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Nonlocal properties and local invariants for bipartite systems

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The nonlocal properties for a kind of generic N-dimensional bipartite quantum systems are investigated. A complete set of invariants under local unitary transformations is presented. It is shown that two generic density matrices are locally equivalent if and only if all these invariants have equal values in these density matrices.

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As a fundamental phenomenon in quantum mechanics, nonlocality has been given a lot of attention in foundational considerations, in the discussion of Bell-type inequalities [1] and hidden variable models, see, e.g., Ref. [2]. Nonlocality turns out to be also very important in quantum information processing such as quantum computation [3], quantum teleportation [4–7], dense coding [8], and quantum cryptographic schemes [9–11]. Nonlocal correlations in quantum systems imply a kind of entanglement among the quantum subsystems. The nonlocal properties as well as the entanglement of two parts of a quantum system remain invariant under local transformations of these parts.

The method developed in Refs. [12,13], in principle, allows one to compute all the invariants of local unitary transformations, though it is not easy to perform it operationally. In Ref. [14], the invariants for general two-qubit systems are studied and a complete set of 18 polynomial invariants is presented. It is proven that two-qubit mixed states are locally equivalent if and only if all these 18 invariants have equal values in these states. Therefore, any nonlocal characteristics of entanglement is a function of these invariants. In Ref. [15], three-qubits states are also discussed in detail from a similar point of view.

In the present paper, we discuss the locally invariant properties of arbitrary dimensional bipartite quantum systems. We present a complete set of invariants and show that two generic density matrices with full rank are locally equivalent if and only if all these invariants have equal values in these density matrices.

We first consider the case of pure states. Let H be an N-dimensional complex Hilbert space, with $|i\rangle$, $i=1,\ldots,N$, as an orthonormal basis. A general pure state on $H\otimes H$ is of the form

$$|\Psi\rangle=\sum_{i,j=1}^{N} a_{ij}|i\rangle\otimes|j\rangle, \quad a_{ij}\in\mathbb{C}, \tag{1}$$

with the normalization $\Sigma_{i,j=1}^{N}a_{ij}a_{ij}^{*}=1$ (* denoting complex conjugation).

A quantity is called an invariant associated with the state $|\Psi\rangle$ if it is invariant under all local unitary transformations, i.e., all maps of the form $U\otimes U$ from $H\otimes H$ to itself, where U is a unitary transformation on the Hilbert space H. Let A denote the matrix given by $(A)_{ij}=a_{ij}$. The following quantities are known to be invariants associated with the state $|\Psi\rangle$ given by Eq. (1), see Refs. [16–19]:

$$I_\alpha=\mathrm{Tr}(AA^\dagger)^\alpha, \quad \alpha=1,\ldots,N \tag{2}$$

(with $A^\dagger$ the adjoint of the matrix A).

In terms of the Schmidt decomposition, a given $|\Psi\rangle$ can always be written in the following form, using two suitable orthonormal basis $\{|i\rangle'\}$, $\{|i\rangle''\}$, $i=1,\ldots,N$:

$$|\Psi\rangle=\sum_{i=1}^{N}\sqrt{\Lambda_i}|i\rangle'\otimes|i\rangle'',$$

where $\Sigma_{i=1}^{N}\Lambda_i=1$, $\Lambda_i\geqslant 0$. The Λ_i, $i=1,\ldots,N$, are the eigenvalues of the matrix $AA^\dagger$. As $AA^\dagger$ is self-adjoint, there always exists a unitary matrix V, $VV^\dagger=V^\dagger V=1$, such that $VAA^\dagger V^\dagger=\mathrm{diag}\{\Lambda_1,\ldots,\Lambda_N\}$. Invariants (2) can then be written in the form

$$I_\alpha=\sum_{i=1}^{N}\Lambda_i^\alpha, \quad \alpha=1,\ldots,N.$$

As the eigenvalues of the matrix $AA^\dagger$ are given by the invariants under local unitary transformations, two pure states (on $H\otimes H$) are equivalent under local unitary transformations if and only if they have the same values of the invariants I_α, $\alpha=1,\ldots,N$ [20]. Moreover, two Hermitian $m\times m$ matrices A and B are unitary equivalent (i.e., there exists a unitary matrix u on an m-dimensional complex vector space satisfying $A=uBu^\dagger$) if and only if

$$\mathrm{Tr}(A^\alpha)=\mathrm{Tr}(B^\alpha), \quad \text{for } \alpha=1,\ldots,m. \tag{3}$$

We consider now mixed states on $H\otimes H$. Let ρ be a density matrix defined on $H\otimes H$ with $\mathrm{rank}(\rho)=n\leqslant N^2$. ρ can be decomposed according to its eigenvalues and eigenvectors:

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$$\rho=\sum_{i=1}^{n}\lambda_i|\nu_i\rangle\langle\nu_i|,$$

where λ_i and $|\nu_i\rangle$, $i=1,...,n$, are the nonzero eigenvalues and eigenvectors, respectively, of the density matrix ρ. $|\nu_i\rangle$ has the form

$$|\nu_i\rangle=\sum_{k,l=1}^{N}a_{kl}^i|k\rangle\otimes|l\rangle,\quad a_{kl}^i\in\mathbb{C},\quad \sum_{k,l=1}^{N}a_{kl}^i a_{kl}^{i*}=1,$$
$$i=1,...,n.$$

Let A_i denote the matrix given by $(A_i)_{kl}=a_{kl}^i$. We introduce $\{\rho_i\}$, $\{\theta_i\}$,

$$\rho_i=\mathrm{Tr}_2|\nu_i\rangle\langle\nu_i|=A_iA_i^\dagger,\quad \theta_i=(\mathrm{Tr}_1|\nu_i\rangle\langle\nu_i|)^*=A_i^\dagger A_i,$$
$$i,\ j=1,...,n, \tag{4}$$

Tr_1 and Tr_2 stand for the traces over the first and second Hilbert spaces, respectively, and therefore, ρ_i and θ_i can be regarded as reduced density matrices. Let $\Omega(\rho)$ and $\Theta(\rho)$ be two "metric tensor" matrices, with entries given by

$$\Omega(\rho)_{ij}=\mathrm{Tr}(\rho_i\rho_j),\quad \Theta(\rho)_{ij}=\mathrm{Tr}(\theta_i\theta_j),\quad \text{for } i,\ j=1,...,n \tag{5}$$

and

$$\Omega(\rho)_{ij}=\Theta(\rho)_{ij}=0,\quad \text{for } N^2\geqslant i,\ j>n.$$

We call a mixed state ρ a *generic* one[1] if the corresponding "metric tensor" matrices Ω, Θ satisfy

$$\det[\Omega(\rho)]\neq 0\quad\text{and}\quad \det[\Theta(\rho)]\neq 0. \tag{6}$$

Obviously, a generic state implies $n=N^2$ or $\det(\rho)\neq 0$, namely, a state with full rank. Nevertheless, a fully ranked density matrix may be not generic in the sense of Eq. (6).

Similarly, we also introduce $X(\rho)$ and $Y(\rho)$ as

$$X(\rho)_{ijk}=\mathrm{Tr}(\rho_i\rho_j\rho_k),\quad Y(\rho)_{ijk}=\mathrm{Tr}(\theta_i\theta_j\theta_k),$$
$$i,\ j,\ k=1,...n. \tag{7}$$

Theorem. Two generic density matrices with full rank are equivalent under local unitary transformations if and only if there exists a ordering of the corresponding eigenstates such that the following invariants have the same values for both density matrices:

$$J^s(\rho)=\mathrm{Tr}_2(\mathrm{Tr}_1\rho^s),\quad s=1,...,N^2,$$
$$\Omega(\rho),\quad \Theta(\rho),\quad X(\rho),\quad Y(\rho). \tag{8}$$

Remark 1. It is well known that the set of eigenvalues and corresponding eigenstates is uniquely defined, but not their labeling. However, from the proof below, one can see that two generic density matrices would have the same set of eigenvalues if they share the same values $\{J^s(\rho)\}$. One can uniquely choose the label for the eigenstates with the different eigenvalues. For the case of degenerate eigenvalues, if two generic density matrices ρ and ρ' are equivalent under local unitary transformations, one can always find a kind of label for the eigenstates such that they share the same invariants (8), i.e., under this label, $\Omega(\rho)_{ij}=\Omega(\rho')_{ij}$, $\Theta(\rho)_{ij}=\Theta(\rho')_{ij}$, $X(\rho)_{ijk}=X(\rho')_{ijk}$, $Y(\rho)_{ijk}=Y(\rho')_{ijk}$. This is due to that these invariants are the sufficient and necessary conditions for two generic density matrices to be equivalent under local unitary transformations, see the proof below.

Proof. We first show that the quantities given in Eq. (8) are invariant under local unitary transformations. Let u and w be unitary transformations, $uu^\dagger=u^\dagger u=ww^\dagger=w^\dagger w=1$. Under local unitary transformation $u\otimes w$, we have $\rho\to\rho'=u\otimes w\ \rho\ u^\dagger\otimes w^\dagger$. Correspondingly, we have $|\nu_i\rangle\to|\nu_i'\rangle=u\otimes w|\nu_i\rangle$, or equivalently A_i is mapped to $A_i'=u^tA_iw$, where u^t is the transpose of u. Therefore,

$$\rho_i'=A_i'A_i'^\dagger=u^tA_iA_i^\dagger u^*=u^t\rho_iu^*,$$
$$\theta_i'=A_i'^\dagger A_i'=w^\dagger A_i^\dagger A_iw=w^\dagger\theta_iw. \tag{9}$$

By using Eq. (9), it is straightforward to check the following relations:

$$J^s(\rho)\to J^s(\rho')=\mathrm{Tr}_2\left[\sum_{i=1}^{n}\lambda_i^s\mathrm{Tr}_1(|\nu_i'\rangle\langle\nu_i'|)\right]$$
$$=\mathrm{Tr}_2\left[\sum_{i=1}^{n}\lambda_i^sA_i'A_i'^\dagger\right]=J^s(\rho),$$

$$\Omega(\rho)_{ij}\to\Omega(\rho')_{ij}=\mathrm{Tr}(\rho_i'\rho_j')=\mathrm{Tr}(u^t\rho_i\rho_ju^*)=\Omega(\rho)_{ij},$$

$$\Theta(\rho)_{ij}\to\Theta(\rho')_{ij}=\mathrm{Tr}(\theta_i'\theta_j')=\mathrm{Tr}(w^\dagger\theta_i\theta_jw)=\Theta(\rho)_{ij},$$

$$X(\rho)_{ijk}\to X(\rho')_{ijk}=\mathrm{Tr}(\rho_i'\rho_j'\rho_k')$$
$$=\mathrm{Tr}(u^t\rho_i\rho_j\rho_ku^*)$$
$$=X(\rho)_{ijk},$$

$$Y(\rho)_{ijk}\to Y(\rho')_{ijk}=\mathrm{Tr}(\theta_i'\theta_j'\theta_k')$$
$$=\mathrm{Tr}(w^\dagger\theta_i\theta_j\theta_kw)$$
$$=Y(\rho)_{ijk},$$

where $i,\ j,\ k=1,...,n$. Hence, the quantities in Eq. (8) are invariants of local unitary transformations. If two density matrices are equivalent under local unitary transformations, then their corresponding invariants in Eq. (8) have the same values.

Now suppose conversely that the states $\rho=\Sigma_{i=1}^n\lambda_i|\nu_i\rangle\langle\nu_i|$ and $\rho'=\Sigma_{i=1}^n\lambda_i'|\nu_i'\rangle\langle\nu_i'|$ give the same values to the invariants in Eq. (8). We are going to prove that ρ and ρ' are equivalent under local unitary transformations.

(a) As

[1]These states are all the ones but a set of measure zero: $\{\rho|\ \det[\Omega(\rho)]=0,\det[\Theta(\rho)]=0\}$.

$$J^s(\rho)=\mathrm{Tr}_2\left(\sum_{i=1}^{n}\lambda_i^s \mathrm{Tr}_1(|\nu_i\rangle\langle\nu_i|)\right)$$
$$=\mathrm{Tr}_2\left(\sum_{i=1}^{n}\lambda_i^s A_i A_i^\dagger\right)$$
$$=\sum_{i=1}^{n}\lambda_i^s,$$

from $J^s(\rho')=J^s(\rho)$, we have

$$\sum_{i=1}^{n}\lambda_i'^s=\sum_{i=1}^{n}\lambda_i^s,\quad \forall\quad s=1,...,N^2.$$

From Eq. (3), we have that ρ' and ρ have the same nonzero eigenvalues, i.e., $\lambda_i'=\lambda_i$, $i=1,...,n$.

(b) From Eq. (5), the generic condition $\det[\Omega(\rho)]\neq 0$ implies that $\{\rho_i\}$, $i=1,...,n(=N^2)$, span the space of $N\times N$ matrices and

$$\rho_i\rho_j=\sum_{k=1}^{n}C_{ij}^k\rho_k,\quad C_{ij}^k\in\mathbb{C}. \tag{10}$$

Taking trace of Eq. (10) and using the condition $\mathrm{Tr}\rho_i=1$, one gets

$$\Omega_{ij}=\sum_{k=1}^{n}C_{ij}^k. \tag{11}$$

From Eqs. (11) and (7), we obtain

$$X_{ijk}=\sum_{l=1}^{n}C_{ij}^l\Omega_{lk}.$$

Therefore,

$$C_{ij}^{\prime}=\sum_{k=1}^{n}X_{ijk}\Omega^{lk}. \tag{12}$$

where the matrices Ω^{ij} is the corresponding inverses of the matrices Ω_{ij} [which exist due to the assumption (6)]. Equation (12) implies that the coefficients C_{ij}^l are given by $\{\Omega_{ij},X_{ijk}\}$. From Eq. (5), the generic condition (6) implies that $\{\rho_i\}$ forms an irreducible N-dimensional representation of the algebra $\mathrm{gl}(N,\mathbb{C})$ with the generators $\{e_i,i=1,...,N^2\}$ satisfying

$$[e_i,e_j]=\sum_{k=1}^{N^2}f_{ij}^k e_k, \tag{13}$$

where $f_{ij}^k=C_{ij}^k-C_{ji}^k$. More explicitly, $\pi\ (e_i)=\rho_i$, $i=1,...,N^2$, where π is the representation of $\mathrm{gl}(N,\mathbb{C})$.

The generic condition $\det[\Omega(\rho')]\neq 0$ implies that $\{\rho_i'\}$, $i=1,...,N^2$, also span the space of $N\times N$ matrices,

$$\rho_i'\rho_j'=\sum_{k=1}^{n}C_{ij}'^k\rho_k',\quad C_{ij}'^k\in\mathbb{C}. \tag{14}$$

If $\Omega(\rho')=\Omega(\rho)$ and $X(\rho')=X(\rho)$, we have $C_{ij}'^l=C_{ij}^l$. Therefore, $\{\rho_i\}$ and $\{\rho_i'\}$ [if one chooses $\pi'(e_i)=\rho_i'$] are two irreducible N-dimensional representation of $\mathrm{gl}(N,\mathbb{C})$ (13). It is well known that all the Casimir operators of the algebra $\mathrm{gl}(N,\mathbb{C})$ can be expressed in terms of homogeneous polynomials of e_i's (for example, the first Casimir operator C_2 can be written as a quadratic polynomial of e_i's). Moreover, Casimir operators are algebraically independent and give rise to a complete set of generators for the center of the universal enveloping algebra of $\mathrm{gl}(N,\mathbb{C})$. They take scalar values on an irreducible representation of $\mathrm{gl}(N,\mathbb{C})$ (from Schur's Lemma) and become the characters of the irreducible representations [21]. Due to the fact that the trace of every polynomial of $\{\rho_i\}$ and $\{\rho_i'\}$ can be represented in terms of $\{\Omega_{ij}(\rho),X_{ijk}(\rho)\}$, and $\{\Omega_{ij}(\rho'),Y_{ijk}(\rho')\}$, respectively (see below Remark 2), we conclude that the values of all the Casimir operators given by the two N-dimensional representations $\{\rho_i\}$ and $\{\rho_i'\}$ are equal, from the conditions $\Omega(\rho)=\Omega(\rho')$ and $X_{ijk}(\rho)=X_{ijk}(\rho')$. Hence, the two sets of representations (primed and unprimed) of the algebra $\mathrm{gl}(N,\mathbb{C})$ are equivalent, i.e.,

$$\rho_i'=u^t\rho_i u^*, \tag{15}$$

for some $u\in\mathcal{U}$.

Similarly, from $\Theta(\rho)=\Theta(\rho')$ and $Y_{ijk}(\rho)=Y_{ijk}(\rho')$, we can deduce that

$$\theta_i'=w^\dagger\theta_i w,\quad \text{for some}\quad w\in\mathcal{U}. \tag{16}$$

From the *Singular value decomposition* of matrices [22], we have $|\nu_i'\rangle=u\otimes w|\nu_i\rangle$, $i=1,...,N^2$, and $\rho'=u\otimes w\rho u^\dagger\otimes w^\dagger$. Hence, ρ' and ρ are equivalent under local unitary transformations. ■

Remark 2. For a degenerate state ρ, $\det[\Omega(\rho)]=0$ (respectively, $\det[\Theta/(\rho)]=0$), the above invariants (8) are not complete in the sense that two degenerate density matrices cannot be equivalent under local unitary transformations even if they give the same values to the invariants in Eq. (8). This is due to the fact that there exist null vectors for the degenerate state. For example, in the case $\det[\Omega(\rho)]=0$, there exists at least one Hermitian matrix B which satisfies $\mathrm{Tr}(B\rho_i)=0$ for $i=1,...,n$. Hence, $\Omega(\rho')_{ij}=\Omega(\rho)_{ij}$ and $X(\rho')_{ijk}=X(\rho)_{ijk}$ are not enough to get the first equivalence relation (15). In this case, some new invariants have to be introduced to get a complete set of invariants. From the algebraic relations (10) and formula (12), other generalized invariants such as $\mathrm{Tr}[(\rho_i)^{m_i}(\rho_j)^{m_j}\cdots(\rho_k)^{m_k}]$ and $\mathrm{Tr}[(\theta_i)^{m_i}(\theta_j)^{m_j}\cdots(\theta_k)^{m_k}]$, $i,j,\ldots,k=1,...,n$; m_i, $m_j,\ldots,m_k\in\mathbb{N}$, can be represented in terms of $\{\Omega_{ij},X_{ijk}\}$ and $\{\Theta_{ij},Y_{ijk}\}$ for a generic state with full rank, for example,

$$\mathrm{Tr}(\rho_{i_1}\rho_{i_2}\cdots\rho_{i_m}) = \sum_{\{\alpha_1,\ldots,\alpha_{m-2}\}} C^{\alpha_1}_{i_1\,i_2} C^{\alpha_2}_{\alpha_1\,i_3} \cdots C^{\alpha_{m-2}}_{\alpha_{m-3}\,i_{m-1}} \Omega_{\alpha_{m-2}\,i_m}.$$

Hence, by doing so we do not get new invariants.

To summarize, we have discussed here the local invariants for arbitrary dimensional bipartite quantum systems and have presented a set of invariants of local unitary transformations. The set of invariants is not necessarily independent (they could be represented by each other in some cases) but it is complete in the sense that two generic density matrices are equivalent under local unitary transformations if and only if all these invariants have equal values for these density matrices.

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