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NONNEGATIVE MATRICES WITH NONNEGATIVE INVERSES

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ABSTRACT. We generalize a result stating that a nonnegative finite square matrix has a nonnegative inverse if and only if it is the product of a permutation matrix by a diagonal matrix. We consider column-finite infinite matrices and give a simple proof using elementary ideas from the theory of partially ordered linear algebras.

In **[1]** the authors show that a nonnegative square matrix has a nonnegative inverse if and only if its entries are all zero except for a single positive entry in each row and column. In this note we generalize this result and simplify the proof as well.

Let A denote the real linear algebra of all column-finite infinite matrices with real entries. We partially order A as follows: $[\alpha_{ij}]\leqq[\beta_{ij}]$ if and only if $\alpha_{ij}\leqq\beta_{ij}$ for all i, j. Thus, A is a partially ordered linear algebra (pola) and if 1 denotes the identity matrix, then $0\leqq 1$. See **[2]** for the precise definition of a pola. An example will illustrate the result to be obtained. Let $x=[\alpha_{ij}]$ and $y=[\beta_{ij}]$ be defined as follows: $\alpha_{ij}=1$ if $i=j+1$ and is zero otherwise; $\beta_{ij}=1$ if $j=i+1$ and is zero otherwise. Thus, $0\leqq x$, $0\leqq y$ and $0\leqq xy\leqq 1\leqq yx$. Note that each column of x contains exactly one positive entry and each row of x contains at most one positive entry.

THEOREM. *Let A be the pola described above. If $x, y\in A$, $0\leqq x$, $0\leqq y$ and $0\leqq xy\leqq 1\leqq yx$, then each column of x contains exactly one positive entry and each row of x contains at most one positive entry. The conclusion applies to the matrix y if we interchange the words "row" and "column".*

PROOF. Define $d=yx-1\geqq 0$ and note that $1+d\leqq(1+d)^2=yxyx\leqq yx=1+d$ since $xy\leqq 1$. Hence, $1+2d\leqq(1+d)^2\leqq 1+d$, which means $d\leqq 0$. Thus $d=0$ and $yx=1$, which means that y is a left inverse for x. Hence, each column of x must contain at least one positive entry. Next construct a matrix z so that $0\leqq z\leqq x$ and each column of z has only one positive entry and this entry is equal to the corresponding entry in the matrix x.

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Note that $0 \leqq zy \leqq xy \leqq 1$, which means that zy and xy are diagonal matrices. Hence, $(zy)(xy)=(xy)(zy)$. Now $z=(zy)(xy)x=(xy)(zy)x=x(yz)$ and $0 \leqq yz \leqq yx=1$, which means that yz is a diagonal matrix. Using elementary properties of matrix multiplication and the fact that x and z have one positive entry in common in each column we see that $yz=1$ and therefore $x=z$. Hence, x has exactly one positive entry in each column.

The example above shows that some rows of x may contain only zeros. We show that x has at most one positive entry in each row. Let us now construct a matrix w so that $0 \leqq w \leqq x$ and each row of w has only one positive entry if the same row of x has a positive entry in it and this entry is equal to the corresponding entry in the matrix x. Now $w=(wy)x$ and since $0 \leqq wy \leqq xy \leqq 1$, we see that wy is a diagonal matrix. The same reasoning applied above to the matrix z shows that $w=x$. Hence, x has at most one positive entry in each row.

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