The Annals of Statistics
1977, Vol. 5, No. 5, 1027-1038

NONPARAMETRIC ESTIMATION OF A VECTOR-VALUED BIVARIATE FAILURE RATE

BY IBRAHIM A. AHMAD AND PI-ERH LIN[1]

Memorial University of Newfoundland and Florida State University

Let $\mathbf{X} = (X_1, X_2)'$ be a bivariate random vector distributed according to an absolutely continuous distribution function $F(\mathbf{x})$ which has first partial derivatives. Let $\bar{F}(\mathbf{x}) = P(X_1 > x_1, X_2 > x_2)$. The vector-valued bivariate failure rate is defined as $\mathbf{r}(\mathbf{x}) = (r_1(\mathbf{x}), r_2(\mathbf{x}))'$, where $r_i(\mathbf{x}) = -\partial \ln \bar{F}(\mathbf{x})/\partial x_i$ $(i = 1, 2)$. In this paper, we propose a smooth nonparametric estimate $\hat{\mathbf{r}}(\mathbf{x})$ of $\mathbf{r}(\mathbf{x})$ using Cacoullos' (*Ann. Inst. Statist. Math.* **18** (1966), 181-190) multivariate density estimate. Regularity conditions are obtained under which $\hat{\mathbf{r}}(\mathbf{x})$ is shown to be pointwise strongly consistent. A set of sufficient conditions is given for the strong uniform consistency of $\hat{\mathbf{r}}(\mathbf{x})$ over a subset S of R^2 where $\bar{F}(\mathbf{x})$ is bounded below by $\varepsilon > 0$. The joint asymptotic normality of the estimate evaluated at q distinct continuity points of the failure rate is established. The methods and results presented in this paper can be generalized to any finite dimensional case in a straightforward manner.

1. Introduction. The univariate failure rate (also known as hazard rate, mortality rate, etc.) and the role it plays in reliability theory are well known. Probability models of monotone failure rate have been studied extensively (see, for example, Chapter 3 of Barlow and Proschan (1975)). Nonparametric estimation of the failure rate has been considered by Watson and Leadbetter (1964a, b), and Barlow and van Zwet (1971), among others. More recently, two different multivariate analogs of failure rate have been proposed. Basu (1971), Cox (1972), and Puri and Rubin (1974) consider a scalar-valued multivariate failure rate. Block (1973), Johnson and Kotz (1975), and Marshall (1975) study a vector-valued multivariate failure rate which is also called hazard gradient. The importance of the hazard gradient concept is reflected in equation (2.2) of Marshall (1975). It establishes an important relationship, which is well known in the univariate case, between the survival probability and the hazard gradient. A similar relationship seems lacking between the survival probability and the scalar-valued failure rate. The usefulness of the vector-valued failure rate is further demonstrated by Johnson and Kotz (1975) through the use of examples. The above mentioned authors are mostly concerned with properties and characterizations of either multivariate failure rate.

Received January 1976; revised January 1977.

[1] Research supported in part by the National Institute of General Medical Sciences through Training Grant 5T01 GM-913.

AMS 1970 *subject classifications*. Primary 62G05, 62G20, 62E20; Secondary 62N05.

Key words and phrases. Bernstein's inequality, Cacoullos' multivariate density estimate, hazard gradient, kernel function, limiting distribution of the estimate, strong consistency, and strong uniform consistency.

It is the purpose of this study to propose a smooth nonparametric estimate for the vector-valued bivariate failure rate. Important asymptotic properties of the estimate are obtained and the limiting distribution of the estimate evaluated at a finite set of continuity points is established. More specifically, let $\mathbf{X} = (X_1, X_2)'$ be a bivariate random vector distributed according to an absolutely continuous distribution function (df) F having the probability density function (pdf) f. Let F_i and f_i be the marginal df and pdf of X_i $(i = 1, 2)$. The vector-valued bivariate failure rate is given by

$$\mathbf{r}(\mathbf{x}) = [r_1(\mathbf{x}), r_2(\mathbf{x})]' , \tag{1.1}$$

where

$$r_i(\mathbf{x}) = \frac{-\partial}{\partial x_i} \ln \bar{F}(\mathbf{x}) = \frac{G_i(\mathbf{x})}{\bar{F}(\mathbf{x})} , \qquad i = 1, 2 , \tag{1.2}$$

with

$$\bar{F}(\mathbf{x}) = P(\mathbf{X} > \mathbf{x}) > 0 ,$$

and

$$G_1(\mathbf{x}) = \textstyle\int_{x_2}^{\infty} f(x_1, v)\, dv \quad \text{and} \quad G_2(\mathbf{x}) = \int_{x_1}^{\infty} f(u, x_2)\, du . \tag{1.3}$$

Based on a random sample $\{\mathbf{X}_j\}_1^n = \{(X_{1j}, X_{2j})\}_1^n$ of size n from F, Cacoullos' (1966) type estimates for $\bar{F}$ and G_i $(i = 1, 2)$ may be obtained via the use of kernel functions. These estimates are then substituted into (1.2) and (1.1) to obtain an estimate $\hat{\mathbf{r}}(\mathbf{x})$ of $\mathbf{r}(\mathbf{x})$.

Some preliminary results on the estimates of $\bar{F}$ and G_i $(i = 1, 2)$ are given in Section 2. Sufficient conditions for pointwise strong consistency and for strong uniform consistency of $\hat{\mathbf{r}}(\mathbf{x})$ are obtained in Section 3. Finally in Section 4, the joint asymptotic normality of $\hat{\mathbf{r}}'(\mathbf{x}_1), \cdots, \hat{\mathbf{r}}'(\mathbf{x}_q)$ is established where $\mathbf{x}_1, \cdots, \mathbf{x}_q$ are q distinct continuity points of the vector-valued bivariate failure rate $\mathbf{r}$. The results and methods of proof employed here can be generalized to any finite dimensional case in a straightforward manner. It is clear that similar estimates and asymptotic results can be obtained for the scalar-valued bivariate failure rate. However, for reasons mentioned in Paragraph 1, estimation of the scalar-valued bivariate failure rate will not be studied here.

In the remainder of this section, a kernel function will be defined and the proposed estimate $\hat{\mathbf{r}}(\mathbf{x})$ given. A kernal function $k(t)$ is a known pdf satisfying the conditions

$$\sup_t k(t) < \infty , \tag{1.4}$$

and

$$\lim_{|t|\to\infty} |t| k(t) = 0 . \tag{1.5}$$

Let $k_1(t)$ and $k_2(t)$ be two (possibly different) kernel functions. Define

$$\bar{K}_i(z) = a_n \textstyle\int_z^{\infty} k_i(v)\, dv , \qquad i = 1, 2 , \tag{1.6}$$

where $\{a_n\}$ is a nonincreasing sequence of positive real numbers converging to

0 as $n \to \infty$. Then, similar to Cacoullos' (1966) estimate for a multivariate density, the estimates for $\bar{F}$ and G_i ($i = 1, 2$) are given, respectively, by

$$\hat{\bar{F}}(\mathbf{x}) = \frac{1}{na_n^2} \sum_{j=1}^n \bar{K}_1\left(\frac{x_1 - X_{1j}}{a_n}\right) \bar{K}_2\left(\frac{x_2 - X_{2j}}{a_n}\right), \tag{1.7}$$

and

$$\hat{G}_i(\mathbf{x}) = \frac{1}{na_n^2} \sum_{j=1}^n k_i\left(\frac{x_i - X_{ij}}{a_n}\right) \bar{K}_{i'}\left(\frac{x_{i'} - X_{i'j}}{a_n}\right), \tag{1.8}$$

$$i, i' = 1, 2;\ i \neq i'.$$

Using (1.7) and (1.8) it is proposed to estimate $\mathbf{r}(\mathbf{x})$ by

$$\hat{\mathbf{r}}(\mathbf{x}) = (\hat{r}_1(\mathbf{x}), \hat{r}_2(\mathbf{x}))', \tag{1.9}$$

where

$$\hat{r}_i(\mathbf{x}) = \frac{\hat{G}_i(\mathbf{x})}{\hat{\bar{F}}(\mathbf{x})}, \qquad i = 1, 2. \tag{1.10}$$

With appropriate choice of kernel functions, such as the standard normal pdf, the denominator of $\hat{r}_i(\mathbf{x})$, $i = 1, 2$, will be strictly positive with probability one. Moreover, the estimate $\hat{r}_i(\mathbf{x})$ may be expressed as $-\partial \ln \hat{\bar{F}}(\mathbf{x})/\partial x_i$ ($i = 1, 2$), a form similar to (1.2) for $r_i(\mathbf{x})$.

2. Preliminary results. Asymptotic properties of the estimates of $\bar{F}$, G_1, and G_2 will be obtained in this section. They will play an important role in the next section where the consistency results of $\hat{\mathbf{r}}(\mathbf{x})$ are established. Lemma 2.1 is concerned with the asymptotic mean and variance of $\hat{\bar{F}}$, and Lemma 2.2 with those of $\hat{G}_i$ ($i = 1, 2$). Lemma 2.3 deals with the pointwise strong consistency of $\hat{G}_i$ ($i = 1, 2$) and of $\hat{\bar{F}}$, and Lemma 2.4 provides sufficient conditions for the strong uniform consistency of $\hat{G}_i$ ($i = 1, 2$) and of $\hat{\bar{F}}$ in R^2.

LEMMA 2.1. *Let* $\mathbf{x}$ *be a continuity point of* $\bar{F}$. *Then, as* $n \to \infty$,

(i) $E\hat{\bar{F}}(\mathbf{x}) \to \bar{F}(\mathbf{x})$,

and

(ii) $n \operatorname{Var} \hat{\bar{F}}(\mathbf{x}) \to \bar{F}(\mathbf{x})(1 - \bar{F}(\mathbf{x}))$.

PROOF. Upon an application of Theorem 2.1 of Cacoullos (1966) the lemma follows immediately. □

From Lemma 2.1 both $\hat{\bar{F}}(\mathbf{x})$ and $\bar{F}_n(\mathbf{x})$, the proportion of $\mathbf{X}_j > \mathbf{x}$, are consistent estimates of $\bar{F}(\mathbf{x})$ with the same rate of convergence.

LEMMA 2.2. *Assume that* f *is continuous and bounded. Then, as* $n \to \infty$,

(i) $E\hat{G}_i(\mathbf{x}) \to G_i(\mathbf{x})$,

and

(ii) $na_n \operatorname{Var} \hat{G}_i(\mathbf{x}) \to \beta_i$,

where

$$\beta_i = G_i(\mathbf{x}) \int k_i^2(u)\, du, \qquad i = 1, 2. \tag{2.1}$$

PROOF. Only the asymptotic results of $\hat{G}_1$ are presented. Those of $\hat{G}_2$ can be obtained by symmetry.

(i) An application of Fubini's theorem gives

$$E\hat{G}_1(\mathbf{x}) = \int_{x_2}^{\infty}\left[\frac{1}{a_n^2}\int\int k_1\left(\frac{x_1-u}{a_n}\right)k_2\left(\frac{s-v}{a_n}\right)f(u, v)\,du\,dv\right]ds\,.$$

Since f is continuous and bounded, Theorem 2.1 of Cacoullos (1966) may be applied to conclude that the expression in the bracket above converges to $f(x_1, s)$, as $n \to \infty$. Hence (i) is established.

(ii) Since $k_1[(x_1 - X_{1j})/a_n]\bar{K}_2[(x_2 - X_{2j})/a_n]$, $j = 1, \cdots, n$, are i.i.d., it follows that

$$\begin{aligned} na_n \operatorname{Var} \hat{G}_1(\mathbf{x}) &= \frac{1}{a_n} E\left\{k_1^2\left(\frac{x_1 - X_{11}}{a_n}\right)\left[\frac{1}{a_n}\bar{K}_2\left(\frac{x_2 - X_{21}}{a_n}\right)\right]^2\right\} - a_n[E\hat{G}_1(\mathbf{x})]^2 \\ &= \int k_1^2(u) \int \left[\frac{1}{a_n}\bar{K}_2\left(\frac{x_2 - v}{a_n}\right)\right]^2 f(x_1 - a_n u, v)\,du\,dv + O(a_n)\,. \end{aligned}$$

Since $a_n^{-1}\bar{K}_2[(x_2 - v)/a_n] \to 0$ or 1 depending on whether $v < x_2$ or $v > x_2$, and since both k_1 and f are continuous and bounded, the Lebesgue dominated convergence theorem may be applied to establish (ii). □

LEMMA 2.3. *Assume that f is continuous and bounded. If, for every $c > 0$, $\sum_{n=1}^{\infty} \exp(-cna_n^2) < \infty$, then*

(i) $\hat{G}_i(\mathbf{x}) \to G_i(\mathbf{x})$, *with probability one, as* $n \to \infty$ $(i = 1, 2)$,

and

(ii) $\hat{\bar{F}}(\mathbf{x}) \to \bar{F}(\mathbf{x})$, *with probability one, as* $n \to \infty$.

PROOF. Only the proof of (i) with $i = 1$ will be given. The case $i = 2$ and part (ii) can be proved along the same lines.

(i) Let $W_{nj}(\mathbf{x}) = k_1[(x_1 - X_{1j})/a_n]\bar{K}_2[(x_2 - X_{2j})/a_n]$, $j = 1, \cdots, n$. Note that $W_{nj}(\mathbf{x})$, $j = 1, \cdots, n$, are i.i.d. uniformly bounded random variables. Set

$$S_n(\mathbf{x}) = \sum_{j=1}^{n} W_{nj}(\mathbf{x}) = na_n^2\hat{G}_1(\mathbf{x}) \tag{2.2}$$

and, for any $\mathbf{x}$, let

$$M = M(\mathbf{x}) = \sup_j |W_{nj}(\mathbf{x}) - EW_{nj}(\mathbf{x})|\,. \tag{2.3}$$

Then it follows from Bernstein's inequality (see, e.g., Bennett (1962)) that, for all n sufficiently large and for any $t > 0$,

$$P[|S_n(\mathbf{x}) - ES_n(\mathbf{x})| > t\sigma_n] \leqq 2\exp\left(-\frac{t^2}{2(1 + Mt/3\sigma_n)}\right), \tag{2.4}$$

where

$$\sigma_n^2 = \operatorname{Var} S_n(\mathbf{x}) = n \operatorname{Var} W_{n1}(\mathbf{x}) \simeq na_n^3\beta_1\,. \tag{2.5}$$

Thus, for any $\varepsilon > 0$ and all n sufficiently large, we have

$$P[|\hat{G}_1(\mathbf{x}) - E\hat{G}_1(\mathbf{x})| > \varepsilon] \leqq 2\exp\left(-\frac{\varepsilon^2 na_n^2}{2(a_1\beta_1 + M\varepsilon/3)}\right).$$

This, together with the Borel-Cantelli lemma and Lemma 2.2 (i) establishes (i). □

LEMMA 2.4. *Assume that f is uniformly continuous and that k_i is a uniformly continuous function of bounded variation with finite first absolute moment* ($i = 1, 2$). *If, for every $c > 0$, $\sum_{n=1}^{\infty} \exp(-cna_n^2) < \infty$, then*

(i) $\sup_{\mathbf{x} \in R^2} |\hat{G}_i(\mathbf{x}) - G_i(\mathbf{x})| \to 0$, *with probability one, as $n \to \infty$*, ($i = 1, 2$), *and*

(ii) $\sup_{\mathbf{x} \in R^2} |\hat{\bar{F}}(\mathbf{x}) - \bar{F}(\mathbf{x})| \to 0$, *with probability one, as $n \to \infty$.*

PROOF. The proof is similar to that of Theorem 1 of Nadaraya (1965) where the strong uniform consistency is established for the univariate density estimate. However, nontrivial modifications are necessary in proving the strong uniform consistency for $\hat{G}_i$ and $\hat{\bar{F}}$. We will thus present a proof of part (i) with $i = 1$ here. Part (ii) can be established similarly. For notational convenience we will write sup for the supremum over $\mathbf{x} \in R^2$. Note that

$$\sup |\hat{G}_1(\mathbf{x}) - G_1(\mathbf{x})| \leqq \sup |\hat{G}_1(\mathbf{x}) - E\hat{G}_1(\mathbf{x})| + \sup |E\hat{G}_1(\mathbf{x}) - G_1(\mathbf{x})| = \Delta_{1n} + \Delta_{2n}, \quad \text{say.} \tag{2.6}$$

Upon integration by parts $E\hat{G}_1(\mathbf{x})$ and $\hat{G}_1(\mathbf{x})$ may be written as

$$E\hat{G}_1(\mathbf{x}) = \frac{1}{a_n^2} \int\int F(u, v)\, dk_1\left(\frac{x_1 - u}{a_n}\right) d\bar{K}_2\left(\frac{x_2 - v}{a_n}\right), \tag{2.7}$$

and similarly

$$\hat{G}_1(\mathbf{x}) = \frac{1}{a_n^2} \int\int F_n(u, v)\, dk_1\left(\frac{x_1 - u}{a_n}\right) d\bar{K}_2\left(\frac{x_2 - v}{a_n}\right), \tag{2.8}$$

where $F_n(u, v)$ is the empirical df. Now it follows from (2.7) and (2.8) that

$$\Delta_{1n} = \sup |E\hat{G}_1(\mathbf{x}) - \hat{G}_1(\mathbf{x})| \leqq (\nu/a_n) \sup |F(\mathbf{x}) - F_n(\mathbf{x})|,$$

where $\nu = \int dk_1$ is the variation of k_1. Thus, Theorem 1 of Kiefer and Wolfowitz (1958) implies that, for any $\varepsilon > 0$,

$$P(\Delta_{1n} > \varepsilon) \leqq P[\sup |F_n(\mathbf{x}) - F(\mathbf{x})| > \varepsilon a_n/\nu] \leqq C \exp(-\varepsilon_1 n a_n^2), \tag{2.9}$$

where $\varepsilon_1 = (\varepsilon/\nu)^2$ and $0 < C < \infty$. Inequality (2.9), together with the Borel-Cantelli lemma, implies that $\Delta_{1n} \to 0$, with probability one, as $n \to \infty$. It remains to show that

$$\Delta_{2n} = \sup |E\hat{G}_1(\mathbf{x}) - G_1(\mathbf{x})| \to 0, \qquad \text{as} \quad n \to \infty. \tag{2.10}$$

Let $\delta > 0$ be given. Then, it is clear that

$$\begin{aligned} \Delta_{2n} &\leqq \sup \int\int_{(x_2 - v)/a_n}^{\infty} k_2(y_2)\, dy_2 \frac{1}{a_n} \int k_1\left(\frac{y_1}{a_n}\right) |f(x_1 - y_1, v) - f(x_1, v)|\, dy_1\, dv \\ &\quad + \sup \left|\int\int_{(x_2 - v)/a_n}^{\infty} k_2(y_2)\, dy_2 \cdot f(x_1, v)\, dv - \int_{x_2}^{\infty} f(x_1, v)\, dv\right| \\ &\leqq \sup \int \left\{\int_{|y_1| \leqq \delta} + \int_{|y_1| > \delta}\right\} \frac{1}{a_n} k_1\left(\frac{y_1}{a_n}\right) |f(x_1 - y_1, v) - f(x_1, v)|\, dy_1\, dv \\ &\quad + \sup \int \left|\int_{x_2 - a_n y_2}^{x_2} k_2(y_2) f(x_1, v)\, dv\right| dy_2\,. \end{aligned} \tag{2.11}$$

The second inequality of (2.11) is obtained by noting that k_2 is a density in the first term and by changing the order of integration in the second. Now let η be an arbitrary small positive quantity. By choosing δ sufficiently small and then with so chosen δ choosing n sufficiently large, we can make the first term in the last expression of (2.11) less than $\eta/2$ upon an application of (7) and (8) of Nadaraya (1965) with obvious modification where f is uniformly continuous. By the mean-valued theorem, the second term is bounded above by $a_n \cdot \sup f(\mathbf{x}) \cdot \int |u| k_2(u)\,du$ which can be made smaller than $\eta/2$ for sufficiently large n. This establishes (2.10) and (i). □

3. Consistency of vector-valued bivariate failure rate estimate. Results of Section 2 will be utilized here to study asymptotic properties of $\hat{\mathbf{r}}(\mathbf{x})$ proposed in (1.9). In particular, the pointwise strong consistency and strong uniform consistency of $\hat{\mathbf{r}}(\mathbf{x})$ will be established.

THEOREM 3.1. *Assume that the conditions of Lemma* 2.3 *hold. Then*

$$\hat{\mathbf{r}}(\mathbf{x}) \to \mathbf{r}(\mathbf{x}) , \quad \textit{with probability one,} \quad \textit{as} \quad n \to \infty .$$

PROOF. This follows immediately from Lemma 2.3. □

The following theorem provides a set of sufficient conditions for strong uniform consistency of $\hat{\mathbf{r}}(\mathbf{x})$ in a subset S of R^2.

THEOREM 3.2. *Assume that the following conditions hold*:

(i) *The* pdf $f(\mathbf{x})$ *is uniformly continuous and* $\bar{F}(\mathbf{x})$ *is bounded below by* $\varepsilon > 0$ *on a subset* S *of* R^2.

(ii) *The kernel function* $k_i(u)$ *is a uniformly continuous function of bounded variation with finite first absolute moment* $(i = 1, 2)$.

(iii) *For every* $c > 0$, $\sum_{n=1}^{\infty} \exp(-cna_n^2) < \infty$.

Then, for $i = 1, 2$,

$$\sup_{\mathbf{x} \in S} |\hat{r}_i(\mathbf{x}) - r_i(\mathbf{x})| \to 0 , \quad \textit{with probability one,} \quad \textit{as} \quad n \to \infty .$$

PROOF. The theorem follows directly from Lemma 2.4 because: Let $\{\phi_{in}(\mathbf{x})\}$, $i = 1, 2$, be sequences of real functions on R^2 and assume that

$$\sup_{\mathbf{x} \in R^2} |\phi_{in}(\mathbf{x}) - \phi_i(\mathbf{x})| \to 0 , \qquad \text{as} \quad n \to \infty .$$

Assume further that ϕ_2 is bounded below by $\varepsilon > 0$ on a subset S of R^2. Then

$$\sup_{\mathbf{x} \in S} \left| \frac{\phi_{1n}(\mathbf{x})}{\phi_{2n}(\mathbf{x})} - \frac{\phi_1(\mathbf{x})}{\phi_2(\mathbf{x})} \right| \to 0 , \qquad \text{as} \quad n \to \infty . \qquad \square$$

4. Limiting distribution of the estimate. Let $\mathbf{x}_\alpha' = (x_{1\alpha}, x_{2\alpha})$, $\alpha = 1, \cdots, q$, be q distinct continuity points of $\mathbf{r}$ such that $\sum_{i=1}^{2} \sum_{\alpha=1}^{q} G_i(\mathbf{x}_\alpha) > 0$, and $\bar{F}(\mathbf{x}_\alpha) > 0$, $\alpha = 1, \cdots, q$. In this section the limiting distribution of $(\hat{\mathbf{r}}'(\mathbf{x}_1), \cdots, \hat{\mathbf{r}}'(\mathbf{x}_q))'$ is derived. The proof is based on a generalization of Theorem (iii) of Rao (1965), page 322.

THEOREM 4.1. *Assume the following conditions*:

(i) $\int uk_i(u)\,du = 0$ *and* $\int u^2k_i(u)\,du < \infty$, $(i = 1, 2)$;
(ii) $na_n \to \infty$ *and* $na_n^4 \to 0$, *as* $n \to \infty$; *and*
(iii) $G_i(\mathbf{x})$ *has bounded partial derivatives of first and second orders* $(i = 1, 2)$.

If $\prod_{i=1}^2 \prod_{\alpha\neq\beta=1}^q (x_{i\alpha} - x_{i\beta}) \neq 0$, *then the limiting distribution of* $[\hat{\mathbf{r}}'(\mathbf{x}_1), \cdots, \hat{\mathbf{r}}'(\mathbf{x}_q)]'$ *is a* $2q$*-variate normal distribution with mean vector* $\mathbf{u}' = \{\mathbf{r}'(\mathbf{x}_1), \cdots, \mathbf{r}'(\mathbf{x}_q)\}$ *and covariance matrix* $(na_n)^{-1}\Sigma$ *with* $\Sigma = (\sigma_{i\alpha j\beta})$ *where*

$$
\begin{aligned}
\sigma_{i\alpha j\beta} &= \lim_{n\to\infty} na_n \operatorname{Cov}[\hat{r}_i(\mathbf{x}_\alpha), \hat{r}_j(\mathbf{x}_\beta)] \qquad i, j = 1, 2\,; \quad \alpha, \beta = 1, \cdots, q \\
&= G_i(\mathbf{x}_\alpha)[\bar{F}(\mathbf{x}_\alpha)]^{-2} \int k_i^2(u)\,du + r_i^2(\mathbf{x}_\alpha)[1 - \bar{F}(\mathbf{x}_\alpha)][\bar{F}(\mathbf{x}_\alpha)]^{-1}, \\
&\qquad\qquad \text{for } \quad i = j \quad \text{and} \quad \alpha = \beta \\
&= r_i(\mathbf{x}_\alpha)r_j(\mathbf{x}_\alpha)[1 - \bar{F}(\mathbf{x}_\alpha)][\bar{F}(\mathbf{x}_\alpha)]^{-1}, \qquad \text{for } \quad i \neq j \quad \text{and} \quad \alpha = \beta \\
&= r_i(\mathbf{x}_\alpha)r_j(\mathbf{x}_\beta)[\bar{F}(\mathbf{x}_\alpha)\bar{F}(\mathbf{x}_\beta)]^{-1}\gamma_{\alpha\beta} \qquad \text{for } \quad i, j = 1, 2\,; \quad \alpha \neq \beta
\end{aligned} \tag{4.1}
$$

with

$$
\begin{aligned}
\gamma_{\alpha\beta} &= \bar{F}(\mathbf{x}_{\max(\alpha,\beta)}) - \bar{F}(\mathbf{x}_\alpha)\bar{F}(\mathbf{x}_\beta) \qquad \text{and} \\
\mathbf{x}'_{\max(\alpha,\beta)} &= (x_{\alpha\max}, x_{\beta\max}) = (\max(x_{1\alpha}, x_{1\beta}), \max(x_{2\alpha}, x_{2\beta}))\,, \\
&\qquad\qquad \text{for} \quad \alpha \neq \beta = 1, \cdots, q\,.
\end{aligned} \tag{4.2}
$$

REMARK. For simplicity the theorem will be shown only for the case $q = 2$. The following lemma will be used in the proof; it also may have independent interest.

LEMMA 4.2. *Let* $\mathbf{x}_\alpha' = (x_{1\alpha}, x_{2\alpha})$, $\alpha = 1, 2$, *be two distinct continuity points of* $G_i(\mathbf{x})$, $i = 1, 2$, *such that* $\sum_{i=1}^2 \sum_{\alpha=1}^2 G_i(\mathbf{x}_\alpha) > 0$. *Define*

$$
\begin{aligned}
Y_{\alpha n} &= a_n^{\frac{1}{2}}[\hat{G}_1(\mathbf{x}_\alpha) - E\hat{G}_1(\mathbf{x}_\alpha)]\,, \\
Y_{\alpha+2\,n} &= a_n^{\frac{1}{2}}[\hat{G}_2(\mathbf{x}_\alpha) - E\hat{G}_2(\mathbf{x}_\alpha)]\,, \\
Y_{\alpha+4\,n} &= \hat{\bar{F}}(\mathbf{x}_\alpha) - E\hat{\bar{F}}(\mathbf{x}_\alpha)\,, \qquad \alpha = 1, 2\,.
\end{aligned} \tag{4.3}
$$

If assumption (ii) *of Theorem* 4.1 *holds, then*

$$
n^{\frac{1}{2}}(Y_{1n}, \cdots, Y_{6n})' \sim AN(\mathbf{0}, \Gamma)\,, \tag{4.4}
$$

where

$$
\Gamma = \operatorname{diag}(\Gamma_1, \Gamma_2, \Gamma_3) \tag{4.5}
$$

with

$$
\begin{aligned}
\Gamma_i &= \int k_i^2(u)\,du \times \operatorname{diag}(G_i(\mathbf{x}_1), G_i(\mathbf{x}_2))\,, \qquad i = 1, 2 \\
\Gamma_3 &= \begin{pmatrix} \bar{F}(\mathbf{x}_1)(1 - \bar{F}(\mathbf{x}_1)) & \gamma_{12} \\ \gamma_{12} & \bar{F}(\mathbf{x}_2)(1 - \bar{F}(\mathbf{x}_2)) \end{pmatrix}.
\end{aligned} \tag{4.6}
$$

REMARK. The asymptotic covariance matrix Γ given in (4.5) is obtained via the asymptotic variances and covariances of Y_{in}, $i = 1, \cdots, 6$. They are listed in (4.13).

PROOF. Define, for $j = 1, \cdots, n$,

$$V_{nj}(\mathbf{x}) = \frac{1}{a_n^{\frac{3}{2}}}\left[k_1\left(\frac{x_1 - X_{1j}}{a_n}\right)\bar{K}_2\left(\frac{x_2 - X_{2j}}{a_n}\right) - Ek_1\left(\frac{x_1 - X_{1j}}{a_n}\right)\bar{K}_2\left(\frac{x_2 - X_{2j}}{a_n}\right)\right],$$

$$W_{nj}(\mathbf{x}) = \frac{1}{a_n^{\frac{3}{2}}}\left[\bar{K}_1\left(\frac{x_1 - X_{1j}}{a_n}\right)k_2\left(\frac{x_2 - X_{2j}}{a_n}\right) - E\bar{K}_1\left(\frac{x_1 - X_{1j}}{a_n}\right)k_2\left(\frac{x_2 - X_{2j}}{a_n}\right)\right], \tag{4.7}$$

and

$$Z_{nj}(\mathbf{x}) = \frac{1}{a_n^{2}}\left[\bar{K}_1\left(\frac{x_1 - X_{1j}}{a_n}\right)\bar{K}_2\left(\frac{x_2 - X_{2j}}{a_n}\right) - E\bar{K}_1\left(\frac{x_1 - X_{1j}}{a_n}\right)\bar{K}_2\left(\frac{x_2 - X_{2j}}{a_n}\right)\right].$$

Then it follows from (1.8) and (4.3) that, for $\alpha = 1, 2$,

$$Y_{\alpha n} = n^{-1}\sum_{j=1}^{n} V_{nj}(\mathbf{x}_\alpha), \qquad Y_{\alpha+2n} = n^{-1}\sum_{j=1}^{n} W_{nj}(\mathbf{x}_\alpha), \qquad Y_{\alpha+4n} = n^{-1}\sum_{j=1}^{n} Z_{nj}(\mathbf{x}_\alpha). \tag{4.8}$$

The lemma will be proved if we can show that, for any real constants c_i, $i = 1, \cdots, 6$, the linear combination

$$S_{nn} = n^{\frac{1}{2}}\sum_{i=1}^{6} c_i Y_{in} \tag{4.9}$$

has an asymptotic normal distribution with mean 0 and variance $\mathbf{C}'\Gamma\mathbf{C}$ where $\mathbf{C}' = (c_1, \cdots, c_6)$. Making use of the representation (4.8) it is clear that the linear combination (4.9) may be rewritten as

$$S_{nn} = n^{-\frac{1}{2}}\sum_{j=1}^{n} T_{nj}, \tag{4.10}$$

where

$$T_{nj} = \sum_{\alpha=1}^{2}\left[c_\alpha V_{nj}(\mathbf{x}_\alpha) + c_{\alpha+2}W_{nj}(\mathbf{x}_\alpha) + c_{\alpha+4}Z_{nj}(\mathbf{x}_\alpha)\right], \qquad j = 1, \cdots, n. \tag{4.11}$$

Since, for fixed $\mathbf{x}_1$, $\mathbf{x}_2$, and n, the random variables $T_{n1}, \cdots, T_{nn}$ are i.i.d., a sufficient condition under which S_{nn} converges to a normal variate, in distribution, is given by (see, e.g., Loève (1963), page 277)

$$\frac{n^{-\frac{1}{2}}E|T_{n1}|^3}{(\operatorname{Var} S_{nn})^{\frac{3}{2}}} \to 0, \qquad \text{as} \quad n \to \infty. \tag{4.12}$$

To evaluate the asymptotic variance of S_{nn} using (4.9) it is necessary to obtain the asymptotic variances and covariances of Y_{in}'s. They turn out to be as follows:

(I) $n \operatorname{Var} Y_{\alpha n} = G_1(\mathbf{x}_\alpha)\int k_1^2(u)\,du + O(a_n^2)$, $\alpha = 1, 2$;

$n \operatorname{Var} Y_{\alpha+2n} = G_2(\mathbf{x}_\alpha)\int k_2^2(u)\,du + O(a_n^2)$, $\alpha = 1, 2$;

$$
\begin{aligned}
&\text{(II)}\quad n \operatorname{Var} Y_{\alpha+4n} = \bar{F}(\mathbf{x}_\alpha)[1-\bar{F}(\mathbf{x}_\alpha)]\,, \quad \alpha = 1, 2\,;\\
(4.13)\qquad &\text{(III)}\quad n \operatorname{Cov}(Y_{1n}, Y_{2n}) = O(a_n)\,, \qquad n \operatorname{Cov}(Y_{3n}, Y_{4n}) = O(a_n)\,;\\
&\text{(IV)}\quad n \operatorname{Cov}(Y_{5n}, Y_{6n}) = \bar{F}(\mathbf{x}_{\max(1,2)}) - \bar{F}(\mathbf{x}_1)\bar{F}(\mathbf{x}_2)\,;\\
&\text{(V)}\quad n \operatorname{Cov}(Y_{\alpha n}, Y_{\alpha+2\,n}) = O(a_n)\,, \quad \alpha = 1, 2\,;\\
&\qquad\ n \operatorname{Cov}(Y_{1n}, Y_{4n}) = O(a_n)\,; \qquad n \operatorname{Cov}(Y_{2n}, Y_{3n}) = O(a_n)\,;\\
&\text{(VI)}\quad n \operatorname{Cov}(Y_{\alpha n}, Y_{\beta n}) = O(a_n^{\frac{1}{2}})\,, \quad \alpha = 1, 2, 3, 4;\ \beta = 5, 6\,.
\end{aligned}
$$

Results (I) and (II) may be obtained directly from Lemma 2.2 (ii) and Lemma 2.1 (ii) respectively. We will only sketch the proof of (III) here since the others may be established similarly. For (III) note that

$$
\begin{aligned}
& n \operatorname{Cov}(Y_{1n}, Y_{2n})\\
&\quad = n a_n \operatorname{Cov}[\hat{G}_1(\mathbf{x}_1), \hat{G}_1(\mathbf{x}_2)]\\
&\quad = \frac{1}{a_n^3} \iint \left\{ k_1\left(\frac{x_{11}-u_1}{a_n}\right)\left[\int_{x_{21}}^{\infty} k_2\left(\frac{t-u_2}{a_n}\right) dt\right] k_1\left(\frac{x_{12}-u_1}{a_n}\right)\right.\\
&\qquad \left. \times \left[\int_{x_{22}}^{\infty} k_2\left(\frac{s-u_2}{a_n}\right) ds\right] f(u_1, u_2)\right\} du_1\, du_2 - a_n E\hat{G}_1(\mathbf{x}_1) E\hat{G}_1(\mathbf{x}_2)\\
(4.14)\qquad &\quad = \iint \left\{ k_1(y_1)\left[\int_{(x_{21}-u_2)/a_n}^{\infty} k_2(y_2)\, dy_2\right] k_1\left(\frac{x_{12}-x_{11}}{a_n} + y_1\right)\right.\\
&\qquad \left. \times \left[\int_{(x_{22}-u_2)/a_n}^{\infty} k_2(z)\, dz\right] f(x_{11} - a_n y_1, u_2)\right\} dy_1\, du_2 + O(a_n)\\
&\quad \simeq \iint_{x_{2\max}}^{\infty} k_1(y_1) k_1\left(\frac{x_{12}-x_{11}}{a_n} + y_1\right) f(x_{11} - a_n y_1, u_2)\, dy_1\, du_2 + O(a_n)\\
&\quad = a_n \lim_{n\to\infty} \sup_{y_1}\left[\frac{1}{a_n} k_1\left(\frac{x_{12}-x_{11}}{a_n} + y_1\right)\right]\\
&\qquad \times \iint_{x_{2\max}}^{\infty} k_1(y_1) f(x_{11} - a_n y_1, u_2)\, dy_1\, du_2 + O(a_n)\,.
\end{aligned}
$$

The third equality in (4.14) is obtained by the change of variables $y_1 = (x_{11} - u_1)/a_n$, $y_2 = (t - u_2)/a_n$, $z = (s - u_2)/a_n$ and $u_2 = u_2$ with the Jacobian of the transformation being $-a_n^3$. The fourth expression is obtained since $\int_{(x_{2\alpha}-u_2)/a_n}^{\infty} k_2(y)\, dy \to 1$ or 0 according to whether $u_2 > x_{2\alpha}$ or $u_2 < x_{2\alpha}$, respectively, ($\alpha = 1, 2$), as $n \to \infty$. The integral in the final expression is finite while $\limsup a_n^{-1} k_1[(x_{12} - x_{11})/a_n + y_1] = 0$ due to (1.5). This establishes the first part of (III). The second part can be proved similarly.

Now, summarizing results (I) through (VI), it is clear that

$$
(4.15)\qquad \lim_{n\to\infty} \operatorname{Var} S_{nn} < \infty\,.
$$

It remains to show that

$$
(4.16)\qquad n^{-\frac{1}{2}} E|T_{n1}|^3 \to 0\,, \qquad \text{as} \quad n \to \infty\,.
$$

Using (4.11) and the c_r-inequality of Loève (1963, page 155) repeatedly, it follows that

$$
\begin{aligned}
E|T_{n1}|^3 \leqq 16 \textstyle\sum_{\alpha=1}^{2} \{&|c_\alpha|^3 E|V_{n1}(\mathbf{x}_\alpha)|^3\\
&+ 4|c_{\alpha+2}|^3 E|W_{n1}(\mathbf{x}_\alpha)|^3 + 4|c_{\alpha+4}|^3 E|Z_{n1}(\mathbf{x}_\alpha)|^3\}\,.
\end{aligned}
$$

Thus (4.16) will be satisfied if we can show that, for $\alpha = 1, 2$,

$$n^{-\frac{1}{2}} E|U_n(\mathbf{x}_\alpha)|^3 \to 0 , \quad \text{as} \quad n \to \infty , \tag{4.17}$$

where U_n is V_{n1}, W_{n1}, and Z_{n1}. Since $|Z_{n1}(\mathbf{x}_\alpha)|^3 \leqq 8$, with probability one, (4.17) for Z_{n1} follows immediately. The proofs for V_{n1} and W_{n1} are alike. We will only show (4.17) for V_{n1}. Using the c_r-inequality of Loève (1963) and with the same arguments as those employed in deriving (4.14), it follows that

$$n^{-\frac{1}{2}} E|V_{n1}(\mathbf{x}_\alpha)|^3 = O((na_n)^{-\frac{1}{2}}) ,$$

which converges to 0 as $n \to \infty$, by the first part of assumption (ii) in Theorem 4.1. This establishes (4.17). Hence (4.16) is satisfied which, together with (4.15), implies (4.12). □

LEMMA 4.3. *Assume that the conditions of Theorem* 4.1 *hold and* $q = 2$. *Define*

$$\begin{aligned} Z_{\alpha n} &= a_n^{\frac{1}{2}}[\hat{G}_1(\mathbf{x}_\alpha) - G_1(\mathbf{x}_\alpha)] , \\ Z_{\alpha+2n} &= a_n^{\frac{1}{2}}[\hat{G}_2(\mathbf{x}_\alpha) - G_2(\mathbf{x}_\alpha)] , \\ Z_{\alpha+4n} &= \hat{\bar{F}}(\mathbf{x}_\alpha) - \bar{F}(\mathbf{x}_\alpha) , \qquad \alpha = 1, 2 . \end{aligned} \tag{4.18}$$

Then

$$n^{\frac{1}{2}}(Z_{1n}, \cdots, Z_{6n})' \sim AN(\mathbf{0}, \Gamma) , \tag{4.19}$$

where Γ *is given by* (4.5).

PROOF. In view of Lemma 4.2, it suffices to show that, for $i = 1, \cdots, 6$,

$$n^{\frac{1}{2}}|Y_{in} - Z_{in}| \to 0 , \quad \text{in probability,} \quad \text{as} \quad n \to \infty . \tag{4.20}$$

Recalling (1.7), (1.8), and after changing the order of integration, it follows that

$$\begin{aligned} E\hat{\bar{F}}(\mathbf{x}) - \bar{F}(\mathbf{x}) &= \textstyle\int\int [\bar{F}(\mathbf{x} - a_n\mathbf{y}) - \bar{F}(\mathbf{x})]k_1(y_1)k_2(y_2)\, d\mathbf{y} \\ &\simeq \frac{a_n^2}{2} \sum_{i=1}^2 \frac{\partial^2 \bar{F}(\mathbf{x})}{\partial x_i^2} \textstyle\int y_i^2 k_i(y_i)\, dy_i . \end{aligned} \tag{4.21}$$

The last expression of (4.21) is obtained by a Taylor expansion of $\bar{F}(\mathbf{x} - a_n\mathbf{y})$ about $\mathbf{x}$ and by assumptions (i) and (iii) of Theorem 4.1. Similarly, it can be shown that, for $i = 1, 2$,

$$E\hat{G}_i(\mathbf{x}) - G_i(\mathbf{x}) \simeq \frac{a_n^2}{2} \sum_{j=1}^2 \frac{\partial^2 G_i(\mathbf{x})}{\partial x_j^2} \textstyle\int y_j^2 k_j(y_j)\, dy_j .$$

Thus

$$\begin{aligned} n^{\frac{1}{2}}|Y_{in} - Z_{in}| &= O((na_n^5)^{\frac{1}{2}}) , \qquad i = 1, 2, 3, 4 , \\ &= O((na_n^4)^{\frac{1}{2}}) , \qquad i = 5, 6 , \end{aligned}$$

which converges to 0, as $n \to \infty$, by assumption (ii) of the theorem. This establishes (4.20). □

The following lemma is a trivial generalization of Theorem (iii) of Rao (1965), page 322. The proof is omitted.

LEMMA 4.4. *Let* $\mathbf{T}_n = (T_{1n}, \cdots, T_{3q\,n})'$ *be a* $3q$*-dimensional statistic such that the asymptotic distribution of* $(na_n)^{\frac{1}{2}}(T_{1n} - \theta_1), \cdots, (na_n)^{\frac{1}{2}}(T_{2q\,n} - \theta_{2q}), n^{\frac{1}{2}}(T_{2q+1\,n} - \theta_{2q+1}), \cdots, n^{\frac{1}{2}}(T_{3q\,n} - \theta_{3q})$ *is* $3q$*-variate normal with mean vector* $\mathbf{0}$ *and covariance matrix* H, *where* $\theta_i \neq 0$ *for* $i = 2q+1, \cdots, 3q$, *and* $na_n \to \infty$ *as* $n \to \infty$. *Let* $h_{2i-1}(\theta_1, \cdots, \theta_{3q}) = \theta_i/\theta_{2q+i}$ *and* $h_{2i}(\theta_1, \cdots, \theta_{3q}) = \theta_{q+i}/\theta_{2q+i}$, $i = 1, \cdots, q$. *Then the asymptotic distribution of* $(na_n)^{\frac{1}{2}}[h_1(\mathbf{T}_n) - h_1(\boldsymbol{\theta}), \cdots, h_{2q}(\mathbf{T}_n) - h_{2q}(\boldsymbol{\theta})]'$ *is* $2q$*-variate normal with mean vector* $\mathbf{0}$ *and covariance matrix* $H\Gamma H'$, *where* $\boldsymbol{\theta}' = (\theta_1, \cdots, \theta_{3q})$ *and* $H = (\partial h_i/\partial\theta_j)$. *The rank of the distribution is that of* $H\Gamma H'$.

PROOF OF THEOREM 4.1. Using Lemmas 4.3 and 4.4 with $q = 2$, we have $\boldsymbol{\theta}' = (\theta_1, \theta_2, \theta_3, \theta_4, \theta_5, \theta_6)$, where $\theta_\alpha = G_1(\mathbf{x}_\alpha)$, $\theta_{\alpha+2} = G_2(\mathbf{x}_\alpha)$, $\theta_{\alpha+4} = \bar{F}(\mathbf{x}_\alpha) > 0$, $\alpha = 1, 2$. Define, as in Lemma 4.4,

$$h_1(\boldsymbol{\theta}) = \theta_1/\theta_5\,, \qquad h_2(\boldsymbol{\theta}) = \theta_3/\theta_5\,, \qquad h_3(\boldsymbol{\theta}) = \theta_2/\theta_6\,, \qquad h_4(\boldsymbol{\theta}) = \theta_4/\theta_6\,.$$

Then

$$H = \left(\frac{\partial h_i}{\partial \theta_j}\right)$$

$$= \begin{pmatrix} [\bar{F}(\mathbf{x}_1)]^{-1} & 0 & 0 & 0 & -r_1(\mathbf{x}_1)[\bar{F}(\mathbf{x}_1)]^{-1} & 0 \\ 0 & 0 & [\bar{F}(\mathbf{x}_1)]^{-1} & 0 & -r_2(\mathbf{x}_1)[\bar{F}(\mathbf{x}_1)]^{-1} & 0 \\ 0 & [\bar{F}(\mathbf{x}_2)]^{-1} & 0 & 0 & 0 & -r_1(\mathbf{x}_2)[\bar{F}(\mathbf{x}_2)]^{-1} \\ 0 & 0 & 0 & [\bar{F}(\mathbf{x}_2)]^{-1} & 0 & -r_2(\mathbf{x}_2)[\bar{F}(\mathbf{x}_2)]^{-1} \end{pmatrix}.$$

Now the theorem follows directly from Lemma 4.4 with $H\Gamma H' = \Sigma$, as given by (4.1). □

It should be remarked that the assumption $\prod_{i=1}^{2} \prod_{\alpha\neq\beta=1}^{q} (x_{i\alpha} - x_{i\beta}) \neq 0$ seems strong and undesirable. However, the theorem still holds with Σ slightly modified when the ith component of some $\mathbf{x}_\alpha$'s is identical. For example, if $x_{11} = x_{12}$ and $x_{21} \neq x_{22}$ then the only changes needed to be made are entries in the submatrix Γ_i $(i = 1, 2)$ given by (4.6). The modified matrix is then

$$\Gamma_i{}^* = \int k_i{}^2(u)\,du \begin{pmatrix} G_i(\mathbf{x}_1) & G_i(x_{11}, x_{2\max}) \\ G_i(x_{11}, x_{2\max}) & G_i(\mathbf{x}_2) \end{pmatrix}.$$

Now, define Γ^* as in (4.5) with $\Gamma_i{}^*$ replacing Γ_i $(i = 1, 2)$. Then Theorem 4.1 holds with covariance matrix $(na_n)^{-1}\Sigma^*$, where $\Sigma^* = H\Gamma^* H'$.

Acknowledgment. The authors are grateful to the referee for many useful comments and suggestions.

REFERENCES

AHMAD, I. A. and LIN, P. E. (1975). Nonparametric estimation of a vector-valued bivariate failure rate I: Consistency. FSU Statistics Report M363.

BARLOW, R. E. and PROSCHAN, F. (1975). *Statistical Theory of Reliability and Life Testing: Probability Models.* Holt, Rinehart and Winston, New York.

BARLOW, R. E. and VAN ZWET, W. (1971). Comparison of several nonparametric estimators of the failure rate function. *Operations Res. and Reliability* (Crouchko, ed.), 375–399.

BASU, A. P. (1971). Bivariate failure rate. *J. Amer. Statist. Assoc.* **66** 103–104.

BENNETT, G. (1962). Probability inequalities for the sum of independent random variables. *J. Amer. Statist. Assoc.* **57** 33–45.

BLOCK, H. W. (1973). Monotone hazard and failure rates for absolutely continuous multivariate distributions. Research Report No. 73-20, Univ. of Pittsburgh.

CACOULLOS, T. (1966). Estimation of a multivariate density. *Ann. Inst. Statist. Math.* **18** 181–190.

COX, D. R. (1972). Regression models and life-tables. *J. Roy. Statist. Soc.* **34** 187–220.

JOHNSON, N. L. and KOTZ, S. (1975). A vector-valued multivariate hazard rate. *J. Multivariate Anal.* **5** 53–66.

KIEFER, J. and WOLFOWITZ, J. (1958). On the deviation of the empiric distribution function of vector variables. *Trans. Amer. Math. Soc.* **87** 173–186.

LOÈVE, M. (1963). *Probability Theory*, 3rd ed. Van Nostrand, New York.

MARSHALL, A. W. (1975). Some comments on the hazard gradient. *Stochastic Processes and Appl.* **3** 293–300.

NADARAYA, E. A. (1965). On nonparametric estimation of density function and regression curve. *Theor. Probability Appl.* **10** 186–190.

PARZEN, E. (1962). On estimation of probability density and mode. *Ann. Math. Statist.* **33** 1065–1076.

PURI, P. S. and RUBIN, H. (1974). On a characterization of a family of distributions with constant multivariate failure rates. *Ann. Probability* **2** 738–740.

RAO, C. R. (1965). *Linear Statistical Inference and Its Applications*. Wiley, New York.

WATSON, G. S. and LEADBETTER, M. R. (1964a). Hazard analysis I. *Biometrika* **51** 175–184.

WATSON, G. S. and LEADBETTER, M. R. (1964b). Hazard analysis II. *Sankhyā Ser. A* **26** 101–116.

DEPARTMENT OF MATHEMATICS
MCMASTER UNIVERSITY
HAMILTON, ONTARIO
CANADA L8S 4K1

DEPARTMENT OF STATISTICS
FLORIDA STATE UNIVERSITY
TALLAHASSEE, FLORIDA 32306