

# Nonlocality in $\mathcal{PT}$ -symmetric waveguide arrays with gain and loss

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We demonstrate that light propagation in waveguide arrays that include  $\mathcal{PT}$ -symmetric structures can exhibit strongly nonlocal sensitivity to topology of the array at fixed other parameters. We consider an array composed of lossless waveguides, that includes a pair of  $\mathcal{PT}$ -symmetric waveguides with balanced gain and loss, and reveal that  $\mathcal{PT}$ -symmetry breaking thresholds are different for planar and circular array configurations. These results demonstrate that  $\mathcal{PT}$ -symmetric structures can offer new regimes for optical beam shaping compared to conservative structures. © 2012 Optical Society of America

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Photonic structures composed of coupled waveguides with loss and gain regions offer new possibilities for shaping optical beams and pulses compared to conservative structures [1–3]. Such structures can be designed as optical analogues of complex parity-time (or  $\mathcal{PT}$ )-symmetric potentials, which can have a real spectrum corresponding to the conservation of power for optical eigenmodes; however, the beam dynamics can demonstrate unique features distinct from conservative systems due to nontrivial wave interference and phase transition effects [4,5]. Recently,  $\mathcal{PT}$ -symmetric properties in couplers composed of two waveguides with gain and loss have been demonstrated experimentally [6]. Importantly, effects occurring for linear  $\mathcal{PT}$ -symmetric structures can be realized using only waveguides with spatially varying absorption coefficients and no amplification, based on general transformation introduced in Ref. [7].

$\mathcal{PT}$ -symmetric potentials appear in many physical contexts, and one feature actively investigated in the context of quantum theories is the property of *nonlocality*, where  $\mathcal{PT}$  defect dynamics can be sensitive to potential profile at distant locations, raising questions about the observability of such behavior in real physical systems [8,9]. In this work, we present a classical analogue of quantum nonlocality in optical  $\mathcal{PT}$ -symmetric structures with gain and loss where beam dynamics can depend on arbitrarily distant boundaries.

To demonstrate the phenomenon of nonlocality in optical structures, we compare arrays of coupled optical waveguides with planar and circular geometries as illustrated in Fig. 1. The beam profile is determined by the mode amplitudes  $a_n$  at individual waveguides, and mode overlap between waveguides is characterized by coupling coefficients:  $C_2$  between the central waveguides  $n = 0, 1$ , and  $C_1$  between all other neighboring waveguides. We consider a  $\mathcal{PT}$ -symmetric structure composed of waveguide with loss at location  $n = 0$  and with gain at the adjacent waveguide  $n = 1$ . The absolute magnitudes of gain/loss should be equal to satisfy  $\mathcal{PT}$  symmetry condition. We use the coupled-mode equations [6,10,11] to model the beam propagation:

$$i \frac{da_n}{dz} + C_1 a_{n-1} + C_1 a_{n+1} = 0, \quad n \neq 0, 1, \quad (1)$$

$$i \frac{da_0}{dz} + i\rho a_0 + C_1 a_{-1} + C_2 a_1 = 0, \quad (2)$$

$$i \frac{da_1}{dz} - i\rho a_1 + C_2 a_0 + C_1 a_2 = 0, \quad (3)$$

where  $n$  is the waveguide number,  $z$  is the propagation distance,  $a_n$  are the mode amplitudes at waveguides,  $\rho > 0$  defines the rate of loss at 0th and gain at 1st waveguides, and  $C_{1,2}$  are the coupling coefficients between the modes of waveguides. The boundary conditions are zero for a planar structure [Fig. 1(a)],

$$a_{N+2} \equiv 0, \quad a_{-N-1} \equiv 0, \quad (4)$$

and periodic for a circular configuration [Fig. 1(b)],

$$a_{N+2} \equiv a_{-N}, \quad a_{-N-1} \equiv a_{N+1}. \quad (5)$$

The Eqs. (1–3) are linear, since we consider weak optical intensities when the gain saturation and nonlinearity can be neglected. Then, the beam dynamics can be described by analyzing the eigenmodes  $a_n = A_n \exp(i\phi_n + i\beta z)$ , where  $A_n \geq 0$  and real  $\phi_n$  are constant amplitude and phase profiles and  $\beta$  is an eigenvalue (propagation constant).

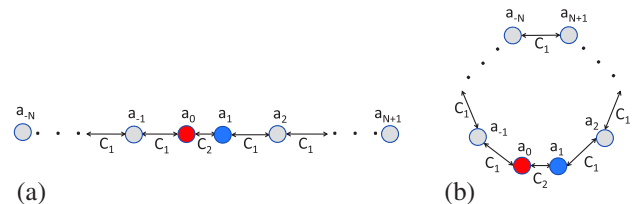


Fig. 1. (Color online) Schematic of (a) planar and (b) circular waveguide array with a pair of  $\mathcal{PT}$ -symmetric waveguides at sites  $n = 0, 1$  with balanced gain and loss.

A key feature of  $\mathcal{PT}$ -symmetric structures is that under certain conditions, the spectrum of all eigenmodes can be real (i.e.,  $\text{Im}(\beta) \equiv 0$ ), meaning that the effects of gain and loss can be compensated on average. On the contrary, if some of the eigenmodes have complex propagation constants, the mode amplitude can grow exponentially fast along the propagation direction as gain cannot be compensated by loss. For a  $\mathcal{PT}$ -symmetric coupler composed of two waveguides [3,6,7], which can be modeled with Eqs. (1–3) by putting  $C_1 = 0$ , the spectrum is real when the value of gain/loss coefficient is below the threshold  $|\rho| < |C_2|$ .

We now derive analytical expressions for  $\mathcal{PT}$ -symmetry breaking thresholds in case of planar and circular array configurations, considering the total number of waveguides  $2N$  to be rather large but finite. For a *planar configuration*, we find that the condition of  $\mathcal{PT}$ -symmetry breaking is

$$|\rho| > \rho_p = |C_2|, \quad (6)$$

and the threshold is the same as for an isolated coupler [6], independent on the coupling coefficient in the rest of the array ( $C_1$ ). This is a surprising result; for example, boundaries can have nontrivial effect on stability for planar structures with periodically placed gain and loss elements [12]. The threshold is shown with solid lines in Figs. 2(a) and (c). To obtain Eq. (6), we note that in the regime of no symmetry breaking for all modes  $d|a_n|/dz = 0$  and obtain from Eq. (1) for  $n \neq 0, 1$

$$C_1|a_n|[\sin(\phi_{n+1} - \phi_n)|a_{n+1}| + \sin(\phi_{n-1} - \phi_n)|a_{n-1}|] = 0. \quad (7)$$

Since there is no energy flow through the array boundaries for planar structure, it follows that  $|a_n a_{n+1}| \sin(\phi_{n+1} - \phi_n) = 0$  for  $|n| \geq 1$ . At  $n = 0, 1$  from Eqs. (2) and (3) we have  $C_2 \sin(\phi_1 - \phi_0)|a_0 a_1| + \rho|a_0|^2 = 0$ ,  $C_2 \sin(\phi_0 - \phi_1)$

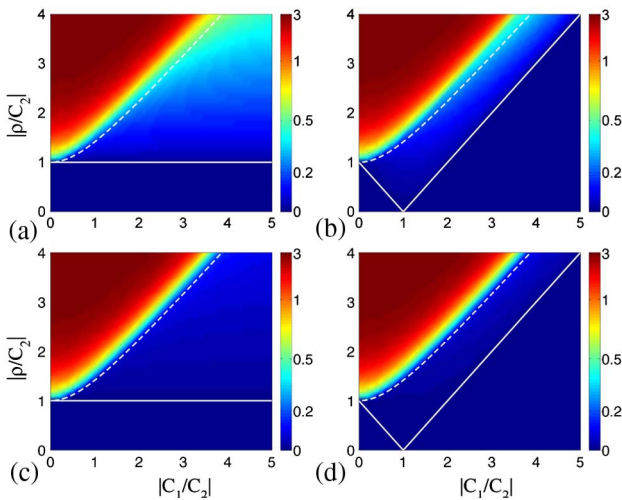


Fig. 2. (Color online) Fastest mode amplification rate in parameter plane  $|\rho/C_2|$  and  $|C_1/C_2|$  for (a), (c) planar and (b), (d) circular waveguide arrays of sizes (a), (b)  $N = 20$  and (c), (d)  $N = 100$ . Solid lines show analytical instability threshold according to (a), (c) Eq. (6) and (b), (d) Eq. (8). Dashed lines show asymptotic instability threshold for infinitely large structures according to Eq. (11).

$|a_0 a_1| - \rho|a_1|^2 = 0$ . It follows that  $|a_0| = |a_1|$  and  $C_2 \sin(\phi_1 - \phi_0) + \rho = 0$ . The latter equality has a solution only if  $|\rho| \leq |C_2|$ ; otherwise for conditions in Eq. (6) the gain and loss cannot be balanced and  $\mathcal{PT}$  symmetry breaking occurs.

For a *circular configuration*, we find that the threshold condition nontrivially depends on all the structure parameters,

$$|\rho| > \rho_c = ||C_1| - |C_2||, \quad (8)$$

and this threshold is shown with solid lines in Figs. 2(b) and (d). Most remarkably, the  $\mathcal{PT}$ -symmetry conditions separating fundamentally different cases of real spectrum, when the power is conserved on average, and complex spectrum, when some guided modes experience amplification, are always different for planar and circular arrays of arbitrary large size ( $N$ )—and this is a manifestation of nonlocality. To explain the stability condition in Eq. (8) we represent the eigenmodes as

$$a_n = F_{\pm} e^{ikn + i\beta z} + B_{\pm} e^{-ikn + i\beta z}, \quad (9)$$

where  $k$  is a wavenumber and subscripts “+” and “−” correspond to  $n \geq 1$  and  $n \leq 0$ , respectively. These expressions satisfy Eq. (1) when the propagation constant is chosen according to the spatial dispersion relation  $\beta = 2C_1 \cos(k)$ , and we then determine the amplitude values, which also fulfill Eqs. (2)–(3) and the boundary conditions at Eq. (4). Equations (2)–(3) yield the following relations between the amplitudes:  $B_- = F_- R_+ + B_+ T$  and  $F_+ = F_- T + B_+ R_- \exp(-2ik)$ , where  $T = 2ie^{-ik} C_1 C_2 \sin(k)/D$  and  $R_{\pm} = [C_2^2 - C_1^2 - \rho^2 \pm 2\rho C_1 \sin(k)]/D$  are the transmission and reflection coefficients for waves incident on  $\mathcal{PT}$ -symmetric pair of waveguides [13], and  $D = C_1^2 e^{-2ik} + \rho^2 - C_2^2$ . Next, we determine the necessary condition for the periodic boundary conditions to be satisfied, which require that the amplitudes of forward ( $F$ ) and backward ( $B$ ) waves are matched as  $F_- = \exp[2ik(N+1)]F_+$  and  $B_- = \exp[-2ik(N+1)]B_+$ . These conditions lead to the following equation for an amplitude ratio  $J = B_+ \exp(-ik)/F_-$ :

$$|J|^2 = \frac{C_1^2 - C_2^2 + \rho^2 - 2C_1[\rho - 2C_2 \text{Im}(J)] \sin(k)}{C_1^2 - C_2^2 + \rho^2 + 2C_1 \rho \sin(k)}. \quad (10)$$

The  $\mathcal{PT}$ -symmetry breaking occurs for a mode with particular  $k$  at such  $\rho$ , when solutions of Eq. (10) disappear as its right-hand side becomes negative. Let us first consider the real  $k$ , which spectrum approaches continuum with  $-\pi < k \leq \pi$  as  $N \rightarrow \infty$ . In this limit a sufficient condition for symmetry breaking corresponds to the case of negative right-hand side of Eq. (10) for any (real)  $k$ , which leads to Eq. (8). We analyze the case of complex  $k$  below, and find that the corresponding threshold [Eq. (11)] is higher than in Eq. (8); therefore, Eq. (8) is a necessary condition for  $\mathcal{PT}$ -symmetry breaking.

From a physical point of view, we should expect that if the *structure size is increased towards infinity*,  $N \rightarrow \infty$ , the type of boundaries should not matter. In this case the  $\mathcal{PT}$ -symmetry breaking would be associated with the amplification at the waveguide with gain, leading to

generation of waves propagating away from the central region. Such eigenmodes would have the form as in Eq. (9), but with vanishing amplitudes of modes coming toward the central region from the boundaries,  $F_- = B_+ = 0$ . The amplitudes  $F_+$  and  $B_-$  would describe either waves propagation away from the defect with  $\text{Im}(k) = 0$  and  $0 \leq k \leq \pi$ , or exponentially decaying waves with  $\text{Im}(k) > 0$ . In the absence of symmetry breaking, the gain and loss in the central waveguides is balanced, such that there are no outward energy flows, which corresponds to  $\text{Im}(k) > 0$  and  $\sin[\text{Re}(k)] = 0$  according to Eq. (7). By analyzing Eqs. (1)–(2), we determine that such modes cease to exist, corresponding to  $\mathcal{PT}$ -symmetry breaking, when

$$|\rho| > \rho_{\text{inf}} = \sqrt{C_1^2 + C_2^2}. \quad (11)$$

We show the instability threshold with dashed lines in Figs. 2(a)–(d). When  $\rho_{p,c} < |\rho| < \rho_{\text{inf}}$  for planar (“p”) or circular (“c”) waveguide arrays, respectively, then there appear unstable modes in arrays of finite length; however, their growth rate reduces to zero as  $O(N^{-1})$ . Here the inverse proportionality relation is estimated due to the fact that the instability should involve propagation of wave from the array center to the boundary, and then back to the central region. Indeed, we see that the amplification rate decreases in this region bounded by solid and dashed lines in Fig. 2 as the structure size is increased from  $N = 20$  [Figs. 2(a) and (b)] to  $N = 100$  [Figs. 2(c) and (d)]. Accordingly, for a particular propagation distance, there will be a structure size when the presence of slowly growing modes would be practically insignificant.

We illustrate our predictions with numerical simulations. As an example, we consider the beam coupled

to waveguide number  $n = 1$  at the input; however, similar scenarios are observed for other input conditions. We first choose the structure parameters  $\rho/C_2 = 0.8$  and  $C_1/C_2 = 1.5$  such that they correspond to a stable region for planar but an unstable region for circular configuration. The plots of beam dynamics presented in Fig. 3(a) show that the power is conserved on average for planar structure. For the circular geometry, Fig. 3(b) demonstrates that power grows exponentially, and we also note that the power increases in “steps” since mode amplification occurs when wave is scattered on the central waveguides, and this happens periodically as beam circulates around. Completely different dynamics are observed for parameters  $\rho/C_2 = 2$  and  $C_1/C_2 = 1.5$ , when symmetry breaking can occur without the effect of boundaries according to condition in Eq. (11). Indeed, we find that the instability development in planar and circular arrays is practically identical, and we show results for circular array in Fig. 3(c). The instability develops in the central region and the power grows at a steady rate, with no effects of boundaries, in agreement with analytical prediction.

In conclusion, we have demonstrated that optical wave dynamics in  $\mathcal{PT}$ -symmetric structures with gain and loss elements can be strongly nonlocal, and can be critically affected by the structure topology. Our results suggest an experimental path toward observation of the fundamental property of the  $\mathcal{PT}$ -symmetric systems associated with their nonlocality, which is an analogue of nonlocality effects raised in the context of quantum theories [8,9].

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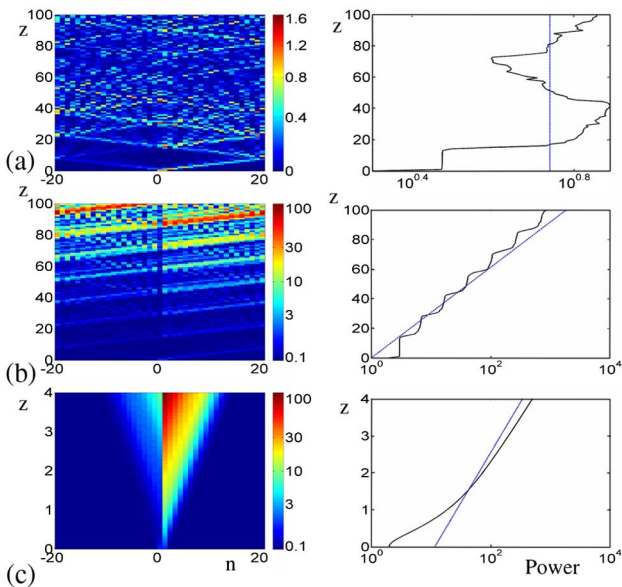


Fig. 3. (Color online) Optical beam dynamics in (a) planar and (b) circular waveguide arrays for  $\rho/C_2 = 0.8$ , and (c) planar or circular (practically identical figures) for  $\rho/C_2 = 2$ . Left plots: the modulus of amplitudes at the waveguides  $|a_n|$  along the propagation distance  $z$ . Right: calculated total power versus distance (solid line) and power trend for the fastest growing eigenmode (dashed blue line). For all the plots,  $C_1/C_2 = 1.5$  and  $N = 20$ .