NON–ORDERED QUANTUM LOGIC AND ITS YES–NO REPRESENTATION

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Summary. It is shown that orthomodular lattice is an ortholattice in which a *unique* operation of bi–implication corresponds to equality relation and that the ordering relation in the binary formulation of quantum logic as well as the operation of implication (conditional) in quantum logic are completely irrelevant for their axiomatization. The soundness and completeness theorems for the corresponding algebraic unified quantum logic are proved. A proper semantics, i.e., a representation of quantum logic is given by means of a new YES–NO relation which enables a proof of the finite model property and decidability of quantum logic. A statistical YES–NO physical interpretation of the quantum logical propositions is provided.

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1. INTRODUCTION

Quantum logic is in the literature considered to be a logic, a partially ordered set, a lattice, a probabilistic structure, a modal structure,... All these structures share one thing: the Hilbert space is their common model. Therefore they are not really varieties of a *basic Hilbertian structure* but only different techniques available in approaching quantum measurements.

The quantum structures differ significantly from the classical ones and therefore it has repeatedly been questioned whether we can smoothly apply logical, probabilistic, lattice and modal technique to quantum measurements. For, in quantum logic the distributivity, modularity, the object language Modus Ponens, and other *classical* "objectives" are lost, in quantum probability theory the Kolmogorovian axioms do not hold, etc., etc. The attempt to overcome the differences by declearing logics and probability theories *empirical* did not help much since that "move" could not make standard logical or probability methods any more applicable to quantum logic or to quantum probability theory. In fact, over the past 20 years we have been piling up more and more unanswered questions and apparently only answering these questions can help us to decide whether we can effectively use logical, modal, or lattice technique in elaborating quantum measurements.

Some of the questions are:

— Is there a *unique* object language operation which can take over the role of the unique classical operation of implication (conditional, set–theoretic inclusion)?

— Is the usual irreflexive and symmetric orthogonality relation appropriate for set–theoretic representation of quantum theoretic measurements? Can such an orthogonality relation provide a relation of accessibility within the modal and Kripkean approach to quantum logic?

— Does quantum logic have the finite model property?

— Is quantum logic decidable?

In this paper we answer most of the questions and obtain a novel representation of quantum logic and quantum measurements.

Essentially, one of the obtained results makes the ordering within quantum sets irrelevant and substitutes the identity for the ordering relation. This renders the usual techniques of logic as a deductive *inferential* theory inappropriate and ascribes quantum deductive logic a particular *equational* meaning. The result is obtained in Sec. 3.

Another result, enables a representation of quantum logic by means of an intransitive and symmetric YES–NO relation (instead of the projectors–stemmed irreflexive and symmetric orthogonality relation) and proves quantum logic decidable as well as possessing the finite model property. This makes the usual modal, Kripkean, and imbedding approaches inapplicable since an intransitive relation does not correspond to any modal formula in the coresponding systems. In a word, the following opinion by Goldblatt turns out to be fully justified: "It is perhaps the first example of a natural and significant logic that leaves the usual methods defeated."[1] The representation is presented in Sec. 4.

In Sec. 5 we provided a physical interpretation of the YES–NO representation based on the statistics of measurements.

2. LATTICE *VS*. LOGICAL APPROACH TO QUANTUM THEORY

Before we dwell, in the next section, on the new results enabled by a departure from the usual techniques we shall first present here some previous recent results that stress particular points at which we have to start the departure and which are mostly concerned with the following two aspects of lattices and logics.

Lattices were formulated in order to describe the set theoretical aspects of a theory of partially ordered sets with a supremum and infimum but so as to keep to the methods of the universal algebras. This was achieved by representing the supremum and infimum with the help of the object language operations of conjunction and disjunction. Partial ordering is then also representable by means of such operations.

Logics, on the other hand, serves us to make empirical claims by means of set–theoretical predicates, to conclude from one statement (proposition) to another (i.e. to *infer* one from another) in a deductive way and to model the obtained structure by lattices as their algebraical models (using classes of equivalence). Logics also rely on the operations of conjunction and disjunction and in particular on the operation of implication (conditional), however, not to "algebraize" the logic but to facilitate deduction and inference. The latter possibility stems from the fact that in classical logic a *unique* operation of implication coresponds to the relation of implication (ordering relation). Therefore, to invoke *an operation of implication* is often considered unavoidable for a proper characterisation of any deductive theory. The modal semantics of the classical logic is but a further characterization of relations between classical "logico–empirical" deductive propositions.

Thus, these "techniques" (lattice and logical methods) are perfectly suited for a description of the classical phase space. But when we try to apply them to the Hilbert space we soon realize that we have to twist the techniques significantly if we want to force them to give us results.

Quantum theory, to start with, generates five different conditionals (in the orthomodular lattice and logic) which reduce to the classical conditional when the propositions are *commensurable*.

We have shown elsewhere[2] that the orthomodularity boils down to the equivalence of all 5 mentioned conditionals with the lattice theoretical conditional (the *relation* of implication) and we also formulated[3,4] unified quantum logic which gives a common and unique axiomatization for all possible conditionals.

Orthomodularity is thus reduced to a connection between object language implication and the model language ordering relation. The unified quantum logic then represents this connection as a connection between the two kinds of truths: the truth of a valuation and the object language defined truth. (Cf. R4 – rule of inference from reference 4.)

Let us introduce unified quantum logic in some details.

Its propositions are based on elementary propositions $p_0, p_1, p_2, ...$ and the following connectives: $\neg$ (negation), $\rightarrow$ (implication), and $\vee$ (disjunction).

The set of propositions Q° is defined formally as follows:
p_j is a proposition for $j = 0, 1, 2, ...$
$\neg A$ is a proposition iff A is a proposition.
$A \rightarrow B$ is a proposition iff A and B are propositions.
$A \vee B$ is a proposition iff A and B are propositions.

The conjuction is introduced by the following definition: $A \wedge B \stackrel{\text{def}}{=} \neg(\neg A \vee \neg B)$.

Our metalanguage consists of axiom schemata from the object language as elementary metapropositions and of compound metapropositions built up by means of the following metaconnectives: $\sim$ (*not*), & (*and*), $\veebar$ (*or*), $\Rightarrow$ (*if..., then*), and $\Leftrightarrow$ (*iff*), with the usual *classical* meaning.

The bi–implication is defined as: $A \leftrightarrow B \stackrel{\text{def}}{=} (A \rightarrow B) \wedge (B \rightarrow A)$.

We define unified quantum logic UQL as the axiom system given below. The sign $\vdash$ may be interpreted as "*it is asserted in* UQL." Connective $\neg$ binds stronger and $\rightarrow$ weaker than $\vee$ and $\wedge$, and we shall occasionally omit brackets under the usual convention. To avoid a clumsy statement of the rule of substitution, we use axiom schemata instead of axioms and from now on whenever we mention axioms we mean axiom schemata.

Axiom Schemata.

A1. $\vdash A \rightarrow A$

A2. $\vdash A \rightarrow \neg\neg A$

A3. $\vdash \neg\neg A \rightarrow A$

A4. $\vdash A \rightarrow A \vee B$

A5. $\vdash B \rightarrow A \vee B$

A6. $\vdash B \rightarrow A \vee \neg A$

Rules of Inference.

R1. $\vdash A \rightarrow B \quad \& \quad \vdash B \rightarrow C \quad \Rightarrow \quad \vdash A \rightarrow C$

R2. $\vdash A \rightarrow B \quad \Rightarrow \quad \vdash \neg B \rightarrow \neg A$

R3. $\vdash A \rightarrow C \quad \& \quad \vdash B \rightarrow C \quad \Rightarrow \quad \vdash A \vee B \rightarrow C$

R4. $\vdash (B \vee \neg B) \rightarrow (A \rightarrow B) \quad \Leftrightarrow \quad \vdash A \rightarrow B$

The operation of implication $A \rightarrow B$ is one of the following:

$$A \rightarrow_1 B \stackrel{\text{def}}{=} \neg A \vee (A \wedge B) \qquad \text{(Mittelstaedt)}$$

$$A \rightarrow_2 B \stackrel{\text{def}}{=} B \vee (\neg A \wedge \neg B) \qquad \text{(Dishkant)}$$

$$A \rightarrow_3 B \stackrel{\text{def}}{=} (\neg A \wedge \neg B) \vee (\neg A \wedge B) \vee \big((\neg A \vee B) \wedge A\big) \qquad \text{(Kalmbach)}$$

$$A \rightarrow_4 B \overset{\text{def}}{=} (A \wedge B) \vee (\neg A \wedge B) \vee \big((\neg A \vee B) \wedge \neg B\big) \qquad \text{(non–tollens)}$$

$$A \rightarrow_5 B \overset{\text{def}}{=} (A \wedge B) \vee (\neg A \wedge B) \vee (\neg A \wedge \neg B) \qquad \text{(relevance)}$$

UQL without the rule R4 is orthologic, also called minimal quantum logic.

To prove that UQL is really quantum logic we have to prove that UQL has an orthomodular lattice as a model. By the orthomodular lattice we mean algebra $L =< L^\circ, ^\perp, \cup, \cap >$ such that the following conditions are satisfied for any $a, b, c \in L^\circ$:

L1.	$a \cup b = b \cup a$
L2.	$(a \cup b) \cup c = a \cup (b \cup a)$
L3.	$a^{\perp\perp} = a$
L4.	$a \cup (b \cup b^\perp) = b \cup b^\perp$
L5.	$a \cup (a \cap b) = a$
L6.	$a \cap b = (a^\perp \cup b^\perp)^\perp$
L7.	$a \supset_i b = c \cup c^\perp \quad \Longrightarrow \quad a \le b \qquad (i = 1,\ldots,5)$

where $a \le b \overset{\text{def}}{=} a \cup b = b$ and $a \supset_i b$ $(i = 1,\ldots,5)$ is defined in a way which is completely analogous to the afore–given one in the logic. From now on we shall use the following denotation: $a \cup a^\perp \overset{\text{def}}{=} 1$ and $a \cap a^\perp \overset{\text{def}}{=} 0$. Of course, L is also orthocomplemented since lattices with unique orthocomplements and orthomodular lattices coincide.[5]

Algebra $< L^\circ, ^\perp, \cup, \cap >$ in which the conditions L1–L6 are satisfied is an ortholattice.

Algebra $< L^\circ, ^\perp, \cup, \cap >$ in which L1–L6 hold and L7 is satisfied by $a \supset b \overset{\text{def}}{=} a^\perp \cup b$ is a distributive lattice with 1 and 0 (Boolean algebra).

That L is really an orthomodular lattice, i.e. that L7 can be used instead of the usual orthomodular law $a \cup b = \big((a \cup b) \cap b^\perp\big) \cup b$, we proved in references 2 and 3.

To prove that the lattice is a model for the unified quantum logic we introduce the following definitions.

Definition 2.1. *We call $\mathcal{L} =< L, h >$ a model of the set Q° if L is an orthomodular lattice and if h: UQL $\mapsto L$ is a morphism in L preserving the operations $\neg$, $\vee$, and $\rightarrow$ while turning them into $^\perp$, $\cup$, and $\supset_i$ $(i = 1,\ldots,5)$, and satisfying $h(A) = 1$ for any $A \in Q^\circ$ for which $\vdash A$ holds.*

Definition 2.2. *We call a proposition $A \in Q^\circ$ true in the the model $\mathcal{L}$ if for any morphism h: UQL $\mapsto L$, $h(A)$=1 holds.*

We prove the soundness of UQL for valid formulas from L by means of the following theorem.

Soundness theorem 2.1. $\vdash A$ *only if* A *is true in any orthomodular model of* UQL.

Proof. By analogy with the binary formulation of quantum logic,[2,6] it is obvious that A1–A6 hold true in any $\mathcal{L}$, and that the statement is preserved by applications of R1–R3. Verification of R4 is also straightforward and we omit it. ■

Some further theorems and formulas for subsequent usage one can find in references 3 and 4, as well as the proofs of the following theorems.

Theorem 2.2. *Let* UQL_i *denotes* UQL *with* $\to = \to_i$, $i = 1,\ldots,5$. *Then in any* UQL_i *we can infer* A1–A6 *and* R1–R4 *for any* $\to_j$, $j = 1,\ldots,5$.

Theorem 2.3. UQL *with* $A \to B \;=\; A \rightharpoonup B \stackrel{\mathrm{def}}{=} \neg A \vee B$ *is classical logic.*

To prove the completeness of UQL for the class of valid formulas of L, we first define relation $\equiv$ and prove some related lemmas.

Definition 2.3. $\qquad A \equiv B \stackrel{\mathrm{def}}{=} \;\vdash A \leftrightarrow B.$

where $\vdash A \leftrightarrow B$ means $\vdash A \to B$ & $\vdash B \to A$.

Lemma 2.1. *The relation* $\equiv$ *is a congruence relation on the algebra of propositions* $\mathcal{A} = < Q^\circ, \neg, \vee, \to >$.

Lemma 2.2. *The Lindenbaum–Tarski algebra* $\mathcal{A}/\equiv$ *is an orthomodular lattice, i. e., the conditions defining the lattice are true for* $\neg/\equiv$, $\vee/\equiv$, *and* $\to /\equiv$ *turning into* $^\perp$, $\cup$, *and* $\supset_i$ *by means of natural isomorphism* $k{:}\mathcal{A} \mapsto \mathcal{A}/\equiv$ *which is induced by the congruence relation* $\equiv$ *and which satisfies* $k(\neg A) = [k(A)]^\perp$, $k(A \vee B) = k(A) \cup k(B)$, *and* $k(A \to B) = k(A) \supset_i k(B)$.

Completeness theorem 2.4. *If* A *is true in any model of* UQL, *then* $\vdash A$.

Proof. The proof is an obvious modification of the analogous proof from reference 3 and we omit it. ■

Taken together, UQL is a proper quantum–logical deductive system so far as its algebraic semantics is concerned.

However, although UQL provides the same axiomatic frame for all five implications it nevertheless splits into five different logics as follows from the following theorem.

Theorem 2.5. *Any orthomodular lattice in which* $a \supset_i b = a \supset_j b$ $(i, j = 1,\ldots,5, i \neq j)$ *is distributive.*

Proof. The proof can be easily carried out for all cases by means of the *commensurability* condition: $a \cap (a^\perp \cup b) \le b$,[2] which is, in effect, Foulis' condition ((iii) of Lemma 2 of reference 7) for Sasaki's *permutability*. Therefore, we shall only examplify it for $i = 1$ and $j = 2$.

We start with $a^\perp \cup (a\cap b) = b\cup(a^\perp \cap b^\perp)$. Using lattice analogues to A1, R1, and R2 we obtain: $b^\perp \cap (a\cup b) \le a$ which boils down to the commensurability of a and b. Since this holds for any $a, b \in L^\circ$ we obtain the distributivity.

In a similar way we proceed for any $i, j = 1,...,5$, $i \neq j$. ■

Corrolary 2.1. *For commensurable elements* $a \supset_i b \;=\; a \supset b \;= a^\perp \cup b, \quad i = 1,...,5.$

Since we can not deal with five logics at once, e.g. already the propositional *ortho–Arguesian law*[8] forces us to make up our mind as to which conditional we should keep to, we shall now dwell to some new results which open a new approach to quantum logic or even more likely the other way round.

3. ALGEBRAIC AXIOMATIZATION OF UNIFIED QUANTUM LOGIC

In the previous section we have shown how both quantum and classical logics are characterized by ascribing at the same time the logical and the object language $(\neg A \vee A)$ truth to the operation of implication within orthologic. The particular feature of classical logic (as opposed to quantum logic) which is "responsible" for the success of its methods is that the ascription for the classical implication is unique. Equivalently, both orthomodular and distributive lattices are characterized by determining, in an ortholattice, the ordering relation with the help of the object language implications being equal to one. And again the particular feature of the Boolean algebra is that such a determination is unique as opposed to orthomodular lattice.

Our idea then was that for orthomodular logic the ordering is not at all so important. The idea proved right through the following theorems which put equation in place of ordering equation (relation of implication) and identity (bi–implication, biconditional) in place of operation of implication (conditional).

Definition 3.1. *We call the expression* $(a \supset_i b) \cap (b \supset_i a)$ $(i = 1,\dots,5)$ *identity and denote it by* $a \equiv b$*. The two elements* a, b *satisfying* $a \equiv b = 1$ *we call identical.*

Definition 3.2. *We call the expression* $(a \supset b) \cap (b \supset a)$ $(i = 1,\dots,5)$ *classical identity and denote it by* $a \equiv_0 b$*. The two elements* a, b *satisfying* $a \equiv_0 b = 1$ *we call classically identical.*

Lemma 3.1. *In any orthomodular lattice:* $a \equiv b = (a \cap b) \cup (a^\perp \cap b^\perp)$.

Proof. We omit the easy proof. To our knowledge the lemma was first mentioned by Hardegree.[9] ■

Lemma 3.2. *In any ortholattice:* $a \equiv_0 b = (a^\perp \cup b) \cap (a \cup b^\perp)$.

Proof. Obvious by definition. ■

The main theorem of this section is the following one. It characterizes orthomodular lattice by means of the operation of identity and the lattice theoretical equation instead of the operation of implication and the lattice theoretical ordering.

Theorem 3.1. *An ortholattice in which any two identical elements are equal, i.e. in which* L7′. $a \equiv b = 1 \quad \Longrightarrow \quad a = b$ *holds, is an orthomodular lattice and vice versa.*

Proof. The vice versa part follows directly from L7 and Def. 1 since right to left meta–equivalence holds in any ortholattice. So we have to prove the orthomodularity condition by means of L1–L6 and L7′. Let us take the following well–known form[2] of the orthomodularity:

$$a \le b \quad \& \quad b^\perp \cup a = 1 \qquad \Longrightarrow \qquad b \le a$$

The first premise can be written as $a\cup b=b$ and as $a\cap b=a$. The former equation can be, by using the lattice analogue for R2, written as $b^\perp=a^\perp\cap b^\perp$. Introducing these $b^\perp$ and a into the second premise the latter reads: $(a^\perp\cap b^\perp)\cup(a\perp b)=1$. Now L7′ gives $a=b$ which is, in effect, the wanted conclusion. ∎

This extraordinary feature of orthomodular lattices and therefore of quantum logic characterizes them in a similar way in which the ordering relation *vs.* the operation of implication characterizes distributive lattices. In other words, the identity which makes two elements both identical and equal in an ortholattice thus making the lattice orthomodular is unique. We could prove this directly but it is much nicer to prove instead, that the classical identity which makes any two elements of an ortholattice both classically identical and equal does not turn the lattice into a distributive one but makes it a lattice which is between being genuinely orthomodular and distributive. (That, by doing so, we at the same time prove the wanted uniqueness of the identity stems from the fact the there are only five implications in an orthomodular lattice which reduce to the classical one for commensurable elements. To our knowledge Hardegree was first who observed[9] that Kotas' theorem[10] on the existence of exactly five (plus classical itself) such implications in any modular lattice is valid for orthomodular lattices as well.[11] It should be noticed at this point that in an ortholattice $a\supset_i b=1$ & $b\supset_i a=1$, $(i=1,\dots,5)$, is equivalent to $(a\supset_i b)\cap(b\supset_i a)=1\ \ (i=1,\dots,5)$.)

Theorem 3.2. *An ortholattice in which any two classically identical elements are equal, i.e. in which* L7″. $a\equiv_0 b=1 \iff a=b$ *holds, is a non–genuine orthomodular lattice which is not distributive.*

Proof. We shall first prove that L7″ implies L7′.

Using $a\cup(b\cap c)\le(a\cup b)\cap(a\cup c)$, which holds in any ortholattice we easily obtain that $(a\cap b)\cup(a^\perp\cap b^\perp)=1$ implies $\big((a^\perp\cap b^\perp)\cup a\big)\cap\big((a^\perp\cap b^\perp)\cup b\big)=1$. Using $a\cup(b\cap c)\le(a\cup b)\cap(a\cup c)$ again for each conjunct of the latter equation we easily obtain that it implies $(a\cup b^\perp)\cap(b\cup a^\perp)=1$. Now L7″ gives $a=b$. Thus $(a^\perp\cap b^\perp)\cup(a\cap b)=1$ implies $a=b$. Hence L7′.

Therefore, a lattice in which L1–L6 and L7″ are satisfied is orthomodular. However, it is not a genuinely orthomodular since L7″ violates most orthomodular lattices from MacLaren's $\mathcal{L}_{10}$ till *Chinese lantern MO2*.

However, such a lattice is not distributive because the distributivity would imply, by L7″ and Theorem 2.3, the validity of the following theorem in classical logic: $\vdash\big((A\wedge B)\rightharpoonup(C\wedge D)\big)\rightharpoonup(A\rightharpoonup C)$. Since this is obviously not a theorem in classical logic we obtain the claim. ∎

The previous theorems enable us to axiomatize unified quantum logic in a completely algebraical way, thus practically identifying quantum logic and its algebraical model — the orthomodular lattice. From this marriage orthomodular lattice gains the ease of infering formulas and availability of different logical semantics as e.g. probabilistic semantics, thus becoming an algebraico–deductive system. Quantum logic, on the other hand, gains a new representation by means of equational algebraical set–theoretical properties thus becoming a decidable system with the finite model property.

More details on all these aspects will be presented in the next section. Here we shall only present the axiomatization itself. The axiomatization is not intended to provide a vehicle for *proving the the old things in a new garment* but simply a novel fact on orthomodular structures and quantum logic and a source for further new results. Therefore we shall next prove its soundness and completeness but we will not burden the reader with the unfamiliar axioms when proving other resluts in the next section.

We define algebraic unified quantum logic AUQL as the axiom system given below.

Axiom Schemata.

AL1. $\vdash A \vee B \leftrightarrow B \vee A$

AL2. $\vdash A \leftrightarrow A \wedge (A \vee B)$

AL3. $\vdash A \leftrightarrow A \wedge (A \vee \neg B)$

AL4. $\vdash (A \vee B) \vee C \leftrightarrow \neg\big((\neg C \wedge \neg B) \wedge \neg A\big)$

Rule of Inference.

RL1. $\vdash (B \vee \neg B) \leftrightarrow (A \leftrightarrow B) \qquad \Longrightarrow \qquad \vdash A \leftrightarrow B$

where the bi–implication is defined as: $A \leftrightarrow B \overset{\text{def}}{=} (\neg A \wedge \neg B) \vee (A \wedge B)$.

Definition 3.3. *We call $\mathcal{L} = < L, h >$ a model of the set Q° (of propositions from* AUQL*) if L is an orthomodular lattice and if h:* AUQL $\mapsto L$ *is a morphism in L preserving the operations $\neg$, $\vee$, and $\leftrightarrow$ while turning them into $^\perp$, $\cup$, and $\equiv$, and satisfying $h(A) = 1$ for any $A \in Q^\circ$ for which $\vdash A$ holds.*

Definition 3.4. *We call a proposition $A \in Q^\circ$ true in the the model $\mathcal{L}$ if for any morphism h:* AUQL $\mapsto L$*, $h(A) = 1$ holds.*

Soundness theorem 3.3. $\vdash A$ *only if A is true in any orthomodular model of* AUQL.

Proof. Sobociński's postulate–system for ortholattices, which we actually translated into the logic, would make our proofs of AL1–AL4 redundant. So, we omit them. The proof of RL1 is with the help of Theorem 1 straightforward and we omit it as well. ■

Lemma 3.3. *The Lindenbaum–Tarski algebra $\mathcal{A}/\leftrightarrow$ is an orthomodular lattice with the natural isomorphism k:$\mathcal{A} \mapsto \mathcal{A}/\leftrightarrow$ which is induced by the congruence relation $\leftrightarrow$ and which satisfies $k(\neg A) = [k(A)]^\perp$, $k(A \vee B) = k(A) \cup k(B)$, and $k(A \leftrightarrow B) = k(A) \equiv k(B)$.*

Completeness theorem 3.4. *If A is true in any model of* AUQL, *then* $\vdash A$.

Proof. The proof is straightforward and we omit it. ■

Remark. As we already stressed above the algebraic unified quantum logic AUQL is not intended to substitute the usual axiomatization but only to provide a distinguishing characterization of quantum logic by means of a unique operation — *bi–implication* — which directly stems from the operations of implication whose classical form — classical implication — in turn serves for a unique characterization of classical logic. This new characterization will in the next section generate some further novel results but in approaching them we shall retain the whole usual logical machinery, in particular Ackermann's binary formulation which Kotas applied to modular and Goldblatt to orthomodular logic, then MacLaren's set–theoretic characterization and Goldblatt's set–theoretic semantics, etc. The reason for that is twofold. First, the main appeal of the mentioned structures lies in the ease of the deriving new formulas, checking on decidability, etc., and this *ease* is based on particular properties of the underlying orthostructure on which orthomodularity or distributivity can be built. Eg., A2, A3, A6, and R2 express orthocomplementarity of an orthostructure and we know that i) a uniquely orthocomplemented ortholattice is an orthomodular lattice,[5,12] and ii) a uniquely complemented ortholattice is a Boolean algebra.[12,13] (Said Greechie on the recent biannual meeting of the International Quantum Structures Association: "I've learned recently that in dealing with quantum structures we should always start from ortho–algebras.") Secondly, we simply could not stand the idea of forcing the reader — and ourselves — through another new axiomatization and formalism.

4. YES–NO REPRESENTATION OF QUANTUM LOGIC

Comparing the representations by means of the operations of implication and bi–implication presented in Sections 2 and 3, respectively, we can easily come to a conjecture that other ordering–like quantum logic concepts can be redefined along a similar line eventually bringing us to a new modelling and proper semantics of quantum logic.

The first concept we should check on is of course the *orthogonality*. We say that elements a and b of an ortholattice are orthogonal and we write $a\perp b$ iff $a \le b^\perp$. This definition which we can *read off* from the algebra of projectors from the Hilbert space is perfectly suited for a representation of orthologic and ortholattices proper.[1,14–16] For, we can rather straightforwardly impose particular conditions on the orthogonality which give us the soundness as well as the completeness of the representation, the orthoframe, the canonical model, the finite model property and the decidability.

Not so when we add the orthomodularity condition to orthologic (ortholattices), i.e. when we deal with quantum logic (the orthomodular lattices). It is then possible to represent the logic by means of conditions imposed on the frame but not by means of the conditions of the first order imposed on the above orthogonality (which appears as the relation of accessibility in the Kripkean, i.e. modal approach) as proved by Goldblatt.[1] Thus it is still not known whether there is a class of orthoframes which determines the logic.[1,6,17]

However, we can approach the whole problem from the *"equational side"* picking up another relation which is not orthogonal but, let us say, *orthogonal–like* relation.

The guideline for the new orthogonal–like relation is the equation $a = b^\perp$ instead of the inequation $a \le b^\perp$. The new relation does not follow the algebra of projectors but the algebra of YES–NO linear subspaces and their orthocomplements. It is given in a set–theoretic way and it is weaker than (i.e. it follows from) MacLaren's orthogonality.[18] We shall call it the YES–NO relation since it perfectly corresponds to YES–NO quantum experiments.

Let us start establishing our representation (semantics) by introducing the YES–NO quantum frame and the YES–NO relation for the algebraic unified quantum logic.

Definition 4.1. *$\mathcal{F} =< X, \ominus >$ is a* YES–NO *quantum frame iff X is a non–empty set, the carrier set of $\mathcal{F}$, and $\ominus$ is a* YES–NO *relation, i.e. $\ominus \subseteq X \times X$ is symmetric and intransitive.*

Of course, the relation is also irreflexive since irreflexivity follows from intransitivity.

Definition 4.2. *Y is said to be a* YES–NO *subset iff*

$$Y \subseteq Z \subset X \quad \Longrightarrow \quad (\forall x \in Z)(x \in Y \ \veebar\ x \ominus Y)$$

where $x \ominus Y \overset{\text{def}}{=} (\forall y \in Y)(x \ominus y)$.

In a pedestrian way we can say that any element of a proper subset of the carrier set X is either belonging to a subset of that subset or to its relative complement. To pick up a *proper* subset is important because a direct reference to X would bring us to the Boolean algebra instead of orthomodular lattice. Thus we rely on the well–known representation of orthomodular structures, by which they can be obtained by gluing together the Boolean algebras, the representation "initiated" by Greechie[19] and nicely formulated by Dietz: "An ortholattice is orthomodular if and only if every its orthogonal subset lies in a maximal Boolean subalgebra (a *block*) of the lattice."[20]

Lemma 4.1. *A* YES–NO *subset* $Y \subseteq Z \subset X$ *is* YES–NO *closed (in* $Z \subset X$*). If we denote* $Y^{\ominus} = \{x : x \ominus y, y \in Y\}$ *then* $Y^{\ominus\ominus} = Y$*.*

Proof. We have to prove:

$$(\forall x \in Z \subset X)\Big[(x \in Y \subseteq Z) \quad \veebar \quad (\exists z \in Z)\big((z \ominus Y) \;\&\; \sim (x \ominus z)\big)\Big].$$

If we assume $x \in Y$ the expression is obviuosly true. Let us suppose $x \notin Y$. According to Def. 2 we have $x \ominus Y$. Then for any $z \ominus Y$ we have got either $z = x$ and in this case the irreflexivity (deducible from the intransitivity) do the job or the intransitivity for any $y \in Y$ gives: $x \ominus y \,\&\, z \ominus y \implies \sim (x \ominus z)$.

Let us now prove $Y^{\ominus\ominus} = Y$. By definition, we have: $Y^{\ominus} = \{x : x \ominus y, y \in Y\}$ and $Y^{\ominus\ominus} = (Y^{\ominus})^{\ominus} = \{z : z \ominus x, x \in Y^{\ominus}\} = \{z : z \ominus x \;\&\; x \ominus y\}$. By intransitivity we get $Y^{\ominus\ominus} = \{z : \sim z \ominus y, y \in Y\}$ and this is nothing but Y by Def. 2. ■

To prove the soundness of our representation we introduce a YES–NO model by the following definition which is actually a modified Goldblatt's definition[6] for the orthomodel reformulated for our YES–NO case. We do so in order to stress the parallelism between the models: the orthogonal one and the YES–NO one.

Definition 4.3. $\mathcal{M} = < X, \ominus, V >$ *is a* YES–NO *quantum model on the* YES–NO *quantum frame* $< X, \ominus >$ *iff* V *is a function assigning to each propositinal variable* p_i *a* YES–NO *subset* $V(p_i) \subset X$*. The truth of a wff* A *at* x *in* $\mathcal{M}$ *is defined recursively as follows. (*$\mathcal{M} : x \models A$ *reads* A *holds at* x *in* $\mathcal{M}$*.)*

$$\begin{array}{lllll}
(1) & x \models p_i & \iff & x \in V(p_i) \\
(2) & x \models A \wedge B & \iff & x \models A \quad \& \quad x \models B \\
(3) & x \models \neg A & \iff & (\forall y)(y \models A \implies x \ominus y)
\end{array}$$

If we denote the set $\{x \in X : x \models A\}$ *by* $\|A\|$ *(or* $\|A\|^{\mathcal{M}}$*), the above reads:*

$$\begin{array}{ll}
(1') & \|p_i\| = V(p_i) \\
(2') & \|A \wedge B\| = \|A\| \bigcap \|B\| \\
(3') & \|\neg A\| = \{x : x \ominus \|A\|\}.
\end{array}$$

*If Γ is a non–empty set of wffs, then Γ **implies** A at x in $\mathcal{M}$, in symbols: $\mathcal{M} : x : \Gamma \models A$, iff $(\exists B \in \Gamma)(\mathcal{M} : x \models B \implies \mathcal{M} : x \models A)$. Γ $\mathcal{M}$–**implies** A, $\mathcal{M} : \Gamma \models A$, iff Γ implies A at all x in $\mathcal{M}$. If $\mathcal{F}$ is a* YES–NO *quantum frame, Γ $\mathcal{F}$–**implies** A, $\mathcal{F} : \Gamma \models A$, iff $\mathcal{M} : \Gamma \models A$ for all models $\mathcal{M}$ on $\mathcal{F}$. If $\mathcal{C}$ is a class of frames, Γ $\mathcal{C}$–**implies** A, $\mathcal{C} : \Gamma \models A$, iff $\mathcal{F} : \Gamma \models A$ for all $\mathcal{F} \in \mathcal{C}$. If $\Gamma = \{A \vee \neg A\}$ then we may simply write $\mathcal{M} \models A$, $\mathcal{F} \models A$, etc., and speak of **truth** of A in $\mathcal{M}$, $\mathcal{F}$–**validity** of A, etc. A class $\mathcal{C}$ of* YES–NO *quantum frames is said to **determine** quantum logic* (AUQL *or* UQL) *iff, for all A, $B \in Q^\circ$, $\vdash A \to B$ iff $\mathcal{C} : A \models B$.*

Lemma 4.2. *If $\mathcal{M}$ is a* YES–NO *model, then for any A set $\|A\|^{\mathcal{M}}$ is* YES–NO *closed.*

Proof. For $\|p_i\| = V(p_i)$, $V(p_i)$ is a YES–NO subset and the result holds by Lemma 1.

Provided that it holds for $\|A\|$ and $\|B\|$ it holds for $\|A \wedge B\|$ as well because the intersection of YES–NO subsets is obviously a YES–NO subset and therefore closed by Lemma 1.

To achieve a general result by induction on the length of formulas the negation remains to be considered. Let us suppose $x \notin \|A\|$. By Def. 2(3&3′) we then obtain $(\exists y)\big[y \models A \ \& \ \sim (x \ominus y)\big]$. Now, if $\mathcal{M} : z \models \neg A$ we get $y \models A \implies z \ominus y$ and by symmetry and the assumed existence of $y \models A$ we get $y \ominus z$. Thus (assuming $y \models A$) we get $y \ominus \|\neg A\|$ and $\sim (x \ominus y)$. So $\|\neg A\|$ is YES–NO closed. ■

Soundness theorem for YES–NO representation of quantum logic 4.1.

$$\vdash \Gamma \to A \quad \implies \quad \mathcal{C} : \Gamma \models A$$

where $\mathcal{C}$ is the class of all YES–NO *quantum frames.*

Proof. Let us first prove the derivability of UQL axioms and rules of inference.

A1 $x \models A \implies x \models A$ is a tautology.

A2 $x \models A \wedge B \iff (x \models A \ \& \ x \models B) \implies x \models A$

A3 $x \models A \wedge B \iff (x \models A \ \& \ x \models B) \implies x \models B$

A4 Let $x \models A$. Then, if $y \models \neg A$, by Def. 3.3, $y \ominus x$, and by symmetry $x \ominus y$ so that the same definition gives $x \models \neg\neg A$.

A5 Let $x \models \neg\neg A$. Then, $y \models \neg A \implies x \ominus y$, i.e. $y \ominus \|A\| \implies x \ominus y$. Since by Lemma 2 $\|A\|$ is YES–NO closed, we have $x \in \|A\|$, i.e., $x \models A$.

A6 $x \models A \wedge \neg A \iff \big[(\forall y)(y \models A \implies y \ominus x) \ \& \ x \models A\big]$. Hence $x \ominus x$ which is in contradiction with irreflexivity of $\ominus$. Thus $(\forall x)(\sim x \models A \wedge \neg A)$ and therefore $(\forall B)(A \wedge \neg A \models B)$.

R1 $\big[(x \models A \implies x \models B) \ \& \ (x \models B \implies x \models C)\big] \implies (x \models A \implies x \models C)$.

R2 Assuming $x \models A \implies x \models B$ and $x \models A \implies x \models C$ we obtain $x \models A \implies (x \models B \ \& \ x \models C)$, which gives $x \models A \implies x \models B \wedge C$.

R3 Suppose $\mathcal{C} : A \models B$, and also $\mathcal{M} : x \models \neg B$. Then $y \models A \implies x \ominus y$ by Def. 2. $A \models B$ gives $y \models A \implies y \models B$. Thus $y \models A \implies x \ominus y$, i.e. $x \models \neg A$.

R4 We have to prove $A \wedge \big(\neg A \vee (A \wedge B)\big) \models B$. Let us start with $A \wedge B \models A$ which is a tautology. Hence for any $\mathcal{M}$ we have $\|A \wedge B\| = \|A\| \bigcap \|B\| \subseteq \|A\|$ by definition of $\|A\|^{\mathcal{M}}$. For $y \in \|A \wedge B\|$, $\|A \wedge B\|$ is YES–NO closed. On the other hand, for $y \notin \|A \wedge B\|$ & $y \in \|A\|$, according to Def. 2, there is at least one y such that $y \ominus \|A \wedge B\|$ and by Lemma 1 $\|A \wedge B\|$ is YES–NO closed. Now, if $x \models A \wedge \neg\big(A \wedge \neg(A \wedge B)\big)$, then $x \in \|A\|$ and $(\forall y)\big[\sim y \in \|A\| \ \veebar\ (\sim y \ominus \|A \wedge B\|) \ \veebar\ x \ominus y\big]$. The latter expression boils down to $\big[(\forall y)(y \in \|A\|)\big] \implies \big[(\sim \exists y)\big((y \ominus \|A \wedge B\|)\ \&\ \sim x \ominus y)\big)\big]$. Thus for all $y \in \|A\|$ there is no one satisfying the second alternative of the YES–NO closure condition and therefore the first one: $x \in \|A \wedge B\|$. Thus $x \models A \wedge B$ and hence the orthomodularity.

The proof of the theorem follows by induction on UQL (AUQL) derivability. ∎

Thus we obtained that quantum logic really does have a YES–NO representation, i.e. a YES–NO model which is of our primary interest here. We are also able to prove the opposite, i.e. that the structure of which the YES–NO representation is a model is exactly quantum logic (UQL, AUQL), but for the proof we refer the reader to reference 21. This is mostly because the result is somewhat less interesting for possible physical applications and for a reconstruction of the Hilbert space as we clarify below.

Completeness theorem for quantum logic 4.2.

$$\mathcal{C} : \Gamma \models A \quad \implies \quad \vdash \Gamma \to A$$

where $\mathcal{C}$ is the class of all YES–NO *quantum frames.*

The completeness is accompanied with the finite model property and decidability of quantum logic (which are both — under particular restrictions — also proved in reference 21).

Decidability boils down to the fact that there is an effective procedure to decide on every non–thesis that it really is a non–thesis and this is very important for any axiomatization because it decides on whether the axiomatization is effective in the sense that it is recursive. The reason why the obtained decidability and the finite model property of quantum logic are not so important for physical applications and the Hilbert space in the present elaboration is the following. Our completeness proof — as opposed to other completeness proofs (given for another representation — by means of the orthogonality) by MacLaren,[22] Goldblatt,[6] Dishkant,[23] Morgan,[24] Iturrioz,[25,26] Nishimura,[15] ... — does provide a proof of the finite model property and the decidability but, on the other hand, they both turn out to be valid only for the finite case, i.e. for the case when there are finitely many *elementary* propositions in the logic. However, a finite propositional lattice (complete orthomodular one of the Jauch–Piron type, i.e. atomistic with the covering property) does not have the Hilbert space as a model, i.e. cannot serve for building up quantum mechanics on it.[27] Thus we have to approach a possible physical interpretation from another side and we will do so in the next section.

5. YES–NO PHYSICAL INTERPRETATION OF QUANTUM LOGIC

As a pedestrian example of a physical interpretation of quantum logic we take the simplest possible experimental situation of measuring spin–1 by a Stern–Gerlach device. Within such an experiment, for example, by opening one channel and blocking the other two on the device we test a proposition (A in the logic, i.e. a in the lattice) (e.g. *'spin–up'*) while by blocking the one and opening the other two we test its orthocomplement ($\neg A$, $a^{\perp}$). This is nicely presented and shown in Figs. 1 and 2 of Hultgren and Shimony's paper.[28]

That such an oversimplified experimental setup can at all be relevant for a general physical interpretation stems from the nature of the interpretation and the kind of meaning we ascribe to propositions.

The analysis of Hultgren and Shimony[28] of the spin–1 case showed that in building a complete Hilbert space edifice we cannot rely *only* on standard outcomes of the experiments carried out on *individual* systems. For, we cannot measure all the states we can describe with the help of the Hilbert space formalism by means of standard individual YES–NO measurements, i.e. there are states which are not eigenstates of the observables we measure. For example, if we decide to orient the measuring device in directions $\mathbf{n}$ in order to measure the spin components of the spin operator $\mathbf{s}$ whose eigenvectors are [1,0,0], [0,1,0], and [0,0,1], then the state $[1/\sqrt{6}, 1/\sqrt{3}, 1/\sqrt{2}]$ can easily be shown not to be an eigenstate of the measured operator $\mathbf{n}\cdot\mathbf{s}$. (We cannot obtain it by applying the rotation matrix on the eigenvectors.)

A possible remedy for such *unrepresentable* states (i.e. the states outside the logic or lattice of standard propositions) seems to be the disputed Jauch's infinite filter procedure for introducing conjunctions (meets, intersections) which cannot be measured directly (either within a single experiment or within a finite number of them) as new elements of the logic (lattice) needed for modelling by the Hilbert space.[28,29] In other words there are infinitely many atoms of the lattice of the subspaces of the Hilbert space which do not belong to the finite lattice of individual YES–NO spin–1 measurements but which can be recovered by the Jauch's procedure. This is not a problem for quantum logic if we look at it as at a structure which corresponds to the Hilbert space because the structure (complete uniquely orthocomplemented[30] atomistic lattice satisfying the covering law) demands by itself an infinite number of atoms.[31] But if we looked at quantum logic as a logic of YES–NO discrete measurements and try to recover the Hilbert space axioms by empirically plausible assumptions then we would obviously try to avoid any *infinitary*[32] procedure which, as Jauch's, in principle simply cannot be substituted by any *arbitrary long* one.[29] Can one offer anything as a substitute for the Jauch's infinitary procedure?

The infinitary Jauch's procedure serves us to obtain the complete Hilbertian structure which bare experimental propositions obtained from a standard experimental setup simply cannot offer — as shown by Hultgren and Shimony.[28] However, Swift and

Wright[33] have shown — answering to a challenge put forward by Hultgren and Shimony — that we can extend the standard experimental setup for measuring spins so as to make *every Hermitian operator acting on the Hilbert space of spin–s particle* measurable. In particular, they employed electric fields in place of magnetic ones thus making electric k–poles (in spin–1 case: quadrupoles) distinguishable. As a result one can offer a *complete* experimental setup for measuring *every* spin operator which is both *finitary* and repeatable, i.e. applicable to individual systems and which offers a complete set of elementary propositions. On the other hand, we can deal not with individual systems but with ensembles and represent states of the disputed kind ($[1/\sqrt{6}, 1/\sqrt{3}, 1/\sqrt{2}]$) as d'Espagnat's mixtures of the second kind.[34] These possibilities immediately address the question of approaching preparation–detection YES–NO procedure. Are we to take the *individual* or the *ensemble* approach?

If we adopt the individual approach, then we bring the old Bohr's "completeness solution" to the stage. That is, given the whole experimental arrangement we can always make an individual system determined by a discrete observable *repeatable*. But in that case we cannot apply the usual policy of standard quantum logicians and claim with them that the system itself is *determined* by its preparation, i.e. that it possesses a corresponding property which we unambiguously recover by a detection procedure. For, only by reffering to the whole experimental procedure — preparation as well as detection — can we say that a *system itself* "possesses" a *projection–*0 *of spin–*1 *property when prepared by a magnetic field* as opposed to an electric field. The system prepared one way or the other will pass the *middle channel* of a detecting device no matter whether that channel used a magnetic or an electric field to detect the state.[33] Since the electrical field is capable of disgusing quadrupole moment while the magnetic field can detect only dipole moments, the probability one ($p = 1$) of passing a particular filter, i.e. the repeatability, therefore has not got a sense without a reference to the whole preparation–detection procedure and the whole experimental setup: without knowing the orthocomplemention we cannot say to which set the measured observable belongs.

If we adopt the ensemble approach, we can apply the statistical approach to the definition of our propositions within the logic we use. In the above example, all channels taken into account within a long run unambiguously decide between dipols and quadrupoles, provided that the ensemble is prepared in a "clean" way and not as a mixture. If it is prepared as a mixture, it will also be unambiguously detected as such.

One can show that the statistical approach is not weaker then the individual approach but is rival with it.[21,35–38] This is not in any contradiction with the usual approach to quantum logic since a proposition can be as *legally* verifiable on a single repeatable individual system as well as on a beam coming out of a repeatable experimental arrangement: we just have to *postulate* whether "it is" one way or the other. Since that is often misunderstood[39] in the literature we shall provide some details here.

Let us take *repeatability* as "measure" of individual as opposed to statistical interpretation.

In order to verify whether an individual observed sytem is in the state $[1/\sqrt{6}, 1/\sqrt{3}, 1/\sqrt{2}]$ or in a completely unprepared state $[1\sqrt{3}, 1\sqrt{3}, 1\sqrt{3}]$ we have to measure not only its beam but also the beams of its orthocomplement, i.e. both *statistical "properties"* in the long run. No such mixture property can be encoded into an individual spin–1 particle. Thus we cannot speak of the repeatability of such systems. On the other hand, continuous observables[40] and discrete observables which do not commute with conserved quantities[41] are both known not to satisfy the repeatability hypothesis. So, for such observables no property can be prepared with certainty.

Apparently, these unrepeatable systems behave differently than the ones characterized by discrete observables.

But there is a way to treat all the observables, continuous as well as discrete, in a common way. Of course, we cannot make mixtures repeatable but we can exclude the repeatability for individual events — leaving only the statistical repeatability which turns into the approximate repeatability for continuous observables. In other words we can exclude the *individual repeatability* even for the discrete observables which undergo measurements of the first kind. In doing so we start with links between propositions and data.

The only way in which quantum theory connects the "elements of the physical reality" (i.e. what we observe) with their "counterparts in the theory"[42] is by means of the Born formula which gives us the probability that the outcome of an experiment will confirm an observable or a property of an *ensemble* of systems.[43] Strictly speaking, what we measure is the mean value of an operator, the scalar product, not the operator, not the state, not the wave function. When we say that a measurements yields the eigenvalue a or the state $|\psi_a>$ this is "slang." We can measure neither A, nor $|\psi_a>$, nor a. What a measurement of the pure state $|\psi_a>$ yileds is $\frac{<\psi_a|A|\psi_a>}{<\psi_a|\psi_a>}$ which is then *equal* to a.

In other words, in case of discrete observables we say that we are able to prepare a property whenever by an appropriate detection (determination, measurement), we can verify the property with certainty — i.e. with probability one / equal to unity,[43,44] i.e. almost certainly, almost sure[45] or "except on a null–event."[46] This means that for repeatable measurements we only know that a property will be verified with certainty (with probability one) — that is *on ensemble*. Whether the property will be verified on *each* so prepared individual system we can only guess. For, there is no "counterpart in the theory" of an individual detection even if it is carried out "with certainty": The Born probabilistic formula — which is the only link between the theory and measurements — refers only to ensembles. However, as shown below, we can consistently *postulate* whether a measurement of the first order is verifying a prepared repeatable property on *each* system or not.

The approach we take is resting on combining the Malus angle (between the preparing and the detecting Stern–Gerlach devices) expressed by probability with that expressed by relative frequency. To connect probability $0 < p < 1$ with the corresponding relative frequency we used the strong law of large numbers for the infinite number of Bernoulli trials which – being independent and exchangeable – perfectly represent quantum measurements on individual quantum systems. These properties of the individual quantum measurements we used to reduce their repeatability to successive measurements but that has no influence on the whole argumentation which rests exclusively on the fact that finitely many experiments out of infinitely many of them may be assumed to fail and to nevertheless build up to probability one.

The argument supporting the statistical interpretation is that probability one of e.g. electrons passing perfectly aligned Stern–Gerlach devices *does imply* that the relative frequency N_+/N of the number N_+ of detections of the prepared property (e.g. *spin–up*) on the systems among the total number N of the prepared systems approaches probability $p = \langle N_+/N \rangle = 1$ almost certainly:

$$P(\lim_{N\to\infty} \frac{N_+}{N} = 1) = 1, \tag{1}$$

but *does not imply* that N_+ analytically equals N, i.e. it does not necessarily follows that the analytical equation $N_+ = N$ should be satisfied.

We therefore must postulate what we want: either $N_+ = N$ *and* (1) or $N_+ \neq N$ *and* (1). We have to stress here that since already the central limit theorem itself, which served us to infer (1), holds only on the open interval $0 < p < 1$, it would be inconsistent to try to *prove* one or the other posssibility.

Of course, the possibility $N_+ \neq N$ doesn't seem very plausible by itself and we therefore used the Malus law to construct the function which reflects the two possibilities and proved a theorem which directly supports another difference between the probability and frequency treatment of individual quantum measurements.

As for the theorem we proved that

$$\lim_{N\to\infty} P(\frac{N_+}{N} = p) = 0, \qquad 0 < p < 1 \tag{2}$$

which expresses randomness of individual results as clustering only around p (*almost* never strictly at p).

As for the function which reflects the two above stated possibilities we will just briefly sketch it here. The reader can find all the relevant theorems and proofs in reference 35, a generalization to spin–s case in reference 37, and a discussion with possible implications on the algebraical structure underlying quantum theory in references 4 and 35. The function refers to the quantum Malus law and reads:

$$G(p) \stackrel{\text{def}}{=} L^{-1} \lim_{N\to\infty} \left[|\alpha(\frac{N_+}{N}) - \alpha(p)| N^{1/2} \right]$$

where α is the angle at which the detection device (a Stern–Gerlach device for spin–s particles, an analyzer for photons) is deflected with regard to the preparation device (another Stern–Gerlach device, polarizer) and where L is a bounded random (stochastic) variable: $0 < L < \infty$. The function is well defined and continuous (or piecewise continuous) on the open interval (0,1). In general it does not correspond to an operator but it does represent a *property* in the sense of von Neumann.[43] For electrons and for projection–0 of spin–1, it is equal to:[35]

$$G(p) = H(p) \stackrel{\text{def}}{=} H[p(\alpha)] = \frac{\sin\alpha}{\sin\alpha}$$

Turning our attention to the probability equal to one we see[35] from the definition of $H(p)$ that H is not defined for the probability equal to one: $H(1) = \frac{0}{0}$. However, its limit exists and equals 1. Thus a continuous extension $\tilde{H}$ of H to [0,1] exists and is given by $\tilde{H}(p) = 1$ for $p \in (0,1)$ and $\tilde{H}(1) = 1$.

We now assume that L is bounded and positive not only for $0 < p < 1$ but for $0 \le p \le 1$ as well.

Thus we are left with the following three possibilities (which hold for an arbitrary spin s too[37]).

1. $G(p)$ is continuous at 1. A necessary and sufficient condition for this is $G(1) = \lim_{p\to 1} G(p)$. In this case we cannot strictly have $N_+{=}N$ since then $G(1) = 0 \ne \lim_{p\to 1} G(p)$ obtains a contradiction.
2. $G(1)$ is undefined. In this case we also cannot have $N_+{=}N$ since the latter equation makes $G(1)$ defined, i.e. equal to zero.
3. $G(1) = 0$. In this case we must have $N_+{=}N$. And *vice versa*: if the latter equation holds we get $G(1) = 0$.

Hence, under the given assumptions a measurement of a discrete observable can be considered repeatable with respect to individual measured systems if and only if $G(p)$ exhibits a jump–discontinuity for $p{=}1$ in the sense of point 3 above.

The interpretative differences between the points are as follows.

1 & 2 admit only the statistical interpretation of the quantum formalism and banish the repeatable measurements on individual systems from quantum mechanics altogether. Of course, the repeatability in the statistical sense remains untouched. Possibility 1 seems to be more plausible than possibility 2 because the assumed continuity of G makes it approach its classical value for large spins.[37] Notably, for a classical probability we have $\lim_{p\to 1} G_{\rm cl}(p) = 0$ and for "large spins" we get $\lim_{s\to\infty} \lim_{p\to 1} G(p) = 0$.

3 admits the individual interpretation of quantum formalism and assumes that the repeatability in the statistical sense implies the repeatability in the individual sense. By adopting this interpretation we cannot but assume that nature differentiate open intervals from closed ones, i.e. distinguishes between two infinitely close points. (The same conclusion about nature we would have to draw if we assumed a sudden *jump* in definition of the random function L leaving $G(1)$ undefined.)

The main consequence of so formally different descriptions of quantum systems is therefore that the interpretations become rivals to each other. And the old problem as to whether an individual quantum system can be considered completely described by the standard formalism or not is given a new aspect: We are forced to make up our mind: either to consider the standard formalism a complete description of an individual quantum system or to understand it as a completely statistical theory.

By keeping to the latter possibility we introduce all the logico–algebraic propositions of the structure (logic, lattice,...) underlying the Hilbertian theory of quantum measurements directly as d'Espagnat's mixtures of the second kind — which cannot be verified on individual systems but can on the appropriate ensemble — and thus we avoid the afore–mentioned infinitary procedure which actually boils down to postulating what we lack to reach the Hilbertian structure.[47]

We have to stress here that by avoiding Jauch's infinitary procedure we did not get rid of any postulation. We only substituted the statistical interpretation postulate for the individual interpretation postulate and the Jauch's infinitary postulate. We did so because we feel that the former postulation is physically more plausible since it fits better into the quantum logic approach and resolves the paradoxes of Hultgren and Shimony[48] by generating all the propositions according to a feasible experimental receipt.

6. CONCLUSIONS

We have thus shown that the logico–algebraic structure underlying quantum measurements and having the Hilbert space as its model can be based on the statistics of the measurements. The propositions in such logic/lattice are formed with the help of *ensembles*, by means of d'Espagnat's mixtures of the second kind, which correspond to a *complete* experimental setup for measuring *every* spin operator. The YES–NO setup is thus interpreted as the one which determine both a proposition and its orthocomplement in the long run.

Quantum logic, whose propositions can therefore be generated by the YES–NO statistical measurements is then shown to reflect the nature of the measurement so as to allow modelling by an ortholattice in which a unique operation of bi–implication corresponds to equality. In other words, the ordering relation turns out to be inessential for orthomodular lattices — quite the other way round then with distributive lattices, the result provided in Se. 3. We could even say that quantum structures are based on equal classes of equivalence while classical structures are based on ordered classes of equivalence.

Such an approach gave us a clue to a representation of quantum logic as well as of orthomodular lattices by means of the YES–NO relation which we provided in Sec. 4. At the same time this embodies a proper semantics for quantum logic which is a rather long wanted result for the finite case since the decidability which the result enables establishes a direct computational approach to quantum measurements, although it is not of particular significance for the Hilbertian modelling.

The fact that orthomodular lattices are characterized by the operation of bi–implication might be significant for a complete axiomatization of *quantum set theory* because it doesn't seem accidental that Takeuti[49] simply dropped the extensionality axiom out of his formulation of *quantum set theory* — the extensionality axiom demands a proper operation of implication.

On the other hand, the fact that orthomodular lattices are essentially *not* characterized by the operation of implication, i.e., that they are essentially *non–ordered* might be significant for possible formulation of the Hilbert space over the non–archimedean, i.e. non–ordered Keller fields.[50]

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[24] C.G. Morgan, Logique et Analyse, **26** *(103–104)*, 323 (1983).

[25] L. Iturrioz, Bull. London Math. Soc. **14,** 542 (1982).

[26] L. Iturrioz, Acta Math. Hung. **47,** 145 (1986).

[27] Actually, only the infinite number of propositions in a Hilbertian lattice *saves* the result of A. Benedetti and G. Teppati, Lett. Nuovo Cim. **2,** 695 (1971) which reads that such a system is undecidable.

[28] B.O. Hultgren III and A. Shimony, J. Math. Phys. **18,** 381 (1977).

[29] A. Shimony, Found. Phys. **1,** 325 (1971).

[30] Uniquely orthocomplemented lattices are orthomodular.[5]

[31] P.-A. Ivert and T. Sjödin, Helv. Phys. Acta, **51,** 635 (1978).

[32] We coined the word *'infinitary'* to mean "incapable of being completed in a finite number of steps" by merging the words *infinite* and *finitary*

[33] A.R. Swift and R. Wright, J. Math. Phys. **21,** 77 (1980).

[34] B. d'Espagnat, in *Preludes in Theoretical Physics,* edited by A. de–Shalit et al. (North–Holland, Amsterdam, 1966), p. 185; B. d'Espagnat, Physics Reports, **110,** *No. 4*, 201 (1984), p. 211. d'Espagnat introduced the mixture of the second kind (*improper* mixtures) in order to take into account the mixture–like data as well as the correlations of the separated subsystems of the Bell–like systems. In our case we deal with the spin detections and the correlations with the spins prepared along some other directions. Since the correlations boil down to the same diagonal elements of the rotation matrix,[37] formally both approaches coincide.

[35] M. Pavičić, Nuovo Cim. **105 B,** 1103 (1990); Errata, *Ibid.* **106 B,** 105 (1991).

[36] M. Pavičić, Nuova Critica, **I–II,** *(Nuova Serie), Quaderno 13–14,* 109 (1990).

[37] M. Pavičić, in Problems in quantum physics; Gdańsk '89, edited by J. Mizerski et al. (World Scientific, Singapore, 1990), p. 440.

[38] M. Jammer, *The Philosophy of Quantum Mechanics* (John Wiley & Sons, New York, 1974), Ch. 10, and L.E. Ballentine, Rev. Mod. Phys. **42,** 358 (1970) seem to purport an opposite view according to which the individual interpretation is contained in the statistical.

[39] Reference 32 draws the reader's attention to some such "sources."

[40] M. Ozawa, J. Math. Phys. **25,** 79 (1984).

[41] H. Araki, and M.M. Yanase, Phys. Rev. **120,** 622 (1960).

[42] A. Einstein, B. Podolsky, and N. Rosen, *Phys. Rev.* **47,** 777 (1935).

[43] J. von Neumann, *Mathematical Foundation of Quantum Mechanics* (Princeton University Press, Princeton, 1955), p. 439.

[44] Reference 41, p. 777.

[45] Reference 42, pp. 250, 439.

[46] Y.S. Chow, and H. Teicher, *Probability Theory: Independence, Interchangeability, Martingales* (Springer–Verlag, New York, 1978), p. 20.

[47] The reason why Hultgren and Shimony[28] could not reproduce *all* propositions is exactly that they kept to *pure* states and redefined propositions which could not be verified on individual systems, i.e. within single experiments: "The most interesting entries are those such that [meets in our lattice]≠[meet in the Hilbertian lattice]. In several cases is [meet in our lattice]=∅, whereas [meet in Hilbertian lattice]≠∅, since the intersection of two two–dimensional subspaces of a three–dimensional Hilbert space is a subspace of dimension at least one (a ray), but this ray may not be spanned by an eigenvector of $\mathbf{n}\cdot\mathbf{s}$ for any $\mathbf{n}$ and hence may not correspond to a verifiable proposition. So then the g.l.b. of [two such propositions] is in [our lattice] ∅."(Ref. 28, p. 387) If they allowed d'Espagnat's mixtures of the second kind they could easily represent, e.g. the state $[1/\sqrt{6},1/\sqrt{3},1/\sqrt{2}]$ as a mixture of the second kind of [1,0,0], [0,1,0], and [0,0,1].

[48] Under "paradoxes" presented by Hultgren and Shimony in reference 28 (although this is most probably not an adequate name) we mean their results according to which: 1) not all propositions can be generated within the standard spin measurements (the usual magnetic field only); 2) the covering property is not satisfied by propositions corresponding to such measurements; 3) propositions corresponding to such measurements form an orthomodular instead of a modular structure.

[49] G. Takeuti, in *Current Issues in Quantum Logic,* edited by E.G. Beltrametti, and B. C. van Fraassen (Plenum Press, New York, 1981), pp. 303.

[50] H.A. Keller, Math. Z. **172,** 41 (1980); H. Gross, Found. Phys. **20,** 529 (1990); H. Gross and U.-M. Künzi, Enseignement Mathématique, **31,** 187 (1985).