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NONOSCILLATORY SOLUTIONS OF SECOND ORDER NONLINEAR DIFFERENTIAL EQUATIONS

LYNN H. ERBE

We consider here a generalization of the equation

$$x'' + a(t)x^{2n+1} = 0$$

where $a(t)$ is a continuous non-negative function on $[0, +\infty)$ and $n \geqq 0$ is an integer. Necessary and sufficient conditions are given for the existence of

(1) a bounded nonoscillatory solution with prescribed limit at ∞;

(2) a nonoscillatory solution whose derivative has a positive limit at ∞.

Specifically, we are concerned with the asymptotic behavior of the solutions of the following second order nonlinear differential equation:

$$x'' + f(t, x)g(x') = 0 . \tag{1}$$

We shall assume the following conditions hold:

(A_0) $f(t, x)$, $g(x')$, and the partial derivative function $f_x(t, x)$ are all continuous for $t \geqq 0$, $x' \geqq 0$, and $|x| < +\infty$.

(A_1) $$f(t, 0) = 0,\ t \geqq 0 .$$

(A_2) $f_x(t, x) \geqq 0$ and is nondecreasing in x for $t \geqq 0$ and $x \geqq 0$.

(A_3) $$g(x') > 0 \text{ for all } x' \geqq 0 .$$

As a special case we have the equation

$$x'' + a(t)x^{2n+1} = 0,\ n \geqq 0 , \tag{2}$$

in which $a(t) \geqq 0$ for $t \geqq 0$ and $g(x') = 1$ for all x'. Oscillatory and nonoscillatory properties of (2) for the case $n \geq 1$ were investigated by Atkinson in [1], Moore and Nehari in [5], and Utz in [9]. Generalizations of equation (2) have been considered by Waltman in [7] and [8], Nehari in [6], Wong in [10], and Macki and Wong in [4].

We shall study equation (1) by considering the equation

$$x'' + f_x(t, \alpha)x = 0 , \tag{3}$$

where α is some real constant depending on solutions of (1). To do this we shall need to establish several lemmas concerning the equation

$$(4) \qquad x'' + p(t)x = 0 ,$$

where $p(t)$ is continuous and satisfies $p(t) \geqq 0$ for $t \geqq 0$.

LEMMA 1.1. *Let $[a, b]$ be a compact interval of the reals and suppose there exists a $\beta(t) \in C^{(2)}$ $[a, b]$ satisfying*

$$\beta(t) > 0 , \quad \beta''(t) + p(t)\beta(t) \leqq 0 , \quad t \in [a, b] .$$

Then $[a, b]$ is an interval of disconjugacy for equation (4). *That is, no nontrival solution of* (4) *has more than one zero on $[a, b]$.*

Proof. If the conclusion is false, then there is a solution $y(t)$ of (4) satisfying $y(t_1) = y(t_2) = 0$ and $y(t) > 0$ on (t_1, t_2), where $a \leqq t_1 < t_2 \leqq b$. It follows that there is a $k > 0$ such that $ky(t) \leqq \beta(t)$ on $[t_1, t_2]$ and $ky(t_0) = \beta(t_0)$ for some $t_1 < t_0 < t_2$. Therefore, $ky'(t_0) = \beta'(t_0)$ and for $t_0 \leqq t \leqq t_2$ we have

$$ky'(t) - \beta'(t) \geqq \int_{t_0}^{t} - p(s)\{ky(s) - \beta(s)\}ds \geqq 0 .$$

Hence,

$$ky(t_2) - \beta(t_2) = \int_{t_0}^{t_2} (ky'(s) - \beta'(s))ds \geqq 0 ,$$

which is a contradiction.

REMARK. If there exists an $\alpha(t) \in C^{(2)}$ $[a, b]$ satisfying

$$\alpha(t) < 0 , \quad \alpha''(t) + p(t)\alpha(t) \geqq 0 , \quad t \in [a, b] ,$$

then the conclusion of the lemma again holds. (Set $\beta(t) = -\alpha(t)$, $t \in [a, b]$.)

Lemma 1.1 is closely related to a theorem of Wintner (see Hartman [2], p. 362, Th. 7.2) and could be obtained directly by setting $z = \beta'/\beta$. Also, a function $\beta(t) \in C^{(2)}$ $[a, b]$ satisfying $\beta''(t) + p(t)\beta(t) \leqq 0$ on $[a, b]$ is just a special case of an upper solution, as defined by Jackson in [3] for general nonlinear second order differential equations. Likewise $\alpha(t) \in C^{(2)}$ $[a, b]$ satisfying $\alpha''(t) + p(t)\alpha(t) \geqq 0$ on $[a, b]$ is a special case of a lower solution.

LEMMA 1.2. *Let $\alpha(t)$, $\beta(t) \in C^{(2)}$ $[a, b]$ and satisfy $\alpha''(t) + p(t)\alpha(t) \geqq 0$, $\beta''(t) + p(t)\beta(t) \leqq 0$, and $0 < \alpha(t) \leqq \beta(t)$ on $[a, b]$. Then for any c, d with $\alpha(a) \leqq c \leqq \beta(a)$, $\alpha(b) \leqq d \leqq \beta(b)$, there is a unique solution $z(t)$ of* (4) *satisfying $z(a) = c$, $z(b) = d$, and $\alpha(t) \leqq z(t) \leqq \beta(t)$ on $[a, b]$.*

Proof. By Lemma 1.1, $[a, b]$ is an interval of disconjugacy for equation (4) so that the BVP

$$x'' + p(t)x = 0 , \quad x(a) = c , \quad x(b) = d$$

has a unique solution $z(t)$ (see for example [2], p. 351). Since $z(t)$ cannot have more than one zero on $[a, b]$ and since initial value problems for (4) have unique solutions, it follows that $z(t) > 0$ on $[a, b]$. If the conclusion of the lemma is false, then assume, to be specific, that $z(t_1) - \beta(t_1) = z(t_2) - \beta(t_2) = 0$ and $z(t) > \beta(t)$ on (t_1, t_2), where $a \leqq t_1 < t_2 \leqq b$. As in Lemma 1.1, there is a $k > 0$, $k < 1$, such that $0 < kz(t) \leqq \beta(t)$ on $[t_1, t_2]$, and $kz(t_0) = \beta(t_0)$, $kz'(t_0) = \beta'(t_0)$ for some $t_1 < t_0 < t_2$. Since $kz(t_2) < z(t_2) = \beta(t_2)$, this leads to a contradiction as in Lemma 1.1. Hence, $z(t) \leqq \beta(t)$ on $[a, b]$. A similar argument shows that $z(t) \geqq \alpha(t)$ on $[a, b]$ and this proves the lemma.

LEMMA 1.3. *Let $\alpha(t), \beta(t) \in C^{(2)}[a, +\infty)$ with $\alpha''(t) + p(t)\alpha(t) \geqq 0$, $\beta''(t) + p(t)\beta(t) \leqq 0$, and $0 < \alpha(t) \leqq \beta(t)$ on $[a, +\infty)$. Then for any $\alpha(a) \leqq c \leqq \beta(a)$ there is a solution $y(t) \in C^{(2)}[a, +\infty)$ of (4) satisfying $y(a) = c$ and $\alpha(t) \leqq y(t) \leqq \beta(t)$ on $[a, +\infty)$.*

Proof. By Lemma 1.2 for each $n \geqq 1$ there is a solution $y_n(t) \in C^{(2)}[a, a + n]$ of (4) satisfying $y_n(a) = c$ and $\alpha(t) \leqq y_n(t) \leqq \beta(t)$ on $[a, a + n]$. Therefore, for each $N \geqq 1$ $|y_n(t)|$ and hence $|y_n''(t)|$ are uniformly bounded on $[a, a + N]$ for all $n = N$. Since $y_n'(t) = y_n'(a) + \int_a^t y_n''$, the $|y_n'(t)|$ are likewise bounded on $[a, a + N]$, uniformly for $n \geqq N$. Now consider the sequence $\{y_n(t)\}_{n=1}^{\infty}$. By the Ascoli-Arzela Theorem there is a subsequence $\{y_n^1(t)\}_{n=1}^{\infty}$ converging to a solution $z_1(t)$ of (4) on $[a, a + 1]$. Inductively, for each $k \geqq 2$ we obtain a subsequence $\{y_n^k(t)\}_{n=1}^{\infty}$ of $\{y_n^{k-1}(t)\}_{n=1}^{\infty}$ which converges to a solution $z_k(t)$ of (4) on $[a, a + k]$. Therefore, the diagonal sequence $\{y_k^k(t)\}_{k=1}^{\infty}$ converges uniformly on each compact subinterval of $[a, +\infty)$. That is,

$$z(t) = \lim_{k\to\infty} y_k^k(t) , \qquad t \in [a, +\infty) ,$$

is the desired solution.

2. After these preliminary lemmas, we are now in a position to establish necessary and sufficient conditions for the existence of certain types of solutions of (1).

THEOREM 2.1. *Assume $A_0 - A_3$ hold and let $\alpha_0 > 0$. Then the following statements are equivalent:*

(a) *For each $0 < \alpha < \alpha_0$ there is a solution $u_\alpha(t)$ of (1) satisfying $\lim_{t\to\infty} u_\alpha(t) = \alpha$.*

(b) $\int^{\infty} tf_y(t, \alpha)dt < +\infty$ *for* $0 < \alpha < \alpha_0$.

Proof. (a) implies (b): Assume $\int^{\infty} tf_y(t, \alpha_1)dt = +\infty$ for some $0 < \alpha_1 < \alpha_0$ and let $\alpha_1 < \beta < \alpha_0$. Let $u_\beta(t)$ be the corresponding solution of (1) with $\lim_{t\to\infty} u_\beta(t) = \beta$. Let $\delta > 0$ be such that $\alpha_1 + \delta < \beta$ and let $T \geqq 0$ be such that $t \geqq T$ implies $u_\beta(t) \geqq \alpha_1 + \delta$. Then for $t \geqq T$

$$u_\beta'' = -f(t, u_\beta)g(u_\beta') \leqq 0$$

so that u_β' decreases to a limit, and this limit clearly must be zero. Therefore, $u_\beta(t) \leqq \beta$ for $t \geqq T$ so that applying the Mean Value Theorem we get

$$f_y(t, \alpha_1) \leqq \frac{f(t, u_\beta(t)) - f(t, \alpha_1)}{u_\beta(t) - \alpha_1} \leqq \frac{f(t, u_\beta(t))}{u_\beta(t) - \alpha_1}$$
$$\leqq \frac{u_\beta(t)}{u_\beta(t) - \alpha_1} \frac{f(t, u_\beta(t))}{u_\beta(t)} \leqq \frac{\beta}{\delta} \frac{f(t, u_\beta(t))}{u_\beta(t)},$$

for $t \geqq T$. Since $\lim_{t\to\infty} u_\beta'(t) = 0$, there is a $T_1 \geqq T$ such that $t \geqq T_1$ implies $g(u_\beta'(t)) \geqq g(0)/2 > 0$. Hence, for $t \geqq T_1$ we have

$$u_\beta''(t) = -f(t, u_\beta(t))g(u_\beta'(t)) \leqq -kf_y(t, \alpha_1)u_\beta(t),$$

where $k = g(0)(\delta/2\beta)$. Also, $\alpha_1'' = 0 \geqq -kf_y(t, \alpha_1)\alpha_1$. Therefore, by Lemma 1.3 there is a solution $z(t)$ of the equation

$$x'' + kf_y(t, \alpha_1)x = 0 \tag{5}$$

satisfying $\alpha_1 \leqq z(t) \leqq u_\beta(t)$ on $[T_1, +\infty)$. Let $w(t) = z(t)\int_{T_1}^t ds/(z(s))^2$ for $t \geqq T_1$. Then $w(t)$ is a solution of (5). Since $z''(t) \leqq 0$ for $t \geqq T_1$, we see that

$$w''(t) = z''(t)\int_{T_1}^t ds/(z(s))^2 \leqq 0$$

for $t \geqq T_1$ and hence $w'(t)$ decreases to a finite nonnegative limit. In fact, we have

$$w'(t) = 1/z(t) + z'(t)\int_{T_1}^t ds/(z(s))^2 \geqq 1/z(t) \geqq 1/\beta$$

for $t \geqq T_1$. Hence, for sufficiently large t, say $t \geqq T_0 \geqq T_1$, we have $w(t) \geqq t/2\beta$. Therefore, for $t \geqq T_0$ we have

$$w'(t) - w'(T_0) = -k\int_{T_0}^t f_y(s, \alpha_1)w(s)ds$$
$$\leqq (-k/2\beta)\int_{T_0}^t sf_y(s, \alpha_1)ds \leqq 0 .$$

Therefore,

$$w'(T_0) \geqq w'(t) + (k/2\beta)\int_{T_0}^{t} sf_y(s, \alpha_1)ds$$

for $t \geqq T_0$, so that

$$\int_{T_0}^{\infty} sf_y(s, \alpha_1)ds < +\infty ,$$

which is the desired contradiction.

Conversely, let $0 < \alpha < \alpha_0$ be given and let

$$M = \max\{g(x') : 0 \leqq x' \leqq \alpha\} .$$

Let $T \geqq 0$ be such that

$$\int_{T}^{\infty}(s - T)f_y(s, \alpha)ds < 1/M \text{ and } \int_{T}^{\infty} f_y(s, \alpha)ds < 1/M .$$

We shall now define a sequence of functions on $[T, +\infty)$ in the following manner:

Let $y_0(t) = \alpha$, $t \geqq T$. Now for $t \geqq T$

$$0 \leqq \int_{t}^{\infty}(s - t)f(s, \alpha)g(0)ds \leqq \alpha\int_{t}^{\infty}(s - t)f_y(s, \alpha)g(0)ds \leqq \alpha ,$$

so that defining $y_1(t) = \alpha - \int_{t}^{\infty}(s - t)f(s, \alpha)g(0)ds$, $t \geqq T$, we have $0 \leqq y_1(t) \leqq \alpha$. Differentiating $y_1(t)$ we have

$$0 \leqq y_1'(t) = \int_{t}^{\infty} f(s, \alpha)g(0)ds \leqq M\alpha\int_{t}^{\infty} f_y(s, \alpha)ds < \alpha .$$

Proceeding inductively, we define for all $k \geqq 1$

$$y_{k+1}(t) = \alpha - \int_{t}^{\infty}(s - t)f(s, y_k(s))g(y_k'(s))ds , \quad t \geqq T ,$$

and obtain $0 \leqq y_k(t)$, $y_k'(t) \leqq \alpha$ for all $k \geqq 1$. It follows that the sequences $y_k(t)$, $y_k'(t)$, and $y_k''(t)$ are uniformly bounded on $[T, T + n]$ for all $n \geqq 1$. The Ascoli-Arzela Theorem and a diagonalization argument yields a subsequence which converges, uniformly on compact subsets of $[T, +\infty)$, to a solution $u_\alpha(t)$ of (1). Obviously, $\lim_{t\to\infty} u_\alpha(t) = \alpha$. This completes the proof of the theorem.

REMARK. If $f(t, x) = -f(t, -x)$ and $g(x') > 0$ and is continuous for $|x'| < +\infty$, then we see that $\int^{\infty} tf_y(t, \alpha)dt < +\infty$ for $0 < |\alpha| < \alpha_0$ if and only if for each $0 < |\alpha| < \alpha_0$ there is a solution $u_\alpha(t)$ of (1) with $\lim_{t\to\infty} u_\alpha(t) = \alpha$.

COROLLARY 2.2. *$\int^\infty t f_y(t, \alpha)dt < +\infty$ for all $\alpha > 0$ if and only if there is a solution $u_\alpha(t)$ of (1) with $\lim_{t\to\infty} u_\alpha(t) = \alpha$ for all $\alpha > 0$.*

COROLLARY 2.3. *If $f(t, x) = \sum_{i=0}^n a_i(t)x^{2i+1}$ where the $a_i(t)$ are continuous nonnegative functions for $t \geqq 0$, then the following statements are equivalent:*

(a) *There is a solution $u_\alpha(t)$ of (1) with $\lim_{t\to\infty} u_\alpha(t) = \alpha$ for all $\alpha \neq 0$.*

(b) $\sum_{i=0}^n \int^\infty t a_i(t)dt < +\infty$.

As examples of equations to which Theorem 2.1 applies but which do not belong to any of the classes of equations considered in references [1], [4] through [8], we have

$$x'' + x(\exp(t(x - \alpha_0)))(1 + x') = 0 \tag{6}$$

$$x'' + x(\exp(t(x^2 - \alpha_0^2) + cx'))(1 + (x')^2) = 0\,, \tag{7}$$

where c is an arbitrary real number. Then for $0 < \alpha < \alpha_0$ there is a solution $u_\alpha(t)$ of (6) with $\lim_{t\to\infty} u_\alpha(t) = \alpha$, and for $0 < |\alpha| < \alpha_0$ there is a solution $y_\alpha(t)$ of (7) with $\lim_{t\to\infty} y_\alpha(t) = \alpha$.

3. In [5] it is shown that equation (2) has solutions for which

$$\lim_{t\to\infty} \frac{y(t)}{t} = \alpha > 0$$

if and only if

$$\int^\infty t^{2n+1} a(t)dt < +\infty\,.$$

In this final section we will show that an analogous result is true for equation (1) provided $f(t, x)$ satisfies the following additional condition.

(A_4) There exist real numbers $c > 0$ and $\lambda > 0$ such that

$$\liminf_{x\to\infty} \frac{f(t, x)}{x f_x(t, cx)} \geqq \lambda > 0, \text{ for all sufficiently large } t.$$

Note that in the case of equation (2) c and λ may be any positive real numbers with $\lambda c^{2n} \leqq 1/(2n + 1)$. We first establish the following lemma.

LEMMA 3.1. *Assume conditions $A_0 - A_3$ hold and let there exist a real number $\beta > 0$ with*

$$\int^\infty t f_y(t, \beta t)dt < +\infty\,.$$

Then there exist solutions to (1), *say* $y(t)$, *such that* $\lim_{t\to\infty} y(t)/t$ *exists and is positive.*

Proof. Let $T > 0$ be such that

$$\int_T^\infty tf_y(t, \beta t)dt < 1/2M ,$$

where $M = \max\{g(x') : 0 \leqq x' \leqq \beta\}$. We define a solution of (1) by

$$u(T) = 0 , \qquad u'(T) = \beta ,$$

and we assert that the solution satisfies $u'(t) \geqq \beta/2$ for $t \geqq T$. Assume, on the contrary, that there is a $\delta > 0$, $\beta/2 > \delta > 0$, and a $t_1 > T$ with $u'(t_1) = \delta$ and $u(t) > 0$ on $(T, t_1]$. Then for $T \leqq t \leqq t_1$ we have

$$u'(T) = u'(t) + \int_T^t f(s, u(s))g(u'(s))ds . \tag{8}$$

Since $u''(t) \leqq 0$ on $(T, t_1]$ and since $u(t)$ is concave it follows that

$$u'(t) \leqq \beta \quad \text{on} \quad (T, t_1) \quad \text{and}$$
$$u(t) \leqq \beta(t - T) \quad \text{on} \quad (T, t_1) .$$

Applying the Mean Value Theorem in (8) we have

$$\begin{aligned}\beta = u'(T) &< u'(t) + M\beta\int_T^t sf_y(s, \beta(s - T))ds \\ &\leqq u'(t) + M\beta\int_T^t sf_y(s, \beta s)ds < u'(t) + \beta/2 .\end{aligned}$$

Hence, $u'(t_1) > \beta/2$, a contradiction. Therefore, $u'(t) \geqq \beta/2$ on $[T, +\infty)$ and hence $\lim_{t\to\infty} u'(t)$ exists and is positive which implies that $\lim_{t\to\infty} u(t)/t$ exists and is positive.

THEOREM 3.2. *Assume conditions* $(A_0) - (A_4)$ *hold. Then* (1) *has solutions, say* $y(t)$, *such that* $\lim_{t\to\infty} y(t)/t$ *exists and is positive if and only if*

$$\int^\infty tf_y(t, \beta t)dt < +\infty \text{ for some } \beta > 0 .$$

Proof. Let $\alpha > 0$ and let $y(t)$ be a solution of (1) with

$$\lim_{t\to\infty} \frac{y(t)}{t} = \alpha .$$

Let $T \geqq 0$ be such that $t \geqq T$ implies $y(t) \geqq \alpha t/2$. Let

$$m_0 = \min\{g(x') : 0 \leqq x' \leqq y'(T) .$$

By condition (A_4) there is a $T_1 \geqq T$ such that $t \geqq T_1$ implies

$$f(t, y(t)) \geqq \lambda y(t) f_y(t, c\alpha t/2) \geqq (kt) f_y(t, c\alpha t/2) ,$$

where $k = \lambda\alpha/2$. Since $0 < y'(t) \leqq y'(T)$ for $t \geqq T$ we have

$$f(t, y(t)) g(y'(t)) \geqq (m_0 kt) f_y(t, c\alpha t/2) , \quad t \geqq T_1 .$$

Therefore,

$$\begin{aligned} y'(T_1) &= y'(t) + \int_{T_1}^{t} f(s, y(s)) g(y'(s)) ds \\ &\geqq y'(t) + \int_{T_1}^{t} (m_0 ks) f_y(s, c\alpha s/2) ds . \end{aligned}$$

Since $\lim_{t\to\infty} y'(t) \geqq 0$, this implies that

$$\int_{T_1}^{\infty} s f_y(s, c\alpha s/2) ds < +\infty ,$$

and this proves the theorem.

As a simple example of an equation to which the previous theorem applies but which is not considered in references [1], [4] through [8], we have

$$x'' + x^2 (\exp(x - \beta t))(1 + x') = 0 , \tag{9}$$

where $\beta > 0$. Condition (A_4) holds for any $0 < c < 1$ and any $\lambda > 0$.

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REFERENCES

1. F. V. Atkinson, *On second order nonlinear oscillations*, Pacific J. Math. **5** (1955), 643-647.
2. P. Hartman, *Ordinary differential equations*, Wiley, 1964.
3. L. K. Jackson, *Subfunctions and second order differential inequalities*, Advances in Math., Academic Press (to appear).
4. J. W. Macki and J. S. W. Wong, *Oscillation of solutions to second order nonlinear differential equations*, Pacificic J. Math. **18** (1968) (to appear)
5. R. A. Moore and Z. Nehari, *Nonoscillation theorems for a class of nonlinear differential equations*, Trans. Amer. Math. Soc. **92-93** (1959), 30-52.
6. Z. Nehari, *On a class of nonlinear second order differential equations*, Trans. Amer. Math. Soc. **94-95** (1960), 101-123.
7. P. Waltman, *Oscillation of solutions of a nonlinear equation*, SIAM Review **5** (1963), 128-130.
8. ———, *Some properties of solution of* $u'' + a(t)f(u) = 0$, Monat. Math. (1963-64), 50-54.
9. W. R. Utz, *Properties of solutions of* $u'' + g(t)u^{2n-1} = 0$, Monat. Math. (1961-62), 55-60.

10. J. S. W. Wong, *A note on second order nonlinear oscillation*, SIAM Review **10** (1968), 88-91.

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