normalization condition

$$\lim_{x\to \infty} I^x_{(M_x)}[u]\, f_X(u) = 1. \tag{60}$$

Using Equation (57), which defines a GF cumulative distribution function $F_X^{(M)}(x)$ (see Definition 10), Equation (60) can be rewritten in the form

$$\lim_{x\to\infty} F_X^{(M)}(x) = 1 \tag{61}$$

This property coincides with property C.

(D) Using (58), one case see that $F_X^{(M)}(x) \in C^1_{-1}(0,\infty)$. For such functions there exists a GF derivative of the Caputo type (24) (see Definition 6).

Using Definition 10, the GF cumulative distribution function $F_X^{(M)}(x)$ is defined by Equation (57). Then, the GFD of the Caputo type of Equation (57) has the form

$$D^{x,*}_{(K_x)}[u]\, F_X^{(M)}(u) = D^{x,*}_{(K_x)}[u]\, I^u_{(M_x)}[w]\, f_X(w), \tag{62}$$

where $f_X(x) \in C_{-1,(K)}(0,\infty)$.

Using the first fundamental theorem for the GFC (Theorem 1) for the GFD of the Caputo type, the equality

$$D^{x,*}_{(K_x)}[u]\, I^u_{(M_x)}[w]\, f_X(w) = f_X(x) \tag{63}$$

is satisfied, if $f_X(x)$ belongs to the set $C_{-1,(K)}(0,\infty)$. Therefore, Equations (62) and (63) give

$$D^{x,*}_{(K_x)}[u]\, F_X^{(M)}(u) = f_X(x). \tag{64}$$

By Definition 8, the GF probability density function $f_X(x)$ satisfies the property

$$f_X(x) \geq 0. \tag{65}$$

Using Equation (64), the inequality (65) can be represented in the form

$$D^{x,*}_{(K_x)}[u]\, F_X^{(M)}(u) \geq 0. \tag{66}$$

This property coincides with property D.

(E) Similar to the proof of property D for the GFD of the Caputo type, the proof for the GFD of the Riemann–Liouville type can be realized. Using the first fundamental theorem of GFC (Theorem 1) for the GFD of the Riemann–Liouville type, the equality

$$D^{x}_{(K_x)}[u]\, I^u_{(M_x)}[w]\, f_X(w) = f_X(x) \tag{67}$$

is satisfied, if $f_X(x)$ belongs to the set $C_{-1}(0,\infty)$. Taking into account the inclusion

$$C_{-1,(K)}(0,\infty) \subset C_{-1}(0,\infty), \tag{68}$$

one can state that the first fundamental theorem for the GFC for the GFD of the Riemann–Liouville type is satisfied for $f_X(x) \in C_{-1,(K)}(0,\infty)$. Therefore, for the GFD of the Riemann–Liouville type, one can obtain

$$D^{x}_{(K_x)}[u]\, F_X^{(M)}(u) \geq 0. \tag{69}$$

This property coincides with property E.

This ends the proof. □

Remark 2. *It should be emphasized that the properties described in Theorem 4 hold for any pair of operator kernels from the Luchko set. It should also be noted that the fact that the GF probability density function belongs to the set* $C_{-1,(K)}(0,\infty) \subset C_{-1}(0,\infty)$ *is important to prove these* *properties. Note that the condition $f_X(x) \in C_{-1,(K)}(0,\infty)$ also guarantees the fulfillment of the first fundamental theorem of GFC.*

Corollary 1 (GF probability density through GF distribution function)**.** *Let a pair $(M_x(x), K_x(x))$ belong to the Luchko set and the function $F^{(M)}_X(x)$ belongs to the set $C^{(M)}_{-1}(0,\infty)$.*

Then, the functions, which are defined by the equations

$$f^{(K_x)}_{X,C}(x) := D^{x,*}_{(K_x)}[u]\, F^{(M)}_X(u), \tag{70}$$

$$f^{(K_x)}_{X,RL}(x) := D^{x}_{(K_x)}[u]\, F^{(M)}_X(u), \tag{71}$$

where $D^{x,*}_{(K_x)}$ is the GFD of the Caputo type and $D^{x}_{(K_x)}$ is the GFD of the Riemann–Liouville type, and are the same

$$f^{(K_x)}_{X,RL}(x) = f^{(K_x)}_{X,C}(x) \tag{72}$$

for $x \in (0,\infty)$, and belong $C^1_{-1}(0,\infty)$. Then, one can use the notation $f^{(K_x)}_X(x)$ or $f_X(x)$ for functions (70) and (71) describe the GF PDF of the random variable X on a positive semi-axis.

Proof. The proof is based on the identity connecting the GF derivatives of two types in the form

$$D^{x}_{(K_x)}[u]\, F^{(M)}_X(u) = D^{x,*}_{(K_x)}[u]\, F^{(M)}_X(u) + K_x(x)\, F^{(M)}_X(0) \tag{73}$$

that is satisfied if $F^{(M)}_X(x) \in C^1_{-1}(0,\infty)$ [87,88] (see Equation (49) in [87], p. 8, and Equation (29) in [88], p. 6).

Using Equations (70) and (71), Equality (73) leads to the equation

$$f^{(K_x)}_{X,C}(x) := f^{(K_x)}_{X,RL}(x) - K_x(x)\, F^{(M)}_X(0). \tag{74}$$

Using that the property of the function $f_X(x) \in C^{(M)}_{-1}(0,\infty)$, in the form (50), Equation (74) gives Equality (72), if $f_X(x) \in C^{(M)}_{-1,(K)}(0,\infty)$.

This ends the proof. □

2.6. General Fractional (GF) Probability for Region $[a,b]$

The GF probability for the region $[a,b] \subset \mathbb{R}_{0,+}$ can be described by an expression similar to Equation (15) in the form

$$P^{(M)}([a,b]) = F^{(M)}_X(b) - F^{(M)}_X(a), \tag{75}$$

where $F^{(M)}_X(x)$ is defined by Equation (25). Equation (75) can be represented in the form

$$P^{(M)}([a,b]) = I^{(M_x)}_{[a,b]}[x]\, f_X(x), \tag{76}$$

where $f_X(x) \in C^{(M)}_{-1}(0,\infty)$ and $I^{(M_x)}_{[a,b]}$ is the GFI that is defined in [95] by the equation

$$I^{(M_x)}_{[a,b]}[x]\, f(x) := I^{b}_{(M_x)}[u]\, f(u) - I^{a}_{(M_x)}[u]\, f(u), \tag{77}$$

if $a > 0$, and, for $a = 0$, by the equation

$$I^{(M_x)}_{[0,x]}[x]\, f(x) := I^{x}_{(M_x)}[u]\, f(u). \tag{78}$$

As a result, one can propose the following definitions.

Definition 11 (GF probability). *Let a pair $(M_x(x), K_x(x))$ belong to the Luchko set, a function $f_X(x)$ belong to the set $C_{-1}^{(M)}(0,\infty)$, and function $F_X^{(M)}(x)$ belong to the set $C_{CDF}^{(M)}(0,\infty)$.*

Then, the real value $P^{(M)}([a,b])$ is defined by the equation

$$P^{(M)}([a,b]) \;=\; I^{(M_x)}_{[a,b]}[x]\, f_X(x) \;=\; F_X^{(M)}(b) \;-\; F_X^{(M)}(a), \tag{79}$$

where $b \;>\; a \;\; \geq\; 0$, is called the GF probability of a random variable X being in the interval $[a,b] \;\subset\; [0,\infty)$.

Remark 3. *It should be emphasized that the GF cumulative distribution function is not non-decreasing (in the standard sense) for all $x > 0$, in the general case. Only the general fractional derivative of this function $F_X^{(M)}(x)$ is non-negative. The first-order derivative of this function must not be nonnegative for all $x > 0$. This means that the function $F_X^{(M)}(x)$ can be decreased at some intervals. For $F_X^{(M)}(x) \in C_{CDF}^{(M)}(0,\infty)$, the non-decreasing function in the standard sense is only the GF integral $I^x_{(K_x)}[u]\, f_X(u)$ for all $x > 0$ since*

$$\frac{d}{dx}\, I^x_{(K_x)}[u]\, F_X^{(M)}(u) \;=\; D^x_{(K_x)}[u]\, F_X^{(M)}(u) \;\geq\; 0. \tag{80}$$

Therefore, there may exist such an interval $[a,b] \subset \mathbb{R}_+$ that the first-order derivative of the function $F_X^{(M)}(x)$ is negative. Then, on this interval, the function $F_X^{(M)}(x)$ decreases in the standard sense, and. This means that

$$F_X^{(M)}(b) \;<\; F_X^{(M)}(a), \tag{81}$$

where $b > a \geq 0$, and the GF probability (79) can be negative

$$P^{(M)}([a,b]) \;=\; F_X^{(M)}(b) \;-\; F_X^{(M)}(a) \;\leq\; 0. \tag{82}$$

At the same time, the non-decreasing condition

$$I^a_{(K_x)}[u]\, F_X^{(M)}(u) \;<\; I^b_{(K_x)}[u]\, F_X^{(M)}(u) \tag{83}$$

should be satisfied and

$$I^{(K_x)}_{[a,b]}[x]\, P^{(M)}((0,x]) \;\geq\; 0 \tag{84}$$

for every $[a,b] \subset [0,\infty)$, since

$$I^{(K_x)}_{[a,b]}[x]\, P^{(M)}((0,x]) \;=\; I^{(K_x)}_{[a,b]}[x]\, F_X^{(M)}(x) \;=\; I^b_{(K_x)}[x]\, F_X^{(M)}(x) \;-\; I^a_{(K_x)}[x]\, F_X^{(M)}(x) \;\geq\; 0 \tag{85}$$

Therefore, it is important to consider not only the general case, in which the GF probability on the interval can be negative, but also the special case, when the GF probability on the interval is non-negative.

2.7. Condition for the GF probability Density Function to be Non-Negative

The GF probability density functions ($f(x) \in C_{-1}^{(M)}(0,\infty)$) are non-negative functions ($f(x) \geq 0$) for all $x > 0$ that satisfy the GF normalization conditions. The GF probability density functions belong to the set $C_{-1,(K)}(0,\infty)$. This property means that the function $f(x)$ can be represented as

$$f(x) \;=\; I^x_{(K_x)}[u]\, \varphi(u) \;=\; (K_x \,*\, \varphi)(x), \tag{86}$$

where $\varphi(x) \;\in\; C_{-1}(0,\infty)$.

The non-negativity of the function $f(x)$ means the non-negativity of the convolution

$$(K_x * \varphi)(x) \geq 0 \quad (\text{for all } x > 0), \tag{87}$$

where the GFD kernel $K_x(x)$ is the non-negative function $K_x(x) \geq 0$ for all $x > 0$.

The properties of the non-negativity of the kernel $K_x(x)$ from the Luchko set and the non-negativity of the convolution (86) do not guarantee the non-negativity of the function $\varphi(x)$. In the general case, the function $\varphi(x)$ need not be non-negative in all points of the positive semi-axis. The function $\varphi(u)$ can also take negative values at some intervals.

Therefore, it is important to consider two following cases for the GF probability densities and GF cumulative distributions:

(A) The function $\varphi(x)$ is non-negative on the positive semi-axis, i.e., the condition

$$\varphi(x) \geq 0 \quad (\text{for all } x > 0) \tag{88}$$

holds. Further, it will be proved that condition (87) will be satisfied in this case.

(B) The function $\varphi(x)$ can be negative on some intervals of the positive semi-axis, i.e., condition (88) is violated. In this case, the condition of non-negativity of the convolution (87) must be satisfied for all $x > 0$.

Theorem 5 (Non-negativity of GFI)**.** *Let a pair $(M_x(x), K_x(x))$ belong to the Luchko set.*

Let $f(x)$ belong to the set $C_{-1,(K)}(0,\infty)$, i.e., the function $f_X(x)$ can be represented as

$$f(x) = I^x_{(K_x)}[u]\, \varphi(u), \tag{89}$$

where $\varphi(x) \in C_{-1}(0,\infty)$.

Then, if the function $\varphi(x)$ is non-negative for all $x > 0$, then the function $f(x)$ is also non-negative for all $x > 0$; that is

$$\varphi(x) \geq 0 \quad (\textit{for all } x > 0) \quad \Rightarrow \quad f(x) \geq 0 \quad (\textit{for all } x > 0). \tag{90}$$

Proof. Using that the kernels $(K_x(x), M_x(x))$, which belong to the Luchko set, are non-negative functions for all $x > 0$ and the assumption that $\varphi(x)$ is the non-negative function for all $x > 0$, the convolution

$$(K_x * \varphi)(x) = I^x_{(K_x)}[u]\, \varphi(u) = \int_0^x K_x(x-u)\, \varphi(u)\, du \tag{91}$$

is the non-negative function for all $x > 0$ by the definition of the integral. Therefore, Function (89) is also non-negative for all $x > 0$.

This ends the proof. □

Note that the statement, which is opposite to the statement of Theorem 5, is not true; that is, the statement that $\varphi(x) \geq 0$ for all $x > 0$, if $f(x) \geq 0$ for all $x > 0$ is not a true statement.

Using Theorem 5, the following property can be proved.

Theorem 6 (Non-negativity of the GF probability density function)**.** *Let a pair $(M_x(x)$, $K_x(x))$ belong to the Luchko set.*

Let $\varphi(x)$ belong to the set $C^{(\{1\})}_{-1}(0,\infty)$, i.e., the function $\varphi(x)$ is a standard probability density function in the sense of Definition 9, and the following conditions are satisfied

$$\varphi(x) \in C_{-1}(0,\infty), \tag{92}$$

$$\varphi(x) \geq 0 \quad (\textit{for all } x > 0), \tag{93}$$

$$\lim_{x \to \infty} \int_0^x \varphi(u)\, du \, = \, 1. \tag{94}$$

Then, function $f(x)$, which can be represented as

$$f(x) \, = \, I^x_{(K_x)}[u]\, \varphi(u) \tag{95}$$

belongs to the set $C^{(M)}_{-1}(0,\infty)$, i.e., the function $f(x)$ is the GF probability density function in the sense of Definition 8.

Proof. (1) If $\varphi(x) \in C^{(\{1\})}_{-1}(0,\infty)$, then $\varphi(x) \in C_{-1}(0,\infty)$. Therefore, Function (95) satisfies the condition $f(x) \in C_{-1,(K)}(0,\infty)$ that follows directly from the definition of the set $C_{-1,(K)}(0,\infty)$.

(2) The statement that the assumption of the non-negativity of a function $\varphi(x)$ for all $x > 0$ leads to the non-negativity of Function (95) is proven as Theorem 5.

(3) The GF integration of Equation (94) gives

$$I^x_{(M_x)}[u]\, f(u) \, = \, I^x_{(M_x)}[u]\, I^u_{(K_x)}[w]\, \varphi(w). \tag{96}$$

Using the associativity of the Laplace convolution and the Sonin condition $((M_x * K_x)(x) = \{1\}$ for all $x > 0)$, one can obtain

$$I^x_{(M_x)}[u]\, I^u_{(K_x)}[w]\, \varphi(w) \, = \, (M_x * (K_x * \varphi))(x) \, =$$

$$((M_x * K_x) * \varphi)(x) \, = \, (\{1\} * \varphi)(x) \, = \, \int_0^x \varphi(u)\, du. \tag{97}$$

Therefore,

$$I^x_{(M_x)}[u]\, f(u) \, = \, \int_0^x \varphi(u)\, du. \tag{98}$$

If $\varphi(x)$ belongs to the set $C^{(\{1\})}_{-1}(0,\infty)$, then the standard normalization condition

$$\lim_{x \to \infty} \int_0^x \varphi(u)\, du \, = \, 1 \tag{99}$$

is satisfied.

Using Equation (98), condition (99) can be written as

$$\lim_{x \to \infty} I^x_{(M_x)}[u]\, f(u)\, du \, = \, 1. \tag{100}$$

This ends the proof. □

Theorem 6 states that

$$\varphi(x) \, \in \, C^{(\{1\})}_{-1}(0,\infty), \quad \text{and} \quad f(x) \, \in \, C_{-1,(K)}(0,\infty) \quad \Rightarrow \quad f(x) \, \in \, C^{(M)}_{-1}(0,\infty). \tag{101}$$

Remark 4. *Note that the statement opposite to the statement of Theorem 6 is not true, since the non-negativity of the function $f_X(x)$ for all $x > 0$ does not lead to the non-negativity of the function $\varphi(x)$ for all $x > 0$.*

2.8. Condition for the GF Probability to be Non-Negative: Complete the GF Probability

Let us describe conditions for the GF probability density functions, for which the GF probability is non-negative.

Definition 12 (Set of functions $C^+_{-1,(K)}(0,\infty)$)**.** *Let functions $M_x(x)$ and $K_x(x)$ be kernels of GFI and GFD, respectively, and let the pair of these kernels belong to the Luchko set.*

Let a function $f(x) \in C_{-1,(K)}(0,\infty)$ satisfy the following condition

$$D^x_{(M_x)}[u]\, f(u) \,\geq\, 0 \quad (\textit{for all } x > 0). \tag{102}$$

Then, the set of such functions is denoted as $C^{+}_{-1,(K)}(0,\infty)$.

The set of functions, for which condition (102) is violated, is denoted as $C^{-}_{-1,(K)}(0,\infty)$.

Theorem 7 (Property of set $C^{+}_{-1,(K)}(0,\infty)$)**.** *Let functions $M_x(x)$ and $K_x(x)$ be kernels of the GFI and GFD, respectively, and let this pair of kernels belong to the Luchko set.*

Let a function $f(x)$ belong to the set $C^{+}_{-1,(K)}(0,\infty)$.

Then, the function $f(x)$ can be represented as a GFI with the kernel $K_x(x)$, such that

$$f(x) \,=\, I^x_{(K_x)}[u]\, \varphi(u) \quad (\textit{for all } x > 0, \tag{103}$$

where $\varphi(x) \in C_{-1}(0,\infty)$ and

$$\varphi(x) \,\geq\, 0 \tag{104}$$

for all $x > 0$.

Proof. If a function $f(x)$ belongs to the set $C^{+}_{-1,(K)}(0,\infty)$, then $f(x)$ satisfies the condition

$$D^x_{(M_x)}[u]\, f(u) \,\geq\, 0 \tag{105}$$

for all $x > 0$. Substitution of Equation (103) into Equation (105) gives

$$D^x_{(M_x)}[u]\, I^u_{(K_x)}[w]\, \varphi(w) \,\geq\, 0. \tag{106}$$

Using the first fundamental theorem of GFC, Equation (106) takes the form $\varphi(w) \geq 0$ for all $x > 0$. □

Definition 13. *[Complete the GF probability density function, and $C^{(M),+}_{-1}(0,\infty)$]*

Let functions $M_x(x)$ and $K_x(x)$ be kernels of the GFI and GFD, respectively, and let this pair of kernels belong to the Luchko set.

Let a function $f(x)$ be the GF probability density function (i.e., $f(x) \in C^{(M)}_{-1}(0,\infty)$) that satisfies the condition

$$f(x) \,\in\, C^{+}_{-1,(K)}(0,\infty) \,\subset\, C_{-1,(K)}(0,\infty). \tag{107}$$

Then, the function $f(x)$ is called the complete GF probability density function, and the set of such functions is denoted as $C^{(M),+}_{-1}(0,\infty)$ or $C^{(M),+}_{PDF}(0,\infty)$.

The function $f(x) \in C^{(M)}_{-1}(0,\infty)$, for which the condition $f(x) \in C^{+}_{-1,(K)}(0,\infty)$ is not satisfied (i.e., $f(x) \in C^{-}_{-1,(K)}(0,\infty)$) will be called the non-complete GF probability density function, and the set of such functions is denoted as $C^{(M),-}_{-1}(0,\infty)$ or $C^{(M),-}_{PDF}(0,\infty)$.

Note that the set $C^{(M),+}_{-1}(0,\infty)$ is the subset of $C^{(M)}_{-1}(0,\infty)$ and

$$C^{+}_{-1,(K)}(0,\infty) \cup C^{-}_{-1,(K)}(0,\infty) \,=\, C_{-1,(K)}(0,\infty). \tag{108}$$

Definition 14 (Complete the GF cumulative distribution function and $C^{(M),+}_{CDF}(0,\infty)$)**.** *Let functions $M_x(x)$ and $K_x(x)$ be kernels of the GFI and GFD, respectively, and let this pair of kernels belong to the Luchko set.*

Let a function $f(x)$ be complete GF probability density function (i.e., $f(x) \in C^{(M),+}_{-1}(0,\infty)$).

Then, the function $F^{(M)}(x)$, which is defined by the equation

$$F_X^{(M)}(x) \,=\, I^x_{(M_x)}[u]\, f(u), \tag{109}$$

is called the complete GF cumulative distribution function and the set of such functions is denoted as $C^{(M),+}_{CDF}(0,\infty)$.

If $f(x) \in C^{(M),-}_{-1}(0,\infty)$), then function (109) is called the non-complete GF cumulative distribution function, and the set of such functions is denoted as $C^{(M),-}_{CDF}(0,\infty)$.

Definition 15 (Complete GF probability)**.** *Let a pair of $(M_x(x), K_x(x))$ belong to the Luchko set, function $f_X(x)$ belongs to the set $C^{(M),+}_{-1}(0,\infty)$, and function $F_X^{(M)}(x)$ belongs to the set $C^{(M),+}_{CDF}(0,\infty)$.*

Then, the real value $P^{(M)}([a,b])$, which is defined by the equation

$$P^{(M)}([a,b]) \,=\, I^{(M_x)}_{[a,b]}[x]\, f_X(x) \,=\, F_X^{(M)}(b) \,-\, F_X^{(M)}(a), \tag{110}$$

where $b > a \geq 0$, is called the complete GF probability of a random variable X being in the interval $[a,b] \subset [0,\infty)$.

If $F_X^{(M)}(x)$ belongs to the set $C^{(M),-}_{CDF}(0,\infty)$, then the value (110) is called the non-complete GF probability of the interval $[a,b] \subset [0,\infty)$.

Let us prove that the complete GF cumulative distribution function is non-decreasing and the complete GF probability of all intervals is non-negative.

Theorem 8 (Non-decreasing GF cumulative distribution function)**.** *Let functions $M_x(x)$ and $K_x(x)$ be kernels of the GFI and GFD, respectively, and let this pair of kernels belong to the Luchko set.*

Let a function $f(x)$ be a complete GF probability density function, i.e., $f(x) \in C^{(M),+}_{-1}(0,\infty)$. Then, the GF cumulative distribution function, which is defined by the equation

$$F_X^{(M)}(x) \,=\, I^x_{(M_x)}[u]\, f(u), \tag{111}$$

satisfies the standard non-decreasing condition in the form

$$\frac{d}{dx} F_X^{(M)}(x) \,\geq\, 0 \tag{112}$$

for all $x > 0$, the GF probability is non-negative

$$P^{(M)}([a,b]) \,\geq\, 0 \tag{113}$$

for all $[a,b] \subset [0,\infty)$, where $b > a \geq 0$.

Proof. If $f(x) \in C^{(M),+}_{-1}(0,\infty)$, then the function $f(x)$ can be represented as

$$f(x) \,=\, I^x_{(K_x)}[u]\, \varphi(u), \tag{114}$$

where $\varphi(x) \in C_{-1}(0,\infty)$, and

$$\varphi(x) \,\geq\, 0 \tag{115}$$

for all $x > 0$. Substitution of Equation (114) into Equation (111) gives

$$F_X^{(M)}(x) \,=\, I^x_{(M_x)}[u]\, I^u_{(K_x)}[w]\, \varphi(w). \tag{116}$$

Using the associativity of the Laplace convolution

$$F_X^{(M)}(x) = (M_x * (K_x * \varphi))(x) = ((M_x * K_x) * \varphi)(x) =$$

$$(\{1\} * \varphi)(x) = \int_0^x \varphi(u)\, du. \tag{117}$$

Then, inequality (112) takes the form

$$\frac{d}{dx} F_X^{(M)}(x) = \frac{d}{dx} \int_0^x \varphi(u)\, du \geq 0 \tag{118}$$

for all $x > 0$.

Using the first fundamental theorem of the standard calculus, the inequality (118) is written as

$$\frac{d}{dx} F_X^{(M)}(x) = \varphi(x) \geq 0 \tag{119}$$

for all $x > 0$.

Inequality (119) means that the GF cumulative distribution Function (111) is a non-decreasing function on the interval $(0, \infty)$. Then,

$$F_X^{(M)}(b) \geq F_X^{(M)}(a), \tag{120}$$

if $b > a \geq 0$.

Using Definition 15, inequality (120) shows that inequality (113) holds for all $[a, b] \subset [0, \infty)$, where $b > a \geq 0$.

□

Corollary 2. *Let functions $M_x(x)$ and $K_x(x)$ be kernels of the GFI and GFD, respectively, and let this pair of kernels belong to the Luchko set.*

Let a function $f(x)$ be a complete GF probability density function, i.e., $f(x) \in C_{-1}^{(M),+}(0, \infty)$, and $F(x)$ be a complete GF cumulative distribution function, i.e., $F(x) \in C_{CDF}^{(M),+}(0, \infty)$, defined by the equation

$$F_X^{(M)}(x) = I^x_{(M_x)}[u]\, f(u). \tag{121}$$

Then, the GF probability

$$P^{(M)}([a, b]) = F_X^{(M)}(b) - F_X^{(M)}(a), \tag{122}$$

where $b > a \geq 0$ satisfies the standard properties of the standard probability theory.

Let A_k, $k \in \mathbb{N}$ be intervals, such that $A_k = [a_k, b_k]$, where $b_k > a_k \geq 0$. Then, the following properties of the complete GF probability density are satisfied.

(1) The non-negativity,

$$P^{(M)}(A_k) \geq 0 \tag{123}$$

for every A_k.

(2) The normalization

$$P^{(M)}((0, \infty)) = 1. \tag{124}$$

(3) If $A_k \subset A_j$, then

$$P^{(M)}(A_k) \leq P^{(M)}(A_j). \tag{125}$$

(4) If $A_k \cap A_j = \varnothing$, then

$$P^{(M)}(A_k \cap A_j) = P^{(M)}(A_k) + P^{(M)}(A_j). \tag{126}$$

(5) *If $A_k \cap A_j \neq \varnothing$, then*

$$P^{(M)}(A_k \cup A_j) = P^{(M)}(A_k) + P^{(M)}(A_j) - P^{(M)}(A_k \cap A_j). \tag{127}$$

(6) *For every A_k and A_j,*

$$P^{(M)}(A_k \cup A_j) \leq P^{(M)}(A_k) + P^{(M)}(A_j). \tag{128}$$

Proof. The proof of these properties follows directly from the properties of the GF cumulative distribution function and Equation (122) that defines the GF probability. □

The conditional GF probability is defined by the equation

$$P^{(M)}(A_k|A_j) = \frac{P^{(M)}(A_k \cap A_j)}{P^{(M)}(A_j)}, \tag{129}$$

where $P^{(M)}(A_j) \neq 0$.

Remark 5. *It should be noted that for the GF probability density functions from a set $C_{-1}^{(M),-}(0,\infty)$, the GF probability on the interval can be negative for some intervals. However, the GF probability*

$$P^{(M)}([0,x]) \geq 0 \tag{130}$$

for all $x > 0$. This property is true because it is described as the Laplace convolution of two non-negative functions

$$P^{(M)}([0,x]) = (M_x * f_X)(x) = \int_0^x M_x(x-u)\, f_X(u)\, du, \tag{131}$$

where the GFI kernel $M_x(x) \geq 0$ for all $x > 0$, and the GF probability density function $f_X(x) \geq 0$ for all $x > 0$. This statement does not depend on which of the two subsets $C_{-1}^{(M),+}(0,\infty)$ or $C_{-1}^{(M),-}(0,\infty)$ is considered.

The negativity of the GF probability on the interval is due to the fact that nonlocality affects the change in the probability density. This influence leads to the fact that the distribution function may decrease in some regions. Such influence of the nonlocality is in some sense similar to the behavior of the Wigner distribution function in quantum statistical mechanics [121,122] and some non-Kolmogorov probability models [123–126]. This property of the nonlocality in the proposed generalization of the standard probability theory should not be excluded from consideration. Because of this, it is proposed in the theory of probability not to be limited only to sets $C_{-1}^{(M),+}(0,\infty)$ and $C_{CDF}^{(M),+}(0,\infty)$. It is useful to study and consider wider sets $C_{-1}^{(M)}(0,\infty)$ and $C_{CDF}^{(M)}(0,\infty)$.

It should be emphasized that the proposed non-local probability theory cannot be reduced to a standard theory that uses classical probability densities and distribution functions. This impossibility is analogous to the fact that fractional calculus and general fractional calculus cannot be reduced to standard calculus, which uses standard integrals and derivatives.

2.9. Operator Kernels in Nonlocal Probability Theory

In the standard probability theory, the dimension of the probability density is always the inverse of the dimension of the random variable

$$[f_X(x)] = [x]^{-1}. \tag{132}$$

The standard cumulative distribution function and probability are dimensionless quantities

$$[F_X(x)] = [P(x)] = [1]. \tag{133}$$

For the correct use of the general fractional calculus in the construction of the nonlocal generalization of probability theory, it is necessary to specify the physical dimensions of the GF integral and GF derivative.

For reasons of convenience, it is proposed to use the following requirement. To preserve the standard physical dimension of quantities, the dimension of the DFD and DFI should coincide with the dimension of the derivative and integral of the first order, respectively. Then, the dimensions of the kernels $M_x(x)$, $M_x(x)$, $M_y(y)$ of the GF integrals and dimensions of the kernels $K_x(x)$, $K_x(x)$, $K_y(y)$ of the GF derivatives are the following

$$[M_x] = [1], \quad [K_x] = [x]^{-1}, \tag{134}$$

where $[1]$ denotes a dimensionless quantity.

The mathematical property that a pair of kernels (M_x, K_x) belongs to the Luchko set, then the kernel pair $(M_{x,new} = K_x,\ K_{x,new} = M_x)$ also belongs to the Luchko set, is violated, if the assumption (134) is used. However, this property of interchangeability of the operator kernels cannot be applied to the physical dimensions of these kernels, since GFI-kernel $M_x(x)$ is $[M_x] = 1$, and GFD-kernel $K_x(x)$ has $[K_x] = [x]^{-1}$.

Therefore, the mathematical property of interchangeability should be somewhat re-formed by using the following property of the variability of the kernel dimension.

The Sonin condition for the kernels $M_x(x)$ and $K_x(x)$ that belong to the Luchko set has the form

$$\int_0^t M_x(x-u)\, K_x(u)\, du = 1. \tag{135}$$

One can see the following property: If the kernel pair (M_x, K_x) belongs to the Luchko set, then the kernel pair $(M_{x,new} = \lambda M_x,\ K_{x,new} = \lambda^{-1} K_x)$ with $\lambda > 0$ also belongs to the Luchko set.

As a result, the following proposition is proved.

Theorem 9 (Interchangeability of Operator Kernels)**.** *Let a kernel pair $(M_x(x), K_x(x))$ belong to the Luchko set.*

Then the kernel pair $(M_{x,new} = \lambda^{-1} K_x(x),\ K_{x,new} = \lambda M_x(x))$ with $\lambda > 0$ and $[\lambda] = [x]^{-1}$ also belongs to the Luchko set.

Let us give examples of kernel pairs $(M_x(x),\ K_x(x))$ that belong to the Luchko set and have physical dimensions $[M_x(x)] = [1]$ and $[K_x(x)] = [x]^{-1}$. In these examples, $\lambda > 0$, $[\lambda] = [x]^{-1}$, $0 < \alpha \leq \beta < 1$, and $x > 0$.

- Example 1. The power law nonlocality:

$$M_x(x) = h_\alpha(\lambda x) = \frac{(\lambda\, x)^{\alpha-1}}{\Gamma(\alpha)}, \quad K_x(x) = \lambda\, h_{1-\alpha}(\lambda x) = \frac{\lambda\, (\lambda\, x)^{-\alpha}}{\Gamma(1-\alpha)}. \tag{136}$$

- Example 2. The Gamma distribution nonlocality:

$$M_x(x) = h_{\alpha,\lambda}(\lambda\, x) = \frac{(\lambda\, x)^{\alpha-1}}{\Gamma(\alpha)} e^{-\lambda\, x}, \quad K_x(x) = \lambda\, h_{1-\alpha,\lambda}(\lambda x) + \frac{\lambda}{\Gamma(1-\alpha)}\, \gamma(1-\alpha, \lambda x). \tag{137}$$

- Example 3. The two-parameter Mittag-Leffler nonlocality:

$$M_x(x) = (\lambda\, x)^{\beta-1}\, E_{\alpha,\beta}[-(\lambda\, x)^\alpha], \quad K_x(x) = \frac{\lambda\, (\lambda\, x)^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)} + \frac{\lambda\, (\lambda\, x)^{-\beta}}{\Gamma(1-\beta+1)}. \tag{138}$$

- Example 4. The Bessel nonlocality:

$$M_x(x) = (\sqrt{\lambda\, x})^{\alpha-1}\, J_{\alpha-1}(2\sqrt{\lambda\, x}), \quad K_x(x) = \lambda\, (\sqrt{\lambda\, x})^{-\alpha}\, I_{-\alpha}(2\sqrt{\lambda\, x}). \tag{139}$$

- Example 5. The hypergeometric Kummer nonlocality:

$$M_x(x) = (\lambda x)^{\alpha-1}\Phi(\beta,\alpha;-\lambda\, x), \quad K_x(x) = \frac{\lambda\, \sin(\pi\alpha)}{\pi}\, (\lambda\, x)^{-\alpha}\, \Phi(-\beta, 1-\alpha; -\lambda\, x). \tag{140}$$

- Example 6. The cosine nonlocality:

$$M_x(x) = \frac{\cos(2\sqrt{\lambda\, x})}{\sqrt{\pi\,\lambda\, x}}, \quad K_x(x) = \frac{\lambda\, \cosh(2\sqrt{\lambda\, x})}{\sqrt{\pi\,\lambda\, x}}. \tag{141}$$

Remark 6. *Note that this list of examples can be expanded by using kernel pairs of the form* $(M_{x,new} = \lambda^{-1}K_x(x), K_{x,new} = \lambda M_x(x))$ *for each pair* $(M_x(x), K_x(x))$ *of examples. For example, using the kernel pair* (138)*, one can consider the following new pair*

$$M_x(x) = \frac{(\lambda\, x)^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)} + \frac{(\lambda\, x)^{-\beta}}{\Gamma(1-\beta+1)}, \quad K_x(x) = \lambda\, (\lambda\, x)^{\beta-1}\, E_{\alpha,\beta}[-(\lambda\, x)^{\alpha}]. \tag{142}$$

In these examples, the following special functions are used: $\gamma(\beta, x)$ is the incomplete gamma function (see Section 9 in [127], pp. 134–142); $E_{\alpha,\beta}[x]$ is the two-parameter Mittag-Leffler function (see Section 3 in [128], pp. 17–54, [129] and Section 1.8 in [4], pp. 40–45); $J_\nu(x)$ is the Bessel function (see Section 7.2.1 in [127], pp. 3–5, and Section 1.7 in [4], pp. 32–39); $I_\nu(x)$ is the modified Bessel function (see Section 7.2.2 in [127], p. 5, and Section 1.7 in [4], pp. 32–39); $\Phi(\beta,\alpha;x)$ is the confluent hypergeometric Kummer function (Section 1.6 in [4], pp. 29–30).

Remark 7. *Note that GFD and GFI, in which kernels are standard probability density functions up to the numerical factors, can be interpreted as integer-order derivatives and integrals with continuously distributed lag [130]. For example, the GFI with the kernels* $M_x(x)$ *and the GFD with kernels* $K_{x,new}(x) = \lambda M_x(x)$*, which are defined in Equations* (137) *and* (138)*, can be used to take into account a continuously distributed lag as a special form of nonlocality.*

Remark 8. *It should be emphasized that kernels* (19) *do not belong to the Luchko set. At the same time, it should be noted that general fractional calculus is a generalization of fractional calculus of the Riemann–Liouville fractional integrals, the Riemann–Liouville and Caputo fractional derivatives of the order* α*. These operators are defined by kernels* (136)*. In the GFC, the kernel pair* (136) *does not belong to the Luchko set, if* $\alpha = 1$*. The GFI with the kernel* $M(x)$ *from the pair* (136) *is the Riemann–Liouville fractional integral of the order* α*:*

$$I^x_{(h_\alpha)}[u]\, f(u) = \lambda^{\alpha-1}\, (I^\alpha_{0+} f)(x) \tag{143}$$

for $x > 0$*.*

Then, the GFD of the RL type is the Riemann–Liouville fractional derivative of the order α*:*

$$D^x_{(h_{1-\alpha})}[u]\, F(u) = \lambda^{-\alpha}\, (D^\alpha_{0+} F)(x). \tag{144}$$

For the Riemann–Liouville fractional integral, Equation (143) *is also used for* $\alpha = 0$ *and* $\alpha = 1$*, where the relations*

$$(I^1_{0+} f)(x) = \int_0^x f(u)\, du, \quad (I^0_{0+} f)(x) = f(x) \tag{145}$$

hold true. For the Riemann–Liouville fractional derivative, Equation (144) is also used for $\alpha = 0$ *and* $\alpha = 1$*, where the relations*

$$(D^1_{0+}F)(x) \;=\; \frac{dF(x)}{dx}, \quad (D^0_{0+}F)(x) \;=\; F(x) \tag{146}$$

also hold true.

Therefore, the function $h_1(x) = \{1\}$ *can be interpreted as the Heaviside step function.*

Therefore, the function $h_0(x)$ *can be interpreted as a kind of Dirac delta function that plays the role of unity with respect to multiplication in form of the Laplace convolution [87], p. 7.*

As a result, using power law kernels (136), we obtain the consideration of a nonlocal probability theory in the framework of a fractional calculus approach, which uses the fractional integral and derivatives of an arbitrary order $\alpha > 0$*. In the framework of this calculus, the standard probability theory can be considered a special case, when the order* α *of operators is equal to integer values.*

It should also be noted that fractional integrals and derivatives of generalized functions and distributions were described in Section 8 of Chapter 2 in book [1], pp. 145–160, including generalized functions on the test function space in the framework of the Schwartz approach.

2.10. Multivariate Probability Distribution

The proposed approach to the consideration of the univariate probability distribution can be extended to multivariate probability distributions.

In the two-dimensional space (x, y), one can consider a multivariate probability distribution consisting of random variables X and Y on the set

$$\mathbb{R}^2_{0,+} \;=\; \{(x,y): \quad x \ge 0, \quad y \ge 0\}. \tag{147}$$

The probability density $f_{XY}(x,y) \;\in\; C_{-1}(\mathbb{R}^2_{0,+})$ is non-negative ($f_{XY}(x,y) \ge 0$) and is normalized

$$\int_0^\infty dx \int_0^\infty dx\, f_{XY}(x,y) \;=\; 1. \tag{148}$$

If $f_{XY}(x,y) \;\in\; C_{-1}(\mathbb{R}^2_{0,+})$, then the cumulative distribution function $F_{XY}(x,y)$ is defined by the integration

$$F_{XY}(x,y) \;=\; \int_0^x dx \int_0^y dy\, f_{XY}(x,y). \tag{149}$$

If $F_{XY}(x,y) \;\in\; C^1_{-1}(\mathbb{R}^2_{0,+})$, then the density $f_{XY}(x,y)$ is defined by the differentiation

$$f_{XY}(x,y) \;=\; \frac{d}{dy}\frac{d}{dx}F_{XY}(x,y). \tag{150}$$

Expressions (25) and (26) can be generalized for the region $W \subset \mathbb{R}^2_+$ such that

$$W \;:=\; \{(u,w): \quad 0 \le u \le x, \quad 0 \le w \le y\}. \tag{151}$$

For region (151), a generalization of Equations (25) and (26) for $\mathbb{R}^2_+$ has the form

$$F^{(M)}_{XY}(x,y) \;=\; I^{(M)}_W[u,w]\, f_{XY}(u,w) \;=\; I^x_{(M_x)}[u] I^y_{(M_y)}[w]\, f_{XY}(u,w), \tag{152}$$

$$f_{XY}(x,y) \;=\; D^{(K)}_W[u,w]\, F^{(M)}_{XY}(u,w) \;=\; D^y_{(K_y)}[w] D^x_{(K_x)}[u]\, F^{(M)}_{XY}(u,w), \tag{153}$$

where $f_{XY}(x,y) \in C^{(M)}_{-1}(\mathbb{R}^2_+)$ and $F^{(M)}_{XY}(x,y) \in C^1_{-1}(\mathbb{R}^2_+)$.

Remark 9. *It should be emphasized that the sequence of actions of the GF derivatives must be the reverse of the action of GF integrals in Equations (152) and (153), i.e., the xy-sequence of GFI and* *yx-sequence of GFD. In general fractional calculus, this requirement is due to the need to fulfill the identity after substituting expression (152) into expression (153) in the form*

$$f_{XY}(x,y) \;=\; D^{y}_{(K_y)}[w] D^{x}_{(K_x)}[u]\, F^{(M)}_{XY}(u,w) \;=$$
$$D^{y}_{(K_y)}[w] D^{x}_{(K_x)}[u]\, I^{u}_{(M_x)}[u']\, I^{w}_{(M_y)}[w']\, f_{XY}(u',w') \;=$$
$$D^{y}_{(K_y)}[w] I^{w}_{(M_y)}[w']\, f_{XY}(x,w') \;=\; f_{XY}(x,y), \tag{154}$$

where the first fundamental theorem of GFC is used twice (first on x, and then on y). Identity (154) holds if the pairs of kernels (M_x, K_x) and (M_y, K_y) belong to the Luchko set and $f_{XY}(x,y)$ belongs to $C_{-1}(\mathbb{R}^2_+)$.

Remark 10. *The action of the GFD of the Caputo type with respect to x on GF distribution function $F_Y(y)$, which depends on y, and vice versa, gives zero*

$$D^{x,*}_{(K_x)}[x']\, F^{(M_y)}_{Y}(y) \;=\; 0, \quad D^{y,*}_{(K_y)}[y']\, F^{(M_x)}_{X}(x) \;=\; 0, \tag{155}$$

since the action of the GFD of the Caputo type on a constant function is equal to zero.

The GFD of the Riemann–Liouville type of a constant function is not equal to zero

$$D^{x}_{(K_x)}[u]\, 1 \;\neq\; 0. \tag{156}$$

For the GFD of the Riemann–Liouville type, the following equation is satisfied

$$D^{x}_{(K_x)}[u]\, 1 \;=\; K_x(x) \tag{157}$$

since

$$D^{x}_{(K_x)}[u]\, 1 \;=\; \frac{d}{dx}\int_0^x K_x(x-u)\, 1\, du \;=\; \frac{d}{dx}\int_0^x K_x(\xi)\, d\xi \;=\; K_x(x).$$

Therefore, the action of the GFD of the Riemann–Liouville type with respect to x on the GF distribution function $F^{(M_y)}_{Y}(y) \neq 0$, and vice versa cannot give zero

$$D^{x}_{(K_x)}[x']\, F^{(M_y)}_{Y}(y) \;=\; F_Y(y)\, D^{x}_{(K_x)}[x']\, 1 \;=\; K_x(x)\, F^{(M_y)}_{Y}(y), \tag{158}$$

$$D^{y}_{(K_y)}[y']\, F^{(M)}_{X}(x) \;=\; F^{(M)}_{X}(x)\, D^{y}_{(K_y)}[y']\, 1 \;=\; K_y(y)\, F^{(M)}_{X}(x). \tag{159}$$

A consequence of this property is the following non-standard equality. If the function $f_{XY}(x,y)$ has the form

$$f(x,y) \;=\; f_1(x) \;+\; f_2(y), \tag{160}$$

then

$$D^{x}_{(K_x)}[x']\, f(x',y) \;=\; D^{x}_{(K_x)}[x']\, f_1(x') \;+\; f_2(y)\, K_x(x), \tag{161}$$

$$D^{y}_{(K_y)}[y']\, f(x,y') \;=\; D^{y}_{(K_y)}[y']\, f_2(y') \;+\; f_1(x)\, K_y(y). \tag{162}$$

These facts should be taken into account for multivariate GF probability distributions.

It should be noted that the standard product (Leibniz) rule is violated for GFD. Therefore, the following inequalities exist

$$D^{x}_{(K_x)}[u]\, f(u)\, g(u) \;\neq\; f(x)\, D^{x}_{(K_x)}[u]\, g(u) \;+\; g(x)\, D^{x}_{(K_x)}[u]\, f(u). \tag{163}$$

The GF derivative of the Caputo type satisfies a similar inequality.

Note that the GF differential equations can describe nonlocality in the space due to the fact that these equations are actually integro-differential, which depends on the region.

2.11. General Fractional Average (Mean) Values

In this subsection, nonlocal generalizations of the standard (local) average value are proposed for continuous distributions on the semi-axis.

First, let us briefly write out the standard formulas that define the average values of the function $A(X)$ of a random variable X, which is distributed with a density $f_X(x)$ on the semi-axis.

Let $f_X(x) \in C_{-1}^{(\{1\})}(0,\infty)$ be a standard probability density function, $A(X)$ be a function of a random variable X, such that $A(x)\, f_X(x) \in C_{-1}(0,\infty)$, and the function

$$F_X(x) = \int_0^x f_X(u)\, du \in C_{CDF}^{(\{1\})}(0,\infty) \quad (164)$$

be the standard cumulative distribution function. Then, the standard average value is described as

$$\langle A(X) \rangle = \mathsf{E}[A(X)] := \lim_{x\to\infty} \int_0^x A(u)\, dF_X(u) = \lim_{x\to\infty} \int_0^x A(u)\, f_X(u)\, du. \quad (165)$$

In constructing definitions of the nonlocal generalizations of the standard expression (165), one should take into account the need to satisfy the following properties in addition to linearity. For the GF average values of the function $A(X)$ of the random variable X on the semi-axis $\mathbb{R}_+$, the following properties should be satisfied.

The first property is the normalization condition for the unit function of a random variable

$$\langle \{1\} \rangle_{(M)} = \mathsf{E}_{(M)}[\{1\}] = 1 \quad (166)$$

that should be satisfied for all types of the average GF values. Equation (166) can be interpreted as a normalization condition of the GF probability density.

The second property is the principle of correspondence with the definition of the standard (local) average value with the GFI kernel $M_x(x)$ equal to unit for all $x > 0$, i.e., $M(x) = \{1\}$ for all $x > 0$,

$$\langle A(X) \rangle_{(\{1\})} = \mathsf{E}_{(\{1\})}[A(X)] = \lim_{x\to\infty} \int_0^x A(x)\, dF_X^{(\{1\})}(x) = \lim_{x\to\infty} \int_0^x A(x)\, f_X(x)\, dx \quad (167)$$

that should be satisfied for all types of GF average values. Note that the operator kernel $M_x(x) = \{1\}$ does not belong to the Luchko set. Therefore, the correspondence principle is verified by substituting the power kernel $M_x(x) = h_\alpha(\lambda x)$, which belongs to the Luchko set together with the kernel $K_x(x) = \lambda h_{1-\alpha}(\lambda x)$, and considering the limit $\alpha \to 1-$.

Let us define three types of GF average values of function $A(X)$, for which the first property (166) and the second property (167) are satisfied. These properties can be easily proven (verified) by direct substitution of the identity function $\{1\}$ for the function $A(X)$ and by the described limit passage $\alpha \to 1+$ for the operator kernel $M_x(x)$.

Definition 16 (GF average values of function $A(X)$)**.** *Let a pair $(M_x(x), K_x(x))$ belong to the Luchko set.*

Let $f_X(x) \in C_{-1}^{(M)}(0,\infty)$ be a GF probability density, $A(X)$ be a function of a random variable X, and the function

$$F_X^{(M)}(x) = I^x_{(M_x)}[u]\, f_X(u) \in C_{CDF}^{(M)}(0,\infty) \quad (168)$$

is the GF cumulative distribution function.

Let $A(x)\, f_X(x) \in C_{-1}(0,\infty)$. Then, the value

$$\langle A(X) \rangle_{T1,(M)} = \mathsf{E}_{T1,(M)}[A(X)] := \lim_{x\to\infty} \int_0^x M_x(x-u)\, A(u)\, dF_X^{(\{1\})}(u) \quad (169)$$

is called the GF average (mean) value of the first type for the function $A(X)$ of the random variable X.

Let $A(x)\,\Big(D^x_{(M_x)}[u]\, f_X(u)\Big) \in C_{-1}(0,\infty)$. Then, the value

$$\langle A(X)\rangle_{T2,(M)} \;=\; \mathsf{E}_{T2,(M)}[A(X)] \;:=\; \lim_{x\to\infty} \int_0^x A(u)\, dF^{(M)}_X(u) \tag{170}$$

is called the GF average (mean) value of the second type for the function $A(X)$ of the random variable X.

Let $A(x)\,\Big(D^x_{(M_x)}[u]\, f_X(u)\Big) \in C_{-1}(0,\infty)$. Then, the value

$$\langle A(X)\rangle_{T3,(M)} \;=\; \mathsf{E}_{T3,(M)}[A(X)] \;:=\; \lim_{x\to\infty} \int_0^x M_x(x-u)\, A(u)\, dF^{(M)}_X(u) \tag{171}$$

is called the GF average (mean) value of the third type for the function $A(X)$ of the random variable X.

The proposed GF average values can be represented by using the notations of the general fractional calculus in the following forms.

(1) For the GF average value of the first type, one can use the fact that the condition $F^{(\{1\})}_X(x) \in C^{(\{1\})}_{CDF}(0,\infty)$ leads to $F^{(\{1\})}_X(x) \in C^1_{-1}(0,\infty)$. Then, the equation

$$F^{(\{1\})}_X(x) \;=\; \int_0^x f_X(u)\, du, \tag{172}$$

leads to the equality

$$\frac{d}{dx} F^{(\{1\})}_X(x) \;=\; \frac{d}{dx} \int_0^x f_X(u)\, du \;=\; f_X(x). \tag{173}$$

Therefore, Equation (169) can be written as

$$\langle A(X)\rangle_{T1,(M)} \;=\; \mathsf{E}_{T1,(M)}[A(X)] \;=\; \lim_{x\to\infty} \int_0^x M_x(x-u)\, A(u)\, f_X(u)\, du \;= \\ \lim_{x\to\infty} I^x_{(M_x)}[u]\, \Big(A(u)\, f_X(u)\Big). \tag{174}$$

(2) For the GF average value of the second type, one can use the fact that the condition $F^{(M)}_X(x) \in C^{(M)}_{CDF}(0,\infty)$ leads to $F^{(M)}_X(x) \in C^1_{-1}(0,\infty)$. Then, Equation (168) gives the equality

$$\frac{d}{dx} F^{(M)}_X(x) \;=\; \frac{d}{dx} I^x_{(M_x)}[u]\, f_X(u) \;=\; D^x_{(M_x)}[u]\, f_X(u), \tag{175}$$

where $D^x_{(M_x)}[u]$ is the FG derivative RL type with the kernel $M_x(x)$ instead of the kernel $K_x(x)$. Therefore, Equation (170) can be written as

$$\langle A(X)\rangle_{T2,(M)} \;=\; \mathsf{E}_{T2,(M)}[A(X)] \;:=\; \lim_{x\to\infty} \int_0^x A(u)\, \Big(D^u_{(M_x)}[w]\, f_X(w)\Big)\, du. \tag{176}$$

(3) For the GF average value of the third type, similarly to the second type, one can use Equation (175). Therefore, Equation (171) can be written as

$$\langle A(X)\rangle_{T3,(M)} \;=\; \mathsf{E}_{T3,(M)}[A(X)] \;=\; \lim_{x\to\infty} \int_0^x M_x(x-u)\, A(u)\, \Big(D^u_{(M_x)}[w]\, f_X(w)\Big)\, du \;= \\ \lim_{x\to\infty} I^x_{(M_x)}[u]\, \Big(A(u)\, \Big(D^u_{(M_x)}[w]\, f_X(w)\Big)\Big). \tag{177}$$

Let us make some remarks about the proposed three types of GF average values.

Using the GF average value of the first type (174), in fact, in addition to the "old density function" $f_X(x)$, "new density function" $f_{new,X}(x) = A(x) f_X(x)$ should be also finite at $x \to \infty$. The following conditions should satisfy at the same time

$$\lim_{x\to\infty} I^x_{(M_x)}[u] \Big(A(u) f_X(u) \Big) < \infty, \tag{178}$$

$$\lim_{x\to\infty} I^x_{(M_x)}[u] f_X(u) < \infty, \tag{179}$$

and

$$\lim_{x\to 0+} I^x_{(M_x)}[u] f_X(u) = 0, \tag{180}$$

for which it is necessary to find the conditions of the parameters. For most GF probability density functions and operator kernels, for which the analytical expressions are known, the average value (174) gives a finite value at $A(X) = \{1\}$ only. Because of this, the GF mean value that is derived by a simple replacement of the first-order integral with a general fractional integral

$$I^x_{(\{1\})}[u] \Big(A(u) f_X(u) \Big) \quad \to \quad I^x_{(M_x)}[u] \Big(A(u) f_X(u) \Big) \tag{181}$$

leads to a not-very useful characteristic of the nonlocal distribution. Such a definition of the GF average value can be used for truncated GF average values over finite intervals $[a, b] \subset \mathbb{R}_+$ of truncated GF distributions. Such distributions and their corresponding to truncated GF average values are discussed in Section 7.

Using the notation of the GF mean value through integration with the GF cumulative distribution functions (see equations (169), (170) and, (171)), it becomes clearer that the second and third types of GF mean value are more adequate generalizations of the standard average values.

Due to the fact that Equations (170) and (171) contain the differentials of the GF cumulative distribution function $dF^{(M)}_X(u)$, the Riemann–Liouville type of GF derivative should be used in Equations (176) and (177).

It should also be emphasized that the GF derivative, which is used in Equations (170) and (171), contains the kernel $M_x(x)$, instead of the GFD kernel $K_x(x)$. Because of this, in the limit case $M_x(x) = \{1\}$, which is described in the correspondence principle, the GF derivative does not give the standard derivative of the first order, but the function itself

$$D^x_{(M_x)}[u] f_X(u) = \frac{d}{dx} \int_0^x M_x(x-u) f_X(u)\, du = \frac{d}{dx} \int_0^x f_X(u)\, du = f_X(x). \tag{182}$$

The proposed three types of average values can be combined into one generalized form with two different operator kernels.

Definition 17 (GF average values with two kernels)**.** *Let two kernel pairs $(M_1(x), K_1(x))$ and $(M_2(x), K_2(x))$ belong to the Luchko set.*

Let $f_X(x) \in C^{(M_2)}_{-1}(0, \infty)$ be a GF probability density, $A(X)$ be a function of a random variable X, such that

$$A(x) \Big(D^x_{(M_2)}[u] f_X(u) \Big) \in C_{-1}(0, \infty), \tag{183}$$

and the function

$$F^{(M_2)}_X(x) = I^x_{(M_2)}[u] f_X(u) \in C^{(M_2)}_{CDF}(0, \infty) \tag{184}$$

is the GF cumulative distribution function.

Then, the value

$$\langle A(X) \rangle_{(M_1),(M_2)} = \mathsf{E}_{(M_1),(M_2)}[A(X)] := \lim_{x\to\infty} \int_0^x M_1(x-u)\, A(u)\, dF^{(M_2)}_X(u) \tag{185}$$

is called the GF average (mean) value with two kernels for the function $A(X)$ of the random variable X.

Equation (185) can be written as

$$\langle A(X)\rangle_{(M_1),(M_2)} = \mathsf{E}_{(M_1),(M_2)}[A(X)] = \lim_{x\to\infty} \int_0^x M_1(x-u)\, A(u) \left(D^u_{(M_2)}[w]\, f_X(w) \right) du =$$
$$\lim_{x\to\infty} I^x_{(M_1)}[u] \left(A(u) \left(D^u_{(M_2)}[w]\, f_X(w) \right) \right). \tag{186}$$

All three types of GF average values are particular cases of their proposed generalization, if we include in the considerations the operator kernel $\{1\}$ as the limiting case of the power law kernels.

(1) If $M_1(x) = M_2(x) = \{1\}$, Equation (185) gives the standard average value (165)

$$\langle A(X)\rangle_{(\{1\}),(\{1\})} = \langle A(X)\rangle. \tag{187}$$

(2) If $M_1(x) = M(x)$ and $M_2(x) = \{1\}$, Equation (185) gives the GF average value of the first type

$$\langle A(X)\rangle_{(M),(\{1\})} = \langle A(X)\rangle_{T1,(M)}. \tag{188}$$

(3) If $M_1(x) = \{1\}$ and $M_2(x) = M(x)$, Equation (185) gives the GF average value of the second type

$$\langle A(X)\rangle_{(\{1\}),(M)} = \langle A(X)\rangle_{T2,(M)}. \tag{189}$$

(4) If $M_1(x) = M_2(x) = M(x)$, Equation (185) gives the GF average value of the third type

$$\langle A(X)\rangle_{(M),(M)} = \langle A(X)\rangle_{T3,(M)}. \tag{190}$$

(5) If $M_1(x) \neq \{1\}$, $M_2(x) \neq \{1\}$ and $M_2(x) \neq M_2(x)$, Equation (185) does not coincide with the three types of average GF values.

The use of two operator kernels $M_1(x)$ and $M_2(x)$ in Definition 17 can be interpreted as follows. The first kernel $M_1(x)$ describes the influence of the nonlocality on the function of random variables (on "classical observable" in the physical interpretation). The second kernel $M_2(x)$ describes the influence of the nonlocality on the probability density (on the distribution of states in the physical interpretation).

For the GF average (mean) value with two kernels $\mathsf{E}_{(M_1),(M_2)}[X^n]$ for the case $M_1(x) \neq \{1\}$, there are problems in finding examples, for which these GF average values are non-zero finite values. Therefore, these GF average values with $M_1(x) \neq \{1\}$ can be used to consider truncated GF distributions on finite intervals $[a,b] \subset \mathbb{R}_+$.

Because of this, it seems that the most interesting for use in applications are the non-truncated GF average values of the second type.

Let us give two examples of average values of the second type.

- Example 1. Using the operator kernels

$$M_x(x) = h_{\alpha,\lambda}(\lambda\, x) = \frac{(\lambda\, x)^{\alpha-1}}{\Gamma(\alpha)} e^{-\lambda\, x}, \quad K_x(x) = \lambda\, h_{1-\alpha,\lambda}(\lambda x) + \frac{\lambda}{\Gamma(1-\alpha)}\, \gamma(1-\alpha, \lambda x), \tag{191}$$

and the GF probability density function

$$f_X(x) = \lambda\, \{1\} \tag{192}$$

that describes the uniform GF distributions (for details see Section 4), the GF average value of the second type has the form

$$\langle X^n\rangle_{T2,(M)} = \mathsf{E}_{T2,(M)}[X^n] = \lim_{x\to\infty} \lambda \int_0^x u^n\, M_X(u)\, du =$$

$$\frac{\lambda^{-n}}{\Gamma(\alpha)} \lim_{x\to\infty} \gamma(\alpha+n,\lambda\, x) \;=\; \frac{\lambda^{-n}\Gamma(\alpha+n)}{\Gamma(\alpha)}. \tag{193}$$

- Example 2. Using the operator kernels

$$M_x(x) \;=\; \frac{(\lambda\, x)^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)} + \frac{(\lambda\, x)^{-\beta}}{\Gamma(1-\beta+1)}, \quad K_x(x) \;=\; \lambda\,(\lambda\, x)^{\beta-1}\, E_{\alpha,\beta}[-(\lambda\, x)^{\alpha}], \tag{194}$$

and the GF probability density function

$$f_X(x) \;=\; \frac{4\lambda^2}{\sqrt{\pi}} \int_0^x (\lambda\, u)^{\beta+1}\, E_{\alpha,\beta}[-(\lambda\, u)^{\alpha}]\, e^{-(\lambda\, u)^2}\, du, \tag{195}$$

the GF average value of the second type has the form

$$\langle X^n \rangle_{T2,(M)} \;=\; \mathsf{E}_{T2,(M)}[X^n] \;=\; \lim_{x\to\infty} \frac{4\lambda}{\sqrt{\pi}} \int_0^x u^n\, e^{-(\lambda\, u)^2}\, du \;=\; \frac{4}{\sqrt{\pi}}\, \Gamma\left(\frac{n+3}{2}\right). \tag{196}$$

Remark 11. *It should be noted that a generalization of the normalization condition and average value by using fractional integration of non-integer order was first proposed in [67–69] and then it was used in papers [67–69] to describe complex physical systems in fractional statistical mechanics [10,131–133]. These generalizations are proposed for the case of the power law nonlocality only.*

The GF characteristic function of the real-valued random variable is defined by the GF probability distribution.

Definition 18 (GF characteristic function)**.** *Let a pair $(M_x(x), K_x(x))$ belong to the Luchko set. Let $f_X(x) \in C_{-1}^{(M)}(0,\infty)$ be a GF probability density, and the function*

$$F_X^{(M)}(x) \;=\; I^x_{(M_x)}[u]\, f_X(u) \;\in\; C_{CDF}^{(M)}(0,\infty) \tag{197}$$

is the GF cumulative distribution function.

Let the following conditions be satisfied

$$\cos(t\, x)\left(D^x_{(M_x)}[u]\, f_X(u)\right) \;\in\; C_{-1}(0,\infty), \tag{198}$$

$$\sin(t\, x)\left(D^x_{(M_x)}[u]\, f_X(u)\right) \;\in\; C_{-1}(0,\infty), \tag{199}$$

for all $t \in \mathbb{R}$.

Then, the value

$$\chi_X(t) \;:=\; \langle \exp\{i\, t\, x\}\rangle_{T2,(M)} \;=\; \mathsf{E}_{T2,(M)}[e^{i\,t\,x}] \;=$$

$$\lim_{x\to\infty} \int_0^x \cos(t\, u)\, dF_X^{(M)}(u) \;+\; i\, \lim_{x\to\infty} \int_0^x \sin(t\, u)\, dF_X^{(M)}(u) \tag{200}$$

is called the GF characteristic function of the second type for the random variable X.

Using the GF probability density function, Equation (200) can be defined by the equation

$$\chi_X(t) \;=\; \lim_{x\to\infty} \int_0^x \cos(t\, u)\left(D^x_{(M_x)}[u]\, f_X(u)\right)du \;+\; i\, \lim_{x\to\infty} \int_0^x \sin(t\, u)\left(D^x_{(M_x)}[u]\, f_X(u)\right)du. \tag{201}$$

As a result, the GF characteristic Function (201) is the Fourier transform of the GF derivative with the kernel $M_x(x)$ of the GF probability density function.

3. Relationship between Local and Nonlocal Quantities

This section will not consider the questions of constructing a nonlocal probability theory. Here, an explanation of the nonlocality will be given.

The purpose of this section is to describe the relationship between nonlocal and local concepts, but, first of all, to point out the differences between nonlocal theory and the standard (local) theory.

In this section, the following relationships are described.

- A relationship between the functions
$$F^{(M)}(x) = \int_0^x M_x(x-u)\, f(u)\, du, \tag{202}$$
and
$$F^{(\{1\})}(x) = \int_0^x f(u)\, du, \tag{203}$$
where $f(x) \in C_{-1}(0,\infty)$.
- A relationship between the functions
$$f_{RL}^{(K)}(x) = \frac{d}{dx}\int_0^x K_x(x-u)\, F(u)\, du, \tag{204}$$
$$f_{C}^{(K)}(x) = \int_0^x K_x(x-u)\, \frac{dF(u)}{du}\, du, \tag{205}$$
where $F(x) \in C^1_{-1}(0,\infty)$ and $F(0+) = 0$.
- A relationship between functions (202) and $f(x) \in C_{-1}(0,\infty)$.
- A relationship between the functions
$$f^{(K)}(x) = \frac{d}{dx}\int_0^x K_x(x-u)\, F(u)\, du, \tag{206}$$
and
$$f(x) = \frac{dF(x)}{dx}, \tag{207}$$
where $F(x) \in C^1_{-1}(0,\infty)$.

For convenience, the description begins with well-known mathematical facts and theorems.

3.1. Mean-Value Theorems for Integrals of the First Order

Let us describe the sets of functions, which are used in this section, and well-known theorems, including the mean-value theorem. The following choice of sets of functions and operator kernels is determined by the general fractional calculus, which will be used to construct a nonlocal probability theory. Let functions $f(x)$ and $f^{(K_x)}(x)$ belong to the set $C_{-1}(0,\infty)$.

The set $C_{-1}(0,\infty)$ is the space of functions that are continuous on the positive real semi-axis and can have an integrable singularity of a power function type at the point zero. The condition $f(x) \in C_{-1}(0,\infty)$ means that $f(x) \in C(0,\infty)$ and it can be represented as $f(x) = x^a f_1(x)$, where $a \in (-1,\infty)$, $f_1(x) \in C[0,\infty)$. Note that there are the following inclusions
$$C[0,\infty) \subset C_{-1}(0,\infty) \subset C(0,\infty). \tag{208}$$

The kernels of integral and integro-differential operators will be assumed to belong to the subset $C_{(-1,0)}(0,\infty)$ of the set $C_{-1}(0,\infty)$. The condition $g(x) \in C_{(-1,0)}(0,\infty)$ means that $g(x) \in C(0,\infty)$ and it can be represented as $g(x) = x^a g_1(x)$, where $a \in (-1,0)$ and $g_1(x) \in C[0,\infty)$.

In standard calculus, the Weierstrass extreme value theorem states that if a real-valued function $f(x)$ is continuous on the closed interval $[a,b]$, then $f(x)$ is bounded on that interval. This means that there exist real numbers m_1 and m_2, such that

$$m_1 \le f(x) \le m_2 \tag{209}$$

for all $x \in [a.b]$ (see Theorem 3 in [134], p. 161). In addition, there is a point on the interval at which the function takes its maximum value and a point where it assumes its minimal value.

To describe connections of the nonlocal quantities with standard (local) quantities one can use the first mean-value theorem for the integral (see Theorem 5 in [134], p. 352). In [134], this name of the theorem is used for the somewhat more general proposition that can be useful for the general fractional integral. Note that the kernels from the Luchko set are nonnegative. The first mean-value theorem for the integral can be formulated in the following form.

Theorem 10. *(*First mean-value theorem for integrals*)*
Let $g(x), f(x)$ be integrable functions on $[a,b] \subset (0,\infty)$,

$$m_1 = \inf_{[a,b]} g(x), \quad m_2 = \sup_{[a,b]} g(x). \tag{210}$$

If $f(x)$ is nonnegative (or non-positive) on $[a,b]$, then

$$\int_a^b g(x)\, f(x)\, dx = \mu \int_a^b f(x)\, dx, \tag{211}$$

where $\mu \in [m_1, m_2]$. If in addition, it is known that $f(x) \in C[a,b]$, then there exists a point $\xi \in [a,b]$, such that

$$\int_a^b g(x)\, f(x)\, dx = g(\xi) \int_a^b f(x)\, dx. \tag{212}$$

This theorem for the integral is proved in [134] (see Theorem 5 in [134], p. 352) for the case $\mathbb{R}$. The above statement of Theorem 10 is given for the positive semi-axis for use in general fractional calculus.

The standard mean-value theorem can be considered as a corollary of Theorem 10 (see, also Corollary 3 in [134], p. 352),

Corollary 3. *Let $g(x)$ be an integrable function on $[a,b] \subset (0,\infty)$. If $g(x) \in C[a,b]$, then there exists a point $\xi \in [a,b]$, such that*

$$\int_a^b g(x)\, dx = g(\xi)\,(b-a). \tag{213}$$

Proof. Let us consider the function $f(x) = 1$ for all $x \in [0,\infty)$. One can see that this function $f(x) = 1$ is a nonnegative function and $f(x) \in C[a,b]$ for $[a,b] \subset (0,\infty)$. Using Theorem 10, Equation (212) with $f(x) = 1$ gives

$$\int_a^b g(x)\, dx = g(\xi) \int_a^b dx = g(\xi)\,(b-a). \tag{214}$$

This ends the proof. □

3.2. Expression of $F^{(M)}(x)$ in Terms of $F(x)$

Let a function $f(x)$ belong to the set $C_{-1}(0,\infty)$, and a kernel $M_x(x)$ belong to the set $C_{(-1,0)}(0,\infty)$, such that $M_x(x) \geq 0$ for all $x > 0$. Then, the following functions are defined as

$$F(x) \; = \; \int_0^x f(u)\, du, \tag{215}$$

$$F^{(M)}(x) \; = \; \int_0^x M_x(x-u)\, f(u)\, du. \tag{216}$$

Let us consider a relationship between function $F^{(M)}(x)$ and function $F(x)$, which are defined by Equations (216) and (215), respectively. Using the mean value theorems and the additivity property of the first-order integral, one can write an equation relating these functions.

Theorem 11 (Function $F^{(M)}(x)$ in terms of $F^{(\{1\})}(x)$)**.** *Let $f(x) \in C_{-1}(0,\infty)$ and $M_x(x) \in C_{(-1,0)}(0,\infty)$ be nonnegative functions for all $x > 0$.*

Then, the function $F^{(M)}(x)$, which is defined by Equation (216), can be described as

$$F^{(M)}(x) \; = \; \lim_{\varepsilon \to 0+} \sum_{k=0}^{n} M_x(x \, - \, \xi_k) \, \Big(F(x_k) \, - \, F(x_{k-1}) \Big), \tag{217}$$

where $\xi_k \in [x_{k-1}, x_k]$ and $0 + \varepsilon = x_0 < x_1 < .. < x_n = x - \varepsilon$, and $\varepsilon > 0$, where $F(x)$ is defined by Equation (215).

If $M_x(x) \; = \; 1$ for all $x \in \mathbb{R}_{0,+}$, then Equation (217) gives

$$F^{(\{1\})}(x) \; = \; \lim_{\varepsilon \to 0+} \Big(F(x-\varepsilon) \, - \, F(0+\varepsilon) \Big). \tag{218}$$

Proof. Using the additivity property of the integral in Equation (216) of the function $F^{(M)}(x)$, one can obtain

$$F^{(M)}(x) \; = \; \int_0^x M_x(x-u)\, f(u)\, du \; = \; \sum_{k=0}^{n} \int_{x_{k-1}}^{x_k} M_x(x-u)\, f(u)\, du, \tag{219}$$

where $0 + \varepsilon = x_0 < x_1 < .. < x_n = x - \varepsilon$.

Then, using Theorem 10 for $g(x) \; = \; M_x(b-x)$ and the non-negativity of the GFI kernel $M_x(x) \geq 0$ for all $x > 0$, integrals (219) can be represented by the equations

$$\int_{x_{k-1}}^{x_k} M_x(x-u)\, f(u)\, du \; = \; M_x(x-\xi_k) \int_{x_{k-1}}^{x_k} f(u)\, du, \tag{220}$$

for all $k = 0, 1, .., n$, where $\xi_k \in [x_{k-1}, x_k]$. Using Equation (215), the integral in Equation (220) can be written in the form

$$\int_{x_{k-1}}^{x_k} f(u)\, du \; = \; \int_{x_{k-1}}^{x_k} dF(u) \; = \; F(x_k) \, - \, F(x_{k-1}). \tag{221}$$

Substitution of Equation (221) into Equation (220), and then the resulting expression into Equation (219) gives

$$\int_0^x M_x(x-u)\, f(u)\, du \; = \; \sum_{k=0}^{n} M_x(x-\xi_k) \, \Big(F(x_k) \, - \, F(x_{k-1}) \Big). \tag{222}$$

Therefore, using the limit $\varepsilon \to 0+$ gives (217).

If $M_x(x) = 1$ for all $x \in \mathbb{R}_{0,+}$, then Equation (222) gives

$$\int_0^x M_x(x-u)\, f(u)\, du \;=\; F(x-\varepsilon) \;-\; F(0+\varepsilon). \tag{223}$$

Therefore, Equation (223) gives (218).
This ends the proof. □

Remark 12. *Equation (217) allows to state that the function*

$$P^{(M)}([a,b]) \;=\; F^{(M)}(b) \;-\; F^{(M)}(a), \tag{224}$$

where $b > a \geq 0$, can be represented in the form

$$P^{(M)}([a,b]) \;=\; \lim_{\varepsilon\to 0+} \sum_{k=0}^{n} \Big(M_x(b \;-\; \xi_k)\,(F(x_k) \;-\; F(x_{k-1})) \;-$$

$$M_x(a \;-\; \eta_k)\,(F(x'_k) \;-\; F(x'_{k-1}))\Big), \tag{225}$$

if $f(x) \in C^{(M)}_{-1}(0,\infty)$, where $\xi_k \in [x_{k-1}, x_k]$ and $0+\varepsilon = x_0 < x_1 < .. < x_n = b-\varepsilon$ $\eta_k \in [x'_{k-1}, x'_k]$ and $0+\varepsilon = x'_0 < x'_1 < .. < x'_n = a-\varepsilon$.

Equation (225) means that $P^{(M)}([a,b])$ depends on the "trajectory" of changes in the function $F(x)$ in space, and not only on the initial and final points as in the standard (local) case

$$P^{(\{1\})}([a,b]) \;=\; F(b) \;-\; F(a), \tag{226}$$

in which $M_x(x) = 1$ for all $x > 0$.

3.3. Expression of $f^{(K_x)}(x)$ in Terms of $F(x)$

Let a function $F(x)$ belong to the set $C^1_{-1}(0,\infty)$, and a kernel $K_x(x)$ belong to the set $C_{(-1,0)}(0,\infty)$, such that $K_x(x) \geq 0$ for all $x > 0$. Then, the following functions can be defined.

$$f(x) \;=\; \frac{d}{dx} F(x), \tag{227}$$

and

$$f_C^{(K_x)}(x) \;=\; \int_0^x du\, K_x(x-u)\, \frac{dF(u)}{du}, \tag{228}$$

$$f_{RL}^{(K_x)}(x) \;=\; \frac{d}{dx} \int_0^x du\, K_x(x-u)\, F(u). \tag{229}$$

Equations (228) and (229) can be represented as a sum that is described by the following Theorem.

Theorem 12. (Function $f^{(K_x)}(x)$ in terms of $F(x)$)
Let $F(x) \in C^1_{-1}(0,\infty)$, $F(0+) = 0$, and $K_x(x) \in C_{(-1,0)}(0,\infty)$ be nonnegative functions for all $x > 0$.

Then, functions (228) and (229) can be represented in the form

$$f^{(K_x)}(x) \;=\; \lim_{\varepsilon\to 0+} \sum_{k=0}^{n} K_x(x-\eta_k)\, \Big(F(x_k) \;-\; F(x_{k-1})\Big), \tag{230}$$

where $0+\varepsilon = x_0 < x_1 < \cdots < x_n = x-\varepsilon$, and $\eta_k \in [x_{k-1}, x_k]$.

Proof. Let us consider the function

$$f^{(K_x)}(x) \;=\; \frac{d}{dx}\int_0^x K_x(x-u)\,F(u)\,du, \tag{231}$$

where $F(u) \in C_{-1}(0,\infty)$. Then, one can use the equality

$$\frac{d}{dx}\int_0^x K_x(x-u)\,F(u)\,du \;=\; \int_0^x K_x(x-u)\,F^{(1)}(u)\,du \;+\; K_x(x)\,F(0+), \tag{232}$$

that holds if $F(x) \in C^1_{-1}(0,\infty)$ and $K_x(x) \in C^1_{(-1,0)}(0,\infty)$ [87], where $F^{(1)}(x) = dF(x)/dx$.

If $F^{(1)}(x) \in C_{-1}(0,\infty)$, the additivity property of the integral in Equation (232) can be used to obtain

$$\int_0^x K_x(x-u)\,F^{(1)}(u)\,du \;=\; \lim_{\varepsilon\to 0}\sum_{k=0}^{n}\int_{x_{k-1}}^{x_k} K_x(x-u)\,F^{(1)}(u)\,du, \tag{233}$$

where $0+\varepsilon = x_0 < x_1 < \cdots < x_n = x-\varepsilon$. Then, using Theorem 10 and the non-negativity of the kernel $K_x(x)$, integral (233) is represented by the equation

$$\int_{x_{k-1}}^{x_k} K_x(x-u)\,F^{(1)}(u)\,du \;=\; K_x(x-\eta_k)\int_{x_{k-1}}^{x_k} F^{(1)}(u)\,du, \tag{234}$$

where $\eta_k \in [x_{k-1}, x_k]$. Using Equation (227), the integral in Equation (234) can be written in the form

$$\int_{x_{k-1}}^{x_k} F^{(1)}(u)\,du \;=\; \int_{x_{k-1}}^{x_k} dF(u) \;=\; F(x_k) \;-\; F(x_{k-1}). \tag{235}$$

Substituting of Equation (235) into Equation (234), and then the resulting expression into Equation (233) gives

$$f^{(K_x)}(x) \;=\; \frac{d}{dx}\int_0^x K_x(x-u)\,F(u)\,du \;=$$

$$\lim_{\varepsilon\to 0}\sum_{k=0}^{n} K_x(x-\eta_k)\,\Big(F(x_k) \;-\; F(x_{k-1})\Big) \;+\; K_x(x)\,F(0+). \tag{236}$$

Using that $F(0+) = 0$, one can obtain (230).

This ends the proof. □

Remark 13. *In the general case, one can use the $F^{(M)}(x)$ instead of $F(x)$ in Equation (230) of Theorem 12.*

3.4. Expression of $F^{(M)}(x)$ through $f(x)$

Using the mean value theorems and the additivity property of the first-order integral, in addition to Theorem 11, one can prove an equation relating the functions $F^{(M)}(x)$ through $f(x)$.

Theorem 13 (Expression of $F^{(M)}(x)$ through $f(x)$)**.** *Let $f(x) \in C_{-1}(0,\infty)$ and $M_x(x) \in C_{(-1,0)}(0,\infty)$ be nonnegative functions for all $x > 0$.*

Then, the function $F^{(M)}(x)$, which is defined by Equation (216), is described by the equation

$$F^{(M)}(x) \;=\; \lim_{\varepsilon\to 0+}\sum_{k=0}^{n} M_x(x \;-\; \xi_k)\,f(\xi_k)\,\Delta x_k, \tag{237}$$

where

$$\Delta x_k \;=\; x_k \;-\; x_{k-1} \tag{238}$$

and $\xi_k \in [x_{k-1}, x_k]$ with $0 + \varepsilon = x_0 < x_1 < .. < x_n = x - \varepsilon$.

If $f^{(K)}(x) \in C_{-1}(0, \infty)$, then the function

$$F^{(M)}(x) \;=\; \int_0^x M_x(x-u)\, f^{(K)}(u)\, du \tag{239}$$

can also be described by Equation (237) in the form

$$F^{(M)}(x) \;=\; \lim_{\varepsilon \to 0+} \sum_{k=0}^{n} M_x(x \,-\, \xi_k)\, f^{(K_x)}(\xi_k)\, \Delta x_k. \tag{240}$$

Proof. Using the additivity property of the first-order integral, Equation (216) can be represented in the form

$$F^{(M)}(x) \;=\; \lim_{\varepsilon \to 0+} \sum_{k=0}^{n} \int_{x_{k-1}}^{x_k} M_x(x-u)\, f(u)\, du, \tag{241}$$

where $0 + \varepsilon = x_0 < x_1 < .. < x_n = x - \varepsilon$.

Then, using Corollary 3 for the function

$$g(u) \;=\; M_x(x-u)\, f(u) \tag{242}$$

with $a = x_{k-1}$ and $b = x_k$, one can obtain

$$\int_{x_{k-1}}^{x_k} M_x(x-u)\, f(u)\, du \;=\; M_x(x-\xi_k)\, f(\xi_k) \int_{x_{k-1}}^{x_k} du \;=\; M_x(x-\xi_k)\, f(\xi_k)\, \Delta x_k, \tag{243}$$

where $\xi_k \in [x_{k-1}, x_k]$ and $\Delta x_k \;=\; x_k \;-\; x_{k-1}$.

Substitution of (243) into Equation (241) gives Equation (240).

This ends the proof. □

3.5. Expression of Function $f^{(K_x)}(x)$ Via $F(X)$

Let us consider the relationship between the functions

$$f^{(K)}(x) \;=\; \frac{d}{dx} \int_0^x K_x(x-u)\, F(u)\, du, \tag{244}$$

and

$$f(x) \;=\; \frac{dF(x)}{dx}, \tag{245}$$

where $F(x) \in C^1_{-1}(0, \infty)$.

Theorem 14. (Expression of function $f^{(K_x)}(x)$ via $f(x)$)

Let $F(x) \in C^1_{-1}(0, \infty)$, $F(0+) = 0$, and $K_x(x) \in C_{(-1,0)}(0, \infty)$ be nonnegative functions on $\mathbb{R}_{0,+}$.

Then, the function $f^{(K_x)}(x)$, which is defined by Equation (244), is described by the equation

$$f^{(K_x)}(x) \;=\; \lim_{\varepsilon \to 0+} \sum_{k=0}^{n} K_x(x-\xi_k)\, f(\xi_k)\, \Delta x_k, \tag{246}$$

$0 + \varepsilon = x_0 < x_1 < \cdots < x_n = x - \varepsilon$*, and* $\xi_k \in [x_{k-1}, x_k]$.

Proof. Using Equations (245) and (232), the conditions $F(x) \in C^1_{-1}(0,\infty)$, $F(0+) = 0$, and the function Equation (231) gives

$$f^{(K_x)}(x) \,=\, \frac{d}{dx}\int_0^x K_x(x-u)\,F(u)\,du \,=$$

$$\int_0^x du\,K_x(x-u)\,\frac{dF(u)}{du} \,+\, K_x(x)\,F(0+) \,=$$

$$\int_0^x K_x(x-u)\,f(u)\,du \,+\, K_x(x)\,F(0+) \,=\, \int_0^x K_x(x-u)\,f(u)\,du. \tag{247}$$

As a result, it is proved the equation

$$f^{(K_x)}(x) \,=\, \int_0^x K_x(x-u)\,f(u)\,du, \tag{248}$$

where $f(x)$ is defined by Equation (245).

Then, the additivity property of the integral in Equation (248) can be used to obtain

$$\int_0^x K_x(x-u)\,f(u)\,du \,=\, \lim_{\varepsilon\to 0+}\sum_{k=0}^{n}\int_{x_{k-1}}^{x_k} K_x(x-u)\,f(u)\,du, \tag{249}$$

where $0+\varepsilon = x_0 < x_1 < \cdots < x_n = x-\varepsilon$.

Then, the mean value theorem is used to obtain

$$f^{(K_x)}(x) \,=\, \lim_{\varepsilon\to 0+}\sum_{k=0}^{n}\int_{x_{k-1}}^{x_k} K_x(x-u)\,f(u)\,du \,=\, \lim_{\varepsilon\to 0+}\sum_{k=0}^{n} K_x(x-\xi_k)\,f(\xi_k)\,\Delta x_k, \tag{250}$$

where $\xi_k \in [x_{k-1}, x_k]$.

This ends the proof. □

4. Uniform and Degenerate GF Distributions

4.1. Uniform GF Distributions

Let us consider the function

$$f_X(x) \,=\, c \,=\, \mathrm{const} \tag{251}$$

for $x \geq 0$, and $f_X(x) \,=\, 0$ for $x < 0$, i.e., $f_X(x) = x\,\{1\}$, where $0 \,<\, c \,<\, \infty$.

In the standard probability theory, Function (251) cannot be considered a probability density function, since the normalization condition is violated

$$\lim_{x\to\infty}\int_0^x f_X(u)\,du \,=\, \lim_{x\to\infty} c\,x \,=\, +\infty. \tag{252}$$

In the NPT, Function (251) can be used. Using the definition of the GF cumulative distribution function $F^{(M)}_X(x)$ for Function (251), one can obtain

$$F^{(M)}_X(x) \,=\, I^x_{(M_x)}[u]\,f_X(u) \,=\, \int_0^x M_x(x\,-\,u)\,f_X(u)\,du \,=$$

$$c\int_0^x M_x(x\,-\,u)\,1\,du \,=\, c\int_0^x M_x(w)\,dw.$$

As a result, if the kernel $M_x(x)$ of the GFI satisfies the conditions

$$\lim_{x\to 0}\int_0^x M_x(u)\,du \,=\, 0. \tag{253}$$

$$\lim_{x \to \infty} \int_0^x M_x(u)\, du \;=\; \frac{1}{c}, \tag{254}$$

then Function (251) describes a GF analog of uniform distribution.

Definition 19 (Uniform GF distribution)**.** *Let pair of kernels $(M_x(x), K_x(x))$ belong to the Luchko set. If the kernel $M_x(x)$ of the GFI belongs to the set $C_{-1}^{(\{1\})}(0,\infty)$, then the function*

$$f_X(x) \;=\; c\,\{1\} \tag{255}$$

belongs to the set $C_{-1}^{(M)}(0,\infty)$, i.e., it is the GF probability density function, where $c \in \mathbb{R}_+$.

Then, such functions are called the GF probability density functions of uniform GF distributions.

It can be seen that the conditions on the kernel of the GFI actually mean that this kernel must describe the standard probability density function on the positive semi-axis up to a constant. Obviously, not all operator kernels satisfy these properties.

4.2. Uniform GF Distribution for Gamma Type of Nonlocality

As an example of a GFI kernel, one can consider the kernel of the GFI in the following pair (137) of the Luchko set in the form

$$M_x(x) \;=\; h_{\alpha,\lambda}(\lambda\, x) \;=\; \frac{(\lambda x)^{\alpha-1}}{\Gamma(\alpha)} e^{-\lambda\, x}, \tag{256}$$

where $0 < \alpha < 1$ and $\lambda > 0$.

The standard PDF for the gamma distribution is described by the function

$$\lambda\, h_{\alpha,\lambda}(\lambda\, x) \;=\; \lambda\, \frac{(\lambda x)^{\alpha-1}}{\Gamma(\alpha)} e^{-\lambda\, x}. \tag{257}$$

Therefore, the normalization condition of the uniform GF distribution

$$c \int_0^{\infty} M_x(x)\, dx \;=\; \frac{c}{\lambda} \int_0^{\infty} \frac{\lambda^{\alpha} x^{\alpha-1}}{\Gamma(\alpha)} e^{-\lambda\, x}\, dx \;=\; \frac{c}{\lambda} \tag{258}$$

gives $c = \lambda$. As a result, if nonlocality is described by the kernel pair (137), then the uniform GF distribution is described by the functions

$$f_X(x) \;=\; \lambda\,\{1\}, \tag{259}$$

where $\lambda > 0$.

The GF cumulative distribution function $F_X^{(M)}(x)$ for the GF probability density (259) has the form

$$F_X^{(M)}(x) \;=\; I^x_{(M_x)}[u]\, f_X(u) \;=\; \frac{1}{\Gamma(\alpha)}\, \gamma(\alpha, \lambda\, x), \tag{260}$$

where $\gamma(\beta, x)$ is the incomplete gamma function (see Section 9 in [127], pp. 134–142).

As a result, one can give the following definition.

Definition 20 (Uniform GF distribution for Gamma distribution of the nonlocality)**.** *Nonlocality is described by the kernel pair of the Luchko set*

$$M_x(x) \;=\; h_{\alpha,\lambda}(\lambda\, x) \;=\; \frac{(\lambda\, x)^{\alpha-1}}{\Gamma(\alpha)} e^{-\lambda\, x}, \tag{261}$$

$$K_x(x) \;=\; \lambda\, h_{1-\alpha,\lambda}(\lambda\, x) \;+\; \frac{\lambda}{\Gamma(1-\alpha)}\, \gamma(1-\alpha, \lambda\, x), \tag{262}$$

where $\gamma(\beta, x)$ is the incomplete gamma function.

The GF probability density function

$$f_X(x) \;=\; \lambda\, \{1\}. \tag{263}$$

The GF cumulative distribution function

$$F_X^{(M)}(x) \;=\; \frac{1}{\Gamma(\alpha)}\, \gamma(\alpha, \lambda\, x). \tag{264}$$

The parameter values

$$\alpha \;\in\; (0,1), \quad \lambda \;>\; 0. \tag{265}$$

4.3. Uniform GF Distribution for Alpha-Exponential Nonlocality

Let us consider the GFI and GFD kernel pair (138) that belongs to the Luchko set. The GFI kernel is

$$M_x(x) \;=\; (\lambda\, x)^{\beta-1}\, E_{\alpha,\beta}[-\, (\lambda\, x)^{\alpha}], \tag{266}$$

where $0 < \alpha \leq \beta < 1$,

The GF probability density function is considered in form (251), i.e., $f_X(x) \;=\; c$ for all $x > 0$. The GF normalization condition

$$\lim_{x\to\infty} \int_0^x M_x(x-u)\, f_X(u)\, du \;=\; \lim_{x\to\infty} \int_0^x M_x(u)\, f_X(x-u)\, du \;=\; 1 \tag{267}$$

has the form

$$\lim_{x\to\infty} \int^x c\, \lambda^{\beta-1}\, u^{\beta-1}\, E_{\alpha,\beta}[\,-\, (\lambda\, u)^{\alpha}]\, du \;=\; 1. \tag{268}$$

Using Equation (4.4.4) of [128], p. 61, in the form

$$\int_0^x u^{\beta-1}\, E_{\alpha,\beta}[\,-\, (\lambda\, u)^{\alpha}]\, du \;=\; x^{\beta}\, E_{\alpha,\beta+1}[\,-\, (\lambda\, x)^{\alpha}], \tag{269}$$

and GF normalization condition (268) takes the form

$$\lim_{x\to\infty} c\, \lambda^{\beta-1}\, x^{\beta}\, E_{\alpha,\beta+1}[\,-\, (\lambda\, x)^{\alpha}] \;=\; 1. \tag{270}$$

Note that the GF normalization condition (268) can be represented in the form

$$\lim_{x\to\infty} F_X^{(M)}(x) \;=\; 1, \tag{271}$$

where $F_X^{(M)}(x)$ is the GF cumulative distribution function

$$F_X^{(M)}(x) \;=\; \lambda\, \lambda^{\beta-1}\, x^{\beta}\, E_{\alpha,\beta+1}[\,-\, (\lambda\, x)^{\alpha}]. \tag{272}$$

Using Theorem 4.3 of [128], p. 64, the asymptotic equation for the function $E_{\alpha,\beta+1}[-\, z]$ has the form

$$E_{\alpha,\beta+1}[-\, z] \;=\; \frac{1}{\Gamma(\beta\, -\, \alpha\, +\, 1)}\, \frac{1}{z}\, +\, O(|z|^{-2}) \quad (|z|\, \to\, \infty), \tag{273}$$

which holds for $0 \;<\; \alpha \;<\; 1$. Therefore,

$$c\, \lambda^{\beta-1}\, x^{\beta}\, E_{\alpha,\beta+1}[\,-\, (\lambda\, x)^{\alpha}] \;=\; x^{\beta} \left(\frac{c\, \lambda^{\beta-1}}{\Gamma(\beta\, -\, \alpha\, +\, 1)}\, \frac{1}{(\lambda\, x^{\alpha}}\, +\, O(|x|^{-2\alpha}) \right) =$$

$$=\; \frac{c\, \lambda^{\beta-1-\alpha}}{\Gamma(\beta\, -\, \alpha\, +\, 1)}\, x^{\beta-\alpha}\, +\, O(|x|^{\beta-2\alpha}) \quad (x \to \infty) \tag{274}$$

Then, GF normalization condition (270) takes the form

$$\lim_{x \to \infty} \left(\frac{c\, \lambda^{\beta-1-\alpha}}{\Gamma(\beta - \alpha + 1)}\, x^{\beta-\alpha} + O(|x|^{\beta-2\alpha}) \right) = 1. \tag{275}$$

As a result, the GF normalization condition holds, if

$$\beta - \alpha = 0, \qquad 0 < \alpha < 1, \qquad c = \lambda^{\alpha-\beta+1}\, \Gamma(\beta - \alpha + 1) = \lambda. \tag{276}$$

In this case, the kernel of GFI has the form

$$M_x(x) = (\lambda\, x)^{\alpha-1}\, E_{\alpha,\alpha}[-(\lambda\, x)^{\alpha}], \tag{277}$$

where $\alpha \in (0,1)$.

The GF probability density function is $f_X(x) = \lambda$ for $x > 0$. Function (277) is also called the alpha-exponential function [4], pp. 50–53. Note that the pair of kernels (138) belongs to the Luchko set, if $\alpha = \beta \in (0,1)$.

The GF cumulative distribution function is defined by Equation (272) with $\alpha = \beta \in (0,1)$.

As a result, one can give the following definition.

Definition 21 (Uniform GF distribution for alpha-exponential nonlocality)**.** *Nonlocality is described by the kernel pair of the Luchko set*

$$M_x(x) = (\lambda\, x)^{\beta-1}\, E_{\alpha,\beta}[-(\lambda\, x)^{\alpha}] \tag{278}$$

$$K_x(x) = \frac{\lambda\, (\lambda\, x)^{\alpha-\beta}}{\Gamma(1+\alpha-\beta)} + \frac{\lambda\, (\lambda\, x)^{-\beta}}{\Gamma(1-\beta)}, \tag{279}$$

where $0 < \alpha \leq \beta < 1$,

The GF probability density function

$$f_X(x) = \lambda\, \{1\}. \tag{280}$$

The GF cumulative distribution function

$$F_X^{(M)}(x) = \lambda\, \lambda^{\alpha-1}\, x^{\alpha}\, E_{\alpha,\alpha+1}[\,-(\lambda\, x)^{\alpha}]. \tag{281}$$

The parameter values

$$0 < \alpha = \beta < 1. \tag{282}$$

4.4. Degenerate GF Distribution (GF Delta Distribution)

The Heaviside step function (or the unit step function) is a piecewise function that can be defined as

$$h(x) = \begin{cases} 1 & x > 0, \\ 0 & x \leq 0. \end{cases} \tag{283}$$

The Dirac delta function can be interpreted as the derivative of the Heaviside function

$$\delta(x) = \frac{dh(x)}{dx}. \tag{284}$$

Therefore the Heaviside function can be considered the integral of the Dirac delta function in the form

$$h(x) = \int_{-\infty}^{x} \delta(u)\, du. \tag{285}$$

At point $x = 0$, expression (285) can make sense only for some forms of defining the integration of the delta function.

By virtue of what has been said, the expressions (284) and (285) sometimes are interpreted as the generalized probability density function and the cumulative distribution function. In this interpretation, the Heaviside function is the cumulative distribution function of a constant random variable, which is almost everywhere, and is zero.

As for the standard (local) case of a real-valued random variable, the degenerate distribution is a one-point distribution, localized at a point $x_0 \in (-\infty, +\infty)$, [113], p. 83. The cumulative distribution function of this distribution is described by the Heaviside step function

$$F_X(x) \;=\; h(x - x_0) \;=\; \begin{cases} 1 & x \geq x_0, \\ 0 & x < x_0. \end{cases} \tag{286}$$

Let us note that functions $M_x(x)$ and $K_x(x)$, which belong to the Luchko set, satisfy the Sonin condition (1), which can be written as

$$(M_x \,*\, K_x)(x) \;:=\; \int_0^x M_x(x-u)\, K_x(u)\, du \;=\; h(x) \;=\; \begin{cases} 1 & x > 0, \\ 0 & x \leq 0. \end{cases} \tag{287}$$

In the Luchko papers, the function $h(x)$ is denoted as $\{1\}$. Using the Sonin condition (287), it is easy to prove the following proposition.

Theorem 15. *(*Property of degenerate GF distribution*)*

Let a kernel pair $(M_x(x), K_x(x))$ belong to the Luchko set.

Then for each pair of kernels that belongs to the Luchko set, there is one GF probability density function that is defined by equation

$$f_X(x) \;=\; \begin{cases} K_x(x) & x > 0, \\ 0 & x \leq 0. \end{cases} \tag{288}$$

Function (288) satisfies the following conditions.

(1) $f_X(x)$ is a GF continuous function on the positive semi-axis $(0, \infty)$, such that $f_X(x) \in C_{-1}(0, \infty)$.
(2) $f_X(x)$ is a non-negative function ($f_X(x) \geq 0$) for all $x > 0$.
(3) The function $f_X(x)$ satisfies the GF normalization condition

$$\lim_{x \to \infty} I^x_{(M_x)}[u]\, f_X(u) \;= 1. \tag{289}$$

The GF cumulative distribution function has the form

$$F^{(M)}_X(x) \;=\; \begin{cases} 1 & x > 0, \\ 0 & x = 0. \end{cases} \tag{290}$$

Note that if a kernel pair $(M_x(x), K_x(x))$ belongs to the Luchko set, then the kernel pair $(M_{x,new}(x) \;=\; \lambda^{-1} K_x(x), K_{x,new}(x) \;=\; \lambda M_x(x))$ belongs to the Luchko set. For this kernel pair, the GF probability density function is defined by equation

$$f_X(x) \;=\; \begin{cases} \lambda\, M_x(x) & x > 0, \\ 0 & x \leq 0. \end{cases} \tag{291}$$

The GF cumulative distribution function has the form

$$F^{(M)}_X(x) \;=\; \begin{cases} 1 & x > 0, \\ 0 & x = 0. \end{cases} \tag{292}$$

Function (288) can be interpreted as a GF probability density function and (290) can be interpreted as a GF cumulative distribution function of a degenerate GF distribution on the semi-axis $[0, \infty)$.

As a result, one can give the following definition.

Definition 22 (Degenerate GF distribution for $M(x)$ nonlocality)**.** *Nonlocality is described by the kernel pair of the Luchko set*

$$M_x(x),\, K_x(x) \quad \textit{any kernel pair that belongs to the Luchko set}. \tag{293}$$

The GF probability density function

$$f_X(x) \;=\; K_x(x), \quad (x \,>\, 0). \tag{294}$$

The GF cumulative distribution function

$$F_X^{(M)}(x) \;=\; 1 \quad (x \,>\, 0) \quad \textit{and} \quad F_X^{(M)}(0) \;=\; 0. \tag{295}$$

The parameter values are defined by the condition that the kernel pair belongs to the Luchko set.

5. Special Functions in General Fractional Distributions

5.1. GF Distributions with Mittag-Leffler and Power Law Functions

Consider the following two examples of GF distributions, in which the probability distribution function $f_X(x)$ and the kernel $M_x(x)$ of the GF integral operator actually change places. In these cases, conditions on the parameters will be searched, under which the functions $f_X(x)$ satisfy the conditions imposed on the GF probability density.

(1) The first example is described by the GFI and GFD kernels

$$M_x(x) \;=\; \frac{(\lambda\, x)^{\mu-1}}{\Gamma(\mu)} \tag{296}$$

$$K_x(x) \;=\; \frac{\lambda(\lambda x)^{-\mu}}{\Gamma(1-\mu)}, \tag{297}$$

where $\mu \in (0,1)$ in order to $M_x(x),\, K_x(x) \in C_{(-1,0)}(0,\infty)$, and the GF probability density

$$f_X(x) \;=\; \lambda\, (\lambda\, x)^{\beta-1}\, E_{\alpha,\beta}[-\, (\lambda\, x)^{\alpha}], \tag{298}$$

where $\alpha > 0$, and $\beta \in \mathbb{R}$. In order for a Function (298) to belong to a set $C_{-1}(0,\infty)$, the condition $\beta > 0$ must be satisfied.

(2) The second example is described by the GFI and GFD kernels (139) in the form

$$M_x(x) \;=\; (\lambda\, x)^{\beta-1}\, E_{\alpha,\beta}[-\, (\lambda\, x)^{\alpha}] \tag{299}$$

$$K_x(x) \;=\; \frac{\lambda\, (\lambda\, x)^{\alpha-\beta}}{\Gamma(1+\alpha-\beta)} \;+\; \frac{\lambda\, (\lambda\, x)^{-\beta}}{\Gamma(1-\beta)}, \tag{300}$$

where $0 < \alpha \leq \beta < 1$, and the GF probability density

$$f_X(x) \;=\; \lambda\, \frac{(\lambda\, x)^{\mu-1}}{\Gamma(\mu)}. \tag{301}$$

In order for Function (301) to belong to the set $C_{-1}(0,\infty)$, the condition $\mu > 0$ must be satisfied.

The GF cumulative distribution function $F_X^{(M)}(x)$, which is defined as

$$F_X^{(M)}(x) \,=\, I^x_{(M_x)}[u]\, f_X(u) \,=\, \int_0^x M_x(x-u)\, f_X(u)\, du, \tag{302}$$

is described by the expression

$$F_X^{(M)}(x) \,=\, \int_0^x \frac{\lambda^{\mu+\beta-1}}{\Gamma(\mu)}\, (x-u)^{\mu-1}\, u^{\beta-1}\, E_{\alpha,\beta}[-(\lambda\, u)^{\alpha}]\, du. \tag{303}$$

Note that the commutativity of the Laplace convolution

$$\int_0^x M_x(x-u)\, f_X(u)\, du \,=\, \int_0^x M_x(u)\, f_X(x-u)\, du \tag{304}$$

allows us to state that Equation (303) describes the GF cumulative distribution function for both of these examples, if the parameters satisfy the conditions under which the functions (298) and (301) belong to the set $C^{(M)}_{-1}(0,\infty)$.

Then, using Equation (4.4.5) of [128], p. 61, in the form

$$\frac{1}{\Gamma(\mu)} \int_0^x (x-u)^{\mu-1}\, u^{\beta-1}\, E_{\alpha,\beta}[-\eta\, u^{\alpha}]\, du \,=\, x^{\beta-1+\mu}\, E_{\alpha,\beta+\mu}[-\eta\, x^{\alpha}], \tag{305}$$

where $\mu > 0$, $\beta > 0$, Equation (303) takes the form

$$F_X^{(M)}(x) \,=\, (\lambda\, x)^{\beta-1+\mu}\, E_{\alpha,\beta+\mu}[-(\lambda\, x)^{\alpha}], \tag{306}$$

where it is assumed that the parameters satisfy the conditions

$$\mu > 0, \quad \alpha > 0, \quad \beta > 0. \tag{307}$$

Let us find the constraints on the parameters α, β, μ under which conditions

$$\lim_{x\to 0+} F_X^{(M)}(x) \,=\, 0, \tag{308}$$

$$\lim_{x\to\infty} F_X^{(M)}(x) \,=\, 1 \tag{309}$$

are satisfied for Function (306).

Using the definition of the two-parameter Mittag-Leffler function by Equation (4.1.1) of [128], p. 56, in the form

$$E_{\alpha,\beta}[z] \,=\, \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha\, k + \beta)} \,=\, \frac{1}{\Gamma(\beta)} + \sum_{k=1}^{\infty} \frac{z^k}{\Gamma(\alpha\, k + \beta)}, \tag{310}$$

where $\alpha > 0$ and $\beta \in \mathbb{R}$, one can see that

$$\lim_{x\to 0+} F_X^{(M)}(x) \,=\, \lim_{x\to 0+} (\lambda\, x)^{\beta-1+\mu}\, \frac{1}{\Gamma(\beta)} + \lim_{x\to 0+} \sum_{k=1}^{\infty} \frac{(-1)^k\, (\lambda\, x)^{k\alpha+\beta-1+\mu}}{\Gamma(\alpha\, k + \beta)}. \tag{311}$$

Therefore, property (308) is satisfied, if the inequality

$$\beta - 1 + \mu > 0 \tag{312}$$

holds.

To prove property (309), one can use Theorem 4.3 of [128], p. 64, which gives the asymptotic equation

$$E_{\alpha,\beta}[-z] \,=\, -\sum_{k=1}^{m} \frac{1}{\Gamma(\beta-k\,\alpha)}\,\frac{1}{(-z)^k} \,+\, O(|z|^{-m-1}) \quad (|z| \,\to\, \infty) \tag{313}$$

that holds for $0 \,<\, \alpha \,<\, 1$.

Using (313), Function (306) satisfies the following asymptotic equation

$$F_X^{(M)}(x) \,=\, (\lambda\,x)^{\beta-1+\mu}\, E_{\alpha,\beta+\mu}[-\,(\lambda\,x)^{\alpha}] \,=\,$$
$$\frac{1}{\Gamma(\beta+\mu\,-\,\alpha)}\,\frac{(\lambda\,x)^{\beta-1+\mu}}{(\lambda\,x)^{\alpha}} \,+\, O(x^{\beta-1+\mu-2\alpha}) \,=\,$$
$$\frac{1}{\Gamma(\beta+\mu\,-\,\alpha)}\,(\lambda\,x)^{\beta-1+\mu-\alpha} \,+\, O(x^{\beta-1+\mu-2\alpha}) \tag{314}$$

for $x \to \infty$.

As a result, property (309) holds, if the following equality is satisfied

$$\beta\,-\,1\,+\,\mu\,-\,\alpha\,=\,0. \tag{315}$$

Using condition (315), Equation (314) takes the form

$$F_X^{(M)}(x) \,=\, \frac{1}{\Gamma(1)}\,(\lambda\,x)^0 \,+\, O(x^{\beta-1+\mu-2\alpha}) \,=\, 1 \,+\, O(x^{\beta-1+\mu-2\alpha}) \tag{316}$$

for $x \,\to\, \infty$, where $\beta-1+\mu-2\alpha \,=\, -\alpha \,<\, 0$. Therefore, property (309) is satisfied, and $F_X^{(M)}(x) \,\to\, 1$ at $x \to \infty$.

For case (315), inequality (312), which is used for $F_X^{(M)}(0+) = 0$, is satisfied, since

$$\beta\,-\,1\,+\,\mu\,=\,\alpha\,>\,0. \tag{317}$$

In the first example, the conditions on the parameters have the form

$$0\,<\,\mu\,<\,1, \quad 0\,<\,\alpha\,<\,1, \quad 0\,<\,\beta\,<\,2, \tag{318}$$

such that

$$\beta\,-\,1\,+\,\mu\,-\,\alpha\,=\,0. \tag{319}$$

Note that, for GF probability density (298), one can use not only the values $\alpha, \beta \in (0,1)$, and $\alpha < \beta$, but also all values $\alpha \in (0,1)$ and $\beta > 0$, such that

$$\beta\,-\,\alpha\,\in\,(0,1), \tag{320}$$

since $\mu \in (0,1)$.

Condition (320) allows us to consider a wider class of probability distributions with $0 < \beta < 2$. For example, $\mu = 0.1$, $\alpha = 0.9$, $\beta = 1.8$. Note that the GF probability density (298) cannot be considered for the case $\beta = \alpha$. Function (298) with $\beta = \alpha$ describes the standard probability density.

In the second example, the conditions on the parameters have the form

$$0\,<\,\alpha\,\leq\,\beta\,<\,1, \quad 0\,<\,\alpha\,\leq\,\mu\,\leq\,1, \tag{321}$$

such that

$$\beta\,-\,1\,+\,\mu\,-\,\alpha\,=\,0. \tag{322}$$

Note that for the power law GF probability density (301) one can use $\beta = \alpha$ and $\mu = 1$. For $\mu = 1$, the GF probability density (301) describes the uniform GF distribution.

It should be emphasized that GF probability density functions are not standard probability density functions, in general. For example, Function (298) is a standard PFD only for $\beta = \alpha$. Note that Function (301) cannot be considered a standard PFD on the positive semi-axis.

As a result, one can give the following definitions.

Definition 23 (GF distribution of the Mittag-Leffler type for power law nonlocality)**.** *Nonlocality is described by the kernel pair of the Luchko set*

$$M_x(x) \;=\; \frac{(\lambda\, x)^{\mu-1}}{\Gamma(\mu)} \tag{323}$$

$$K_x(x) \;=\; \frac{\lambda(\lambda x)^{-\mu}}{\Gamma(1-\mu)}, \tag{324}$$

where $0 \;<\; \mu\, 1$.

The GF probability density function

$$f_X(x) \;=\; \lambda\,(\lambda\, x)^{\beta-1}\, E_{\alpha,\beta}[-\,(\lambda\, x)^{\alpha}], \tag{325}$$

where $\alpha > 0$*, and* $\beta \in \mathbb{R}$*.*

The GF cumulative distribution function

$$F^{(M)}_X(x) \;=\; (\lambda\, x)^{\beta-1+\mu}\, E_{\alpha,\beta+\mu}[-\,(\lambda\, x)^{\alpha}]. \tag{326}$$

The parameter values

$$0 \;<\; \mu \;<\; 1, \quad 0 \;<\; \alpha \;<\; 1, \quad 0 \;<\; \beta \;<\; 2, \tag{327}$$

such that

$$\beta \,-\, 1 \,+\, \mu \,-\, \alpha \;=\; 0. \tag{328}$$

Definition 24 (The GF power law distribution for nonlocality of the Mittag-Leffler type)**.** *Nonlocality is described by the kernel pair of the Luchko set*

$$M_x(x) \;=\; (\lambda\, x)^{\beta-1}\, E_{\alpha,\beta}[-\,(\lambda\, x)^{\alpha}] \tag{329}$$

$$K_x(x) \;=\; \frac{\lambda\,(\lambda\, x)^{\alpha-\beta}}{\Gamma(1+\alpha-\beta)} \;+\; \frac{\lambda\,(\lambda\, x)^{-\beta}}{\Gamma(1-\beta)}, \tag{330}$$

where $0 < \alpha \le \beta < 1$*.*

The GF probability density function

$$f_X(x) \;=\; \lambda\, \frac{(\lambda\, x)^{\mu-1}}{\Gamma(\mu)}, \tag{331}$$

where $\mu > 0$*.*

The GF cumulative distribution function

$$F^{(M)}_X(x) \;=\; (\lambda\, x)^{\beta-1+\mu}\, E_{\alpha,\beta+\mu}[-\,(\lambda\, x)^{\alpha}]. \tag{332}$$

The parameter values

$$0 \;<\; \alpha \;\le\; \beta \;<\; 1, \quad 0 \;<\; \alpha \;\le\; \mu \;\le\; 1, \tag{333}$$

such that

$$\beta - 1 + \mu - \alpha \;=\; 0. \tag{334}$$

5.2. GF Distributions with Prabhakar Function

Let us consider the power law nonlocality that is described by the kernel pair (136) that belongs to the Luchko set, where GFI kernel is

$$M_x(x) \;=\; h_\mu(\lambda\, x) \;=\; \frac{(\lambda\, x)^{\mu-1}}{\Gamma(\mu)} \tag{335}$$

with $\mu \in (0,1)$.

Let us consider the function

$$f(x) \;=\; C_x\, x^{\beta-1}\, E^{\gamma}_{\alpha,\beta}[-\eta\, x^{\alpha}], \tag{336}$$

where $\mu > 0$, $\alpha > 0$, $\beta > 0$, $\eta > 0$ ($\eta \neq \lambda$) and $E^{\gamma}_{\alpha,\beta}[z]$ is the Prabhakar function [135] that is also called the three-parametric Mittag-Leffler function (see Section 5.1 in [129], p. 115-128).

The Prabhakar function is defined as

$$E^{\gamma}_{\alpha,\beta}[z] \;=\; \sum_{n=0}^{\infty} \frac{(\gamma)_n}{n!\,\Gamma(\alpha\, n+\beta)}\, z^n, \tag{337}$$

where $\alpha > 0$, $\beta > 0$, $\gamma > 0$, and $(\gamma)_n$ is the Pochhammer symbol that is defined for any non-negative integer n as

$$(\gamma)_n \;=\; \frac{\Gamma(\gamma+n)}{\Gamma(\gamma)}. \tag{338}$$

The Prabhakar function with $\gamma = 1$ is the two-parametric Mittag-Leffler function, and the Prabhakar function with $\beta = \gamma = 1$ is the classical Mittag-Leffler function

$$E^{\gamma}_{\alpha,\beta}[z] \;=\; E_{\alpha,\beta}[z], \quad E^{\gamma}_{\alpha,1}[z] \;=\; E_{\alpha}[z]. \tag{339}$$

In order for Function (336) to be a GF probability density on $\mathbb{R}_+$, the GF normalization condition should be satisfied. The GF normalization condition for Function (336) has the form

$$\lim_{x\to\infty} I^{x}_{(M_x)}[u]\, f(u) \;=\; 1, \tag{340}$$

where the kernel $M_x(x)$ is defined by (335).

In addition to this normalization condition, it is important to check the property

$$\lim_{x\to 0+} I^{x}_{(M_x)}[u]\, f(u) \;=\; 0. \tag{341}$$

The restrictions on the parameters α, β, γ, and C_x should be derived from conditions (340) and (341).

The GF integral $I^{x}_{(M_x)}$ with kernel (335) is expressed through the Riemann–Liouville fractional integral

$$I^{x}_{(h_\mu)}[u]\, f(u) \;=\; \lambda^{\mu-1}\, (I^{\mu}_{0+} f)(x), \tag{342}$$

where $0 \;<\; \mu \;<\; 1$.

In fractional calculus [1,4], the Riemann–Liouville fractional integral is defined for all $\mu > 0$.

Let us define a function $F(x)$ that is the GF integral (342) of Function (336) in the form

$$F(x) \;=\; \lambda^{\mu-1}\, I^{x}_{(h_\mu)}[u]\, f(u) \;=\; \lambda^{\mu-1} \int_0^x \frac{(x-u)^{\mu-1}}{\Gamma(\mu)}\, f(u)\, du. \tag{343}$$

Using Equation (5.1.47) of Theorem 5.5 in [129], p. 125, in the form

$$\int_0^x \frac{(x-u)^{\mu-1}}{\Gamma(\mu)}\, u^{\beta-1}\, E^{\gamma}_{\alpha,\beta}[-\eta\, u^{\alpha}]\, du \;=\; x^{\beta+\mu-1}\, E^{\gamma}_{\alpha,\beta+\mu}[-\eta\, x^{\alpha}], \tag{344}$$

where $\mu > 0$, $\alpha > 0$, $\beta > 0$, $\eta \in \mathbb{R}$, Function (343) takes the form

$$F(x) \;=\; C_x\, \lambda^{\mu-1}\, x^{\beta+\mu-1}\, E^{\gamma}_{\alpha,\beta+\mu}[-\eta\, x^{\alpha}]. \tag{345}$$

Using Equation (337), one can see

$$F(x) \;\sim\; \frac{C_x}{\Gamma(\beta)}\, \lambda^{\mu-1}\, x^{\beta+\mu-1} \qquad (x \;\to\; 0+). \tag{346}$$

Therefore, condition (341) holds, if

$$\beta \,+\, \mu \,-\, 1 \,>\, 0. \tag{347}$$

An asymptotic expansion can be considered for real positive parameters [136,137]. Using Equation (5.1.31) of Theorem 5.4 in [129], p. 121, (see also [138,139]) for $0 < \alpha < 2$, the following asymptotic expansion holds

$$E^{\gamma}_{\alpha,\beta}[-x] \;\sim\; h(x) \;\sim\; \frac{x^{-\gamma}}{\Gamma(\beta-\alpha\gamma)}, \tag{348}$$

where $h(x)$ is defined (see Equation (5.1.25) in [129], p. 119) in the form

$$h(x) \;\sim\; \frac{x^{-\gamma}}{\Gamma(\gamma)} \sum_{k=0}^{\infty} \frac{(-1)^k\, \Gamma(\gamma+k)}{\Gamma(\beta-\alpha(\gamma+k))}\, x^{-k}. \tag{349}$$

Then, one can obtain

$$F(x) \;\sim\; \frac{C_x}{\Gamma(\beta+\mu-\alpha\,\gamma)}\, \lambda^{\mu-1}\, x^{\beta+\mu-1}\, (\eta\, x^{\alpha})^{-\gamma} \;=\; \frac{C_x \lambda^{\mu-1}}{\eta^{\gamma}\Gamma(\beta+\mu-\alpha\,\gamma)}\, x^{\beta+\mu-1-\alpha\,\gamma} \qquad x \;\to\; \infty. \tag{350}$$

The GF normalization condition (340) is satisfied, if

$$\beta \,+\, \mu \,-\, 1 \,-\, \alpha\,\gamma \;=\; 0. \tag{351}$$

$$C_x \;=\; \lambda^{1-\mu}\, \eta^{\gamma}\, \Gamma(\beta+\mu-\alpha\,\gamma) \;=\; \lambda^{1-\mu}\, \eta^{\gamma}. \tag{352}$$

Note that Function (336) is completely monotonic [140] (see also [129], p. 124), for the following values of the parameters

$$0 \,<\, \alpha \,\le\, 1, \quad 0 \,<\, \alpha\,\gamma \,\le\, \beta \,\le\, 1, \quad \eta \,>\, 0. \tag{353}$$

Therefore, Function (345) is completely monotonic for the case

$$0 \,<\, \alpha \,\le\, 1, \quad 0 \,<\, \alpha\,\gamma \,\le\, \beta + \mu \,\le\, 1, \quad \eta \,>\, 0. \tag{354}$$

The complete monotonicity of a function $F(x)$ means that $F(x)$ is continuous on $(0,\infty)$, infinitely differentiable on $(0,\infty)$, and the condition $(-1)^n d^n F(x)/dx^n \geq 0$ is satisfied for all $n \in \mathbb{N}$ and all $x > 0$. Because of this, for a completely monotonic function, there is a first-order derivative $(d/dz)E^{\gamma}_{\alpha,\beta+\mu}[-\eta\, z]$. The first derivative of the function $F(x)$ has the form

$$\frac{d}{dx}F(x) = \lambda^{\mu-1} (\beta+\mu-1)x^{\beta+\mu-2} E^{\gamma}_{\alpha,\beta+\mu}[-\eta\, x^{\alpha}] + \lambda^{\mu-1} x^{\beta+\mu+\alpha-2} \alpha \left(\frac{dE^{\gamma}_{\alpha,\beta+\mu}[-\eta\, z]}{dz} \right)_{z=x^{\alpha}} =$$
$$\lambda^{\mu-1} x^{\beta+\mu+\alpha-2} \left((\beta+\mu-1)\, E^{\gamma}_{\alpha,\beta+\mu}[-\eta\, x^{\alpha}] + \alpha \left(\frac{dE^{\gamma}_{\alpha,\beta+\mu}[-\eta\, z]}{dz} \right)_{z=x^{\alpha}} \right). \quad (355)$$

Then, Function (345) belongs to the set $C^1_{-1}(0,\infty)$, and, therefore, $f_X(x)$ belongs to the set $C_{-1,(K)}(0,\infty)$, if the parameters satisfy the conditions

$$\beta + \mu - 1 > 0, \quad \beta + \mu + \alpha - 1 > 0. \quad (356)$$

For the function $f_X(x)$ that belongs to the set $C_{-1,(K)}(0,\infty)$ one can use the Luchko theorem. Then, the function $f_X(x)$ that belongs to the set $C_{-1,(K)}(0,\infty)$ has the properties

$$I^x_{(M_x)}[u]\, f(u) \in C^1_{-1}(0,\infty) \quad \text{and} \quad \lim_{x\to 0+} I^x_{(M_x)}[u]\, f(u) = 0. \quad (357)$$

Therefore, the following theorem is proved.

Theorem 16 (GF distribution of Prabhakar type)**.** *Let a kernel pair $M_x(x), K_x(x)$ belong to the Luchko set and a function $f_X(x)$ be defined by Equation (336) with the Prabhakar Function (337).*

If the kernel pair $M_x(x), K_x(x)$ is described by equation (136), then Function (336) describes the GF cumulative distribution function for $\beta + \mu - 1 - \alpha\,\gamma = 0$, $C_x = \lambda^{1-\mu}\,\eta^{\gamma}$.

As a result, one can give the following definition.

Definition 25 (The GF distribution of Prabhakar type for power law nonlocality)**.** *Nonlocality is described by the kernel pair of the Luchko set*

$$M_x(x) = h_{\mu}(\lambda\, x) = \frac{(\lambda\, x)^{\mu-1}}{\Gamma(\mu)}, \quad (358)$$

$$K_x(x) = \lambda\, h_{1-\mu}(\lambda\, x) = \frac{\lambda(\lambda\, x)^{-\mu}}{\Gamma(1-\mu)}, \quad (359)$$

where $\mu \in (0,1)$.

The GF probability density function

$$f_X(x) = \lambda^{1-\mu}\,\eta^{\gamma}\, x^{\beta-1}\, E^{\gamma}_{\alpha,\beta}[-\eta\, x^{\alpha}], \quad (360)$$

where $\mu > 0$, $\alpha > 0$, $\beta > 0$, $\eta > 0$.

The GF cumulative distribution function

$$F^{(M)}_X(x) = \eta^{\gamma}\, x^{\alpha\,\gamma}\, E^{\gamma}_{\alpha,\alpha\,\gamma+1}[-\eta\, x^{\alpha}]. \quad (361)$$

The parameter values

$$\alpha \in (0,1), \quad \beta + \mu - 1 - \alpha\,\gamma = 0. \quad (362)$$

5.3. GF Distributions with Kilbas–Saigo Function

Let us consider the power law nonlocality that is described by the kernel pair (136) that belongs to the Luchko set, where the GFI kernel is

$$M_x(x) = h_{\alpha}(\lambda\, x) = \frac{(\lambda\, x)^{\alpha-1}}{\Gamma(\alpha)} \quad (363)$$

with $\alpha \in (0,1)$.

Let us consider the function

$$f(x) \;=\; C_x\, x^{\alpha\, c}\, E_{\alpha,b,c}[-\,\eta\, x^{\alpha\, b}], \tag{364}$$

where function $E_{\alpha,b,c}[z]$ is called the Kilbas–Saigo function [129]. It is a generalization of the classical Mittag-Leffler function that is proposed by Kilbas and Saigo [141], (see also [4], p. 48, and Section 5.2 in [129], pp. 128–147) that is defined by the series

$$E_{\alpha,b,c}[z] \;=\; \sum_{k=0}^{\infty} a_k(\alpha,b,c)\, z^k, \tag{365}$$

with

$$a_0(\alpha,b,c) \;=\; 1, \quad a_k(\alpha,b,c) \;=\; \prod_{j=0}^{k-1} \frac{\Gamma(1+\alpha(jb+c))}{\Gamma(1+\alpha(jb+c+1))} \quad (k \in \mathbb{N}), \tag{366}$$

where $\alpha > 0$, $b > 0$, $c \in \mathbb{R}$ such that $\alpha(k\, b + c) \neq -1, -2, \ldots$

The Kilbas–Saigo function with $b = 1$ gives (see Equation (5.2.5) in [129], p. 129) the two-parametric Mittag-Leffler function

$$E_{\alpha,1,c}[z] \;=\; \Gamma(\alpha\, c\, +\, 1)\, E_{\alpha,\alpha\, c\, +1}[z], \tag{367}$$

and

$$E_{\alpha,1,0}[x] \;=\; E_{\alpha}[x]. \tag{368}$$

In order for Function (364) to be a GF probability density on $\mathbb{R}_+$ the GF normalization condition should be satisfied. The GF normalization condition for the GF probability density Function (364) has the form

$$\lim_{x\to\infty} I^x_{(M_x)}[u]\, f(u) \;=\; 1, \tag{369}$$

where the kernel $M_x(x)$ is defined by (363).

In addition to this normalization condition, it is important to check the property

$$\lim_{x\to 0+} I^x_{(M_x)}[u]\, f(u) \;=\; 0. \tag{370}$$

The restrictions on the parameters α, b, c, and C_x should be derived from conditions (369) and (370).

The GF integral $I^x_{(M_x)}$ with kernel (363) is expressed through the Riemann–Liouville fractional integral

$$I^x_{(h_\alpha)}[u]\, f(x) \;=\; \lambda^{\alpha-1}\, (I^{\alpha}_{0+}\, f)(x), \tag{371}$$

where $0 \;<\; \alpha \;<\; 1$.

In fractional calculus [1,4], the Riemann–Liouville fractional integral is defined for all $\alpha > 0$.

In order to prove that the function $f(x)$ belongs to the set $C_{-1,(K)}(0,\infty)$ one can use the Luchko theorem. According to this theorem, if a function $f(x)$ satisfies the conditions

$$I^x_{(M_x)}[u]\, f(u) \;\in\; C^1_{-1}(0,\infty) \quad \text{and} \quad \lim_{x\to 0+} I^x_{(M_x)}[u]\, f(u) \;=\; 0, \tag{372}$$

then the function $f_X(x)$ belongs to the set $C_{-1,(K)}(0,\infty)$.

Theorem 17 (GF distributions of the Kilbas–Saigo type)**.** *Let a kernel pair $M_x(x), K_x(x)$ belong to the Luchko set and a function $f_X(x)$ be defined by Equation* (364) *with the Kilbas–Saigo Function* (365)*.*

If the kernel pair $M_x(x), K_x(x)$ is described by Equation (136), then Function (364) describes the GF probability distribution for $c = b-1$, $C_x = \eta\,\lambda^{1-\alpha}$.

The GF probability density function has the form

$$f_X(x) = \eta\,\lambda^{1-\alpha}\,x^{\alpha\,(b-1)}\,E_{\alpha,b,b-1}[-\eta\,x^{\alpha\,b}], \tag{373}$$

where

$$\alpha \in (0,1], \quad b > 0. \tag{374}$$

The GF cumulative distribution function has the form

$$F_X^{(M)}(x) = 1 - E_{\alpha,b,b-1}[-\eta\,x^{\alpha\,b}], \tag{375}$$

Proof. An explicit expression for the GF cumulative distribution function can be derived by using Equation (5.2.48) of Theorem 5.32 in [129], p. 141, in the form

$$\int_0^x \frac{(x-u)^{\alpha-1}}{\Gamma(\alpha)}\,u^{\alpha\,c}\,E_{\alpha,b,c}[\eta\,x^{\alpha\,b}]\,du = \frac{1}{\eta}\,u^{\alpha(c-b+1)}\left(E_{\alpha,b,c}[\eta\,x^{\alpha\,b}] - 1\right) \tag{376}$$

for $\alpha > 0$, $b > 0$, $c > -1/\alpha$ and $\eta \neq 0$.

Let us define the function

$$F_X^{(M)}(x) = I^x_{(h_\alpha)}[u]\,f(u) = \lambda^{\alpha-1}\,(I^\alpha_{0+}f)(x) = \lambda^{\alpha-1}\int_0^x \frac{(x-u)^{\alpha-1}}{\Gamma(\alpha)}\,u^{\alpha\,c}\,E_{\alpha,b,c}[\eta\,x^{\alpha\,b}]\,du. \tag{377}$$

Then, using Equation (376), the function $F_X^{(M)}(x)$ has the form

$$F_X^{(M)}(x) = \frac{C_x\,\lambda^{\alpha-1}}{\eta}\,x^{\alpha(c-b+1)}\left(1 - E_{\alpha,b,c}[-\eta\,x^{\alpha\,b}]\right). \tag{378}$$

Using Equation (378), condition (369) can be written as

$$\lim_{x\to\infty} F_X^{(M)}(x) = \frac{C_x\lambda^{\alpha-1}}{\eta}\,\lim_{x\to\infty} x^{\alpha(c-b+1)}\left(1 - E_{\alpha,b,c}[-\eta\,x^{\alpha\,b}]\right) = 1. \tag{379}$$

In addition to this normalization condition, it should be considered the condition

$$\lim_{x\to 0+} F_X^{(M)}(x) = \frac{C_x\lambda^{\alpha-1}}{\eta}\,\lim_{x\to 0+} x^{\alpha(c-b+1)}\left(1 - E_{\alpha,b,c}[-\eta\,x^{\alpha\,b}]\right) = 0. \tag{380}$$

An asymptotic formula for Function (378) can be derived by using the results of Boudabsa, Simon, and Vallois in the works [142,143]. The following three cases should be considered:

(1) In the first case, one can consider the Kilbas–Saigo function with $c = b-1$ and $b > 0$. In Theorem 2 of [142], p. 9. (see also Proposition 4.12. in [143], p. 31), one can see the following inequality

$$\frac{1}{1 + x\,\Gamma(1-\alpha)} \leq E_{\alpha,b,b-1}[-x] \leq \frac{1}{1 + x\,\dfrac{\Gamma(\alpha\,(b-1)+1)}{\Gamma(\alpha\,b+1)}} \tag{381}$$

for all $x \geq 0$ and every $\alpha \in (0,1]$, $b > 0$.

Using Remark 4 of [142], p. 9, (see also Remark 4.13. in [143], p. 32), one can use the asymptotic behaviors

$$1 - E_{\alpha,b,b-1}[-x] \sim \frac{\Gamma(1+\alpha\,(b-1))}{\Gamma(1+\alpha\,b)}\,x \qquad (x \to 0), \tag{382}$$

and

$$E_{\alpha,b,b-1}[-x] \sim \frac{1}{\Gamma(1-\alpha)\,x} \qquad (x \to \infty). \tag{383}$$

Then, Function (378) with $c = b-1$ is described as

$$F_X^{(M)}(x) = \frac{C_x\lambda^{\alpha-1}}{\eta}\left(1 - E_{\alpha,b,b-1}[-\eta\,x^{\alpha\,b}]\right) \sim C_x\,\lambda^{\alpha-1}\frac{\Gamma(1+\alpha\,(b-1))}{\Gamma(1+\alpha\,b)}\,x^{\alpha\,b} \qquad (x \to 0) \tag{384}$$

$$F_X^{(M)}(x) = \frac{C_x\,\lambda^{\alpha-1}}{\eta}\left(1 - E_{\alpha,b,b-1}[-\eta\,x^{\alpha\,b}]\right) \sim \frac{C_x\lambda^{\alpha-1}}{\eta}\left(1 - \frac{1}{\Gamma(1-\alpha)\,\eta}\,x^{-\alpha\,b}\right) \qquad (x \to \infty). \tag{385}$$

Then, conditions (369) and (370) are satisfied for the parameters $\alpha\,b > 0$. As a result, one can obtain

$$\alpha \in (0,1], \quad b > 0, \quad c = b-1, \quad C_x = \eta\,\lambda^{1-\alpha}. \tag{386}$$

(2) In the second case, one can consider the Kilbas–Saigo function with $c = b - 1/\alpha$ and $b > 0$. In this case,

$$\alpha\,(c - b + 1) = \alpha\left(1 - \frac{1}{\alpha}\right) = \alpha - 1.$$

Using Remark 4.13 in [143], p. 32, the following asymptotic equation is proved

$$1 - E_{\alpha,b,b-1/\alpha}[-x] \sim \frac{\Gamma(\alpha\,b)}{\Gamma(\alpha\,(b+1))}\,x \qquad (x \to 0). \tag{387}$$

In Remark 8 of [142], p. 18, the following asymptotic equation is proved

$$E_{\alpha,b,b-1/\alpha}[-x] \sim A(\alpha,b)\,x^{-1-1/b} \qquad (x \to \infty), \tag{388}$$

where

$$A(\alpha,b) = (\alpha\,b)^{\alpha/b}\,\Gamma(\alpha+1)\,G(1-\alpha;\alpha\,b)\,G(1+\alpha;\alpha\,b), \tag{389}$$

and $G(a;b)$ is the Barnes double Gamma function (see Appendix A in [142], pp. 22–23). Then, Function (378) with $c = b - 1/\alpha$ is described as

$$\begin{aligned} F_X(x) &= \frac{C_x\lambda^{\alpha-1}}{\eta}\,x^{\alpha-1}\left(1 - E_{\alpha,b,b-1/\alpha}[-\eta\,x^{\alpha\,b}]\right) \sim \\ &\frac{C_x\lambda^{\alpha-1}}{\eta}\,x^{\alpha-1}\,\frac{\Gamma(\alpha\,b)\eta}{\Gamma(\alpha\,(b+1))}\,\eta\,x^{\alpha\,b} \sim \\ &C_x\,\lambda^{\alpha-1}\,\frac{\Gamma(\alpha\,b)}{\Gamma(\alpha\,(b+1))}\,x^{\alpha\,b+\alpha-1)} \qquad (x \to 0), \end{aligned} \tag{390}$$

and

$$\begin{aligned} F_X^{(M)}(x) &= \frac{C_x\lambda^{\alpha-1}}{\eta}\,x^{\alpha-1}\left(1 - E_{\alpha,b,b-1/\alpha}[-\eta\,x^{\alpha\,b}]\right) \sim \\ &\frac{C_x\lambda^{\alpha-1}}{\eta}\left(1 - A(\alpha,b)\,(\eta\,x^{\alpha\,b})^{-1-1/b}\right) \sim \\ &\frac{C_x\lambda^{\alpha-1}}{\eta}\,x^{\alpha-1}\left(1 - \eta^{-1-1/b}\,A(\alpha,b)\,x^{-\alpha\,(b+1)}\right) \qquad (x \to \infty). \end{aligned} \tag{391}$$

Then, conditions (370) and (370) are satisfied for the parameters

$$\alpha - 1 = 0, \quad \alpha\,(b+1) > 0. \tag{392}$$

As a result, one can obtain

$$\alpha - 1 = 0, \quad b > 0, \quad c = b - 1. \tag{393}$$

(3) In the third case, one can consider the Kilbas–Saigo function with $c > b - 1/\alpha$ and $\alpha \in [0,1]$, $b > 0$.
In Conjecture 4 of [142], p. 16, one can see the following inequality. For every $\alpha \in (0,1]$, $b > 0$, $c > b - 1/\alpha$ and $x \geq 0$ one has

$$\frac{1}{1 + x \, \dfrac{\Gamma(1+\alpha\,(c-b))}{\Gamma(1+\alpha\,(c+1-b))}} \leq E_{\alpha,b,c}[-x] \leq \frac{1}{1 + x \, \dfrac{\Gamma(1+\alpha\,c)}{\Gamma(1+\alpha\,(c+1))}} \tag{394}$$

for all $x \geq 0$.
In Proposition 6 of [142], p. 16, the following asymptotic equation is proved

$$E_{\alpha,b,c}[-x] \sim \frac{\Gamma(1+\alpha(c+1-b))}{\Gamma(1+\alpha(c-b))\,x} \qquad (x \to \infty) \tag{395}$$

for $\alpha \in [0,1]$, $b > 0$, $c > b - 1/\alpha$. Then, Function (378) with $c > b - 1/\alpha$ is described as

$$F^{(M)}_X(x) = \frac{C_x\,\lambda^{\alpha-1}}{\eta}\, x^{\alpha(c-b+1)} \left(1 - E_{\alpha,b,c}[-\eta\, x^{\alpha\, b}]\right) \sim$$
$$\frac{C_x \lambda^{\alpha-1}}{\eta}\, x^{\alpha(c-b+1)} \left(1 - \frac{\Gamma(1+\alpha(c+1-b))}{\Gamma(1+\alpha(c-b))\eta}\, x^{-\alpha\, b}\right) \qquad (x \to \infty). \tag{396}$$

Then, conditions (370) and (370) are satisfied for the parameters

$$\alpha\,(c - b + 1) = 0, \quad \alpha\, b > 0, \quad C_x = \eta\,\lambda^{1-\alpha}. \tag{397}$$

As a result, one can obtain

$$c = b - 1, \quad b > 0, \quad 0 < \alpha \leq 1 \quad C_x = \eta\,\lambda^{1-\alpha}. \tag{398}$$

Using representation (371), the first of two conditions (357) can be written as

$$\frac{d}{dx} F^{(M)}_X(x) \in C_{-1}(0,\infty). \tag{399}$$

The fulfillment of condition (399) and condition (370) allows us to state that the function $f_X(x)$ belongs the set $C_{-1,(K)}(0,\infty)$.

It is known that the Kilbas–Saigo function $E_{\alpha,b,c}[-x]$ is completely monotonic for some values of the parameters. The complete monotonicity of a function $f(x)$ means that $f(x)$ is continuous on $(0,\infty)$, infinitely differentiable on $(0,\infty)$, and the inequality $(-1)^n d^n f(x)/dx^n \geq 0$ is satisfied for all $n \in \mathbb{N}$ and all $x > 0$.

Using Theorem 1 in [142], p. 5, (see also Proposition 5.31 of [129], p. 141), one can see that the Kilbas–Saigo function $E_{\alpha,b,c}[-x]$ with $\alpha > 0$, $b > 0$ and $c > -1/\alpha$ is completely monotonic on $(0,\infty)$ if and only if $\alpha \leq 1$ and $c > b - 1/\alpha$.

The first derivative of function $F^{(M)}_X(x)$ of the form (375) belongs to the set $C_{-1}(0,\infty)$, the condition $\alpha\, b > 0$ and $\alpha(c - b + 1) + \alpha b - 1 > -1$ should be satisfied. The fulfillment of this condition and condition (370) leads to the statement that the function $f_X(x)$ belongs the set $C_{-1,(K)}(0,\infty)$, if $\alpha \in (0,1]$, $b > 0$, $c = b - 1$.

This ends the proof. □

As a result, one can give the following definition.

Definition 26 (The GF distribution of Kilbas–Saigo type for power law nonlocality)**.** *Nonlocality is described by the kernel pair of the Luchko set*

$$M_x(x) \,=\, h_\alpha(\lambda\, x) \,=\, \frac{(\lambda\, x)^{\alpha-1}}{\Gamma(\alpha)}, \tag{400}$$

$$K_x(x) \,=\, \lambda\, h_{1-\alpha}(\lambda\, x) \,=\, \frac{\lambda(\lambda\, x)^{-\alpha}}{\Gamma(1-\alpha)}. \tag{401}$$

The GF probability density function

$$f_X(x) \,=\, \eta\, \lambda^{1-\alpha}\, x^{\alpha\,(b-1)}\, E_{\alpha,b,b-1}[-\eta\, x^{\alpha\, b}]. \tag{402}$$

The GF cumulative distribution function

$$F^{(M)}_X(x) \,=\, 1 \,-\, E_{\alpha,b,b-1}[-\eta\, x^{\alpha\, b}]. \tag{403}$$

The parameter values

$$\alpha \,\in\, (0,\, 1), \quad b \,>\, 0. \tag{404}$$

Remark 14. *In the case of the kernels* (136)*, the GFD of the RL type is the Riemann–Liouville fractional derivative*

$$D^x_{(h_{1-\alpha})}[x']\, f(x') \,=\, \lambda^{1-\alpha}\, (D^\alpha_{RL,0+}\, f)(x). \tag{405}$$

The linear GF differential equation

$$D^x_{(M_x)}[u]\, f(u) \,=\, \eta_x\, V(x)\, f(u) \tag{406}$$

with the fractional derivative (405) *take the form*

$$(D^\alpha_{RL,0+} f)(x) \,=\, \eta\, V(x)\, f(x), \tag{407}$$

where $\eta = \lambda^{\alpha-1}\, \eta_x$.

If $V(x) \in L_\infty(0, x_0)$ or if $V(x)$ is bounded on $[0, x_0]$, then the Cauchy type problem for the fractional differential Equation (407) *and the condition $(I^{1-\alpha}_{RL,0+} f)(0+) = C_x \in \mathbb{R}$ has a unique solution $f(x)$ in the space $L^\alpha(0, x_0)$. This statement is proved in [4], p. 158, as Corollary 3.5. For example, one can consider Equation* (407) *with $V(x) = x^\beta$.*

In particular, there exists a unique solution $f(x) \in L^\alpha(0, x_0)$ of the Cauchy type problem for the equation

$$(D^\alpha_{RL,0} f)(x) \,=\, -\,\eta\, x^\beta\, f(x), \tag{408}$$

and $(I^{1-\alpha}_{RL,0+} f)(0+) = C_x \in \mathbb{R}$, where $x \in (0, x_0)$ with $\eta \in \mathbb{R}$ and $\beta \geq 0$.

The exact analytical solution of Equation (408) *is given in [4], p. 227, as Example 4.3. Therefore, the solution of Equation* (408) *has the form*

$$f(x) \,=\, \frac{C_x}{\Gamma(\alpha)}\, x^{\alpha-1} E_{\alpha,\beta/\alpha+1,(\beta-1)/\alpha+1}[-\,\eta\, x^{\alpha+\beta}], \tag{409}$$

if $\alpha \in (0,1)$, $\beta > -\alpha$ and $\eta \in \mathbb{R}$. If $\alpha = 1$, then $\beta > 0$.

5.4. Convolutional GF Distributions from Standard Distributions

Using Definition 9 of the standard PDF and Theorem 6, one can prove the following statement.

Theorem 18 (GF distribution from standard distribution)**.**
Let a pair $(M_x(x), K_x(x))$ belong to the Luchko set.
Let $\varphi(x)$ be a standard probability density function in the sense of Definition 9.
Then, the functions

$$f_X^{(K)}(x) \;=\; I^x_{(K_x)}[u]\,\varphi(u) \;=\; \int^x K_x(x-u)\,\varphi(u)\,du, \tag{410}$$

$$f_X^{(\lambda M)}(x) \;=\; I^x_{(\lambda M_x)}[u]\,\varphi(u) \;=\; \lambda \int^x M_x(x-u)\,\varphi(u)\,du \tag{411}$$

are the GF probability density functions in the sense of Definition 8 for the kernel pairs $(M_x(x)$, $K_x(x))$ and $(M_{x,new} = \lambda^{-1} K_x(x), K_{x,new} = \lambda M_x(x))$, respectively.

Proof. For Function (410), the statement of Theorem 18 is a direct consequence of Theorem 6 and Definition 9.

For Function (411), one must additionally use the statement of Proposition 9 according to which the kernel pair $(M_{x,new} = \lambda^{-1} K_x(x), K_{x,new} = \lambda M_x(x))$ with $\lambda > 0$ belongs to the Luchko set, if the pair $(M_x(x), K_x(x))$ belongs to the Luchko set.

This ends the proof. □

As a result, one can give the following definition.

Definition 27. *The GF probability density functions of the form (410) and (411) are called the complete GF probability density functions. The distribution functions (410) corresponding to them are called the complete GF cumulative distributions functions.*

Remark 15. *Note that the statement of Theorem 18 is a direct consequence of Theorem 6. It is separated into a special statement in order to emphasize the constructive nature of this statement. Theorem 18 allows one to obtain (construct) GF probability density functions through the Laplace convolutions of standard probability density functions on the semi-axis and operator kernels from the Luchko set.*

It should also be noted that when constructing the GF probability density functions, the condition of non-negativity of the function $\varphi(x)$ can be weakened. In order for functions (410) and (411) to be GF probability density functions, it is sufficient to use the condition of non-negativity of the convolution of function φ and the GFD kernel for all $x > 0$, instead of the requirement of non-negativity of the function φ.

For example, one can consider the standard PDF of the Gamma distribution [113], p. 47, on the positive semi-axis

$$\varphi(x) \;=\; \frac{\lambda^\beta}{\Gamma(\beta)}\, x^{\beta-1}\, e^{-\lambda\, x}, \tag{412}$$

where $x > 0$, and $\lambda > 0$ is the rate parameter, $\beta > 0$ is the shape parameter. Then, for any kernel pair $(M_x(x), K_x(x))$ that belongs to the Luchko set, one can define the GF probability density Functions (410) and (411) by the equations

$$f_X^{(K)}(x) \;=\; \frac{\lambda^\beta}{\Gamma(\beta)} \int^x K_x(x-u)\, u^{\beta-1}\, e^{-\lambda\, u}\, du. \tag{413}$$

$$f_X^{(\lambda M)}(x) \;=\; \frac{\lambda^{\beta+1}}{\Gamma(\beta)} \int^x M_x(x-u)\, u^{\beta-1}\, e^{-\lambda\, u}\, du. \tag{414}$$

In a particular case, for the Gamma distribution nonlocality (137) in the form

$$M_x(x) \;=\; h_{\alpha,\lambda}(\lambda\, x) \;=\; \frac{(\lambda\, x)^{\alpha-1}}{\Gamma(\alpha)} e^{-\lambda\, x}, \quad K_x(x) \;=\; \lambda\, h_{1-\alpha,\lambda}(\lambda x) \;+\; \frac{\lambda}{\Gamma(1-\alpha)}\, \gamma(1-\alpha, \lambda x), \tag{415}$$

where $\alpha \in (0,1)$, $\lambda > 0$, and the standard PDF in the form of the Gamma distribution (412), the GF probability density functions (410) and (411) have the form

$$f^{(K)}_X(x) \,=\, \lambda\, h_{1-\alpha+\beta,\lambda}(\lambda x) \,+\, \frac{\lambda^{\beta+1}}{\Gamma(1-\alpha)\Gamma(\beta)} \int^x \gamma(1-\alpha, \lambda(x-u))\, u^{\beta-1}\, e^{-\lambda\, u}\, du, \quad (416)$$

$$f^{(\lambda M)}_X(x) \,=\, \frac{\lambda^{\alpha+\beta}}{\Gamma(\alpha+\beta)}\, x^{\alpha+\beta-1}\, e^{-\lambda\, x}, \quad (417)$$

where $\alpha \in (0,1)$, $\beta > 0$, $\lambda > 0$, and $x > 0$. Here, the following property [113], p. 47, is used in the form

$$(h_{\alpha,\lambda}(\lambda x) \,*\, h_{\beta,\lambda}(\lambda x)) \,=\, \frac{1}{\lambda} h_{\alpha+\beta,\lambda}(\lambda x). \quad (418)$$

Remark 16. *Note that the GF probability density functions* (416) *and* (414) *correspond to different nonlocalities, namely, to two different pairs of operator kernels* $(M_x(x), K_x(x))$ *and* $(M_{x,new} = \lambda^{-1}K_x(x), K_{x,new} = \lambda M_x(x))$*, respectively.*

As the next example, one can consider the alpha-exponential function, which is described in Section 4.3 as a standard PDF on the positive semi-axis in the form

$$\varphi(x) \,=\, \lambda\, (\lambda\, x)^{\alpha-1}\, E_{\alpha,\alpha}[-\, (\lambda\, x)^{\alpha}], \quad (419)$$

where $\lambda > 0$ and $\alpha > 0$.

Then, for any pair $(M_x(x), K_x(x))$ that belongs to the Luchko set, one can define the GF probability density functions (410) and (411) by the equations

$$f^{(K)}_X(x) \,=\, \lambda^{\alpha} \int^x K_x(x-u)\, u^{\alpha-1}\, E_{\alpha,\alpha}[-\, (\lambda\, u)^{\alpha}]\, du, \quad (420)$$

$$f^{(\lambda M)}_X(x) \,=\, \lambda^{\alpha+1} \int^x M_x(x-u)\, u^{\alpha-1}\, E_{\alpha,\alpha}[-\, (\lambda\, u)^{\alpha}]\, du. \quad (421)$$

As a standard probability distribution, for example, the following probability distributions on the semi-axis can be considered.

- For the chi-squared distribution, see Chapter 11 of [144], pp. 69–73.
- For the Erlang distribution, see Chapter 15 of [144], pp. 84–85, and Section 3.11 in of [145], pp. 145–153.
- For the exponential distribution, see Chapter 17 of [144], pp. 88–92, and Section 3.9 of [145], pp. 133–136.
- For the Fisher–Snedecor distribution, see Chapter 20 of [144], pp. 102–106.
- For the Gamma distribution, see Chapter 22 of [144], pp. 109–113, and Section 3.10 of [145], pp. 136–142.
- For the inverse Gaussian (Wald) distribution, see Chapter 25 of [144], pp. 120–121, and Sections 3.22 and 3.24a of [145], pp. 194–199, pp. 206–209.
- For the Rayleigh distribution, see Chapter 39 of [144], pp. 173–175, and Section 3.15 of [145], pp. 168–175.
- For the Weibull–Gnedenko distribution, see Chapter 46 of [144], pp. 193–201, and Section 3.12 of [145], pp. 153–159.
- For the Nakagami distribution, see Section 3.18 of [145], pp. 179–182.
- For the Beta prime distribution (beta distribution of the second kind), see Section 3.19 of [145], pp. 182–186.
- For the Maxwell–Boltzmann distribution, see Section 3.17 of [145], pp. 175–179.

As a result, one can give the following definitions.

Definition 28 (The GF convolutional f_X-distributions for M and K nonlocalities)**.** *Nonlocality is described by the kernel pair of the Luchko set*

$$M_x(x),\, K_x(x) \quad \textit{any kernel pair that belongs to the Luchko set} \tag{422}$$

$$\lambda^{-1}\, K_x(x),\, \lambda\, M_x(x) \quad \textit{any kernel pair that belongs to the Luchko set.} \tag{423}$$

The GF probability density function

$$f_X^{(K)}(x) \;=\; \int_0^x K_x(x-u)\, f_{St}(u)\, du, \tag{424}$$

$$f_X^{(\lambda M)}(x) \;=\; \lambda \int_0^x M_x(x-u)\; f_{St}(u)\, du. \tag{425}$$

The GF cumulative distribution function

$$F_X^{(M)}(x) \;=\; \int_0^x f_{St}(u)\, du, \tag{426}$$

$$F_X^{(\lambda^{-1}K)}(x) \;=\; \int_0^x f_{St}(u)\, du. \tag{427}$$

The parameter values are defined by the condition that the kernel pair belongs to the Luchko set, and $f_{St}(x)$ belongs to the set $C_{-1}^{(\{1\})}(0,\infty)$.

Definition 29 (The GF convolutional Gamma distributions for Gamma nonlocalities)**.** *Nonlocality is described by the kernel pair of the Luchko set*

$$M_x(x) \;=\; h_{\alpha,\lambda}(\lambda\, x) \;=\; \frac{(\lambda\, x)^{\alpha-1}}{\Gamma(\alpha)} e^{-\lambda\, x}, \tag{428}$$

$$K_x(x) \;=\; \lambda\, h_{1-\alpha,\lambda}(\lambda x) \;+\; \frac{\lambda}{\Gamma(1-\alpha)}\, \gamma(1-\alpha, \lambda x). \tag{429}$$

The GF probability density function

$$f_X^{(K)}(x) \;=\; \lambda\, h_{1-\alpha+\beta,\lambda}(\lambda x) \;+\; \frac{\lambda^{\beta+1}}{\Gamma(1-\alpha)\Gamma(\beta)} \int^x \gamma(1-\alpha, \lambda(x-u))\, u^{\beta-1}\, e^{-\lambda\, u}\, du, \tag{430}$$

$$f_X^{(\lambda M)}(x) \;=\; \lambda\, \frac{(\lambda\, x)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} e^{-\lambda\, x}. \tag{431}$$

The GF cumulative distribution function

$$F_X^{(M)}(x) \;=\; F_X^{(\lambda^{-1}K)}(x) \;=\; \frac{1}{\Gamma(\beta)} \gamma(\beta, \lambda x). \tag{432}$$

The parameter values

$$\alpha \in (0,1), \quad \beta > 0, \quad \lambda > 0. \tag{433}$$

5.5. GF Probability Density for Power Law Nonlocality

Let us consider the GF differential equation, the solution for which is the GF probability density function.

Let nonlocality be described by the kernel pair (137) that belongs to the Luchko set, where the GFI kernel has the form

$$M_x(x) \;=\; h_\alpha(\lambda x) \;=\; \frac{(\lambda x)^{\alpha-1}}{\Gamma(\alpha)}, \tag{434}$$

where $\alpha \in (0,1)$.

In this case, the GFD of the RL type is the Riemann–Liouville fractional derivative

$$D^x_{(h_{1-\alpha})}[x'] \, f(x') \; = \; \lambda^{1-\alpha} \, (D^\alpha_{RL,0+} \, f)(x). \tag{435}$$

$$D^{x,*}_{(h_{1-\alpha})}[x'] \, f(x') \; = \; \lambda^{1-\alpha} \, (D^\alpha_{C,0+} \, f)(x). \tag{436}$$

Let us consider the simplest case of the linear fractional differential equation with the Caputo fractional derivative. To solve these equations, one can use the results described in [4].

The exact analytical solution of the equation with the Caputo fractional derivative

$$(D^\alpha_{C,0+} f)(x) \; = \; -\,\eta \, f(x), \tag{437}$$

where $\eta = \eta_x \, \lambda^{1-\alpha}$ and condition $f(0) = C_x \in \mathbb{R}$ is given in [4], p. 312, as Theorem 5.12.

If $\alpha \in (0,1)$ and $\eta \in \mathbb{R}$, then the solution of Equation (437) has the form

$$f(x) \; = \; f(0) \, E_\alpha[-\,\eta \, x^\alpha], \tag{438}$$

where $E_\alpha[z]$ is the classical Mittag-Leffler function (see Equation (3.1.1) in [128], p. 17) that is defined as

$$E_\alpha[z] \; = \; \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha \, k + 1)}, \tag{439}$$

where $\alpha > 0$ (in general, $\alpha \in \mathbb{C}$). For $\alpha = 1$, solution (438) takes the well-known form

$$f(x) \; = \; f(0) \, \exp(-\,\eta \, x). \tag{440}$$

For Function (438), the probability density, which is a solution of Equation (437), has the form

$$f_X(x) \; = \; N(M_x, f) \, f(x) \; = \; N(M_x, f) \, f(0) \, E_\alpha[-\,\eta \, x^\alpha], \tag{441}$$

where $\eta > 0$.

Let us prove the following properties of the GF cumulative distribution function

$$F^{(M)}_X(x) \; = \; I^x_{(M_x)}[u] \, f_X(u) \, du \tag{442}$$

in the form

$$F^{(M)}_X(0+) \; = \; \lim_{x \to 0} F^{(M)}_X(x) \; = \; 0, \tag{443}$$

$$F^{(M)}_X(+\infty) \; = \; \lim_{x \to \infty} F^{(M)}_X(x) \; = \; 1. \tag{444}$$

The GF normalization condition for probability density can be considered in the form (444). For this purpose, the following well-known facts will be used.

- Using Equation (3.7.44) (Proposition 3.25) in [128], p. 50, the following equation is satisfied

$$(I^\alpha_{0+} E_\alpha[\eta \, u^\alpha])(x) \; = \; \frac{1}{\eta} \Big(E_\alpha[\eta \, x^\alpha] \; - \; 1 \Big), \tag{445}$$

  where $\eta \neq 0$ and $\alpha > 0$.
- Using Equation (3.4.15) (Proposition 3.6) in [128], p. 26, the following equation for the asymptotics is satisfied

$$E_\alpha[-z] \; = \; -\sum_{k=1}^{n} \frac{(-z)^k}{\Gamma(1 - \alpha k)} \; + \; O(|z|^{-n-1}), \quad |z| \to \infty \tag{446}$$

  for $\alpha \in (0,2)$ and n is an arbitrary positive integer number.

Using (446), the limit $z \to \infty$ has the form

$$\lim_{x \to \infty} E_{\alpha}[-x] = 0. \tag{447}$$

Using (445) and (446), the limit has the form

$$\lim_{x \to \infty} (I^{\alpha}_{0+}[u] E_{\alpha}[\eta u^{\alpha}])(x) = -\frac{1}{\eta}. \tag{448}$$

For the GFI kernel (434), the Riemann–Liouville fractional integration gives

$$F^{(M)}_X(x) = \int_0^x M_x(x-x')\, f_X(x')\, dx' =$$
$$\int_0^x M_x(x-x')\, f(0)\, N(M_x, f)\, E_{\alpha}[-\eta\, (x')^{\alpha}]\, dx' =$$
$$\lambda^{\alpha-1}\, f(0)\, N(M_x, f)\, (I^{\alpha}_{0+}\, E_{\alpha}[-\eta\, (x')^{\alpha}])(x) =$$
$$\lambda^{\alpha-1}\, f(0)\, \frac{-1}{\eta} \Big(E_{\alpha}[-\eta\, x^{\alpha}] - 1 \Big). \tag{449}$$

As a result, it is proven that

$$F^{(M)}_X(x) = \frac{\lambda^{\alpha-1}\, f(0)}{\eta}\, N(M_x, f) \Big(1 - E_{\alpha}[-\eta\, x^{\alpha}] \Big), \tag{450}$$

where $\eta = \eta_x \lambda^{\alpha-1}$.

Let us consider the limit

$$F^{(M)}_X(+\infty) = \lim_{x \to \infty} F^{(M)}_X(x) =$$
$$\lim_{x \to \infty} \frac{\lambda^{\alpha-1}\, f(0)}{\eta}\, N(M_x, f) \Big(1 - E_{\alpha}[-\eta\, x^{\alpha}] \Big) =$$
$$\frac{\lambda^{\alpha-1}\, f(0)}{\eta}\, N(M_x, f). \tag{451}$$

As a result, $F^{(M)}_X(+\infty)$ should be equal to the unit, the normalizing coefficient is defined as

$$N(M_x, f) = \frac{\eta}{\lambda^{\alpha-1}\, f(0)} = \frac{\eta_x\, \lambda^{\alpha-1}}{\lambda^{\alpha-1}\, f(0)} = \frac{\eta_x}{f(0)}, \tag{452}$$

where $f(0) \neq 0$.

As a result, the GF probability density (441), which is a solution of Equation (437), has the form

$$f_X(x) = N(M_x, f)\, f(x) = \frac{\eta}{\lambda^{\alpha-1}}\, E_{\alpha}[-\eta\, x^{\alpha}], \tag{453}$$

and the GF cumulative distribution function

$$F^{(M)}_X(x) = 1 - E_{\alpha}[-\eta\, x^{\alpha}], \tag{454}$$

where $\alpha \in (0, 1)$, and $\eta = \eta_x \lambda^{\alpha-1} > 0$.

For $\alpha = 1$, probability density (453) has the form

$$f_X(x) = \eta\, E_1[-\eta\, x] = \eta\, \exp(-\eta\, x), \tag{455}$$

$$F_X(x) = 1 - \exp(-\eta\, x), \tag{456}$$

that describes the well-known exponential distribution, where $\eta = \eta_x > 0$.

To prove (443), one can use definition (439) to obtain

$$\lim_{x \to 0+} E_{\alpha}[-\eta\, x^{\alpha}] \;=\; 1. \tag{457}$$

Then, using Equation (454), one can see that

$$F_X^{(M)}(0+) \;=\; 0. \tag{458}$$

As a result, one can give the following definition.

Definition 30 (The GF Mittag-Leffler distribution for power law nonlocality)**.** *Nonlocality is described by the kernel pair of the Luchko set*

$$M_x(x) \;=\; h_{\alpha}(\lambda x) \;=\; \frac{(\lambda x)^{\alpha-1}}{\Gamma(\alpha)}, \quad K_x(x) \;=\; \lambda\, h_{1-\alpha}(\lambda x) \;=\; \frac{\lambda (\lambda x)^{-\alpha}}{\Gamma(1-\alpha)}. \tag{459}$$

The GF probability density function

$$f_X(x) \;=\; \frac{\eta}{\lambda^{\alpha-1}}\, E_{\alpha}[-\eta\, x^{\alpha}]. \tag{460}$$

The GF cumulative distribution function

$$F_X^{(M)}(x) \;=\; 1 \;-\; E_{\alpha}[-\eta\, x^{\alpha}]. \tag{461}$$

The parameter values

$$\alpha \;\in\; (0,1), \quad \eta \;=\; \eta_x\, \lambda^{\alpha-1} \;>\; 0. \tag{462}$$

5.6. Non-Equivalence of Equations with GFD and Their Solutions in Different Spaces

Let us consider the space with new coordinates, such that

$$x_{new} \;=\; x^2. \tag{463}$$

In the standard (local) probability theory, equations for x_{new} and x are equivalent due to the chain rule,

$$\frac{\partial}{\partial x_{new}} \;=\; \frac{\partial}{\partial x^2} \;=\; \frac{1}{2\,x}\,\frac{\partial}{\partial x} \tag{464}$$

with an appropriate definition of the probability density function $f_{X,new}(x)$.

It is obvious that the equation for $f_{X,1}(x)$ that have the form

$$\frac{d f_{X,1}(x)}{dx} \;=\; -\,2\,\eta\, x\, f_{X,1}(x), \tag{465}$$

and the equation for $f_{X,2}(x^2)$ in the form

$$\frac{d f_{X,2}(x)}{dx^2} \;=\; -\,\eta\, f_{X,2}(x^2), \tag{466}$$

are equivalent due to the chain rule (464). In view of this equivalence, it is not necessary to use the x_{new}-space as something new, since it is enough to work in the standard x-space.

The solutions of Equations (465) and (466) have the form

$$f_{X,1}(x) \;=\; f_{X,2}(x^2) \;=\; N_x\, \exp\Big(-\,\eta\, x^2\Big), \tag{467}$$

where N_x is normalization coefficient.

It can be seen that the solutions of the equations in different spaces (x-space and x_{new}-space, where $x_{new} = x^2$) are the same.

A completely different situation in fractional calculus, where the standard chain rule is violated.

For fractional calculus, the chain rule similar to (464) is violated [109]. Therefore, in the nonlocal probability theory, it should be considered non-equivalent fractional differential equations, and, in general, the following inequalities are satisfied for solutions

$$f_{X,1}(x) \neq f_{X,2}(x^2). \tag{468}$$

Let us consider the x_{new}-space and the fractional differential equation

$$(D^{\alpha}_{C,0+} f_X)(x^2) = -\eta\, \lambda^{\alpha-1}\, f_{X,2}(x^2), \tag{469}$$

where $x_{new} = x^2$.

The solution of Equation (469) has the form

$$f_X(x^2) = \eta\, E_{\alpha}[-\eta\, \lambda^{\alpha-1}\, x^{2\alpha_x}], \tag{470}$$

Note that $E_1[z] = \exp(z)$ for $\alpha_x = 1$.

It should be emphasized that the solutions of fractional analogs of the Equations (465) and (466) do not coincide in the general case. These solutions coincide only if the orders of these equations are integers.

For example, $x = p$ and

$$\eta = \frac{1}{KT}, \quad \lambda^{\alpha_x - 1}\, \gamma_y = \frac{1}{2m} \tag{471}$$

expression (470) looks similar to the standard form of the probability density of the standard form of the Maxwell distribution

$$f_P(p^2) = \frac{1}{(\pi\, m\, k\, T)^{1/2}} \exp\Big(-\frac{p^2}{2\, m\, k\, T}\Big), \tag{472}$$

where $p \in (0, \infty)$ instead of the standard $p \in (-\infty, \infty)$, and, therefore, there is no 2 in the denominator of the normalized coefficients.

Remark 17. *Note that one can also consider a more general case of a space, namely, a fractional space, in which the coordinates of this space are*

$$x_{new} = q^{\alpha}, \quad y_{new} = p^{\alpha}, \tag{473}$$

where $x = q \geq 0$ and $y = p \geq 0$, such that $(x_{new}, y_{new}) \in \mathbb{R}^2_{0,+}$.

Such spaces and dynamic systems in them were proposed in 2004 [67–69] and then used to describe non-Hamiltonian dynamics in [10,131–133]. The use of such a space was also justified by the use of fractional integral operators whose kernels have a power law form. Fractional generalization of average values and reduced distribution functions are defined in these works. These papers consider dynamical systems that are described by fractional powers of variables. The fractional powers are considered as convenient ways to describe systems in the fractional dimensional space. Dynamical systems, which are Hamiltonian systems in the space (x_{new}, y_{new}), are non-Hamiltonian systems in the standard space (x, y). Generalizations of the Liouville and Bogoliubov hierarchy equations for such systems are proposed. The generalized Fokker–Planck equation, generalized transport equation, and generalized Chapman–Kolmogorov equation are derived from Liouville and Bogoliubov equations for systems in space (x^{α}, y^{α}).

Remark 18. *Note that a similar situation in the nonlocal (general fractional) vector calculus. The violation of the standard chain rule leads to the fact that operators defined in different coordinate systems (Cartesian, cylindrical, and spherical) cannot be related to each other by coordinate transformations. The GF integral and GF differential vector operators in the different orthogonal curvilinear* *coordinates (OCC) should be defined separately. The mutual consistency of these GF integral and GF differential operators are expressed in the fulfillment of vector analogs of the fundamental theorems of GFC, such as the GF gradient theorem, the GF Stock theorem, and the GF divergence (Gauss–Ostrogradsky) theorem. These GF vector operators are suggested in [95] for OCC through the Lame coefficients and these definitions can be used for all OCC. Equations for spherical, cylindrical, and Cartesian coordinates are particular forms of equations written with the Lame coefficients, but these expressions cannot be related to each other by coordinate transformations.*

Note that one can consider probability distributions in cylindrical, spherical, and other OCC by using equations that are proposed in [95]. For example, the proposed formulas allowed can be used to calculate the probability of spherical regions with spherical symmetry of the nonlocality and GF probability distribution.

6. General Fractional Distribution of Exponential Type

To simplify further constructions, let us first consider the exponential distribution in the framework of standard probability theory.

6.1. Standard Exponential Distribution

The probability density function $f_X(x)$ of exponential distribution $X \sim \mathrm{Exp}(\lambda)$ has the form

$$f_X(x) \;=\; \lambda\, \exp(-\lambda\, x) \tag{474}$$

for $x \geq 0$, and $f_X(x) \;=\; 0$ for $x < 0$, where $\lambda > 0$ is the rate parameter.

The cumulative distribution function $F_X(x)$ of exponential distribution is

$$F_X(x) \;=\; 1 \,-\, \exp(-\lambda\, x). \tag{475}$$

Function (474) can be considered as a solution of the linear differential equation of the first-order

$$\frac{d}{dx} f_X(x) \;=\; -\lambda\, f_X(x). \tag{476}$$

Taking into account the standard normalization condition in the form

$$\lim_{x \to \infty} \int_0^x f_X(u)\, du \;=\; 1, \tag{477}$$

one can obtain the normalization coefficient $N = \lambda$ and solution in form (474).

Therefore, the probability density of the standard exponential distribution can be defined as a solution of linear differential Equation (476), which satisfies normalization condition (477) and property $F_X(x) \to 0$ at $x \to 0+$.

Integrating Equation (476) in the form

$$\int_0^x f_X^{(1)}(u)\, du \;=\; -\lambda \int_0^x f_X(u)\, du, \tag{478}$$

and using the second fundamental theorem of the mathematical analysis (standard calculus) in the form

$$\int_0^x f_X^{(1)}(u)\, du \;=\; f_X(x) \,-\, f_X(0), \tag{479}$$

where $f^{(1)}(x) = df(x)/dx$, one can obtain

$$f_X(x) \,-\, f_X(0) \;=\; -\lambda \int_0^x f_X(u)\, du. \tag{480}$$

Then, using the definition of the standard cumulative distribution function $F_X(x)$ in the form

$$F_X(x) \;=\; \int_0^x f_X(u)\, du, \tag{481}$$

Equation (480) takes the form

$$f_X(x) - f_X(0) = -\lambda F_X(x). \tag{482}$$

Using Function (481), the normalization condition (477) can be written as

$$\lim_{x\to\infty} F_X(x) = 1. \tag{483}$$

Note that the condition

$$\lim_{x\to 0+} F_X(x) = 0 \tag{484}$$

is satisfied if Equation (482) holds for all $x > 0$.

6.2. Approach to Nonlocal Analog of Exponential Distribution

To construct a nonlocal analog of the exponential distribution by using the methods of GFC, one can consider the linear GF differential equation

$$D^{x,*}_{(K_x)}[u]\, f_X(u) = -\eta\, f_X(x), \tag{485}$$

where $D^{x,*}_{(M_x)}$ is the GFD for the Caputo type. The solution of Equation (485), which satisfies the normalization condition

$$\lim_{x\to\infty} I^x_{(M_x)}[u]\, f_X(u) = 1, \tag{486}$$

can be considered as a nonlocal analog of the standard exponential distribution.

Definition 31. *The GF probability density function, which is a solution of the linear GF differential Equation (485), which satisfies normalization condition (486) and the property*

$$\lim_{x\to 0+} I^x_{(M_x)}[u]\, f_X(u) = 0 \tag{487}$$

is called the GF probability density of GF distribution of exponential type.

The GF integration of Equation (485) in the form

$$I^x_{(M_x)}[s]\, D^{s,*}_{(K_x)}[u]\, f_X(u) = -\eta\, I^x_{(M_x)}[u]\, f_X(u), \tag{488}$$

and the second fundamental theorem of GFC written as

$$I^x_{(M_x)}[s]\, D^{s,*}_{(K_x)}[u]\, f_X(u) = f_X(x) - f_X(0), \tag{489}$$

gives the equation

$$f_X(x) - f_X(0) = -\eta\, I^x_{(M_x)}[u]\, f_X(u). \tag{490}$$

Using the definition of the GF cumulative distribution function

$$F^{(M)}_X(x) = I^x_{(M_x)}[u]\, f_X(u), \tag{491}$$

Equation (490) is written as

$$f_X(x) - f_X(0) = -\eta\, F^{(M)}_X(x). \tag{492}$$

For Equation (492), it is immediately clear that the condition

$$\lim_{x\to 0+} F^{(M)}_X(x) = 0 \tag{493}$$

is satisfied if $f_X(0+) = f_X(0)$.

As a result, the following statement was proved.

Theorem 19. *[General fractional distribution of the exponential type].*

Let a kernel pair $(M_x(x), K_x(x))$ belong to the Luchko set.

If a GF probability density function $f_X(x) \in C_{-1,(K)}(0,\infty)$ satisfies the GF differential equation

$$D^{x,*}_{(K_x)}[u]\, f_X(u) \;=\; -\,\eta\, f_X(x), \tag{494}$$

where $D^{x,*}_{(K_x)}$ is GFD of the Caputo type and $\eta \neq 0$, then the GF cumulative distribution function $F^{(M)}_X(x)$ has the form

$$F^{(M)}_X(x) \;=\; -\,\eta^{-1} \Big(f_X(x) \;-\; f_X(0) \Big). \tag{495}$$

The GF probability $P^{(M)}_X[a,b]$ is given by the equation

$$P^{(M)}_X[a,b] \;=\; -\,\eta^{-1} \Big(f_X(b) \;-\; f_X(a) \Big), \tag{496}$$

where $b \;>\; a \;\;\geq\; 0$.

The statement for GF differential equations with GF derivatives of Riemann–Liouville type is proved similar to the proof of Theorem 19.

Theorem 20 (GF probability for the GF distribution of the exponential type)**.** *Let a kernel pair $(M_x(x), K_x(x))$ belong to the Luchko set.*

If a GF probability density function $f_X(x) \in C_{-1,(K)}(0,\infty)$ satisfies the GF differential equation

$$D^{x}_{(K_x)}[u]\, f_X(u) \;=\; -\,\eta\, f_X(x), \tag{497}$$

where $D^{x}_{(K_x)}$ is the GFD of the RL type and $\lambda \neq 0$, then the GF cumulative distribution function $F^{(M)}_X(x)$ has the form

$$F^{(M)}_X(x) \;=\; -\,\eta^{-1}\, f_X(x). \tag{498}$$

The GF probability $P^{(M)}_X[a,b]$ is

$$P^{(M)}_X[a,b] \;=\; -\,\eta^{-1}\Big(f_X(b) \;-\; f_X(a) \Big), \tag{499}$$

where $b \;>\; a \;\;\geq\; 0$.

The solutions of the GF differential Equations (494) and (497) can be obtained by using the methods of Luchko's general operational calculus, which was suggested in works [89,100].

6.3. Solution of the Linear GF Differential Equations

The solutions of linear GF differential equations can be derived by using the Luchko operational calculus [89]. These solutions are expressed in terms of functions (see Equations (416) and (5.7) [89], pp. 360, 365), which will be called Luchko functions.

Definition 32. *[First and Second Luchko functions]*

Let a kernel pair $(M_x(x), K_x(x))$ belong to the Luchko set, and $M^{*,j}(x)$ be the convolution j-power

$$M^{*,j}(x) \;:=\; (M_{x,1} \,*\, \ldots \,*\, M_{x,j})(x), \tag{500}$$

where $M_{x,k}(x) = M_x(x)$ for all $k = 1, \ldots, j$, and $x \in (0,\infty)$.

Then, the function

$$\mathbb{F}(M_x,\eta,x) \;=\; \sum_{j=0}^{\infty} M^{*,j}(x)\,\eta^{j-1} \tag{501}$$

is called the first Luchko function.

The function

$$\mathbb{L}(M_x,\eta,x) \;:=\; I^x_{(K_x)}[u]\,\mathbb{F}(M_x,\eta,u) \;=\; \int_0^x du\,K_x(x-u)\,\mathbb{F}(M_x,\eta,u), \tag{502}$$

where $I^x_{(K_x)}[u]$ is the GF integral with the kernel $K_x(x)$, and is called the second Luchko function.

Note that Equation (502) contains the GFI with kernel $K_x(x)$ instead of the kernel $M_x(x)$.

The following theorem states that the first Luchko function (as a convolution series) is convergent.

Theorem 21 (Convergence of the first Luchko function)**.** *Let a kernel pair $(M_x(x), K_x(x))$ belong to the Luchko set, and the power series*

$$f(z,\eta) \;=\; \sum_{j=0}^{\infty} \eta^{j-1}\,z^j \tag{503}$$

has non-zero convergence radius $r = |\eta|^{-1}$, if $\eta \neq 0$.

Then, Function (501) as a convolution series is convergent for all $x \in (0,\infty)$, and the function $\mathbb{F}(z,\eta,x)$ belongs to the ring $\mathcal{R}_{-1} = (C_{-1}(0,\infty),*,+)$, where the multiplication $*$ is the Laplace convolution and $+$ the standard addition of functions.

Theorem 21 is proved in [89] (see Theorem 4.4 in [89], p. 359, and comments on page 360 of [89]).

The function $\mathbb{F}(z,\eta,x)$ belongs to the triple $\mathcal{R}_{-1} = (C_{-1}(0,\infty),*,+)$ that is a commutative ring without divisors of zero [87].

Examples of the first and second Luchko functions are proposed [89], pp. 361, 366–368.

Remark 19. *Note that the second Luchko function $\mathbb{L}(M_x,\eta,x)$ does not depend on the kernel $K_x(x)$ due to the Sonin condition*

$$(M_x * K_x)(x) \;=\; \{1\}, \tag{504}$$

where $\{1\}$ denotes the function that is identically equal to 1 for all $x \in [0,\infty)$.

Using condition (504), the convolution of GFD kernel $K_x(x)$ and the first Luchko function $\mathbb{F}(M_x,\eta,x)$ can be written as

$$\mathbb{L}(M_x,\eta,x) \;=\; (K_x * \mathbb{F})(x) \;=\; \sum_{j=1}^{\infty}\Big(K_x * M^{*,j}\Big)(x)\,\eta^{j-1} \;=$$

$$\sum_{j=1}^{\infty}\Big(K * M * M^{*,j-1}\Big)(x)\,\eta^{j-1} \;=\; \sum_{j=1}^{\infty}\Big(\{1\} * M^{*,j-1}\Big)(x)\,\eta^{j-1} \;=$$

$$\{1\} + \left(\{1\} * \sum_{j=2}^{\infty}\Big(M^{*,j-1}\,\eta^{j-1}\Big)\right)(x) \;=\; \{1\} + \left(\{1\} * \sum_{j=1}^{\infty}\Big(M^{*,j}\,\eta^{j}\Big)\right)(x), \tag{505}$$

where $\eta^0 = \{1\}$.

As a result, the second Luchko function $\mathbb{L}(M_x, \eta, x)$ can be represented in the form

$$\mathbb{L}(M_x, \eta, x) = \int_0^x K_x(x-u)\, \mathbb{F}(M, \eta, u)\, du = \{1\} + \int_0^x \Big(\sum_{j=1}^{\infty} M^{*,j}(u)\, \eta^j \Big)\, du. \tag{506}$$

One can see that the second Luchko function $\mathbb{L}(M_x, \eta, x)$ is independent of the kernel $K_x(x)$ since the Sonin condition $(K_x * M_x)(x) = \{1\}$ are satisfied for all $x \in (0, \infty)$.

The second Luchko Function (502) is used [89] in solutions of equations with GFD, which is defined by the kernel $K_x(x)$ associated with the kernel $M_x(x)$ of the GFI.

If a kernel pair $(M_x(x), K_x(x))$ belongs to the Luchko set, then $\mathbb{F}(M_x, \eta, x) \in C_{-1}(0, \infty)$ and $\mathbb{L}(M_x, \eta, x) \in C_{-1}(0, \infty)$. Therefore, these Luchko functions belong to the ring $\mathcal{R}_{-1}$. These statements are based on the fact that GFI $I^x_{(K_x)}[u]$ is the operator on $C_{-1}(0, \infty)$, [89].

Using the first Luchko function $\mathbb{F}(M_x, \eta, x)$ and second Luchko function $\mathbb{L}(M_x, \eta, x)$, one can propose solutions of the linear GF differential equations for the GF probability density functions.

To obtain the solution of the GF differential equation for the GF probability density, Theorem 5.1 of [89], p. 366, should be used.

Theorem 22 (Unique solution of the linear GF differential equation)**.** *Let $f(x) \in C^1_{-1}(0, \infty)$, and the pair $(M_x(x), K_x(x))$ belong to the Luchko set, and η be a bounded nonzero parameter.*

Then, the GF differential equation

$$D^{x,*}_{(K_x)}[u]\, f(u) = \eta\, f(x), \tag{507}$$

where $\eta \neq 0$, has the unique solution

$$f(x) = \mathbb{L}(M_x, \eta, x) f(0), \tag{508}$$

where the function $\mathbb{L}(M_x, \eta, x)$ is defined by Equation (502).

This theorem is proved in [89] (see Theorem 5.1 in [89], p. 366.)

In the next subsection, some examples of linear GF differential equations and solutions are proposed.

6.4. GF Distribution of the Exponential Type from Equations with GFD of Caputo Type

Let $f_X(x) \in C_{-1}(0, \infty)$, a pair $(M_x(x), K_x(x))$ belong to the Luchko set. Then, the GF differential equations

$$D^{x,*}_{(K_x)}[u]\, f_X(u) = -\eta\, f_X(x), \tag{509}$$

have the unique solutions

$$f_X(x) = \mathbb{L}(M_x, -\eta, x)\, f_X(0), \tag{510}$$

where $\mathbb{L}(M_x, -\eta, x)$ is the second Luchko function.

In order for Function (510) to be a probability density function, i.e., $f_X(x) \in C^{(M)}_{-1}(0, \infty)$, some conditions of the second Luchko Function (510) should be satisfied.

Let us prove that the condition

$$f_X(x) \in C_{-1,(K)}(0, \infty) \tag{511}$$

is satisfied for Function (510).

Theorem 23 (Property of second Luchko function)**.** *Let a kernel pair $(M_x(x), K_x(x))$ belong to the Luchko set.*

Then, the second Luchko Function (502) *and solution* (510) *of Equation* (509) *belong to the set* $C_{-1,(K)}(0,\infty)$*, i.e.,*

$$\mathbb{L}(M_x,\eta,x) \in C_{-1,(K)}(0,\infty). \tag{512}$$

Proof. Using Theorem 21 (see also Theorem 4.4 in [89], p. 359), the first Luchko function $\mathbb{F}(M_x,\eta,x)$ is an element of the ring $\mathcal{R}_{-1} = (C_{-1}(0,\infty),*,+)$. Therefore, the $\mathbb{F}(M_x,\eta,x)$ belongs to the set $C_{-1}(0,\infty)$.

Using Equation (502) of Definition 32, the second Luchko function $\mathbb{L}(M_x,\eta,x)$ is represented in the form

$$\mathbb{L}(M_x,\eta,x) := I^x_{(K_x)}[u]\,\mathbb{F}(M_x,\eta,u), \tag{513}$$

where $I^x_{(K_x)}[u]$ is the GF integral with the kernel $K_x(x)$.

Equation (513) means that the second Luchko function can be represented in the form $I^x_{(K_x)}[u]\,\varphi(u)$ for all $x>0$, where $\varphi(x)\in C_{-1}(0,\infty)$, where $\varphi(x)=\mathbb{F}(M_x,\eta,x)$.

As a result, the second Luchko function and solution (510) of Equation (509) belong to the set $C_{-1,(K)}(0,\infty)$.

This ends the proof. □

Corollary 4. *Let a kernel pair* $(M_x(x),K_x(x))$ *belong to the Luchko set.*

Then, the second Luchko Function (502) *satisfies the condition*

$$\lim_{x\to 0+} I^x_{M_x}[u]\,\mathbb{L}(M_x,-\eta,x) = 0. \tag{514}$$

Proof. Using the Luchko Theorem (Theorem 3) one can state that if $f_X(x) \in C_{-1,(K)}(0,\infty)$, then the condition

$$\lim_{x\to 0+} I^x_{M_x}[u]\,f_X(x) = 0 \tag{515}$$

is satisfied.

Therefore, using the Luchko theorem (Theorem 3) and the fact that $\mathbb{L}(M_x,\eta,x) \in C_{-1,(K)}(0,\infty)$, we obtain that the property (514) is satisfied.
□

In addition to conditions (512) and (514), the GF normalization condition for solution (510) must also be satisfied. The GF normalization condition can be represented by using the GF cumulative distribution function

$$F^{(M)}_X(x) = f_X(0)\,\Big(M_x * \mathbb{L}\Big)(x) = f_X(0)\,I^x_{(M_x)}[u]\,\mathbb{L}(M_p,\eta,u). \tag{516}$$

Using the associativity of the Laplace convolution and the equation

$$\mathbb{L}(M_x,-\eta,x) = \Big(K_x * \mathbb{F}\Big)(x), \tag{517}$$

one can obtain

$$F^{(M)}_X(x) = f_X(0)\,\Big(M_x * \mathbb{L}\Big)(x) = f_X(0)\,\Big(M_x * K_x * \mathbb{F}\Big)(x) =$$

$$f_X(0)\,\Big(\{1\} * \mathbb{F}\Big)(x) = f_X(0)\int_0^x \mathbb{F}(M_x,-\eta,u)\,du. \tag{518}$$

As a result, the GF cumulative distribution function is represented in the form

$$F^{(M)}_X(x) = f_X(0)\int_0^x \mathbb{F}(M_x,-\eta,u)\,du. \tag{519}$$

Equation (519) can be interpreted as a condition that the first Luchko function must be a standard probability density function for the positive semi-axis, if $\mathbb{F}(M_x, -\eta, u) \geq 0$ for all $x > 0$. For example, the standard normalization condition ($F_X^{(M)}(+\infty) = 1$) for the solution has the form

$$f_X(0) \int_0^{\infty} \mathbb{F}(M_x, -\eta, x)\, dx \,=\, 1, \tag{520}$$

where the first Luchko function $\mathbb{F}(M_x, -\eta, x)$ belongs to the set $C_{-1}(0, \infty)$, i.e., $\mathbb{F}(M_x, -\eta, x) \in C_{-1}(0, \infty)$.

As a result, the following proposition is proved.

Theorem 24 (GF probability density function as a solution of the GF differential equation)**.** *Let a pair $(M_x(x), K_x(x))$ belong to the Luchko set.*

Let a function $f_X(x)$ belong to the set $C^1_{-1}(0, \infty)$ and satisfy the GF differential equation

$$D^{x,*}_{(K_x)}[u]\, f_X(u) \,=\, -\,\eta\, f_X(x). \tag{521}$$

Then, the function $f_X(x)$ is the GF probability density function up to a numerical factor $f_X(0)$, if the first Luchko function satisfies the standard normalization condition (520) up to a numerical factor $f_X(0)$. This normalization condition means that the function

$$F_X^{(M)}(x) \,:=\, f_X(0) \int_0^x \mathbb{F}(M_x, -\eta, u)\, du \tag{522}$$

must satisfy the conditions

$$\lim_{x \to \infty} F_X^{(M)}(x) \,=\, 1. \tag{523}$$

As a particular case, if $\mathbb{F}(M_x, -\eta, x) \,\in\, C^{(\{1\})}_{-1}(0, \infty)$, then $f_X(x) \,\in\, C^{(M)}_{-1}(0, \infty)$.

Note that the following conditions

$$F_X^{(M)}(0+) \,\in\, C^1_{-1}(0, \infty), \quad F_X^{(M)}(0+) \,=\, 0 \tag{524}$$

are satisfied since the second Luchko function belongs to the set $C_{-1,(K)}(0, \infty)$,

It is obvious that not all operator kernels, whose pairs belong to the Luchko set, satisfy the condition that the first Luchko function belongs to the set of standard probability density functions. Such kernels form a subset of the Luchko set. In the next subsection, it will be shown that such a subset is not empty.

As a result, one can give the following definition.

Definition 33 (The GF distributions of the exponential type)**.** *Nonlocality is described by the kernel pair of the Luchko set*

$$M_x(x),\, K_x(x) \quad \textit{any kernel pair that belongs to the Luchko set} \tag{525}$$

The GF probability density function

$$f_X(x) \,=\, N^{-1}\, \mathbb{L}(M_x, -\eta, x)\, f_X(0), \tag{526}$$

where $\mathbb{L}(M_x, -\eta, x)$ is the second Luchko function.

The GF cumulative distribution function

$$F_X^{(M)}(x) \,=\, N^{-1} \int_0^x \mathbb{F}(M_x, -\eta, u)\, du, \tag{527}$$

$$N \,=\, \lim_{x \to \infty} \int_0^x \mathbb{F}(M_x, -\eta, u)\, du \,<\, \infty, \tag{528}$$

where $\mathbb{F}(M_x, -\eta, x)$ is the first Luchko function.

The parameter values are defined by the condition that the kernel pair belongs to the Luchko set and that $\mathbb{F}(M_x, -\eta, u)$ belongs to the set $C_{-1}^{(\{1\})}(0, \infty)$.

6.5. Example of GF Distribution of the Exponential Type

Let us consider the power law nonlocality that is described by the kernel pair (136) from the Luchko set in the form

$$M_x(x) \,=\, h_\alpha(\lambda\, x) \,=\, \lambda^{\alpha-1}\, h_\alpha(x) \,=\, \frac{(\lambda\, x)^{\alpha-1}}{\Gamma(\alpha)}, \quad K_x(x) \,=\, \lambda \frac{(\lambda\, x)^{-\alpha}}{\Gamma(1-\alpha)}, \tag{529}$$

where $x > 0$, and $0 < \alpha < 1$.

(1) Let us derive the first Luchko function. Using the equality

$$(h_\alpha(x) \,*\, h_\beta(x)) \,=\, h_{\alpha+\beta}(x) \tag{530}$$

for $\alpha > 0$, $\beta > 0$, and $h_\alpha(\lambda\, x)) = \lambda^{\alpha-1}\, h_\alpha(x)$ one can obtain

$$(h_{k\,\alpha}(\lambda\, x) \,*\, h_\alpha(\lambda\, x)) \,=\, \lambda^{(k+1)\,\alpha-2}\,(h_{k\,\alpha}(x) \,*\, h_\alpha(x)) \,=\, \lambda^{(k+1)\alpha-2}\, h_{(k+1)\alpha}(x) \,=\, \lambda^{-1}\, h_{(k+1)\alpha}(\lambda\, x). \tag{531}$$

Then, the convolution j-power

$$M^{*,j}(x) \,=\, \lambda^{j(\alpha-1)}\, h_{j\alpha}(x) \,=\, \lambda^{1-j} \frac{(\lambda\, x)^{j\alpha-1}}{\Gamma(j\alpha)}, \tag{532}$$

and the first Luchko function has the form

$$\mathbb{F}(M, \eta, x) \,=\, \sum_{j=1}^{\infty} \eta^{j-1}\, M^{*,j}(x) \,=\, \sum_{j=1}^{\infty} \eta^{j-1}\, \lambda^{1-j} \frac{(\lambda\, x)^{j\alpha-1}}{\Gamma(j\alpha)} \,=\, (\lambda\, x)^{\alpha-1} \sum_{j=0}^{\infty} \frac{(\eta\, \lambda^{-1})^j\, (\lambda\, x)^{j\alpha}}{\Gamma(j\alpha+\alpha)} \,=\, (\lambda\, x)^{\alpha-1}\, E_{\alpha,\alpha}[(\eta/\lambda)\, (\lambda\, x)^\alpha]. \tag{533}$$

Here, $E_{\alpha,\beta}(z)$ is the two-parameter Mittag-Leffler function [128] that is defined as

$$E_{\alpha,\beta}(z) \,:=\, \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+\beta)}, \tag{534}$$

where $\alpha > 0$, $\beta \in \mathbb{R}$.

(2) Let us derive the second Luchko function. Using Equation (502) and $\beta\,\Gamma(\beta) = \Gamma(\beta + 1)$, the second Luchko function $\mathbb{L}(M, \lambda, x)$ is written as

$$\mathbb{L}(M_x, \eta, x) \,=\, \{1\} \,+\, \int_0^x \Big(\sum_{j=1}^{\infty} M^{*,j}(u)\, \eta^j\Big) du \,=\, \{1\} \,+\, \int_0^x \Big(\sum_{j=1}^{\infty} \lambda^{j(\alpha-1)}\, \eta^j \frac{u^{j\alpha-1}}{\Gamma(j\alpha)}\Big) du \,=\, \{1\} \,+\, \sum_{j=1}^{\infty} \lambda^{j(\alpha-1)}\, \eta^j\, \frac{x^{j\alpha}}{\Gamma(j\alpha+1)} \,=\, \{1\} \,+\, \sum_{j=1}^{\infty} \lambda^{-j}\, \eta^j\, \frac{(\lambda\, x)^{j\alpha}}{\Gamma(j\alpha+1)} \,=\, \{1\} + \sum_{j=1}^{\infty} \frac{1}{\Gamma(j\alpha+1)} \Big(\frac{\eta}{\lambda}\, (\lambda\, x)^\alpha\Big)^j \,=\, E_\alpha[(\eta/\lambda)\, (\lambda\, x)^\alpha], \tag{535}$$

where $E_\alpha(z) = E_{\alpha,1}(z)$ is the Mittag-Leffler function [128].

As a result, the GF differential equation

$$D^{x,*}_{(K)}[u]\, f(u) \;=\; -\,\eta\, f(x) \tag{536}$$

has the solution

$$f(x) \;=\; E_{\alpha}[\,-\,(\eta/\lambda)\,(\lambda\, x)^{\alpha}]\, f(0). \tag{537}$$

(3) Let us consider the normalization condition for solution (537).
Using Equation (4.4.4) of [128], p. 61, in the form

$$\int_0^x u^{\beta-1}\, E_{\alpha,\beta}[\,-\,\gamma\, u^{\alpha}]\, du \;=\; x^{\beta}\, E_{\alpha,\beta+1}[\,-\,\gamma\, x^{\alpha}], \tag{538}$$

the GF cumulative distribution Function (519) is equal to

$$F^{(M)}_X(x) \;=\; f_X(0)\int_0^x \mathbb{F}(M_x,\,-\,\eta,u)\, du \;=\; f_X(0)\int_0^x (\lambda\, u)^{\alpha-1} E_{\alpha,\alpha}[\,-\,(\eta/\lambda)\,(\lambda\, u)^{\alpha}]\, du \;=\;$$
$$\frac{f_X(0)}{\lambda}\,(\lambda\, x)^{\alpha} E_{\alpha,\alpha+1}[\,-\,(\eta/\lambda)\,(\lambda\, x)^{\alpha}]. \tag{539}$$

(I) Using the definition of the Mittag-Leffler function

$$F^{(M)}_X(x) \;=\; \frac{f_X(0)}{\lambda}\, \sum_{j=0}^{\infty}\Big(-\,\frac{\eta}{\lambda}\Big)^j \frac{(\lambda\, x)^{(j+1)\alpha}}{\Gamma((j+1)\alpha+1)} \;=$$
$$\frac{f_X(0)}{\lambda}\, \frac{(\lambda\, x)^{(\alpha}}{\Gamma(\alpha+1)} \;+\; \frac{f_X(0)}{\lambda}\, \sum_{j=1}^{\infty}\Big(-\,\frac{\eta}{\lambda}\Big)^j \frac{(\lambda\, x)^{(j+1)\alpha}}{\Gamma((j+1)\alpha+1)}, \tag{540}$$

one can see that $F^{(M)}_X(0+) = 0$ for $\alpha > 0$.

(II) Using Equation (1.8.28) of [4] in the form

$$E_{\alpha,\beta}[z] \;=\; -\,\frac{1}{\Gamma(\beta-\alpha)}\,\frac{1}{z} \;+\; O(z^{-1}), \tag{541}$$

which holds for $\alpha \in (0,2)$ at $x \;\to\; +\infty$, one can obtain

$$F^{(M)}_X(x) \;=\; \frac{f_X(0)}{\lambda}\,\frac{\lambda}{\eta}\,\frac{1}{\Gamma(\beta-\alpha)} \;+\; O((\lambda\, x)^{-\alpha}), \tag{542}$$

where $\beta = \alpha + 1$, $\Gamma(1) = 1$, and $\alpha \in (0,1)$ should be used.

As a result, one can see

$$\lim_{x\to+\infty} F^{(M)}_X(x) \;=\; \frac{f_X(0)}{\eta}. \tag{543}$$

Then, the GF normalization condition gives

$$f_X(0) \;=\; \eta. \tag{544}$$

As a result, the GF cumulative distribution function has the form

$$F^{(M)}_X(x) \;=\; \frac{\eta}{\lambda}\,(\lambda\, x)^{\alpha} E_{\alpha,\alpha+1}[\,-\,(\eta/\lambda)\,(\lambda\, x)^{\alpha}]. \tag{545}$$

The GF probability density

$$f_X(x) \;=\; \eta\, \mathbb{L}(M_x,-\eta,x) \;=\; \eta\, E_{\alpha}[\,-\,(\eta/\lambda)\,(\lambda\, x)^{\alpha}] \tag{546}$$

is the unique solution of the GF differential equation

$$D^{x,*}_{(K_x)}[x']\, f_X(x') \;=\; -\,\eta\, f_X(x). \tag{547}$$

Remark 20. *Using the equality*

$$z\, E_{\alpha,\alpha+\beta}[z] \;=\; E_{\alpha,\beta}[z] \;-\; \frac{1}{\Gamma(\beta)} \tag{548}$$

for $z = -\,(\eta/\lambda)\,(\lambda\, x)^{\alpha}$, *and* $E_{\alpha,1}[z] = E_{\alpha}[z]$, *Equation* (545) *can be written as*

$$F^{(M)}_X(x) \;=\; 1 \;-\; E_{\alpha}[\,-\,(\eta/\lambda)\,(\lambda\, x)^{\alpha}]. \tag{549}$$

where $E_{\alpha}[z]$ *is the Mittag-Leffler function [129].*

7. Truncated GF Distributions and Average Values

In the standard probability theory of distributions on positive semi-axis, truncated distributions, and truncated moment functions are considered [113], pp. 279–284, [146]. Truncated distributions are derived from probability distributions by restrictions of theirs domains.

In this section, truncated GF probability density functions, truncated GF cumulative distribution functions, and truncated GF average values of random variables are suggested.

7.1. Truncated GF Probability Density Function

Let us consider a GF distribution of a random variable X on the positive semi-axis $(0,\infty)$ to define truncated GF distributions. Let $f_X(x)$ be a GF probability density function, and $F^{(M)}_X(x)$ be a cumulative distribution function on the positive semi-axis. One can consider an interval $[a,b] \subset (0,\infty)$. In order to obtain the probability density of a random variable X on the interval $[a,b] \subset (0,\infty)$, we should use a new normalization condition. In this case, one can say that it is a GF distribution of a random variable X on domain $[a,b]$.

A truncated GF probability density function can be defined in the following form.

Definition 34 (Truncated GF probability density function)**.** *Let a pair* $(M_x(x), K_x(x))$ *belong to the Luchko set.*

Let $f_X(x) \in C^{(M)}_{-1}(0,\infty)$ *be a GF probability density, and* $F^{(M)}_X(x) \in C^{(M)}_{CDF}(0,\infty)$ *be the GF cumulative distribution function.*

Then, the function

$$f_{[a,b]}(x) \;=\; \frac{f_X(x)}{F^{(M)}_X(b) \;-\; F^{(M)}_X(a)}, \tag{550}$$

where $F^{(M)}_X(b) \;>\; F^{(M)}_X(a)$ *and* $b \;>\; a \;\geq\; 0$*, is called the truncated GF probability density function.*

Remark 21. *In the standard probability theory, the truncated probability density function is defined as*

$$f^{St}_{[a,b]}(x) \;=\; \begin{cases} f_X(x) & x \in (a,b], \\ 0 & x \notin (a,b]. \end{cases} \tag{551}$$

Note that Function (551) *does not belong to the set* $C_{-1}(0,\infty)$ *in contrast to Function* (550)*, which belongs to the set* $C_{-1}(0,\infty)$*.*

If $f_X(x) \in C^{(M)}_{-1}(0,\infty)$, then it is obvious that the following properties are satisfied for the truncated GF probability density Function (550) in the form

$$f_{[a,b]}(x) \;\in\; C_{-1,(K)}(0,\infty), \tag{552}$$

$$f_{[a,b]}(x) \geq 0. \tag{553}$$

The normalization condition (550) for the truncated GF probability density functions is changed. To give a correct normalization condition for (550), the GFI for the interval $[a,b] \subset \mathbb{R}_+$ should be used. The GFI for the interval $[a,b] \subset \mathbb{R}_+$ is defined [95] by the equation

$$I^{(M_x)}_{[a,b]}[x]\, f(x) \;=\; I^{b}_{(M_x)}[x]\, f(x) \;-\; I^{a}_{(M_x)}[x]\, f(x), \tag{554}$$

if $b > a > 0$. For $a = 0$, the GFI is

$$I^{(M_x)}_{[0,b]}[x]\, f(x) \;=\; I^{b}_{(M_x)}[x]\, f(x). \tag{555}$$

Let us prove the following theorem about the GF normalization condition.

Theorem 25. (The GF normalization condition for the truncated GF probability density) *Let a pair $(M_x(x), K_x(x))$ belong to the Luchko set.*

Let $f_X(x) \in C^{(M)}_{-1}(0,\infty)$ be a GF probability density, and $F^{(M)}_X(x) \in C^{(M)}_{CDF}(0,\infty)$ be the GF cumulative distribution function.

Then, the truncated GF probability density Function (550) satisfies the normalization condition in the form

$$I^{(M_x)}_{[a,b]}[u]\, f_{[a,b]} \;=\; 1, \tag{556}$$

where $I^{(M_x)}_{[a,b]}[u]$ is the GFI defined by Equation (554).

Proof. The action of the GFI (554) on the truncated GF probability density Function (550) gives

$$\begin{aligned} I^{(M_x)}_{[a,b]}[u]\, f_{[a,b]}(u) \;&=\; I^{b}_{(M_x)}[u]\, f_{[a,b]}(u) \;-\; I^{a}_{(M_x)}[u]\, f_{[a,b]}(u) \;= \\ &\frac{1}{F^{(M)}_X(b) \;-\; F^{(M)}_X(a)} \Big(I^{b}_{(M_x)}[u]\, f_X(u) \;-\; I^{a}_{(M_x)}[u]\, f_X(u) \Big) \;= \\ &\frac{1}{F^{(M)}_X(b) \;-\; F^{(M)}_X(a)} \Big(F^{(M)}_X(b) \;-\; F^{(M)}_X(a) \Big) \;=\; 1. \end{aligned} \tag{557}$$

Therefore, the GF normalization condition for the function $f_{[a,b]}(x)$ has the form

$$I^{(M_x)}_{[a,b]}[u]\, f_{[a,b]} \;=\; 1. \tag{558}$$

This is the end of the proof. □

7.2. Truncated GF Cumulative Distribution Function

Let us define a GF cumulative distribution function for the truncated GF distributions on the positive semi-axis.

Definition 35 (Truncated GF cumulative distribution function)**.** *Let a pair $(M_x(x), K_x(x))$ belong to the Luchko set.*

Let $f_X(x) \in C^{(M)}_{-1}(0,\infty)$ be a GF probability density, and

$$F^{(M)}_X(x) \;=\; I^{x}_{(M_x)}[u]\, f_X(u) \;\in C^{(M)}_{CDF}(0,\infty) \tag{559}$$

is the GF cumulative distribution function.

Let a truncated GF distribution on the interval $[a,b] \subset \mathbb{R}_+$ be described by the truncated GF probability density function

$$f_{[a,b]}(x) \;=\; \frac{f_X(x)}{F^{(M)}_X(b) \;-\; F^{(M)}_X(a)}, \tag{560}$$

where $b \;>\; a \;\geq\; 0$ and $F^{(M)}_X(b) \;>\; F^{(M)}_X(a)$.

Then, the function

$$F^{(M)}_{[a,b]}(x) \;=\; \frac{F^{(M)}_X(x) \;-\; F^{(M)}_X(a)}{F^{(M)}_X(b) \;-\; F^{(M)}_X(a)}. \tag{561}$$

is called the truncated GF cumulative distribution function.

It is obvious that the following properties are satisfied

$$F^{(M)}_{[a,b]}(a) \;=\; 0, \tag{562}$$

$$F^{(M)}_{[a,b]}(b) \;=\; 1, \tag{563}$$

$$F^{(M)}_{[a,b]}(b) \;\in\; C^1_{-1}(0,\infty), \tag{564}$$

$$D^{x,*}_{(K_x)}[u]\, F^{(M)}_{[a,b]}(b) \;\geq\; 0. \tag{565}$$

These properties directly follow from Definition (35) and properties of $f_X(x) \in C^{(M)}_{-1}(0,\infty)$. Note that inequality (565) contains GFD of the Caputo type only.

Let us consider a connection between truncated GF cumulative distribution Function (561) and truncated GF probability density (560).

Theorem 26 (Truncated GF cumulative distribution via truncated GF probability density)**.** *Let $f_X(x) \in C^{(M)}_{-1}(0,\infty)$, a pair $(M_x(x), K_x(x))$, belong to the Luchko set.*

Then, the truncated GF cumulative distribution Function (561) on the interval $[a,b] \subset \mathbb{R}_+$ is connected with the truncated GF probability density Function (550) by the equation

$$D^{x,*}_{(K_x)}[u]\, F^{(M)}_{[a,b]}(u) \;=\; f_{[a,b]}(x), \tag{566}$$

where $D^{x,*}_{(K_x)}[u]$ is the GFD of the Caputo type.

Proof. Using the GFD of the Caputo type and the first fundamental theorem of the GFC, one can obtain

$$\begin{aligned} D^{x,*}_{(K_x)}[u]\, F^{(M)}_{[a,b]}(u) \;&=\; \frac{1}{F^{(M)}_X(b) \;-\; F^{(M)}_X(a)} \Big(D^{x,*}_{(K_x)}[u]\, F^{(M)}_X(u) \;-\; D^{x,*}_{(K_x)}[u]\, F^{(M)}_X(a) \Big) \;= \\ &\frac{1}{F^{(M)}_X(b) \;-\; F^{(M)}_X(a)}\, f_X(x) \;=\; f_{[a,b]}(x), \end{aligned} \tag{567}$$

where the property of the equality to zero of the Caputo fractional derivative of a constant value is used in the form

$$D^{x,*}_{(K_x)}[u]\, F^{(M)}_X(a) \;=\; F^{(M)}_X(a)\, D^{x,*}_{(K_x)}[u]\, 1 \;=\; 0. \tag{568}$$

This is the end of the proof. □

Remark 22. *Note that the GFD of the Riemann–Liouville type of a constant function is not equal to zero*

$$D^x_{(K_x)}[u]\, F^{(M)}_X(a) \;=\; F^{(M)}_X(a)\, D^x_{(K_x)}[u]\, 1 \;=\; F^{(M)}_X(a)\, K_x(x). \tag{569}$$

Property (569) gives the inequality

$$D^x_{(K_x)}[u]\, F^{(M)}_{[a,b]}(u) \;=\; \frac{f_X(x) \,-\, F^{(M)}_X(a)\, K_x(x)}{F^{(M)}_X(b) \,-\, F^{(M)}_X(a)} \;\neq\; f_{[a,b]}(x). \tag{570}$$

Therefore, the GFD of the Riemann–Liouville type cannot be used for the truncated GF distributions, since $F^{(M)}_X(a) \neq 0$ is in this case.

7.3. Truncated GF Average Values

Let us give a definition of the truncated GF average values.

Definition 36 (Truncated GF average value of function $A(X)$)**.** *Let a pair $(M_x(x), K_x(x))$ belong to the Luchko set.*

Let $f_X(x) \in C^{(M)}_{-1}(0,\infty)$ be a GF probability density, $A(X)$ be a function of a random variable X such that $A(x)\, f_X(x) \in C_{-1}(0,\infty)$, and

$$F^{(M)}_X(x) \;=\; I^x_{(M_x)}[u]\, f_X(u) \;\in\; C^{(M)}_{CDF}(0,\infty) \tag{571}$$

is the GF cumulative distribution function.

Then, the function

$$\mathsf{E}^x_{T1,(M_x)}[A(X)] \;:=\; I^x_{(M_x)}[u] \Big(A(u)\, f_X(u) \Big) \;=\; \int_0^x M_x(x-u)\, A(u)\, f_X(u)\, du \tag{572}$$

is called the truncated GF average value of the first type of function $A(X)$ on the interval $[0,x] \subset \mathbb{R}_+$.

Let a truncated GF distribution on the interval $[a,b] \subset \mathbb{R}_+$ be described by the truncated GF probability density function

$$f_{[a,b]}(x) \;=\; \frac{f_X(x)}{F^{(M)}_X(b) \,-\, F^{(M)}_X(a)}, \tag{573}$$

where $b \,>\, a \;\geq\; 0$ and $F^{(M)}_X(b) \,>\, F^{(M)}_X(a)$.

Then, the truncated GF average value of the first type of a function $A(X)$ of a random variable X on the interval $[a,b] \subset \mathbb{R}_+$ is given by the equation

$$\mathsf{E}^{(M_x)}_{[a,b]}[A(X)] \;:=\; I^{(M_x)}_{[a,b]}[u] \Big(A(u)\, f_{[a,b]}(u) \Big), \tag{574}$$

where $[0,x] \subset \mathbb{R}_+$.

Then, the value

$$\mathsf{E}^{(M_1),(M_2)}_{[a,b]}[A(X)] \;:=\; \lim_{x\to\infty} I^{(M_1)}_{[a,b]}[u] \left(A(u) \left(D^{u,*}_{(M_2)}[w]\, f_[a,b](w) \right) \right) \tag{575}$$

is called the truncated GF average value with two kernels for the function $A(X)$ of the random variable X, where $D^{u,*}_{(M_2)}$ is the GFD of the Caputo type with the kernel $M_2(x)$.

Note that the truncated GF average value (574) is expressed through the truncated GF average value (572) by the equation

$$\mathsf{E}^{(M_x)}_{[a,b]}[A(X)] \, = \, \frac{1}{F^{(M)}_X(b) \, - \, F^{(M)}_X(a)} \Big(\mathsf{E}^{b}_{(M_x)}[A(X)] \, - \, \mathsf{E}^{a}_{(M_x)}[A(X)] \Big). \tag{576}$$

Remark 23. *For a truncated GF probability density $f_{[a,b]}(x)$, truncated GF average value (574) of the first type of the function $A(x) = 1$ for all $x \in \mathbb{R}_+$, is equal to one*

$$\mathsf{E}^{(M_x)}_{[a,b]}[\,1\,] \, = \, 1. \tag{577}$$

Equation (577) can be interpreted as a GF normalization condition of the truncated GF probability density $f_{[a,b]}(x)$.

A similar interpretation exists for the truncated GF average value with two kernels (575).

It should also be noted the truncated GF average value (572) of the function $A(X) = 1$ (for all $x > 0$) is equal to the GF cumulative distribution function $\mathsf{E}^{x}_{(M_x)}[1] \, = \, F^{(M)}_X(x)$.

7.4. First Example of Calculation of Truncated GF Average Value

Let us consider the uniform GF distribution with the Gamma distribution of the nonlocality considered in Section 4.2. Then, the kernel pair of the Luchko set is described by expressions (301) with the GFI kernel

$$M_x(x) \, = \, h_{\alpha,\lambda}(\lambda\, x) \, = \, \frac{(\lambda\, x)^{\alpha-1}}{\Gamma(\alpha)} e^{-\lambda\, x}, \tag{578}$$

where $\lambda \, > \, 0$ and $\gamma(\beta, x)$ is the incomplete gamma function (see Section 9 in [127], pp. 134–142).

The GF probability density of the uniform GF distribution for nonlocality in form (578), is described by the function

$$f_X(x) \, = \, \lambda\, \{1\}. \tag{579}$$

The function $A(X) = X^n$ of a random variable X is considered in the form

$$A(x) \, = \, x^n, \tag{580}$$

where $n \in \mathbb{N}$.

The GF cumulative distribution function $F^{(M)}_X(x)$ for the GF probability density (579) has the form

$$F^{(M)}_X(x) \, = \, I^{x}_{(M_x)}[u] \, f_X(u) \, = \, \frac{1}{\Gamma(\alpha)} \, \gamma(\alpha, \lambda\, x). \tag{581}$$

Then, the truncated GF average value of function $A(X) \, = \, X^n$ on the interval $[0, x] \subset \mathbb{R}_+$ is

$$\mathsf{E}^{x}_{(M_x)}[X^n] \, = \, I^{x}_{(M_x)}[u] \, \Big(A(u) \, f_X(u) \Big) \, = \, \frac{\lambda^{\alpha-1}}{\Gamma(\alpha)} \int_0^x (x-u)^{\alpha-1} \, e^{-\lambda\,(x-u)} \, u^n \, \lambda \, du \, =$$

$$\frac{\lambda^{\alpha}}{\Gamma(\alpha)} \int_0^x (x \, - \, u)^n \, u^{\alpha-1} \, e^{-\lambda\, u} \, du. \tag{582}$$

Using the equality

$$(x \, - \, u)^n \, = \, \sum_{k=0}^{n} \binom{n}{k} x^{n-k} u^k, \tag{583}$$

Equation (582) can be written as

$$\mathsf{E}^{x}_{(M_x)}[X^n] \;=\; \frac{\lambda^{\alpha}}{\Gamma(\alpha)} \sum_{k=0}^{n} \binom{n}{k} x^{n-k} \int_0^x u^{\alpha+k-1}\, e^{-\lambda\, u}\, du. \tag{584}$$

Using the equation

$$\int_0^x u^{\alpha+k-1}\, e^{-\lambda\, u}\, du \;=\; \lambda^{-\alpha-k}\, \gamma(\alpha+l, \lambda\, x), \tag{585}$$

where $\alpha\,+\,k\;>\;0$, Equation (584) takes the form

$$\mathsf{E}^{x}_{(M_x)}[X^n] \;=\; \frac{\lambda^{\alpha}}{\Gamma(\alpha)} \sum_{k=0}^{n} \binom{n}{k} x^{n-k}\lambda^{-\alpha-k}\, \gamma(\alpha+k, \lambda\, x) \;=\;$$

$$\sum_{k=0}^{n} \binom{n}{k} \frac{\lambda^{-k}}{\Gamma(\alpha)}\, x^{n-k}\, \gamma(\alpha+k, \lambda\, x). \tag{586}$$

Using Equation (576), the truncated GF average value of function $A(X) = X^n$ on the interval $[a,b] \subset \mathbb{R}_+$ has the form

$$\mathsf{E}^{(M_x)}_{[a,b]}[X^n] \;=\; \frac{1}{F^{(M)}_X(b)\;-\;F^{(M)}_X(a)} \Big(\mathsf{E}^{b}_{(M_x)}[b^n]\;-\;\mathsf{E}^{a}_{(M_x)}[a^n]\Big). \tag{587}$$

Substitution of Equations (581) and (586) into expression (587) gives

$$\mathsf{E}^{(M_x)}_{[a,b]}[X^n] \;=\; \frac{1}{\gamma(\alpha,\lambda\, b)\;-\;\gamma(\alpha,\lambda\, a)} \sum_{k=0}^{n} \binom{n}{k} \lambda^{-k} \Big(b^{n-k}\, \gamma(\alpha+k, \lambda\, b)\;-\;a^{n-k}\, \gamma(\alpha+k, \lambda\, a), \tag{588}$$

where $b\;>\;a\;\;\geq\;0$.

7.5. Second Example of the Calculation of the Truncated GF Average Value

In this subsection, it is considered an example of the calculation of the truncated GF average values of the first type for the GF distribution that is described as the second example in Section 5.1. Note that the first example of Section 5.1 cannot be used for this purpose since Equation (305) (Equation (4.4.5) of [128], p. 61) can be used for the case $u^{\beta+\gamma-1}\, E_{\alpha,\beta}[-\,\eta\, u^{\alpha}]$ only if $\gamma = 0$.

Consider the kernel pair (138), in which the GFI kernel has the form

$$M_x(x) \;=\; (\lambda\, x)^{\beta-1}\, E_{\alpha,\beta}[-\,(\lambda\, x)^{\alpha}] \tag{589}$$

where $0 < \alpha \leq \beta < 1$, the GF probability density

$$f_X(x) \;=\; \lambda\, \frac{(\lambda\, x)^{\mu-1}}{\Gamma(\mu)}, \tag{590}$$

and the function $A(X)$ of the random variable

$$A(X) \;=\; X^{\gamma}, \tag{591}$$

where $\gamma \in \mathbb{R}$.

In order for the function $f_X(x)$ and the product $A(x)\, f_X(x)$ to belong to a set $C_{-1}(0,\infty)$, the following condition must be satisfied

$$\mu\;>\;0, \quad \mu+\gamma\;>\;0. \tag{592}$$

The truncated GF average value of the function $A(X) = X^{\gamma}$ of the random variable X has the form

$$\mathsf{E}^{x}_{(M_x)}[X^{\gamma}] \,=\, I^{x}_{(M_x)}[u] \left(A(u)\, f_X(u) \right) \,=$$

$$\int_0^x \frac{\lambda^{\beta+\mu-1}}{\Gamma(\mu)} \, u^{\mu+\gamma-1} \, (x \,-\, u)^{\beta-1} \, E_{\alpha,\beta}[-\, (\lambda\, (x \,-\, u))^{\alpha}] \, du. \tag{593}$$

Then, using Equation (4.4.5) of [128], p. 61, in the form

$$\frac{1}{\Gamma(\mu)} \int_0^x u^{\mu+\gamma-1} \, (x-u)^{\beta-1} \, E_{\alpha,\beta}[-\, \eta\, (x-u)^{\alpha}] \, du \,=$$

$$\frac{\Gamma(\mu+\gamma)}{\Gamma(\mu)} x^{\beta+\gamma+\mu-1} \, E_{\alpha,\beta+\mu+\gamma}[-\, \eta\, x^{\alpha}], \tag{594}$$

where $\mu+\gamma > 0$, $\beta > 0$, Equation (593) takes the form

$$\mathsf{E}^{x}_{(M_x)}[X^{\gamma}] \,=\, \lambda^{\beta+\mu-1} \, \frac{\Gamma(\mu+\gamma)}{\Gamma(\mu)} x^{\beta+\mu-1+\gamma} \, E_{\alpha,\beta+\mu+\gamma}[-\, (\lambda\, x)^{\alpha}], \tag{595}$$

where it is assumed that the parameters satisfy the conditions

$$\alpha \,>\, 0, \quad \beta \,>\, 0, \quad \gamma \,>\, -\,\mu, \quad \mu \,>\, 0. \tag{596}$$

The GF probability density (590) satisfies the normalization condition if equality (315) is satisfied in the form

$$\beta \,+\, \mu \,-\, 1 \,=\, \alpha \,>\, 0. \tag{597}$$

As a result, the conditions on the parameters have the form (321) with $\gamma > -\mu$

$$0 \,<\, \alpha \,\leq\, \beta \,<\, 1, \quad 0 \,<\, \alpha \,\leq\, \mu \,\leq\, 1, \tag{598}$$

such that equality (597) holds.

Using Equation (597), the truncated GF average value (595) of the function $A(X) = X^{\gamma}$ of the random variable X has the form

$$\mathsf{E}^{x}_{(M_x)}[X^{\gamma}] \,=\, \lambda^{\alpha} \, \frac{\Gamma(\mu+\gamma)}{\Gamma(\mu)} \, x^{\alpha+\gamma} \, E_{\alpha,\alpha+\gamma+1}[-\, (\lambda\, x)^{\alpha}]. \tag{599}$$

The GF cumulative distribution Function (306) has the form

$$F^{(M)}_X(x) \,=\, (\lambda\, x)^{\alpha} \, E_{\alpha,\alpha+1}[-\, (\lambda\, x)^{\alpha}]. \tag{600}$$

Using Equation (576), the truncated GF average value of function $A(X) = X^{\gamma}$ on the interval $[a,b] \subset \mathbb{R}_+$ has the form

$$\mathsf{E}^{(M_x)}_{[a,b]}[X^{\gamma}] \,=\, \frac{1}{F^{(M)}_X(b) \,-\, F^{(M)}_X(a)} \left(\mathsf{E}^{b}_{(M_x)}[b^{\gamma}] \,-\, \mathsf{E}^{a}_{(M_x)}[a^{\gamma}] \right). \tag{601}$$

Substitution (599) and (600) into expression (601) gives

$$\mathsf{E}^{(M_x)}_{[a,b]}[X^{\gamma}] \,=\, \frac{1}{b^{\alpha} \, E_{\alpha,\alpha+1}[-\, (\lambda\, b)^{\alpha}] \,-\, a^{\alpha} \, E_{\alpha,\alpha+1}[-\, (\lambda\, a)^{\alpha}]} \cdot$$

$$\frac{\Gamma(\mu+\gamma)}{\Gamma(\mu)} \left(b^{\alpha+\gamma} \, E_{\alpha,\alpha+\gamma+1}[-\, (\lambda\, b)^{\alpha}] \,-\, a^{\alpha+\gamma} \, E_{\alpha,\alpha+\gamma+1}[-\, (\lambda\, a)^{\alpha}] \right), \tag{602}$$

where $b > a \geq 0$.

8. Conclusions

In this paper, a nonlocal generalization of the standard probability theory of the continuous distribution of the semi-axis is formulated by using general fractional calculus (GFC) in the Luchko form as a mathematical tool.

Let us briefly list the most important results proposed in this paper.

(1) Basic concepts of the nonlocal probability theory, nonlocality, described by the pairs of Sonin kernels that belong to the Luchko set, are suggested. Nonlocal (GF) generalizations of the probability density function, the cumulative distribution function, probability, average values, and characteristic functions are proposed. The properties of these functions are described and proved.

(2) Nonlocal (general fractional) distributions are suggested and their properties are proved. Among these distributions, the following distributions are described:
   (a) Nonlocal analogs of uniform and degenerate distributions;
   (b) Distributions with special functions, namely with the Mittag-Leffler function, the power law function, the Prabhakar function, the Kilbas–Saigo function;
   (c) Convolutional distributions that can be represented as a convolution of the operator kernels and standard probability density;
   (d) Distributions of the exponential types are suggested as generalizations of the standard exponential distributions by using solutions of linear general fractional differential equations.

(3) The truncated GF probability density function, truncated GF cumulative distribution function, and truncated GF average values are considered. Examples of the calculation of the truncated GF average value are given.

It should be emphasized that the proposed nonlocal probability theory cannot be reduced to a standard theory that uses classical probability densities and distribution functions. This impossibility is analogous to the fact that fractional calculus and the general fractional calculus cannot be reduced to standard calculus, which uses standard integrals and integral derivatives.

Obviously, all aspects and questions of the nonlocal probability theory could not be considered in one article. Generalizations of all the concepts and methods of the standard theory of probabilities for a nonlocal case could not be proposed here. Moreover, it is obvious that only one type of nonlocality is considered in this work. Nonlocality is actually described by the Laplace convolution only. Many important and interesting questions and problems have not been resolved in this work and require further study and research in the future.

As a further development of the nonlocal probability theory, the following directions of its expansion seem important.

First, it is important to expand the types of nonlocalities for which a mathematically correct NPT can be constructed. For example, in addition to the nonlocalities described by the Laplace convolution, it is important to study the nonlocalities described by the Mellin convolution. However, for this type of nonlocality, unfortunately, a general fractional calculus similar to Luchko's GF calculus has not yet been created.

Secondly, it is important to describe discrete theories of non-local probabilities, which make it possible to correctly describe nonlocal discrete distributions. Unfortunately, a discrete analog of the general fractional calculus in the Luchko form has not yet been created.

Thirdly, it is interesting to further develop the approach proposed in this paper, including a description of the properties of the proposed probability distributions. For example, it is important to write (mathematically accurately) the descriptions of the nonlocal probabilities for piecewise continuous distributions and probability distributions on the entire real axis, and not just on the positive semi-axis. Such formulations clearly go beyond the function spaces used in the GFC in the Luchko form. One can assume that the piecewise continuous case can be described by using some of the tools used in [95]. Note that the general fractional calculus of many variables, which is partially described in [95], can be used for the detailed study of GF distributions in multidimensional spaces.

The proposed mathematical theory can be used primarily to describe the nonlocal models of statistical mechanics [147–151], physical kinetics of plasma-like media [65,152–155], non-Markovian quantum physics of open systems [66,97], statistical optics [156,157], and statistical radiophysics [158]. This may be due to the role of non-standard spatial and frequency dispersions. A nonlocal theory of probability can be an important tool for describing complex processes in the economy, in technical and computer sciences, where nonlocality can make a significant contribution to the studied processes and phenomena.

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