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NONNEGATIVE SOLUTIONS TO A SEMILINEAR DIRICHLET PROBLEM IN A BALL ARE POSITIVE AND RADIALLY SYMMETRIC*

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Abstract

We prove that nonnegative solutions to a semilinear Dirichlet problem in a ball are positive, and hence radially symmetric. In particular this answers a question in [3] where positive solutions were proven to be radially symmetric. In section 4 we provide a sufficient condition on the geometry of the domain which ensures that nonnegative solutions are positive in the interior.

1. INTRODUCTION

Let $f:R\to R$ be a smooth function, Ω a smooth bounded region in R^n, and $u:\Omega\to R$ a classical solution to

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$$-\Delta u = f(u) \quad \text{in} \quad \Omega \tag{1.1}$$

$$u = 0 \quad \text{on} \quad \partial\Omega. \tag{1.2}$$

In Theorem 1 of [G-N-N] it was proven that if Ω is a ball and $u > 0$ in Ω then u is radially symmetric. Moreover (see [3], pp. 220), it was observed that if $f(0) \geq 0$ and $u \geq 0$, $u \not\equiv 0$ in Ω then $u > 0$ in Ω, and hence radially symmetric. Further also in [3] an example of a linear problem where $f(0) < 0$, and a nonnegative solution u with interior zeros is given. In this example $n = 1$. Also see [1] for examples of nonlinear problems with $f(0) < 0$ and nonnegative solutions with interior zeros. Here we prove that these phenomena happen only in dimension $n = 1$. Indeed, here we prove:

Theorem A: <u>If</u> Ω <u>is a ball in</u> R^n <u>with</u> $n > 1$, u <u>is a nonnegative solution of</u> (1.1) <u>and</u> $u \not\equiv 0$ <u>in</u> Ω <u>then</u> $u > 0$ <u>in</u> Ω <u>and hence radially symmetric</u>.

Our motivation for this study was the observation in [2] that when $f(0) < 0$, if u is radially symmetric, nonnegative, and $n > 1$, then $u > 0$ in Ω. Our proofs are based on the maximum principle, and reflexion arguments introduced in [3]. Finally, since Theorem A was established for the case $f(0) \geq 0$ in [3], here we assume throughout that

$$f(0) < 0. \tag{1.3}$$

2. PRELIMINARY LEMMAS AND THEIR PROOFS

First we recall some notations following [3]. Let

$$T_t := \{x \in R^n \ ; \ x.e = t\},$$
$$\Sigma_t := \{x \in \Omega \ ; \ x.e > t\}$$

where $e = (1,0,\ldots,0)$, and $\cdot$ denotes the usual inner product in R^n and let Σ_t' be the reflexion of Σ_t across T_t. For $x \in R^n$ we will denote by x^t the reflexion of x across T_t. Also u_i will denote the partial derivative of u with respect to x_i, similarly u_{ij}. Further for $x \in \partial\Omega$ we will denote by $\nu(x) = (\nu_1(x),\ldots,\nu_n(x))$ the outward unit normal and by the ε-neighbourhood theorem there exists $\gamma > 0$ such that for every $0 < \varepsilon < \gamma$, we can define (see [4])

$$D_\varepsilon = \{x \in \Omega \ ; \ \text{dist.}(x,\partial\Omega) < \varepsilon\}$$
$$= \{x + t\nu(x) \ ; \ x \in \partial\Omega \text{ and } t \in (-\varepsilon,0)\}.$$

We now state:

Lemma P: _Let_ $u:\bar\Omega \to R$ _be a nonnegative solution to_ (1.1)-(1.3). _If for some_ $t \in (0,1)$ _we have_

$$u_1(x) \le 0 \qquad \forall x \in \Sigma_t, \tag{2.1}$$

$$u(x) \le u(x^t) \quad \forall x \in \Sigma_t, \tag{2.2}$$

and

$$u(\alpha) < u(\alpha^t), \text{ for some } \alpha \in \Sigma_t, \tag{2.3}$$

then

$$u(x) < u(x^t) \quad \forall x \in \Sigma_t, \tag{2.4}$$

and

$$u_1(x) < 0 \quad \forall x \in T_t \cap \Omega. \tag{2.5}$$

Proof: see proof of Lemma 2.2 in [G-N-N] and observe that one requires that only $u \ge 0$.

Next we state and prove:

Lemma Q: _Let_ $u:\bar\Omega \to R$ _be a solution to_ (1.1)-(1.3) _and_ $z \in \partial\Omega$. _If_ $\nu_1(z) > 0$ _and there exists_ $\varepsilon > 0$ _such that_ $u(x) \ge 0$ _for_ $x \in \bar\Omega$ _with_

$\|x - z\| < \varepsilon$, then there exists $\varepsilon_1 \in (0,\varepsilon)$ such that $u_1(x) < 0 \ \forall x \in \Omega$ with $\|x - z\| < \varepsilon_1$.

Proof: Since $\nu_1(z) > 0$, without loss of generality we can assume that $\nu_1(x) > 0 \ \forall x \in \partial\Omega$ with $\|x - z\| < \varepsilon$. This and the assumption that $u \geq 0$ on $\|x - z\| < \varepsilon$ imply that

$$u_1(x) \leq 0 \quad \forall x \in \partial\Omega \quad \text{with } \|x - z\| < \varepsilon. \tag{2.6}$$

If $u_1(z) < 0$ then the existence of ε_1 follows from the continuity of u_1. On the other hand if $u_1(z) = 0$, then $\nabla u(z) = 0$. Now for $k \in \{2,\ldots,n\}$ we let $y = (-\nu_k(z),0,\ldots,0,\nu_1(z),0,\ldots,0)$. Since y is tangent to $\partial\Omega$ at z we can find $\phi:(-1,1) \to \partial\Omega$ such that $\phi(0) = z$ and $\phi'(0) = y$. Since now $\nabla u(z) = 0$ we see that $u_1(\phi(t))$ has a local maximum at $t = 0$. In particular,

$$0 = \frac{d}{dt}\left(u_1(\phi(t))\right)\Big|_{t=0} = \nabla u_1(z).y \tag{2.7}$$

$$= u_{11}(z)\left(-\nu_k(z)\right) + u_{1k}(z)\nu_1(z).$$

On the other hand, because $u(\phi(t)) \equiv 0$, computing the second derivative of u with respect to t at $t = 0$, we have

$$0 = \sum_{i,j=1}^{n} u_{ij}(z)\phi_i'(0)\phi_j'(0) + \nabla u(z).\phi''(0) \tag{2.8}$$

$$= u_{11}(z)\left(\nu_k(z)\right)^2 - 2u_{1k}\nu_1(z)\nu_k(z) + u_{kk}(z)\left(\nu_1(z)\right)^2,$$

where the ϕ_k's are the components of ϕ. Now from (2.7) and (2.8) we have for $k = 2,\ldots,n$

$$u_{kk}(z)\big(\nu_1(z)\big)^2 = u_{11}(z)\big(\nu_k(z)\big)^2, \tag{2.9}$$

and since $\nu_1(z) > 0$ we obtain,

$$\begin{aligned} -f(0) &= \Delta u(z) \\ &= u_{11}(z)\{1 + \sum_{k=2}^{n} \big(\nu_k(z)/\nu_1(z)\big)^2\}. \end{aligned} \tag{2.10}$$

But $f(0) < 0$. Thus by (2.10) and the continuity of u_{11} we see that there exists $\varepsilon_1 \in (0,\varepsilon)$ such that $u_{11}(x) > 0$ for $\|x-z\| < \varepsilon_1$ with $x \in \bar{\Omega}$. Now since $\nu_1(x) > 0$ $\forall x \in \partial\Omega$ with $\|x-z\| < \varepsilon_1$, we can further assume that if $\|x-z\| < \varepsilon_1$ with $x \in \Omega$, then there exists $b(x) \in \partial\Omega$ of the form $b(x) = x + \delta e$ $(\delta > 0)$ with $x + se \in \Omega$ $\forall s \in (0,\delta)$. Hence

$$0 > u_1\big(b(x)\big) - \int_0^{\delta} u_{11}(x + se)ds = u_1(x), \tag{2.11}$$

and Lemma Q is proven.

Remark: Lemma Q extends Lemma 2.1 of [3] in that we do not assume $u > 0$ in Ω.

Corollary R: *If u is a solution to (1.1)-(1.3) and $u \geq 0$ on a neighbourhood of $\partial\Omega$, then there exists $\mu > 0$ such that $u > 0$ on D_μ.*

Proof: Let $Z \in \partial\Omega$. Since Δ is invariant under rotations and translations, without loss of generality, we can assume that $Z = \nu(Z) = e$. By Lemma Q there exists $\varepsilon_1 > 0$ such that $u(x) > 0$ if $x \in \Omega$ and $\|x-Z\| < \varepsilon_1(Z)$. Since $\partial\Omega$ is compact we can find $x_1, x_2, \ldots, x_m$ such that $x_i \in \partial\Omega$ and such that

$$W = B(x_1, \varepsilon(x_1)) \cup B(x_2, \varepsilon(x_2)) \cup \ldots \cup B(x_m, \varepsilon(x_m)) \supset \partial\Omega.$$

Since W is open in R^n and $\partial\Omega$ is closed, there exists $\mu > 0$ such that $\{x \in R^n;\ d(x,\partial\Omega) < \mu\} \subset W$. In particular, if $x \in \Omega$ and $d(x,\partial\Omega) < \mu$ we have $u(x) > 0$.

3. PROOF OF THEOREM A

Suppose now there exists $y \in B$ such that $u(y) = 0$. By Corollary R we have $\|y\| \leq 1 - \mu$. Without loss of generality, we can assume y to have maximal norm. Further, because Δ is invariant under rotations, we can assume that

$$y = (\delta, 0, \ldots, 0) \quad \text{with} \quad \delta \in [0, 1-\mu]. \tag{3.1}$$

Now since $u(y) = 0$, by Corollary R there exists $t^* \in [(1/2), 1)$ for which (2.4) does not hold. Let ζ be defined by

$$\zeta = \inf\{s \in [0,1);\ (2.1)\text{-}(2.3) \text{ hold on } \Sigma_t\ \forall t \in (s,1]\}. \tag{3.2}$$

By Lemma Q there exists $k \geq 2$ such that

$$\zeta \leq 1 - (\varepsilon_1/k). \tag{3.3}$$

Also since $\delta \geq 0$ we see that (using Corollary R)

$$\zeta \geq 1/2. \tag{3.4}$$

By the continuity of u and u_1 we see that (2.1) and (2.2) hold on Σ_ζ. If (2.3) does not hold on Σ_ζ, then by Lemma P we have $u(x) = u(x^\zeta)$ $\forall x \in \Sigma_\zeta$. In particular, by the continuity of u we have

$$u(x^\zeta) = 0 \quad \text{if} \quad \|x\| = 1 \tag{3.5}$$

Thus, setting $\sigma = \mu/m$ where μ is as in Corollary R and $m > 2\zeta$ is large enough so that $\zeta + \sigma < 1$, we obtain,

$$\begin{aligned} 0 &= u(\zeta+\sigma,\sqrt{1-(\zeta+\sigma)^2},0,\ldots,0) \\ &= u(\zeta-\sigma,\sqrt{1-(\zeta+\sigma)^2},0,\ldots,0). \end{aligned} \tag{3.6}$$

But

$$\begin{aligned} \|(\zeta-\sigma,\sqrt{1-(\zeta+\sigma)^2},0,\ldots,0\|^2 &= (\zeta-\sigma)^2 + 1 - (\zeta+\sigma)^2 \\ &= 1 - 2\zeta\sigma \\ &> 1 - m\sigma \\ &= 1 - \mu, \end{aligned}$$

hence (3.6) contradicts Corollary R. Thus (2.3) hold on Σ_ζ.

Now we claim that:

> there exists $\eta > 0$ such that if $\|x\| < 1$ and $x.e > \zeta-\eta$
> then $u_1(x) < 0$. (3.7)

If not there exists a sequence $\{x_n\}$ with $\|x_n\| < 1$, $\{x_n.e\}$ converging to ζ, and $u_1(x_n) \geq 0$. Thus, without loss of generality, we can assume that $\{x_n\}$ converges to $z \in T_\zeta \cap \bar{B}$. If $z \in T_\zeta \cap B$ then by (2.5) we have

$$0 > u_1(z) = \lim u_1(x_n) \geq 0, \tag{3.8}$$

which is a contradiction. On the other hand if $z \in T_\zeta \cap \partial B$, by Lemma Q for n large $u_1(x_n) < 0$ which contradicts the definition of $\{x_n\}$, and (3.7) is proven.

Next we show that:

there exists $\Upsilon \in (0,\eta)$ such that if $t \in (\zeta-\Upsilon,\zeta)$, $x \in \Sigma_t$ then
$u(x^t) \geq u(x)$. (3.9)

In fact, if not there exists a sequence (t_n,x_n) with $t_n \to \zeta$, $x_n \in \Sigma_{t_n}$ such that $u(x_n^{t_n}) < u(x_n)$. Without loss of generality we can assume that $\{x_n\}$ converges to z. Since $t_n \to \zeta$ we see that $z \in \bar{\Sigma}_\zeta$. If $z \in \Sigma_\zeta$ we have

$$u(z^\zeta) = \lim u(x_n^{t_n}) \leq \lim u(x_n) = u(z) \tag{3.10}$$

contradicting that (2.4) holds on Σ_ζ. Thus either $\|z\| = 1$ or $z \in T_\zeta \cap B$. If $z \in T_\zeta \cap B$ then $x_n^{t_n} \to z$. By the mean value theorem there exists a sequence $\{y_n\}$ with y_n in the segment joining $x_n^{t_n}$ with x_n such that $u_1(y_n) \geq 0$. Since for n large $y_n.e > \zeta-\eta$, by (3.7) we have a contradiction. Thus $\|z\| = 1$ and $z.e > \zeta$. Now for $x \in \bar{\Sigma}_\zeta$ we define $w(x) = u(x^\zeta) - u(x)$. Since (2.4) hold on Σ_ζ we have $w > 0$ on Σ_ζ. Since, also by the continuity of u we have

$$u(z^t) = \lim u(x_n^{t_n}) \leq \lim u(x_n) = u(z) = 0, \tag{3.11}$$

and hence $w(z) = 0$. Thus by Hopf's maximum principle (see Lemma H in [G-N-N]) we have $0 > w_1(z)$. Since $u \geq 0$, by (3.11) we have $u_1(z^\zeta) = 0$. Further since $u \geq 0$, $u_1(z) \leq 0$. Hence

$$0 > w_1(z) = -u_1(z^\zeta) - u_1(z) \geq 0. \tag{3.12}$$

This contradiction proves (3.9), and we have shown that for some $\theta < \zeta$, (2.2) holds for all $t \in [\theta, \zeta)$. By taking $\theta \in (\zeta - \eta, \zeta)$, from (3.7) we see that if $t \in [\theta, \zeta)$ then (2.1) as well as (2.3) hold. Hence this contradicts the definition of ζ. Thus there does not exist y B such that $u(y) = 0$. That is, every nonnegative solution to (1.1)-(1.3) is positive in B, and by Theorem 1 of [3] it is also radially symmetric, which proves Theorem A.

4. EXTENSIONS

Doublechecking the proof of Theroem A it is readily seen that nonnegative solutions to (1.1)-(1.2) are actually positive in the interior if Ω satisfies the following geometrical condition.

For each $y \in \Omega$ there exists $x \in \partial\Omega$ such that (4.1)

(a) $\{z;\ (z - x) \cdot \nu(x) > 0\} \cap \Omega = \phi$,

(b) For some $t < 0$, $y \in \Sigma_{t,x} \cup \Sigma'_{t,x}$, where $\Sigma_{t,x}$ is the connected component of $\{z \in \Omega;\ (z - x) \cdot \nu(x) > t\}$ containing x in its closure and $\Sigma'_{t,x}$ the reflexion of $\Sigma_{t,x}$ across the plane $\{z;\ (z - x) \cdot \nu(x) = t\}$,

(c) for all $z \in \partial\Omega \cap \partial\Sigma_{t,x}$ $\nu(z) \cdot \nu(x) > 0$, and $\Sigma_{t,x} \cup \Sigma_{t,x} \subset \Omega$.

Examples of regions satisfying (4.1) include balls, the region between two balls, and the union of two balls. The condition that Ω be bounded is necessary. Indeed, if $\Omega = \{(x,y);\ 0 < x < 1,\ y \in R\}$, v is a nonnegative solution to $-v'' = f(v)$, $v(0) = v(1) = 0$ with zeroes in

(0,1) (see [1], [2], [3]) then defining $u(x,y) = v(x)$ we have a solution to (1.1)-(1.2) with interior zeroes.

Finally, as noted in the introduction, examples of non-negative solutions in bounded regions with interior zeroes can be found in [1] and [3].

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