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NONOSCILLATORY SOLUTIONS OF DIFFERENTIAL EQUATIONS WITH DEVIATING ARGUMENT

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In the paper a result of J. Ohriska in [3] concerning oscillation of the second order linear differential equation with delay is extended to an n-th order differential equation with deviating argument. The main tool in establishing the results are Kiguradze lemmas.

We consider the differential equation

(1) $$L_n\, y(t) + f(t, y[g(t)]) = 0$$

where $n > 1$,

$$L_n\, y(t) = p_n(t)\,[p_{n-1}(t)\,(\ldots\,[p_1(t)\,(p_0(t)\,y(t))']'\,\ldots)']',$$

p_i, $i = 0, 1, \ldots, n$ are positive and continuous functions on $\langle t_0, \infty)$, f is real valued and continuous on $D = \langle t_0, \infty) \times R$, $g\colon \langle t_0, \infty) \to \langle t_0, \infty)$ is continuous and $t_0 \in R$.

The expressions

(2) $$L_0\, y(t) = p_0(t)\, y(t)\,, \quad L_i\, y(t) = p_i(t)\,[L_{i-1}\, y(t)]'\,, \quad i = 1, 2, \ldots, n\,,$$

are called *the quasi-derivatives of y at the point $t \in \langle t_0, \infty)$*. We restrict our considerations to those solutions of (1) which exist on some ray $\langle T_y, \infty)$ and satisfy the condition

(3) $$\sup\,\{|y(t)|\colon t_1 \leqq t < \infty\} > 0 \quad \text{for any} \quad t_1 \in \langle T_y, \infty)\,.$$

Such a solution is called *oscillatory* if it has arbitrarily large zeros. Otherwise a solution is called *nonoscillatory*. The equation (1) is called *oscillatory* if all solutions of (1) are oscillatory.

Sometimes we will require the following conditions to be satisfied:

(4) $\int^\infty p_i^{-1}(t)\,dt = \infty\,, \quad i = 1, 2, \ldots, n-1\,;$

(5) $y\,f(t, y) \geqq 0$ for all $(t, y) \in D$, and for any interval of the form $\langle t_1, \infty)$ with $t_1 \geqq t_0$ and any function $h \in C(\langle t_1, \infty))$, $f[t, h(t)] \equiv 0$ implies $h(t) \equiv 0$;

(6) $y\,f(t, y) \leqq 0$ for all $(t, y) \in D$, and for any interval of the form $\langle t_1, \infty)$ with $t_1 \geqq t_0$ and any function $h \in C(\langle t_1, \infty))$, $f[t, h(t)] \equiv 0$ implies $h(t) \equiv 0$;

(7) $\lim_{t\to\infty} g(t) = \infty$.

To obtain the main results we need three lemmas. The first is adapted from the papers [1], [4], [5], [7] and contains a generalization of the well-known first two Kiguradze lemmas.

Lemma 1. *Let the condition* (4) *be satisfied and let* y *be a positive function on the interval* $\langle t_1, \infty)$, $t_1 \geqq t_0$, *such that* $L_n y$ *exists on* $\langle t_1, \infty)$, *is of constant sign and is not identically zero on any interval of the form* $\langle t_2, \infty)$, $t_2 \geqq t_1$.

Then there exists an integer l, $0 \leqq l \leqq n$, *with* $n + l$ *odd for* $L_n y \leqq 0$ *or* $n + l$ *even for* $L_n y \geqq 0$, *such that*

$$l \leqq n - 1 \quad \textit{implies} \quad (-1)^{l+j} L_j\, y(t) > 0 \quad \textit{for every} \quad t \geqq t_1$$ 3

$$(j = l, l + 1, \ldots, n - 1),$$

$$l > 1 \quad \textit{implies} \quad L_i\, y(t) > 0 \quad \textit{for all large } t \quad (i = 1, 2, \ldots, l - 1).$$

Further, for every $i = 0, 1, \ldots, n - 1$, $\lim_{t\to\infty} L_i\, y(t)$ *exists in the extended real line* $R^* = R \cup \{-\infty, \infty\}$ *whereby*

$$\textit{for} \quad l \leqq n - 1, \quad \lim_{t\to\infty} L_l\, y(t) = c_l \geqq 0 \quad \textit{is finite},$$

$$\textit{for} \quad l \leqq n - 2, \quad \lim_{t\to\infty} L_j\, y(t) = 0 \quad (j = l + 1, \ldots, n - 1),$$

$$\textit{for} \quad l \geqq 2, \quad \lim_{t\to\infty} L_i\, y(t) = \infty \quad (i = 0, \ldots, l - 2)$$

Remark. If $1 \leqq l \leqq n - 1$, the lemma gives no exact result about c_l and $c_{l-1} = \lim_{t\to\infty} L_{l-1}\, y(t)$. If $c_l > 0$, then $c_{l-1} = \infty$. For $c_l = 0$, $c_{l-1} > 0$ may be finite or infinite as the example of functions

$$y_1(t) = \operatorname{arc\,tg} t, \quad y_1'(t) = 1/(1 + t^2), \quad y_1''(t) = -2t/(1 + t^2)^2,$$

$$y_2(t) = \ln t, \quad y_2'(t) = 1/t, \quad y_2''(t) = -1/t^2$$

with $l = 1, n = 2, p_0 \equiv p_1 \equiv p_2 \equiv 1$ shows. Similarly, if $l = n$, then $\lim_{t\to\infty} L_{n-1}\, y(t) = c_{n-1} > 0$ may be finite or infinite.

Define functions

$$(8) \qquad I_0 \equiv 1, \quad I_k(t, a; p_{i_1}, p_{i_2}, \ldots, p_{i_k}) =$$

$$= \int_a^t p_{i_1}^{-1}(t_{i_1}) \int_a^{t_{i_1}} p_{i_2}^{-1}(t_{i_2}) \int_a^{t_{i_2}} \ldots \int_a^{t_{i_{k-1}}} p_{i_k}^{-1}(t_{i_k})\, \mathrm{d}t_{i_k}\, \mathrm{d}t_{i_{k-1}} \ldots \mathrm{d}t_{i_1},$$

$$1 \leqq k \leqq n - 1, \quad t_0 \leqq a \leqq t < \infty.$$

Then the functions

$$(9) \qquad x_j(t, a) = p_0^{-1}(t)\, I_{j-1}(t, a; p_1, p_2, \ldots, p_{j-1}), \quad j = 1, 2, \ldots, n,$$

form a fundamental system of solutions of the equation $L_n x(t) = 0$ in $\langle a, \infty)$ and

(10) $$L_{j-1}x_j(t, a) \equiv 1\,, \quad L_i x_j(t, a) \equiv 0 \quad \text{for} \quad i \geqq j\,,$$
$$L_i x_j(t, a) > 0 \quad \text{in} \quad (a, \infty) \quad \text{for} \quad i < j - 1\,.$$

For the sake of brevity, denote

(11) $$P_0(t, a) \equiv 1\,, \quad P_j(t, a) = I_j(t, a; p_1, \ldots, p_j)\,,$$
$$j = 1, 2, \ldots, \quad n - 1\,, \quad t_0 \leqq a \leqq b < \infty$$

and

(12) $$Q_n(t, a) \equiv 1\,, \quad Q_j(t, a) = I_{n-j}(t, a; p_{n-1}, p_{n-2}, \ldots, p_j)\,,$$
$$j = 1, 2, \ldots, n - 1\,, \quad t_0 \leqq a \leqq t < \infty\,.$$

In the case all $p_i \equiv 1$

$$P_j(t, a) = \frac{(t - a)^j}{j!}\,, \quad j = 0, 1, \ldots, n - 1$$

and

$$Q_j(t, a) = \frac{(t - a)^{n-j}}{(n - j)!}\,, \quad j = 1, \ldots, n - 1, n\,.$$

Remark. If l from Lemma 1 satisfies $0 \leq l \leq n - 1$, then by the variation of constants formula ([1], p. 96, (9_{01})) with a sufficiently great and with respect to (11) and

$$L_0\, y(t) = \sum_{j=0}^{l} L_j\, y(a)\, P_j(t, a) + \int_a^t p_{l+1}^{-1}(s)\, L_{l+1}\, y(s)\, P_l(t, s)\, \mathrm{d}s\,, \quad t \geqq a$$

where $L_j\, y(a) > 0$, $j = 0, 1, \ldots, l$, and $L_{l+1}\, y(t) \leqq 0$, $t \geqq a$, we get that

$$L_0\, y(t) \leqq \sum_{j=0}^{l} L_j\, y(a)\, P_j(t, a)\,.$$

In the general case, when y is either positive in a neighbourhood of infinity or negative, we come to the inequality

(13) $$|L_0\, y(t)| \leqq \sum_{j=0}^{l} |L_j\, y(a)|\, P_j(t, a)\,.$$

Further, (4) implies that

$$\lim_{t \to \infty} \frac{P_j(t, a)}{P_{l+1}(t, a)} = 0\,, \quad j = 0, 1, \ldots, l$$

and hence, by (13),

(14) $$\lim_{t \to \infty} \frac{L_0\, y(t)}{P_k(t, a)} = 0\,, \quad k = l + 1, \ldots, n\,.$$

In the case $l = 0$, by Lemma 1, $|L_0 y|$ is a nonincreasing function and hence there exists

(15) $$\lim_{t\to\infty} \frac{L_0\, y(t)}{P_0(t, a)} = c_0 .$$

In the case $1 \leqq l \leqq n - 1$, by Lemma 2.1 ([6], p. 298),

(16) $$\lim_{t\to\infty} \frac{L_0\, y(t)}{P_j(t, a)} = \lim_{t\to\infty} L_j\, y(t) = c_j\,, \quad j = 0, \ldots, l$$

whereby $|c_j| = \infty$ for $j = 0, \ldots, l - 2$, and $0 < |c_{l-1}| \leqq \infty$. Thus the following statement is true:

If the conditions of Lemma 1 are satisfied and there is an integer k, $0 \leqq k \leqq n - 2$, such that

$$\lim_{t\to\infty} \frac{L_0\, y(t)}{P_k(t, a)} > 0\,, \quad \lim_{t\to\infty} \frac{L_0\, y(t)}{P_{k+1}(t, a)} = 0\,,$$

then by the former relation $k \leqq l$ and by the latter $k \geqq l - 1$, hence

k is either $l - 1$ or l.

U. Elias in [2] has generalized the third Kiguradze lemma. From his results (Theorem 3, case $j = k + 1$) the following lemma is important for our considerations.

Lemma 2. *Let l be an integer, $1 \leqq l \leqq n - 1$, $a \in \langle t_0, \infty)$. If the function y satisfies*

$$L_0\, y(a), \ldots, L_{l-1}\, y(a) \geqq 0\,, \quad L_{l+1}\, y(t) \leqq 0 \quad \textit{for} \quad a \leqq t < \infty\,,$$

then

(a) $$\left(\frac{L_i\, y(t)}{L_i x_{l+1}(t, a)}\right)' \leqq 0\,, \quad i = 0, 1, \ldots, l\,, \quad a < t < \infty$$

and

(b) $$L_i\, y(t) \geqq L_{i+1}\, y(t) \frac{L_i x_{l+1}(t, a)}{L_{i+1} x_{l+1}(t, a)}\,, \quad i = 0, 1, \ldots, l\,, \quad a < t < \infty\,.$$

Hence by (a), (b), (2), (9), (11),

(c) $$\frac{L_0\, y(t)}{P_l(t, a)} \quad \textit{is a nonincreasing function in } (a, \infty)\,,$$

(d) $$L_0\, y(t) \geqq L_i\, y(t) \frac{P_l(t, a)}{L_i x_{l+1}(t, a)}\,, \quad i = 1, \ldots, l\,, \quad a < t < \infty$$

and with respect to (10),

(e) $$L_0\, y(t) \geqq L_l\, y(t)\, P_l(t, a)\,, \quad a \leqq t < \infty\,.$$

Lemma 3. *Let the conditions* (4), (5), (7) (*the conditions* (4), (6), (7)) *be satisfied and let u be a nonoscillatory solution of the equation* (1). *Denote by δ the sign of*

$u(t)$ in a sufficiently small neighbourhood of infinity. Then there exists a number $t_1, t_1 \geqq t_0$, and an integer l, $0 \leqq l \leqq n$, with $n + l$ odd ($n + l$ even), such that

(a) *for $l \leqq n - 1$, $(-1)^{l+j} \delta L_j u(t) > 0$ for every $t \geqq t_1$, $j = l, l + 1, \ldots, n - 1$, and*

$$\lim_{t\to\infty} L_l u(t) = c_l \quad \textit{is finite, whereby} \quad \delta c_l \geqq 0 \, ;$$

(b) *for $l \leqq n - 2$,*

$$\lim_{t\to\infty} L_j u(t) = 0 \, , \quad j = l + 1, \ldots, n - 1 \, ;$$

(c) *for $l \geqq 2$,*

$$\delta L_i u(t) > 0 \quad \textit{for all large } t, \quad i = 1, 2, \ldots, l - 1$$

and

$$\lim_{t\to\infty} \delta L_i u(t) = \infty \, , \quad i = 0, \ldots, l - 2 \, ;$$

(d) *for $l \leqq n - 1$, u is a solution of the integro-differential equation*

$$L_l y(t) = c_l + (-1)^{n-l+1} \int_t^\infty p_n^{-1}(s) f(s, y[g(s)]) \, Q_{l+1}(s, t) \, ds \, . \tag{17}$$

Proof. Suppose u is nonoscillatory and positive in a neighbourhood of infinity. If u is negative, the proof can be done in a similar way. With respect to (7), there exists a t_1, $t_1 \geqq t_0$, such that $u(t) > 0$ and also $u[g(t)] > 0$ for $t \geqq t_1$. Then on the basis of (1), (5) implies ((6) implies) that $L_n u \leqq 0$ ($L_n u \geqq 0$) on $\langle t_1, \infty)$ and $L_n u$ is not identically zero on any interval of the form $\langle t_2, \infty)$, $t_2 \geqq t_1$. Hence Lemma 1 can be applied. By that lemma the statements (a), (b), (c) are true.

Suppose now that $l \leqq n - 1$. If $l = n - 1$, then integrating we obtain

$$L_{n-1} u(t) = c_{n-1} + \int_t^\infty p_n^{-1}(s) f(s, u[g(s)]) \, ds$$

and hence in the case $l = n - 1$, (17) is satisfied by u. When $l \leqq n - 2$, then taking into account (b), by repeated integration we get

$$L_j u(t) = (-1)^{n-j+1} \int_t^\infty p_n^{-1}(s) f(s, u[g(s)]) \, Q_{j+1}(s, t) \, ds \, , \tag{18}$$

$$j = n - 1, \ldots, l + 1 \, .$$

Finally, integrating (18) for $j = l + 1$ we come to the conclusion that u is a solution of (17).

Now we can solve the first problem which is to find a sufficient condition for c_l in Lemma 3 to be zero.

Remark. It is clear that the number l in Lemma 3 is uniquely determined. This justifies the following

Definition 1. Suppose that the conditions (4), (5), (7) (the conditions (4), (6), (7)) are satisfied. Let u be a nonoscillatory solution of the equation (1) and δ its sign in a sufficiently small neighbourhood of ∞. We say that u has *property* P_l *with* $l \in \{0, 1, \ldots, n\}$ *and* $n + l$ *is odd* ($n + l$ *is even*) if it has properties (a), (b), (c) from Lemma 3.

We recall that under the conditions (4), (5), (7) (the conditions (4), (6), (7)) each nonoscillatory solution of (1) has property P_l with some $l \in \{0, 1, \ldots, n\}$.

Theorem 1. *Let the conditions* (4), (5), (7) (*the conditions* (4), (6), (7)) *be satisfied and let* u *be a nonoscillatory solution of the equation* (1) *with property* P_l, *where* $0 \leqq l \leqq n - 1$.

Let there exist a function $G = G(t, y): D_1 = \langle t_0, \infty) \times \langle 0, \infty) \rightarrow \langle 0, \infty)$, *which is continuous, nondecreasing in* y *for each fixed* t *and such that*

$$|f(t, y)| \geqq G(t, |y|) \quad ((t, y) \in D) . \tag{19}$$

Then the condition: For each $k > 0$ *and each* a *from a neighbourhood of* ∞ *either*

$$\int_t^\infty p_n^{-1}(s)\, Q_{l+1}(s, t)\, G(s, k\, p_0^{-1}[g(s)]\, P_l[g(s), a])\, \mathrm{d}s = \infty \tag{20}$$

for all $t \geqq a$ *or*

$$\lim_{t \to \infty} \int_t^\infty p_n^{-1}(s)\, Q_{l+1}(s, t)\, G(s, k\, p_0^{-1}[g(s)]\, P_l[g(s), a])\, \mathrm{d}s > 0 \tag{20'}$$

implies that

$$c_l = \lim_{t \to \infty} L_l\, u(t) = 0 . \tag{21}$$

Remark. Suppose that $m: \langle t_0, \infty) \rightarrow \langle 0, \infty)$ is a continuous function and let us investigate

$$\int_t^\infty p_n^{-1}(s)\, Q_{l+1}(s, t)\, m(s)\, \mathrm{d}s \tag{20''}$$

which represents the general form of the integrals in (20) or (20'). As the function $p_n^{-1}(s)\, Q_{l+1}(s, t)\, m(s)$ is nonincreasing in the variable t for $t < s$, s being fixed, and nonnegative, two cases are possible concerning (20''). Either $\int_t^\infty p_n^{-1}(s)\, Q_{l+1}(s, t) \,.\, m(s)\, \mathrm{d}s = \infty$ for all $t \geqq a$, or there is a t_1, $a \leqq t_1 < \infty$, such that $\int_t^\infty p_n^{-1}(s) \,.\, Q_{l+1}(s, t)\, m(s)\, \mathrm{d}s < \infty$ for all $t \geqq t_1$ and this function is nonincreasing in $\langle t_1, \infty)$. Hence $\lim_{t \to \infty} \int_t^\infty p_n^{-1}(s)\, Q_{l+1}(s, t)\, m(s)\, \mathrm{d}s$ exists and is finite and nonnegative.

In the case $l = n - 1$, the condition (20') cannot hold, because if $\int_t^\infty p_n^{-1}(s)\, m(s)\, \mathrm{d}s$ exists, then $\lim_{t \to \infty} \int_t^\infty p_n^{-1}(s)\, m(s)\, \mathrm{d}s = 0$.

Proof of Theorem 1. Suppose the conditions (4), (5), (7) and (19) are satisfied and $c_l \neq 0$. Then, by Lemma 3, $n + l$ is odd, and hence (17) implies that the equality

$$L_l\, u(t) = c_l + \int_t^\infty p_n^{-1}(s)\, Q_{l+1}(s, t)\, f(s, u[g(s)])\, \mathrm{d}s \tag{22}$$

is true. Denote by δ the sign of u in a sufficiently small neighbourhood of ∞. Let t_1 be such that

$$\begin{aligned} &\delta L_i\, u(t) \quad > 0 \quad \text{in} \quad \langle t_1, \infty), \quad i = 0, \ldots, l, \\ &\delta L_{l+1}\, u(t) \leqq 0 \quad \text{in} \quad \langle t_1, \infty). \end{aligned} \tag{23}$$

By (7), there exists an $a \geqq t_1$ such that $g(t) \geqq t_1$ for each $t \geqq a$. We shall distinguish two cases:

1. $l = 0$.

By (23), $\delta L_0 u$ is nonincreasing in $\langle t_1, \infty)$ and as it converges to δc_0, we have

$$\delta L_0 u \geqq \delta c_0 > 0 .$$

This implies

$$|u(t)| \geqq \frac{|c_0|}{p_0(t)}, \quad t \in \langle t_1, \infty). \tag{24}$$

(22) can be written in the form

$$\delta L_0\, u(t) = \delta c_0 + \delta \int_t^\infty p_n^{-1}(s)\, Q_1(s, t) f(s, u[g(s)])\, \mathrm{d}s . \tag{25}$$

By (5), this means

$$|L_0\, u(t)| = |c_0| + \int_t^\infty p_n^{-1}(s)\, Q_1(s, t)\, |f(s, u[g(s)])|\, \mathrm{d}s \tag{26}$$

and hence, (19) and (24) yield

$$\begin{aligned} |L_0\, u(t)| &\geqq |c_0| + \int_t^\infty p_n^{-1}(s) G(s, |u[g(s)]|)\, Q_1(s, t)\, \mathrm{d}s \geqq \\ &\geqq |c_0| + |c_0| \int_t^\infty p_n^{-1}(s) \frac{G(s, |c_0| p_0^{-1}[g(s)])}{|c_0|}\, Q_1(s, t)\, \mathrm{d}s \end{aligned}$$

which gives

$$1 \geqq \frac{|c_0|}{|L_0\, u(t)|} \left(1 + \int_t^\infty p_n^{-1}(s)\, Q_1(s, t) \frac{G(s, |c_0|\, p_0^{-1}[g(s)])}{|c_0|}\, \mathrm{d}s\right) \tag{27}$$

and this contradicts (20) or (20′) because

$$\lim_{t\to\infty} \frac{|c_0|}{|L_0\, u(t)|} = 1 .$$

2. $1 \leqq l \leqq n - 1$.

By Lemma 2, $\delta L_0\, u(t) \geqq \delta L_l\, u(t)\, P_l(t, a)$ for all $t \geqq a$ and hence, with respect to (22),

$$\begin{aligned} \delta L_0\, u(t) &\geqq \delta L_l\, u(t)\, P_l(t, a) = \\ &= \delta c_l\, P_l(t, a) + P_l(t, a) \int_t^\infty p_n^{-1}(s)\, Q_{l+1}(s, t)\, |f(s, u[g(s)])|\, \mathrm{d}s \geqq \\ &\geqq |c_l|\, P_l(t, a) + P_l(t, a) \int_t^\infty p_n^{-1}(s)\, Q_{l+1}(s, t)\, G(s, |u[g(s)]|)\, \mathrm{d}s, \quad t \geqq a . \end{aligned}$$

Thus

$$(28)\qquad |L_0 u(t)| \geqq |c_l|\, P_l(t, a) + P_l(t, a) \int_t^\infty p_n^{-1}(s)\, Q_{l+1}(s, t)\,.$$

$$.\, G(s, |u[g(s)]|)\, \mathrm{d}s\,, \quad a \leqq t < \infty\,.$$

This implies that

$$|L_0 u(t)| \geqq |c_l|\, P_l(t, a)$$

and

$$|u(t)| \geqq |c_l|\, x_{l+1}(t, a)\,, \quad t \geqq a\,.$$

Therefore

$$|L_0 u(t)| \geqq |c_l|\, P_l(t, a) + |c_l|\, P_l(t, a) \int_t^\infty p_n^{-1}(s)\, Q_{l+1}(s, t)\, \frac{G(s, |c_l|\, x_{l+1}[g(s), a])}{|c_l|}\, \mathrm{d}s$$

for all $t \geqq b$ such that $g(t) \geqq a$ for $t \geqq b$. Then

$$(29)\qquad 1 \geqq \frac{|c_l|\, P_l(t, a)}{|L_0 u(t)|} \left(1 + \int_t^\infty p_n^{-1}(s)\, Q_{l+1}(s, t)\, \frac{G(s, |c_l|\, x_{l+1}[g(s), a])}{|c_l|}\, \mathrm{d}s\right).$$

As $\lim_{t\to\infty} L_l u(t) = c_l \neq 0$, Lemma 3, [5], p. 199 yields

$$\lim_{t\to\infty} \frac{L_0 u(t)}{P_l(t, a)} = \lim_{t\to\infty} L_l u(t) = c_l\,,$$

which shows that (29) contradicts (20) or (20′).

Let now the conditions (4), (6), (7), (19) be satisfied. Then, by Lemma 3, $n + l$ is even, and instead of (22) we have

$$(22')\qquad L_l u(t) = c_l - \int_t^\infty p_n^{-1}(s)\, Q_{l+1}(s, t)\, f(s, u[g(s)])\, \mathrm{d}s\,.$$

The relations (23), (24) remain valid.

When $l = 0$, from (22′) we get

$$(25')\qquad \delta L_0 u(t) = \delta c_0 - \delta \int_t^\infty p_n^{-1}(s)\, Q_1(s, t)\, f(s, u[g(s)])\, \mathrm{d}s\,.$$

Again we come to (26) and (27) which implies that (21) is true.

When $1 \leqq l \leqq n - 1$, we obtain (28) and (29). This gives that (20) or (20′) implies (21).

Corollary 1. *Suppose that all assumptions of Theorem 1 are satisfied but (19) is replaced by*

$$(30)\qquad |f(t, y)| \geqq \alpha(t)\, |y| \quad ((t, y) \in D)\,,$$

where $\alpha \in C(\langle t_0, \infty))$ is a nonnegative function. Let u be a nonoscillatory solution of the equation (1) with property P_l and let l satisfy $0 \leqq l \leqq n - 1$.

Then the condition: For each a from a neighbourhood of ∞ *either*

$$(31) \qquad \int_t^\infty p_n^{-1}(s)\, Q_{l+1}(s, t)\, \alpha(s)\, p_0^{-1}[g(s)]\, P_l[g(s), a]\, \mathrm{d}s = \infty$$

for all $t \geqq a$, *or*

$$(31') \qquad \lim_{t\to\infty} \int_t^\infty p_n^{-1}(s)\, Q_{l+1}(s, t)\, \alpha(s)\, p_0^{-1}[g(s)]\, P_l[g(s), a]\, \mathrm{d}s > 0\,,$$

is sufficient for the equality

$$c_l = \lim_{t\to\infty} L_l\, u(t) = 0$$

to hold.

Remark. In the special case $g(t) \equiv t$ the condition (31′) is weaker than the condition (38′) in [7], p. 127,

$$\int_t^\infty \frac{\alpha(s)}{p_0(s)\, p_n(s)}\, P_l(s, t)\, Q_{l+1}(s, t)\, \mathrm{d}s = \infty$$

and hence Corollary 1 improves and generalizes the sufficient condition in Corollary 1 to Theorem 6 in that paper when $h = h(t, y)$.

Denote

$$h(t) = \max\,[t, \max_{a \leqq s \leqq t} g(s)] \quad \text{for all} \quad t \geqq a\,,$$

where a has the same meaning as in Theorem 1. Clearly $h(t) \geqq t$ and h is nondecreasing in $\langle a, \infty)$.

Theorem 2. *Let* $1 \leqq l \leqq n - 1$ *be an integer. Let the conditions* (4), (5), (7), (19), (20) *or* (20′) (*the conditions* (4), (6), (7), (19), (20) *or* (20′)) *be satisfied. Let* $n + l$ *be odd* ($n + l$ *be even*). *Let the function G be such that*

$$(32) \qquad G(t, ky) \geqq k\, G(t, y)$$

for each $k > 0$ *and each* $(t, y) \in D_1$.

Then the condition: For each a from a neighbourhood of ∞ *either*

$$(33) \qquad \int_t^\infty p_n^{-1}(s)\, Q_{l+1}(s, t)\, G\left(s, p_0^{-1}[g(s)]\, \frac{P_l[g(s), a]}{P_l[h(s), a]}\right) \mathrm{d}s = \infty$$

for all $t \geqq a$ *or*

$$(33') \quad \limsup_{t\to\infty} P_l(t, a) \int_t^\infty p_n^{-1}(s)\, Q_{l+1}(s, t)\, G\left(s, p_0^{-1}[g(s)]\, \frac{P_l[g(s), a]}{P_l[h(s), a]}\right) \mathrm{d}s > 1$$

is a sufficient condition that there exists no nonoscillatory solution u of the equation (1) *with property* P_l.

Proof. Let u be an arbitrary nonoscillatory solution of (1) such that the integer l from Lemma 3 satisfies $1 \leqq l \leqq n - 1$. As all assumptions of Theorem 1 are satisfied, $c_l = 0$ in (22) or (22′) and hence u satisfies (28) in the form

$$(28') \qquad |L_0\, u(t)| \geqq P_l(t, a) \int_t^\infty p_n^{-1}(s)\, Q_{l+1}(s, t)\, G(s, |u[g(s)]|)\, ds\,.$$

As $|L_0 u|$ is an increasing function and $h(t) \geqq t$,

$$(34) \qquad |L_0 u[h(t)]| \geqq |L_0\, u(t)| \quad \text{for all} \quad t \geqq a\,.$$

By Lemma 2, $|L_0\, u(t)|/P_l(t, a)$ is a nonincreasing function in (a, ∞) and hence, $g(s) \leqq h(s)$ implies

$$(35) \qquad |u[g(s)]| = \frac{|L_0 u[g(s)]|}{p_0[g(s)]} \geqq \frac{1}{p_0[g(s)]} \frac{P_l[g(s), a]}{P_l[h(s), a]} |L_0 u[h(s)]|\,.$$

Further, h is nondecreasing and therefore

$$(36) \qquad |L_0 u[h(s)]| \geqq |L_0 u[h(t)]|\,, \quad s \geqq t\,.$$

Putting (34), (35) and (36) into (28′) we get

$$|L_0 u[h(t)]| \geqq P_l(t, a) \int_t^\infty p_n^{-1}(s)\, Q_{l+1}(s, t)\, G\left(s, \frac{1}{p_0[g(s)]} \frac{P_l[g(s), a]}{P_l[h(s), a]} |L_0 u[h(t)]|\right) ds\,.$$

Now using (32) we come to the inequality

$$1 \geqq P_l(t, a) \int_t^\infty p_n^{-1}(s)\, Q_{l+1}(s, t)\, G\left(s, p_0^{-1}[g(s)] \frac{P_l[g(s), a]}{P_l[h(s), a]}\right) ds$$

which contradicts (33) or (33′).

Corollary 2. *If the assumptions of Theorem* 2 *are satisfied but* (19) *is replaced by* (30), (20) *or* (20′) *by* (31) *or* (31′) *and* (32) *is omitted, then the condition*: *For each a from a neighbourhood of ∞ either*

$$(37) \qquad \int_t^\infty p_n^{-1}(s)\, Q_{l+1}(s, t)\, \alpha(s)\, p_0^{-1}[g(s)] \frac{P_l[g(s), a]}{P_l[h(s), a]}\, ds = \infty$$

for all $t \geqq a$ or

$$(37') \qquad \limsup_{t \to \infty} P_l(t, a) \int_t^\infty p_n^{-1}(s)\, Q_{l+1}(s, t)\, \alpha(s)\, p_0^{-1}[g(s)] \frac{P_l[g(s), a]}{P_l[h(s), a]}\, ds > 1$$

where a is a sufficiently great number is a sufficient condition that there exists no nonoscillatory solution u of the equation (1) *with property* P_l.

The next theorem concerns all solutions of the equation (1). Similarly as in [1] we introduce the definitions.

Definition 2. The equation (1) is said *to have property* A if for n even each solution u of that equation is oscillatory and for n odd each solution is either oscillatory or satisfies the conditions:

(a) There exists a $t_1, t_1 \geqq t_0$, such that $(-1)^j\, \delta\, L_j\, u(t) > 0$ for every $t \geqq t_1$, $j = 0, 1, \ldots, n-1$

and

(b) $$\lim_{t\to\infty} L_j u(t) = 0, \quad j = 0, 1, \ldots, n - 1.$$

Definition 3. The equation (1) is said *to have property* B if for n even each solution of that equation is either oscillatory or satisfies conditions (a), (b) from Definition 2 or the conditions

(c) There exists a t_2, $t_2 \geqq t_0$, such that $\delta L_i u(t) > 0$ for every $t \geqq t_2$, $i = 0, 1, \ldots$ $\ldots, n - 1$;

(d) $$\lim_{t\to\infty} \delta L_i u(t) = \infty, \quad i = 0, \ldots, n - 1,$$

and for n odd each of its solutions is either oscillatory or satisfies conditions (c) and (d).

In both definitions δ means the sign of the nonoscillatory solution u in a neighbourhood of infinity.

Theorem 3. *Let the conditions* (4), (5), (7), (19) *and* (32) *be satisfied. Further, let the conditions* (20) *or* (20') *and* (33) *or* (33') *be fulfilled for* $l = n - 1, n - 3, \ldots, 1$ *provided* n *is even* ($l = n - 1, n - 3, \ldots, 2$ *provided* n *is odd*).

Then the equation (1) *has property* A.

Proof. Let u be a nonoscillatory solution of the equation (1). Then, by Lemma 3, there exists an integer l, $0 \leqq l \leqq n$, with $n + l$ odd, such that the statement of that lemma is true. Hence l is one of the numbers $n - 1, n - 3, \ldots, 1$ when n is even and l belongs to the set consisting of the numbers $n - 1, n - 3, \ldots, 2, 0$ when n is odd. By Theorem 2, for $l \neq 0$ no such solution exists. Hence, if n is even, each solution u of (1) is oscillatory and if n is odd, u is either oscillatory or possesses properties (a), (b) from Definition 2. Thus the equation (1) has property A.

Corollary 3. *Let the conditions* (4), (5), (7), (30) *be satisfied. Further, let the conditions* (31) *or* (31'), (37) *or* (37') *be fulfilled for* $l = n - 1, n - 3, \ldots, 1$ *provided* n *is even* ($l = n - 1, n - 3, \ldots, 2$ *provided* n *is odd*). *Then the equation* (1) *has property* A.

Remark. In the case $n = 2$, $l = 1$, $p_0 = p_1 = p_2 \equiv 1$, $g(t) \leqq t$, (31) *or* (31') *and* (37) *or* (37') are reduced to the conditions

$$\int_t^\infty p(s) \frac{g(s)}{s} \, ds = \infty$$

and

$$\limsup_{t\to\infty} t \int_t^\infty p(s) \frac{g(s)}{s} \, ds > 1.$$

Hence Corollary 3 generalizes the first part of Theorem 1 in [3].

If instead of (5) we suppose (6) we obtain the following theorem.

Theorem 4. *Let the conditions* (4), (6), (7), (19) *and* (32) *be satisfied. Further, let* *the conditions* (20) *or* (20′) *and* (33) *or* (33′) *be fulfilled for* $l = n - 2, n - 4, \ldots, 2$ *for n even* ($l = n - 2, n - 4, \ldots, 3, 1$ *for n odd*). *Finally, let the condition*

$$\int_a^\infty G\left(t, \frac{c\, P_{n-1}[g(t), a]}{p_0[g(t)]}\right) \mathrm{d}t = \infty \tag{38}$$

be fulfilled for each $c > 0$ *and each sufficiently great* a.

Then the equation (1) *has property* B.

The proof of this theorem proceeds in the same way as that of Theorem 3. Comparing Definition 3 with Lemma 3 yields that the only thing which remains to show is that in the case $l = n$,

$$\lim_{t\to\infty} \delta L_{n-1}\, u(t) = \infty\,. \tag{39}$$

Hence let $l = n$. Then by (6), $\delta L_{n-1} u$ is nondecreasing in a neighbourhood of ∞ and hence, by statement (c) in Lemma 3, there exists a constant $c > 0$ such that

$$\delta L_{n-1}\, u(t) \geqq c > 0 \tag{40}$$

in $\langle a, \infty)$. Using the same lemma we obtain by repeated integration of (40) that

$$\delta L_{n-j}\, u(t) \geqq c\, I_{j-1}(t, a;\, p_{n-j+1}, \ldots, p_{n-1})\,, \quad t \geqq a\,, \quad j = 1, \ldots, n$$

and thus

$$\delta L_0\, u(t) \geqq c\, P_{n-1}(t, a)\,, \quad t \geqq a\,.$$

This implies that

$$|L_n u(t)| = |f(t, u[g(t)])| \geqq G\left(t, \frac{|L_0 u[g(t)]|}{p_0[g(t)]}\right) \geqq G\left(t, \frac{c\, P_{n-1}[g(t), a]}{p_0[g(t)]}\right), \quad t \geqq a$$

and in view of (38), (39) follows.

Corollary 4. *Let the conditions* (4), (6), (7), (30) *be satisfied. Further, let the conditions* (31) *or* (31′) *and* (37) *or* (37′) *be fulfilled for* $l = n - 2, n - 4, \ldots, 2$ *when n is even* ($l = n - 2, n - 4, \ldots, 3, 1$ *when n is odd*). *Finally, let the condition*

$$\int_a^\infty \alpha(t)\, \frac{P_{n-1}[g(t), a]}{p_0[g(t)]}\, \mathrm{d}t = \infty \tag{41}$$

be fulfilled for all sufficiently great a.

Then the equation (1) *has property* B.

Theorem 2 does not say anything about the case $l = 0$. In a special case of the deviating argument the answer is given by

Theorem 5. *Let the conditions* (4), (5), (7), (19) *and* (20) *or* (20′) *for* $l = 0$ (*the conditions* (4), (6), (7), (19) *and* (20) *or* (20′) *for* $l = 0$) *be satisfied. Let n be odd* (*n even*). *For the function g let there exist an increasing sequence of points* $\{t_k\}_{k=1}^{\infty}$ *with* $\lim_{k\to\infty} t_k = \infty$ *such that*

$$g(\langle t_k, \infty)) \subset \langle t_k, \infty)\,, \quad k = 1, 2, \ldots\,. \tag{42}$$

In particular, (42) *is satisfied when* $g(t) \geqq t$ *for* $t \geqq t_0$. *Let there exist a function* $H = H(t, y)$: $D_1 \to \langle 0, \infty)$ *which is continuous, nondecreasing in* y *for each fixed* t *and such that*

$$|f(t, y)| \leqq H(t, |y|) \tag{43}$$

for every $(t, y) \in D$, *and for any sufficiently great number* a *there exists a* k_0, $0 < k_0 < 1$ *such that for each* $k > 0$,

$$\int_t^\infty p_n^{-1}(s)\, Q_1(s, t) \frac{H(s, k\, p_0^{-1}[g(s)])}{k}\, ds \leqq k_0 \tag{44}$$

for all $t \geqq a$.

Then there is no nonoscillatory solution u *of the equation* (1) *with property* P_0.

Proof. Let u be an arbitrary nonoscillatory solution of (1) with property P_0. Since all assumptions of Theorem 1 for $l = 0$ are satisfied, $c_0 = 0$ in (22) or (22′) and hence

$$|L_0\, u(t)| = \int_t^\infty p_n^{-1}(s)\, Q_1(s, t) \left| f\left(s, \frac{L_0 u[g(s)]}{p_0[g(s)]}\right)\right| ds$$

and in view of (43) we have

$$|L_0\, u(t)| \leqq \int_t^\infty p_n^{-1}(s)\, Q_1(s, t)\, H\left(s, \frac{|L_0 u[g(s)]|}{p_0[g(s)]}\right) ds\,. \tag{45}$$

By Lemma 3, $|L_0 u|$ is a nonincreasing function which converges to 0 as $t \to \infty$. Thus for any $\varepsilon > 0$ there exists a $t_k = a$ satisfying (42) and such that

$$|L_0\, u(t)| \leqq |L_0\, u(t_k)| \leqq \varepsilon \tag{46}$$

for all $t \geqq a$. Putting (46) into (45), on the basis of (44) we come to the inequality

$$|L_0\, u(t)| \leqq \int_t^\infty p_n^{-1}(s)\, Q_1(s, t)\, H(s, \varepsilon p_0^{-1}[g(s)])\, ds \leqq k_0 \varepsilon \tag{47}$$

for all $t \geqq a$. Hence the inequality (46) in $\langle a, \infty)$ has led to the inequality (47) in the same interval. Repeating this process p-times we get that

$$|L_0\, u(t)| \leqq k_0^p \varepsilon$$

in $\langle a, \infty)$ which for $p \to \infty$ implies that $L_0\, u(t) \equiv 0$ in $\langle a, \infty)$ which contradicts the condition (3). Hence u with property P_0 does not exist.

Corollary 5. *Let the conditions of Theorem* 3 *be satisfied. Further, if* n *is odd, let* (20) *or* (20′) *for* $l = 0$, (42), (43), (44) *be satisfied. Then each solution of the equation* (1) *is oscillatory.*

Another sufficient condition for the nonexistence of a nonoscillatory solution u of (1) with property P_l is given in the next theorem. As usual, let us denote

$$\gamma(t) = \sup\{s \geqq t_0 : g(s) \leqq t\} \quad \text{for all} \quad t \geqq t_0\,. \tag{48}$$

With help of this function, we define

$$m(t) = \max(\gamma(t), t), \quad t \geqq t_0 .$$

By virtue of (48) and the continuity of g, for each $s > \gamma(t)$ we have $g(s) > t$ and $g[\gamma(t)] = t$. Hence m possesses the following properties:

(49) $$s \geqq m(t) \text{ implies } g(s) \geqq t, \quad m(t) \geqq t \text{ and, further,} \\ \text{if } g(t) \leqq t, \text{ then } m(t) = \gamma(t).$$

Theorem 6. *Let* $1 \leqq l \leqq n - 1$ *be an integer. Let the conditions* (4), (5), (7), (19), (20) *or* (20') (*the conditions* (4), (6), (7), (19), (20) *or* (20')) *be satisfied. Further, let* (32) *be fulfilled. Then the condition*:

For each a from a neighbourhood of ∞ *either*

(50) $$\int_{m(t)}^{\infty} p_n^{-1}(s)\, Q_{l+1}(s, t)\, G(s, p_0^{-1}[g(s)])\, ds = \infty$$

for all sufficiently great t or

(50') $$\limsup_{t\to\infty} P_l(t, a) \int_{m(t)}^{\infty} p_n^{-1}(s)\, Q_{l+1}(s, t)\, G(s, p_0^{-1}[g(s)])\, ds > 1$$

is a sufficient condition that there exists no nonoscillatory solution u of the equation (1) *with property* P_l.

Proof. If u is a nonoscillatory solution of (1) with property P_l, then similarly as in the proof of Theorem 2 we come to the inequality

(28') $$|L_0 u(t)| \geqq P_l(t, a) \int_t^{\infty} p_n^{-1}(s)\, Q_{l+1}(s, t)\, G(s, |u[g(s)]|)\, ds \geqq$$
$$\geqq P_l(t, a) \int_{m(t)}^{\infty} p_n^{-1}(s)\, Q_{l+1}(s, t)\, G(s, |u[g(s)]|)\, ds, \quad t \geqq a .$$

Since $|L_0 u|$ is increasing, (49) implies that

$$|u[g(s)]| = |L_0 u[g(s)]|\, p_0^{-1}[g(s)] \geqq |L_0 u(t)|\, p_0^{-1}[g(s)]$$

for all $s \geqq m(t)$ and the inequality for $L_0 u$ turns into

$$|L_0 u(t)| \geqq P_l(t, a) \int_{m(t)}^{\infty} p_n^{-1}(s)\, Q_{l+1}(s, t)\, G(s, |L_0 u(t)|\, p_0^{-1}[g(s)])\, ds$$

which in view of (32) leads to the inequality

$$1 \geqq P_l(t, a) \int_{m(t)}^{\infty} p_n^{-1}(s)\, Q_{l+1}(s, t)\, G(s, p_0^{-1}[g(s)])\, ds, \quad t \geqq a .$$

This contradicts (50) or (50') and thus Theorem 6 is proved.

Corollary 6. *If the assumptions of Theorem 6 are satisfied but* (19) *is replaced by* (30), (20) *or* (20') *by* (31) *or* (31') *and* (32) *is omitted, then the condition*:

Either

$$\int_{m(t)}^{\infty} p_n^{-1}(s)\, Q_{l+1}(s, t)\, \alpha(s)\, p_0^{-1}[g(s)]\, ds = \infty \tag{51}$$

for all t from a neighbourhood of ∞ *or for all sufficiently great a,*

$$\limsup_{t \to \infty} P_l(t, a) \int_{m(t)}^{\infty} p_n^{-1}(s)\, Q_{l+1}(s, t)\, \alpha(s)\, p_0^{-1}[g(s)]\, ds > 1\,, \tag{51'}$$

is sufficient that there exists no nonoscillatory solution u of the equation (1) *with property* P_l.

If instead of Theorem 2 we use Theorem 6 in the proof of Theorem 3, we get

Corollary 7. *Let the conditions* (4), (5), (7), (19), (32) *be satisfied. Further, let the conditions* (20) *or* (20′) *and* (50) *or* (50′) *be fulfilled for* $l = n - 1, n - 3, \ldots, 1$ *when n is even* ($l = n - 1, n - 3, \ldots, 2$ *when n is odd*).

Then the equation (1) *has property* A.

The next corollary is a modification of Theorem 4.

Corollary 8. *Let the conditions* (4), (6), (7), (19) *and* (32) *be satisfied. Further, let the conditions* (20) *or* (20′) *and* (50) *or* (50′) *be fulfilled for* $l = n - 2, n - 4, \ldots, 2$ *when n is even* ($l = n - 2, n - 4, \ldots, 3, 1$ *when n is odd*). *Finally, let the condition* (38) *be satisfied.*

Then the equation (1) *has property* B.

Remark. In a similar way Corollaries 3, 4 and 5 can be modified.

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