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Nonnegative integral solution of linear equations

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Abstract. A method to obtain a nonnegative integral solution of a system of linear equations, if such a solution exists is given. The method writes linear equations as an integer programming problem and then solves the problem using a combination of artificial basis technique and a method of integer forms.

Keywords. All-integer programming; artificial basis technique; Gomory method of integer forms; linear programming; non-negative integral solution.

1. Introduction

Many problems like those of path-length, fixed-charge, batch-size, transportation and allocation, chemical reactor vessel (Gottfried and Weisman [7]), computer networking involve quantities which can be only nonnegative integers. Such a problem giving rise to linear equations needs nonnegative integral solution.

Hurt and Waid [8] propose a generalized inverse A^- which gives the general integral solution (all the integral solutions) to linear equations (also Ben-Israel and Greville [1]; Marcus and Minc [9]; Sen and Shamim [12]). There seems to be no easy way to seeve out nonnegative integral solutions from the general form $\mathrm{x} = A^-\mathrm{b} + (I - A^-A)\,\mathrm{y}$ where y and I are arbitrary integral n-vector and $n \times n$ unit matrix, respectively. $A\mathrm{x} = \mathrm{b}$ are the equations where A is an $m \times n$ integral matrix.

The method described here investigates equations $A\mathrm{x} = \mathrm{b}$, consistent or not, underdetermined or overdetermined as an all-integer programming (all-ip) problem and gives a nonnegative integral solution x when it exists. To solve the all-ip problem the method involves a particular form of the artificial basis technique (Sen [11]; Chung [2]; Strum [13]) and the Gomory method of integer forms (Gomory [6]; Vajda [15]; Salkin [10]).

2. The problem

Obtain a nonnegative integral solution x of $A\mathrm{x} = \mathrm{b}$ (if it exists) where $A = (a_{ij})$ is a given $m \times n$ integral matrix, $\mathrm{b} = (b_i)$ is a given non-negative integral m-vector and $\mathrm{x} = (x_j)$ is an n-vector. (1)

Note. There is no loss of generality in considering (*i*) $b \geqslant 0$ and (ii) A and b integral. For, if these are not then multiply the equations with negative b_i by -1 and nonintegral equations by suitable scalars.

3. Existence of a nonnegative integral solution

$Ax = b$ has a non-negative solution x if and only if $A^t y \geqslant 0$, $b^t y < 0$ has no solution y. *t* indicates the transpose (Vajda [14]; Farkas [3]).

Equivalently, $Ax = b$ has no nonnegative solution x if and only if $A^t y \geqslant 0$, $b^t y < 0$ has a solution y.

Let A and b be integral and $Ax = b$ be consistent (i.e., $AA^- b = b$). Also, let $Ax = b$ have a nonnegative solution. If A is nonsingular and its inverse is also integral then $Ax = b$ has unique nonnegative integral solution $x = A^{-1} b$. Further, if $x = A^- b$ is not integral then $Ax = b$ has no integral solution (it has non-integral rational solutions though). A^- here is a (reflexive) generalized inverse that satisfies

$$AA^- A = A,\ A^- AA^- = A^-,\ A^- A \text{ and } AA^- \text{ are integral.}$$

These results are not of immediate use. However, the method tells if a nonnegative solution of $Ax = b$ does not exist. In fact, the necessary and sufficient condition for $Ax = b$ to have a nonnegative solution is the method producing one. Further, the sufficient condition that this solution is integral is the (Gomory) method giving one.

4. The method

The method consists of two parts.

Part 1 (*Equivalent ip problem*). Write (1) as an all-ip problem.

Part 2 (*Gomory-artificial-basis technique*). Solve this ip problem using Gomory method of integer forms in which a particular form of the artificial basis technique is embedded.

(i) *Equivalent ip problem*

Let x and A be now extended $(n + m)$-vector and $m \times (n + m)$ matrix, respectively. Further, let the last m columns of A form an $m \times m$ unit matrix. The *ip* problem equivalent to (1) is

Obtain x so that

Min $z = x_{n+1} + \ldots + x_{n+m} = 0$: Objective function

subject to

$Ax = b$: Constraints

$x \geqslant 0$ and integral : Nonnegativity and integrality conditions. (2)

(ii) *Gomory-artificial-basis technique*

Step 1. Solve the *ip* problem as a linear programming (*lp*) problem using the artificial basis technique in 'restricted tableau' (described later). If it is infeasible, so is the *ip* problem—terminate. If the optimal solution is all integer then the *ip* problem is solved—terminate. Otherwise go to step 2.

Step 2. Consider one of the variables which have a fraction* in their value in the optimal (simplex) restricted tableau.

(i) Let the row of such a variable be

$$x \; z_1 \; .. \; z_t \; z_0$$

where z_0 is the present value of the variable x.

(ii) Write every z_t as $L_t + f_t$, where $L_t =$ the largest integer contained in z_t and hence f_t is nonnegative. In particular, f_0 is positive.

(iii) Add to the (simplex) tableau the further row (Gomory constraints)

$$s_1 \; -f_1 \; .. \; -f_t \; -f_0.$$

(iv) Apply the Dual Simplex method (described later) on this tableau. This renders the new variable s_1 non-basic.

Note. The Simplex tableau is already dual feasible, since the final tableau of the artificial basis technique (Simplex method) is reached. So the Dual Simplex method has been used.

Step 3. If the result again contains a basic variable which is not an integer then continue introducing new variables, $s_2, \ldots$ The method terminates in a finite number of steps if the feasible region of the *ip* problem is bounded (sufficient but not necessary).

5. Artificial basis technique in restricted tableau

Step 1. Set up the restricted Simplex tableau for (2), and write the coefficients (in parentheses) which x_j have in the objective function and the last row, i.e., d_j-row using the checking rule (described later) as below

	(0)	.. (0)	.. (0)		
	x_1	x_{j_0}	x_n	b	
(1) x_{n+1}	a_{11}	a_{1j_0}	a_{1n}	b_1	
	$\vdots$				(3)
(1) x_{n+i_0}	a_{i_01}	$a_{i_0j_0}$	a_{i_0n}	b_{i_0}	
	$\vdots$				
(1) x_{n+m}	a_{m1}	a_{mj_0}	a_{mn}	b_m	
	d_1	d_{j_0}	d_n	d_{n+1}	

Step 2. (pivot selection). Let d_{j_0} be positive. Consider then, for all positive a_{ij_0}, the ratios b_i/a_{ij_0} and take a smallest. If this is obtained for i_0 then call $p = a_{i_0j_0}$

* It is generally assumed that convergence is speeded by choosing that cut which bites as deeply as possible. This is usually taken to mean the selection of the row that gives the largest fraction (f_0).

the pivot (marked with a plus). Go to Step 3. Otherwise the present tableau is final and it either indicates no solution of $A\mathrm{x} = \mathrm{b}$ or gives a nonnegative solution.

Step 3 (next-tableau computation). Having interchanged x_{j_0} and x_{n+i_0} obtain the next tableau as follows.

	x_1	..	x_{n+i_0}	..	x_n	b
x_{n+1}			$-a_{1j_0}/p$			
			$\vdots$			
x_{j_0}	a_{i_01}/p		$1/p$		a_{i_0n}/p	b_{i_0}/p
			$\vdots$			
x_{n+m}			$-a_{mj_0}/p$			
			$-d_{j_0}/p$			

(4)

The blank positions are filled in as follows :

$$a_{ij} \leftarrow a_{ij} - a_{i_0j}a_{ij_0}/p$$

$$d_j \leftarrow d_j - a_{i_0j}d_{j_0}/p. \tag{5}$$

Note. '$\leftarrow$' means 'is replaced by'.

(i) The foregoing two 'replacements' are actually identical when we consider the last row (i.e., d_j-row) as just another row like the rows of (a_{ij}).

(ii) The right-hand side elements are the elements of the foregoing tableau throughout the computation.

Step 4 (termination condition). If the bottom row i.e. d_j-row excluding the last element is nonpositive, or if none of $x_{n+1}, \ldots, x_{n+m}$ occurs in the basis with a nonzero value then a nonnegative solution is reached. Otherwise go to step 2.

6. The checking rule for a simplex tableau

Let the *lp* problem be

Minimize $z = \mathrm{c}^t\mathrm{x}$ subject to $A\mathrm{x} = \mathrm{y}$, $\mathrm{x} \geqslant 0$

where

$$\mathrm{c} = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}, \mathrm{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}, \mathrm{b} = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}, A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}.$$

We attach to all variables x_i the coefficient which they have in the objective function. Let, for example, a current 'restricted tableau' be

	(c_1)	(c_6)	(c_2)	(c_4)	
	x_1	x_6	x_2	x_4	
$(c_3)\ x_3$	p_{11}	p_{12}	p_{13}	p_{14}	v_1
$(c_5)\ x_5$	p_{21}	p_{22}	p_{23}	p_{24}	v_2
	d_1	d_2	d_3	d_4	d_5

Then

$$\begin{aligned} c_3p_{11} + c_5p_{21} - c_1 &= d_1 \\ c_3p_{12} + c_5p_{22} - c_6 &= d_2 \\ c_3p_{14} + c_5p_{24} - c_2 &= d_3 \\ c_3v_1 + c_5v_2 \qquad &= d_5. \end{aligned}$$

Such a relationship holds in all tableaux. This relationship is referred to as the *checking rule* for a tableau. Satisfaction of this rule is necessary for a restricted tableau to be correct but it is not sufficient (i.e., the rule may be satisfied even if a computational mistake occurs).

7. The dual simplex method

When to use. Let an *lp* problem be

$$\begin{aligned} \text{Minimize} \quad & z = c_1x_1 + \cdots + c_nx_n \\ \text{subject to} \quad & a_{11}x_1 + \cdots + a_{1n}x_n + a_{1\,n+1}\,x_{n+1} = b_1 \\ & \vdots \qquad\qquad\qquad\qquad\qquad\qquad \vdots \\ & a_{m1}x_1 + \cdots + a_{mn}x_n + a_{mn+1}\,x_{n+m} = b_m \\ & x_i \geqslant 0 \quad i = 1\,(1)\,n + m. \end{aligned}$$

Also, let

$$a_{1n+1} = \cdots = a_{mn+1} = -1$$

and all c_j $j = 1\,(1)\,n$ be non-negative so that, in the first tableau, the first n elements in the bottom row are nonpositive (since we minimize). We call such a tableau *dual feasible*. If, in addition, all b_i $i = 1\,(1)\,m$ are nonnegative then the result is reached. Otherwise apply dual simplex method.

The method

Step 1 (pivot selection). Let b_{i_0} be negative. Consider, for all $a_{i_0j} < 0$, $|\, c_j/a_{i_0j} \,|$ and take a smallest. If this is obtained for j_0 then $a_{i_0j_0}$ is the pivot.

Step 2. (next-tableau computation). Same as in the Simplex algorithm (Vajda [15], Chung [2]; Gass [4]) or as in Step 3 of Sec. 5.

Step 3 (termination condition). If the bottom row (i.e., c_j-row) excluding the last element is nonpositive then the solution is reached—terminate. Otherwise go to step 1.

8. Examples

(i) Obtain a nonnegative integral solution of

$$\begin{bmatrix} -1 & 5 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 5 \\ 8 \end{bmatrix}$$

Restricted tableau 0

	(o)	(o)	(o)	(o)	
	x_1	x_2	x_3	x_4	b
(1) x_5	−1	5+	1	0	5
(1) x_6	1	2	0	1	8
	0	7	1	1	13

Restricted tableau 1

	x_1	x_5	x_3	x_4	b
x_2	−1/5	1/5	1/5	0	1
x_6	7/5+	−2/5	−2/5	1	6
	7/5	−7/5	−2/5	1	6

Restricted tableau 2

	x_6	x_5	x_3	x_4	b
x_2	1/7	1/7	1/7	1/7	13/7
x_1	5/7	−2/7	−2/7	5/7	30/7
	−1	−1	0	0	0

A nonnegative solution is thus x = (30/7 13/7 0 0)'.

Consider the first row as it contains the largest fraction in the value of the variable x_2, viz., 6/7. Generate the new row (Gomory constraint) as in the algorithm and append this row. Thus

Restricted tableau 20

	x_6	x_5	x_3	x_4	b
x_2	1/7	1/7	1/7	1/7	13/7
x_1	5/7	−2/7	−2/7	5/7	30/7
s_1	−1/7	−1/7	−1/7+	−1/7	−6/7
	−1	−1	0	0	0

Restricted tableau 21

	x_6	x_5	s_1	x_4	b
x_2	0	0	1	0	1
x_1	1	0	−2	1	6
x_3	1	1	−7	1	6
	−1	−1	0	0	0

Hence a nonnegative integral solution is x = (6 1 6 0)'.

Note. x = (0 1 0 6)' could also be another nonnegative integral solution.

(ii) *Degenerate case* (redundant equation)

Obtain a nonnegative integral solution (Sen [17]) of

$$\begin{bmatrix} -1 & 2 & 3 & 3 \\ 2 & 5 & 6 & 3 \\ -5 & -8 & -9 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 7 \\ 16 \\ -25 \end{bmatrix}$$

Restricted tableau 0

	(o) x_1	(o) x_2	(o) x_3	(o) x_4	b
(1) x_5	−1	2	3^+	3	7
(1) x_6	2	5	6	3	16
(1) x_7	5	8	9	3	25
	6	15	18	9	48

Restricted tableau 1

	x_1	x_2	x_5	x_4	b
x_3	−1/3	2/3	1/3	1	7/3
x_6	4	1^+	−2	−3	2
x_7	8	2	−3	−6	4
	0	3	−6	−9	6

Note. The last equation has been multiplied by −1 to make b_3 positive.

Restricted tableau 2

	x_1	x_6	x_5	x_4	b
x_3	− 3	−2/3	5/3	3	1
x_2	4	1	−2	−3	2
x_7	0	−2	1	0	0
	−12	−3	0	0	0

The artificial variable x_7 remains in the basis with a zero value. A nonnegative integral solution is $x = (0 \;\; 2 \;\; 1 \;\; 0)'$.

(iii) *Nonnegative nonintegral solution*

Obtain a nonnegative integral solution of

$$\begin{bmatrix} 1 & 2 & 1 \\ -4 & -2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Restricted tableau 0

	(o) x_1	(o) x_2	(o) x_3	b
(1) x_4	1	2	1	1
(1) x_5	−4	−2	3^+	2
	−3	0	4	3

Restricted tableau 1

	x_1	x_2	x_5	b
x_4	7/3	$8/3^+$	−1/3	1/3
x_3	−4/3	−2/3	1/3	2/3
	7/3	8/3	−4/3	1/3

Restricted tableau 2

	x_1	x_4	x_5	b
x_2	7/8	3/8	−1/8	1/8
x_3	−3/4	2/8	1/4	3/4
	0	−1	−1	0

Hence a nonnegative solution is $x = (0 \quad 1/8 \quad 3/4)^t$.

Adding the Gomory constraint we have

Restricted tableau 20

	x_1	x_4	x_5	b
x_2	7/8	3/8	−1/8	1/8
x_3	−3/4	1/4	1/4	3/4
s_1	$-1/4^+$	−1/4	−1/4	−3/4
	0	−1	−1	0

Restricted tableau 21

	s_1	x_4	x_5	b
x_2	7/2	−1/2	-1^+	−5/2
x_3	−3	1	1	3
x_1	−4	1	1	3
	0	−1	−1	0

	s_1	x_4	x_2	b
x_5	−7/2	1/2	−1	5/2
x_3	1/2	1/2	1	1/2
x_1	−1/2	1/2	1	1/2
	−7/2	−1/2	−1	5/2

By adding the Gomory constraint we obtain the last row except the last element (viz., 5/2) nonpositive and one artificial variable, viz., x_5 is still in the basis with the nonzero value 5/2. Hence the equations have no integral solution.

(iv) *Inconsistent equations*

Obtain a nonnegative integral solution (Sen [11]) of

$$\begin{bmatrix} 5 & 3 & 2 \\ 2 & 1 & 2 \\ 4 & 2 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 10 \\ 5 \\ 1 \end{bmatrix}$$

Restricted tableau 0

	(o) x_1	(o) x_2	(o) x_3	b
(1) x_4	5	3	2	10
(1) x_5	2	1	2	5
(1) x_6	4	2^+	4	1
	11	6	8	16

	x_1	x_6	x_3	b
x_4	−1	−3/2	−4	17/2
x_5	0	1	0	9/2
x_2	2	1/2	2	1/2
	−1	−3	−4	13

The last row except the last element (viz, 13) is nonpositive and two artificial variables, viz. x_4 and x_5 are still in the basis with nonzero values. Hence the equations have no nonnegative solution. In fact, the equations have no solution at all.

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