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Nonparametric frontier analysis with multiple constituencies.

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ABSTRACT. We introduce a methodology for generalizing Data Envelopment Analysis to incorporate the role and impact of *constituencies* in the classification of the model's attributes. Constituencies determine whether entities' attributes in a DEA study are treated as desirable or undesirable. This extension of DEA is the basis for a methodology to answer questions that arise such as: Which constituencies find what entities efficient? Which entities are in the efficient frontier for a specified constituency? and What benchmarking prescriptions apply to inefficient entities for a given constituency? Constituencies allow new applications for DEA. Analyses of public projects to determine their impact on voters, marketing studies where a product defined by multiple attributes is analyzed with respect to diverse markets are two examples of the type of application for the new methodology. We introduce a DEA LP especially formulated for this new framework with many desirable properties. The new methodology is motivated and validated with a cost-benefit analysis application for a public project.

Key Words: Nonparametric Efficient Frontiers, Data Envelopment Analysis (DEA), Linear Programming, and Convex Analysis.

Introduction.

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Data Envelopment Analysis (DEA) is a nonparametric tool to assess efficiency of a collection of functionally similar entities, "decision making units" (DMUs), that transform inputs into outputs. DEA clusters the entities as efficient or inefficient depending on their relative location with respect to an efficient frontier. It was introduced by Charnes[1] *et al.* in 1978 and it has evolved to the point where over a thousand articles have been published about the theory and its applications (Seiford[2]). This methodology is nonparametric in the sense that it does not require an assumption about a functional form of the efficient frontier and therefore, no parameter estimation, making it useful in a wide variety of applications.

An important part of building a DEA model is the identification of the entities' list of common attributes and their classification as either inputs or outputs. In many situations, this process involves more than just decisions as to whether to include or exclude individual attributes; it also requires determinations as to whether an attribute will be treated as an input or output. There are applications where particular attributes resist this determination either because it is not obvious or because, simply, to some it is an input and to others an output.

In this paper we study a generalization of standard DEA. The generalization consists of treating each attribute as potentially "desirable" and "undesirable" (or *isotonic* and *anti-isotonic* in the terminology in Dyson,[3] *et al.*). In the traditional application of DEA, attributes are determined to be, *a priori*, either outputs or inputs; i.e., desirable or undesirable, and this determination does not change in the course of the study. We will focus on the consequences of relaxing this rigid aspect of DEA.

An attribute is desirable or undesirable depending on the judgment of a *constituency*. Different constituencies provide different attribute classifications. Note that, in a model with m attributes, there can be as many as 2^m constituencies. Issues that arise under this more general model, include:

i) For each constituency, which are the entities that are DEA efficient?

ii) For each entity, which of the constituencies consider it DEA efficient?

iii) For any entities not efficient by some constituency, what modifications would make them DEA efficient?

Extending DEA to analyze the impact on multiple constituencies presents interesting challenges. Standard LP formulations in DEA, both oriented and unoriented, need to be re-examined under this framework. Complications emerge due to the fact that a decision about orientation for an LP is usually tailored to the specific input/output assignment of attributes and it may not make much sense if applied to a different classification. Unoriented forms such as the "additive" model of Charnes,[4] *et al.*, are not appropriate for benchmarking determinations. There is a need for a consistent unoriented benchmarking approach that works over multiple constituencies. Finally, the analysis generates vast amounts of data. These need to be systematically processed so that they can turned into useful information that will assist the analyst in understanding the roles of – and interactions among – entities, attributes and constituencies. In this paper, we address these challenges and offer tools for an effective and consistent framework for DEA that consolidates the issues that arise when treating multiple constituencies.

The paper makes three basic contributions. The first is a framework for DEA analyses incorporating constituencies with discordant views about the role of particular attributes. Applications where attributes have opposite impacts on different affected groups are now within the scope of nonparametric frontier analysis. Our work formalizes the problem and provides a methodology for this generalization of DEA. The second contribution of this work is a new linear programming formulation for DEA. This LP is motivated by the new framework and has important desirable properties; namely, *i*) it is unoriented; *ii*) it is always feasible and bounded under deleted domain; *iii*) it provides necessary and sufficient conditions for efficiency classification under the variable returns to scale assumption; *iv*) it is translational invariant; and *v*) it provides meaningful benchmarks. Lastly, we introduce procedures for analyzing and interpreting the results generated by the new methodology. These results provide the decision maker (and politician) with tools to make decisions in an environment affected by the interaction of constituencies. The analysis also allows inferences about the popularity of an entity with respect to the constituencies and the amenability of a constituency with respect to the different entities. We motivate and validate the new methodology with an example. The example shows how we can apply our methodology to make a decision about a set of complex alternatives impacting multiple constituencies, in the real sense of the word.

The problem of not having a clear classification of DEA attributes has been addressed before. A related problem is that of "undesirable" outputs. An undesirable DEA output in the sense that less is better or, analogously, a desirable input, presents a modelling and computational complication and is discussed in Dyson,[3] *et al.* and Scheel[5]. This modelling question can be treated as an issue of DEA with multiple constituencies. An output that is the result of a transformation process; e.g., a pollutant in, say, power generation, is a positive measure of productivity and is, in this sense, "desirable." The problem is that there is a significant constituency who considers pollution undesirable; hence the two dissonant interpretations. Our analysis can be used to produce and understand results based on one, or many, such outputs, being either desirable or undesirable or both. As we will see, our approach is equivalent to a transformation used in Scheel[5].

A more obvious and relevant use of the concept of constituencies as we are proposing here appears in Sarrico,[6] *et al.* Here, different student types consider universities' attributes as either inputs or outputs depending on their age, ability, aptitude, future job prospects, etc. For example, an attribute measuring the university's exclusivity is desirable for ambitious and talented students but it is undesirable for students with less aptitude. The study in Sarrico, *et al.* illustrates well the issues that arise when different constituencies weigh in about the desirability of attributes in a DEA study and it represents an excellent example of the type of application for which our model is designed. The concept of *constituency* generalizes all the modelling issues that can arise when attributes can be considered both desirable or undesirable depending on who or how they are viewed.

The development in this paper will be limited to the variable returns to scale (BCC) DEA model of Banker[7] *et al.* This is the most general of the DEA models since the efficient set is a superset for all DEA analyses with the same data under the other returns to scale assumptions; namely, constant, increasing, and decreasing returns to scale.

In the next section, we lay down the formal groundwork on which we will build the new framework. This section contains definitions and some theory. After this, we dedicate a section to make a case for a new DEA unoriented LP formulation that provides classification and geometric information as well as meaningful benchmarking data. After this, we introduce the new DEA LP and investigate its properties. We illustrate the new framework with an application involving a public project: building Interstate-40 through the city of Memphis. We use the example to present a methodology for analyzing and understanding the vast output of the new model. The paper closes with the conclusion that the new framework for DEA with multiple constituencies is a valuable tool in the management scientist toolbox. All proofs have been relegated to an appendix.

The Model.

The model involves $n > 1$ entities (e.g., projects, processes, products, DMUs, etc.) that can be characterized by the same $m > 1$ attributes. The model is defined by a data set consisting of n points, $\{a^1, \ldots, a^n\}$, each with m components: thus, $a^j = (a^j_1, \ldots, a^j_m)$. Components correspond to attributes of the model. The value of each component is the magnitude of the attribute with which it is associated. We invoke the standard assumption that the data set is 'reduced;' i.e., there is no duplication of data points. The third aspect of the model are 'constituencies.' We say that an attribute is 'desirable' for a constituency if the constituency considers greater magnitudes preferable; and conversely, 'undesirable' if less is better. The indexing scheme to identify the different constituencies presented in the following definition offers notational advantages we will appreciate better later.

Definition 1. A *constituency* is an m dimensional vector δ^ℓ composed of 1s and -1s. Component i of δ^ℓ is a '1' if this constituency considers the attribute desirable; conversely, a '-1' indicates that this constituency judges this attribute as undesirable.

A study involving m attributes can have up to 2^m constituencies although most may not exist or be considered in an actual application. As long as more than one constituency is affected, the framework is interesting and useful. In a three dimensional model the eight potential constituencies are $\delta^1 = (1, 1, 1), \delta^2 = (1, 1, -1), \delta^3 = (1, -1, 1), \delta^4 = (1, -1, -1), \delta^5 = (-1, 1, 1), \delta^6 = (-1, 1, -1), \delta^7 = (-1, -1, 1), \delta^8 = (-1, -1, -1)$. Constituency, e.g., $\delta^6 = (-1, 1, -1)$ corresponds to the one where the first and third attributes are desirable and for which more is better (i.e., an "output" in conventional DEA) and attribute 2 is undesirable. Although possible, it is not necessary to provide a formula relating the index ℓ with specific constituencies (e.g., a base 2 equivalent between the components of the vector and the index ℓ).

The following definition adapts the standard variable "BCC" returns to scale model for DEA efficiency of Banker[4] *et al.* to incorporate constituencies:

DEFINITION 2. An entity is *efficient* in a DEA variable returns model with respect to a constituency, δ^ℓ, if it is impossible to find a convex combination of the data of the remaining $n-1$ entities such that:

1. for every desirable attribute, the value of the combination is greater than or equal to that of the entity being tested;

2. for every undesirable attribute, the value of the combination is less than or equal to that of the entity tested; and

3. for at least one attribute, the value of the combination is a strict inequality.

From Definition 2, we can see that each constituency generates a different production possibility set for the same DEA data set. Each production possibility set corresponds to a variable returns envelopment polyhedral set where the efficient frontier is defined by the constituency. We will refer to an inefficient entity for a given constituency as *dominated*. This occurs when there exists a convex combination of the data that satisfies the conditions of the definition. This form of the variable returns definition was chosen because it will simplify our theoretical development when we introduce a new DEA LP formulation below.

We will illustrate the ideas so far with a simple example. Imagine eight automobile models to be marketed and sold in different markets. Table 1 presents the data corresponding to these products. Suppose that markets are defined by their preferences for each of two attributes: size and horsepower. This generates $2^2 = 4$ potential markets which are the 'constituencies' in this example. The constituencies are: $\delta^1 = (1,1)$, $\delta^2 = (1,-1)$, $\delta^3 = (-1,1)$, and $\delta^4 = (-1,-1)$. So, for example, δ^1 is the constituency (market) that values greater size and horsepower in their cars; e.g., the North American market (perhaps from another era), and δ^4 is the constituency (market) that considers both attributes undesirable; e.g., Southern Europe. Figure 1 depicts the production possibility sets for these four constituencies.

Table 1 Here: "Attribute Data for Eight Automobiles".

Figure 1 Here: "The Four PPS for the Two-Attribute Example".

Figures 1a, 1b, 1c, and 1d depict the four variable returns (BCC) production possibility sets for constituencies δ^1, δ^2, δ^3, and δ^4, respectively. Consider constituency δ^1, the production possibility set for which appears in Figure 1a. In this market, bigger size and greater power are better. For this constituency, the two points, a^4 and a^5 are in the efficient frontier of the production possibility set. Each of these points is BCC efficient because no convex combination of the remaining data points will provide a value for the two attributes that are greater than or equal to, with at least one strictly greater, than what it is for these two points.

Table 2 summarizes the status of all the entities with respect to the different constituencies.

Table 2 Here: "Efficiency Report in Terms of Entities".

This example of DEA with multiple constituencies in two dimensions provides insights about the general case. The different situations that occur in this simple example reappear in higher dimensions. This suggests that the new framework for DEA can be used to identify different categories of points as follows:

i) Entities that are efficient for two or more constituencies as in the case of points a^2, a^3, and a^5.

ii) Entities that are efficient for a single constituency as in the case of points a^1, a^4, a^6, and a^9.

iii) Entities that are on the boundary of the production possibility set of some constituency but are dominated. This is the case of points a^4 for constituency δ^2 and a^9 for constituency δ^1.

iv) Entities that are always inefficient independent of the constituency. This is the case of entities a^7 and a^8.

In an application of DEA with multiple constituencies, the decision maker would be interested in the classification of entities using the categories above. Entities that are efficient for many constituencies may be of special interest because of their broad market appeal (Category *i*). Entities that appeal to a single constituency would represent to a marketer a specialty product (Category *ii*). Entities on the boundary that are inefficient for some constituency are equivalent to weak efficient DMUs in DEA (Category *iii*). From past experience with DEA, we may expect analyses to be confounded and complicated by their presence. Entities in the fourth category may appear to be some sort of "middle of the road" compromise (central planners of certain economies seemed to have based policies on such conclusions). In the context of efficient frontiers these entities are nothing more than universally inefficient alternatives and it behooves the decision maker to identify them quickly and discard them from further consideration.

Conversely, an analysis of a DEA with multiple constituencies can focus on the constituencies rather than the entities. Table 3 reports on the matching of each constituency and a corresponding set of efficient entities. We can see in this example that different constituencies play different roles. A classification scheme for constituencies can be based on the number of efficient entities with which they can be matched; for example, a scheme based on just two categories would be as follows:

i) Constituencies with a single, or few, efficient entities as in the case of δ^1 and δ^2.

ii) Constituencies with many efficient entities as in the case of δ^3 and δ^4.

A marketer analyzing his/her markets would identify constituencies δ^1 and δ^2 as more exclusive possibly because they are more selective. Constituencies in the second category are in some sense more amenable since they have the largest number of efficient entities.

Table 3 Here: "Efficiency Report in Terms of Constituencies".

The information in Tables 2 and 3 can be summarized using a matrix as in Table 4. The sum total of the rows in the column labeled "Eff." gives the total number of entities that are efficient for a given constituency and the sum total of columns in row "Total" displays the number of constituencies finding the corresponding entity efficient. This representation permits efficient inferencing about entities and constituencies.

Table 4 Here: "Efficiency Matrix".

The next two sections are dedicated to the theoretical device used to make the determination about the efficiency of entities with respect to constituencies; namely the linear program formulation. First we present arguments for a new DEA LP formulation. After that, we provide theoretical results about this new LP.

A Case for a New DEA LP Formulation.

Several LP formulations have been proposed for traditional DEA. LP formulations fall into two categories: oriented and unoriented. The choice of LP formulation used does not affect the classification of the entities.

Oriented LP forms require a judgment as to which attributes will be fixed and which will be allowed to vary when establishing efficiency and benchmarks (for a compendium of oriented LP DEA formulations, please refer to Seiford,[8] *et al.*). The idea of orientation extends to the multiple constituencies model. Instead of having input and output oriented formulations, we have "desirably" and "undesirably" oriented LP formulations. The best way to present this generalization is to simply consider the attribute mix and, for any constituency (except $\delta^1 = (1, \ldots, 1)$ and $\delta^{2^m} = (-1, \ldots, -1)$), reorder, without loss of generality, the attributes such that undesirable components are first and desirable components are last. With this, we can apply the standard DEA oriented forms to obtain scores in either or both orientations. We can adapt the oriented LP formulations easily to deal with the case of δ^1 and δ^{2^m}. (See, e.g., Thompson[9] *et al.*, for an example of the case where all attributes are inputs and Adolphson[10] *et al.* and Caporaletti[11] *et al.* for an example where all attributes are outputs.)

The decision as to which attribute will be fixed and which attribute will be allowed to vary, is essentially connected to benchmarking. A benchmarking recommendation for an inefficient entity will either be based on a reduction of inputs, or an increase of outputs, and sometimes, the very nature of these attributes makes one or the other impractical. The decision about an oriented form becomes arbitrary in the presence of multiple constituencies since, if there is any reason for selecting a particular orientation for one constituency, there will be another constituency for which this reason will be contradicted. This view is shared by Scheel[5] where he states that the oriented form is "not very appropriate" when the desirability of an attribute is not clearly defined.

DEA also uses unoriented LP formulations. The standard unoriented DEA LP formulation is the additive form introduced in Charnes[4] *et al.* This LP formulation serves the purpose of providing necessary and sufficient conditions for the classification of entities in DEA. The additive model in DEA can be adapted for multiple constituencies. The procedure would be the same as with oriented forms described above. The additive model is formulated to *maximize* the l_1-distance of the point being tested to the efficient frontier (Charnes[4] *et al.*, Tavares[12] *et al.*). The effect is that inefficient entities are benchmarked by points in the efficient frontier which may be "far away" for practical purposes. This problem with the additive model makes it unsuitable for benchmarking purposes.

A positive characteristic of the additive model is that it is translation invariant. An affine displacement of the data does not alter the efficient frontier and the classification of DMUs as efficient or inefficient is invariant to translation of the data. Moreover, unlike oriented forms, the additive model is translation invariant in its scores (Ali,[13] *et al.*).

Finally, both the oriented forms and the unoriented additive form suffer from infeasibility under deleted domain in the envelopment formulation (Dulá[14] *et al.*, Seiford[15] *et al.*). In fact, the LP will always be infeasible under deleted domain when the DMU being scored is extreme efficient for the case of the additive model. Important information such as the multipliers (weights) of a supporting hyperplane for the production possibility set is lost when no feasible solution is available.

The arguments presented above will be used to support a new LP formulation, specifically for DEA with constituencies. The new LP formulation we propose will have advantages of being unoriented; providing useful and intuitive benchmarks; being translation invariant in both its classifications results and scores; and finally, being feasible and bounded under deleted domain.

The New DEA LP Formulation.

Definition 2 establishes how entities will be classified with respect to constituencies. The classification criterion is based on the variable returns to scale assumption. In this section, we present a new DEA LP formulation to make this classification.

The classification requires the optimal solution to an LP. The LP we introduce here incorporates a new element in DEA LP formulations; namely, the use of a specially constructed "auxiliary" vector α^ℓ in $\Re^m$. This vector has a geometric role and economic interpretation.

The primal/dual pair of linear programs follows. Here, j^* is the index of the entity under examination, and α^ℓ is the auxiliary vector.

$$(\mathbf{P}^\ell) \equiv \begin{cases} \min\limits_{\pi,\beta} & w(\ell) = \langle \alpha^\ell, \pi \rangle & + & \beta & & \\ \text{s.t.} & \langle a^j, \pi \rangle & + & \beta & \geq & 0; \quad \forall j \neq j^* \\ & \langle \alpha^\ell - a^{j^*}, \pi \rangle & & & = & 1; \\ & \delta^\ell \pi & & & \leq & 0; \end{cases}$$

$$(\mathbf{D}^\ell) \equiv \begin{cases} \max\limits_{\lambda \geq 0, \theta} & z(\ell) = & & \theta & & \\ \text{s.t.} & \sum\limits_{j \neq j^*} a^j \lambda_j & + & (\alpha^\ell - a^{j^*})\theta & \overset{\ell}{\leftrightarrow} & \alpha^\ell; \\ & \sum\limits_{j \neq j^*} \lambda_j & & & = & 1; \end{cases}$$

where the symbol '$\overset{\ell}{\leftrightarrow}$' in $(\mathbf{D}^\ell)$ is defined as follows:

$$\overset{\ell}{\leftrightarrow} \equiv \begin{cases} \geq, & \text{if } \delta_i^\ell = 1; \\ \leq, & \text{if } \delta_i^\ell = -1. \end{cases}$$

The auxiliary vector α^ℓ is constructed as follows:

$$\alpha_i^\ell = \begin{cases} \min_j a_i^j, & \text{if } \delta_i^\ell = 1; \\ \max_j a_i^j, & \text{if } \delta_i^\ell = -1; \end{cases}$$

and all ties are broken arbitrarily. We assume that α^ℓ is not equal to any of the data points. We will discuss the auxiliary vector further after the presentation of the theoretical properties of the LP formulation $(\mathbf{P}^\ell)/(\mathbf{D}^\ell)$.

The operation applied in the LP formulations $(P^\ell)/(D^\ell)$ to change an attribute's classification from input to output, or vice-versa, is equivalent to negating the attribute's values as we switch from one classification to the other. To see this, note that replacing the coefficients in a row in (P^ℓ) by their negatives and thus changing the attribute's classification, and proceeding to multiply the row by -1, the original coefficients are restored but the inequality will be reversed. This, therefore, is an implementation of the technique for changing an attribute's classification used in Scheel[5].

Denote by $\pi^*(\ell) = (\pi_1^*(\ell), \ldots, \pi_m^*(\ell)), \beta^*(\ell)$ and $\lambda^*(\ell) = (\lambda_1^*(\ell), \ldots, \lambda_n^*(\ell)), \theta^*(\ell)$ two optimal solutions, the first for $(\mathbf{P}^\ell)$ and the other for $(\mathbf{D}^\ell)$. The corresponding optimal objective function values are denoted by $w^*(\ell)$ and $z^*(\ell)$. Also, let T^ℓ be the production possibility set for constituency δ^ℓ and we define $T^\ell/\ a^{j^*}$ as the production possibility set of the data when the point a^{j^*} is excluded from the data set. Note that $T^\ell/\ a^{j^*} \subset T^\ell$.

The following results establish that the primal/dual pair is always feasible and bounded under the deleted domain (Result 1); that the optimal solution provides necessary and sufficient conditions to classify entities as efficient or inefficient for a given constituency (Result 2); and that the LP formulation is translation invariant for an affine displacement in both classification and score (Result 3). Proofs and discussion have been relegated to an appendix.

Result 1. *Both LPs in the pair* $(\mathbf{P}^\ell)/(\mathbf{D}^\ell)$ *are feasible and bounded.*

Proof. See Appendix 1.

Result 2. *The data point* a^{j^*} *corresponds to an inefficient entity for constituency* δ^ℓ *if and only if either*

i) $z^*(\ell) = w^*(\ell) > 1$*; or*

ii) $z^*(\ell) = w^*(\ell) = 1$ *and there exists an optimal solution such that one of the* m *constraints in* $(\mathbf{D}^\ell)$ *is not binding.*

Proof. See Appendix 1.

Note that the classification of the entities based on Result 2 is a variable returns to scale partition into efficient and inefficient entities since the conditions of Definition 2 are satisfied. Next we show that the new LP formulation is translation invariant. Define $(\widehat{\mathbf{D}^\ell})$ as the translated model of $(\mathbf{D}^\ell)$ based on the addition of a common vector, $u \in \Re^m$ to each data point: $\hat{a}^j = a^j + u; \forall j$.

Result 3. *The affinely translated model,* $(\widehat{\mathbf{D}^\ell})$*, is equivalent to* $(\mathbf{D}^\ell)$*.*

Proof. See Appendix 1.

The auxiliary vector α^ℓ plays an important role in the formulation of LPs ($\mathbf{P}^\ell$) and ($\mathbf{D}^\ell$). Geometrically, it is a point constructed to be located deep in the interior of the production possibility set (by our assumption that the data set is not a singleton) with the property that all rays originating there and passing through any data point eventually intersect the boundary. Note that other definitions for α^ℓ besides the one given here may result in infeasible and unbounded LPs. Finding α^ℓ requires scanning the data one attribute at a time to identify the maximum and minimum for each component. This is a simple computational requirement. The auxiliary vector represents, in some sense, a worst performance imaginary entity exhibited by the data for constituency δ^ℓ. Figure 2 depicts the four production possibility sets for the data in Table 1. The four auxiliary vectors $\alpha^1, \alpha^2, \alpha^3$, and α^4 corresponding to the four constituencies are shown as the points at the intersection of the extension of lines from points which have the "worst" of the corresponding attribute value.

Figure 2 Here: "The Four Production Possibility Sets with the Auxiliary Vector".

The auxiliary vector is also important in defining the benchmarks for inefficient entities. The following definition will enable the construction of benchmarks for inefficient entities.

Definition 3. Let a^{j*} be an inefficient entity for constituency δ^ℓ. Then, its *benchmark* is the vector $a^{j^{*\prime}} = \sum_j a^j \lambda^*(\ell)_j$.

From the definition, a benchmark for an inefficient entity using LP formulations primal/dual corresponds to an extension of the ray emanating from the auxiliary vector through the point of the entity being scored to the boundary of the production possibility set. Notice that all extensions are calculated using the convex combinations of data points in the boundary of the production possibility set without incorporating any of the slacks that may be part of an optimal solution. Therefore, although it cannot be guaranteed, we may expect projections to end up as points in the efficient frontier. Figure 3 depicts a production possibility set with the benchmarks for inefficient entities a^4, a^5, a^6, a^7, a^8, and a^9. These benchmarks represent recommendations based on an extension of inefficient entities towards the boundary of the efficient frontier in a direction away from the auxiliary vector α^ℓ. So, in a sense, benchmarks are complete opposites of what can be considered a standard for worst case performance.

Figure 3 Here: "New LP Benchmarks".

We have introduced a new LP formulation for DEA with multiple constituencies. The new LPs are suitable for DEA with multiple constituencies because of their desirable features: they are unoriented; they are always feasible and bounded under deleted domain; they provide necessary and sufficient conditions for classifying entities; and they are translation invariant to affine displacement and score. The benchmarks that these LPs offer represent recommendations based on actions that correspond to the opposite of the worst that the data conveys. In the next section, we illustrate this new framework for DEA by applying it to a public highway project.

Implementation: DEA with Multiple Constituencies for the Memphis I-40 Public Project.

Plans to build the interstate system in Memphis began in the post-war era as they did throughout the country. In 1955 plans were approved to build 68 miles of interstate in Memphis, which lies in Shelby County, Tennessee. As in many other cities, the main route of the interstate was planned to pass straight through the city, and a by-pass was planned to circle the periphery as shown in Figure 4.

Figure 4 Here: "I-40 Project Through Memphis as Planned".

In 1956 the U.S. Bureau of Public Roads (USBPR) approved the location of I-40 through Overton Park, passing through the park at grade level and taking 26 acres from the Park's 342 acres. After plans were made public, citizens began to express concerns about the park route. Not surprisingly, this was primarily from individuals who lived in the vicinity of Overton Park. This area is referred to as "Midtown Memphis" and contains some old and affluent neighborhoods.

Many alternatives were considered over the years that the I-40/Overton Park project was studied. For our purposes, we utilize 26 different projects shown in Table A.2.1 in Appendix 2. It should be noted that not all of these projects were actually considered. We have created a longer list of projects that might have been possibilities, in order to have more projects to compare. Table A.2.1 contains descriptions of the projects. Table A.2.2 in Appendix 2 contains the numeric attributes that form the data set used in our study.

This study provides a good example of a public work with numerous projects and various constituencies which differ in their position on the desirability of the attributes. There are a number of constituencies with different perspectives on this issue. First there are the midtown residents who want neither their neighborhood nor the park bisected. Then there are the residents of other parts of Memphis and Shelby County who want convenient access to downtown and midtown Memphis. This may be for purposes of commuting to work, for shopping, or for recreation. The largest constituency is the American public that uses the Interstate system and wants to be able to pass through Memphis on I-40 safely and without let. Another constituency would be construction companies that want as much construction as possible to enhance their revenue. Along similar lines we also may consider automobile body shops the income of which comes primarily from repairing wrecked cars. So, while it may seem morbid to say there is a constituency that views accidents positively, body shops would have good reasons for this perspective.

Although there are a total of nine attributes which represents a potential of $2^9 = 512$ constituencies, this analysis considers three as "fixed." Attributes "Variance" and "Time" were always classified as 'undesirable' and "Work Trip" as 'desirable' and we assume there is no constituency that would consider them otherwise. This reduces the number of the analyses to $2^6 = 64$ for 26 entities.

The study consists of solving LPs $(\mathbf{P}^\ell)/(\mathbf{D}^\ell)$ once for each entity and each constituency. Each LP solution corresponds to the "scoring" of a project under a given constituency. Each constituency generates the work of a full standard DEA analysis. This means that a total of 1,664 LPs were solved. The value of the objective function at optimality serves to classify the project as either efficient or inefficient. This relation is given by Result 2 above; that is, a project is efficient if the LP optimum objective function value is less than unity or if it is equal to unity and there are no optimal solutions with positive slacks in $(\mathbf{D}^\ell)$; otherwise, it is a dominated (inefficient) project. Note that verifying the presence of positive slacks in *all* optimal solutions is an intractable problem. As is well known in DEA, only rarely is an entity efficient without being extreme-efficient. Therefore we report only extreme-efficiencies.

The LPs were solved using the *Premium LP Solver* in Microsoft Excel 2000 with the aid of a program coded in Visual Basic for Applications. Further details on the actual program as well as more testing are given in Bougnol[16]. The results of these computer runs for the I-40 through Memphis project are given in Table A.2.3 in Appendix 2.

Table A.2.3 answers an important and relevant question: Which projects stand out as "popular" in the sense that they appeal to many constituencies? This is answered by looking at the last row, "Total" in Table A.2.3. These entries are the simple count of the constituencies that find the project corresponding to the column efficient. From here we see that Project 5 has the broadest appeal. Indeed, it is efficient for every constituency in the model. This appears to be due to the fact that Project 5 has relatively low "Variance" values and the lowest "Time" values; both attributes that have been fixed as 'undesirable' in the model. Another interesting project with broad appeal is Project 12 since it is efficient for 48 constituencies. Clearly, constituencies are responding to the relatively low values of the fixed attributes and the high value for attribute "Park" which, of course, becomes a liability in the estimation of constituencies that consider this attribute undesirable; e.g., constituencies 33, 34, and 35. Other interesting projects that generate a consensus across many constituencies are 1, 4, 11, 13, 18, and 23; each with 32 favorable constituencies. At the other end we have Projects 2, 3, 6, 7, 8, 9, and 10. In fact, independent testing of these entities reveals that there are no constituencies that find them efficient even when we allow the three fixed attributes to vary. This was verified when they were uncovered as interior points of the convex hull of the data. These projects should be discarded from further consideration since they will never be efficient to any of the constituencies in the study. The analyst may also want to take a closer look at Project 15 since it attracts the interest of only four constituencies. This may be due to the fact that it has extreme values for many attributes.

The totals for the rows are also useful information. These entries are given by the last column in Table A.2.3. This is information about the nature of the different constituencies. The maximum value here is 12 and it corresponds to constituencies 9 and 41. These constituencies are, in some sense, the most amenable. The analyst may also turn his/her attention towards the opposite end of this "congeniality" scale; e.g., constituencies 30, 32, 62, and 64. These constituencies appear to be the hardest to please since they approve of only five projects. The analyst should focus on constituencies with extreme values in this column. These should trigger questions such as: What is the cardinality of the elements of these constituencies? (small constituencies may generate less concern than larger ones); What are the characteristics (e.g., demographics, level of influence) of the elements in these constituencies?

Further insights into constituencies and projects can be gleaned by comparing the percentage of efficient projects for each constituency and the percentage of projects found efficient across all constituencies. Figures 5 and 6 present this information in a form that a decision maker can readily use. These figures depict data processed from Table A.2.3. Figure 5 exhibits, for each project, the proportion of the constituencies that consider it efficient ranked from high to low. A decision maker would use this information to identify which projects are the most appealing (e.g., Project 5) and which ones are not appealing to any constituencies (e.g., Projects 2, 3, 6, 7, 8, 9, and 10). Figure 6 can be used to learn what percentage of the projects are efficient for each of the constituencies also ranked from high to low. This information will facilitate the identification of constituencies of special interest: the most amenable constituencies such as Constituencies 9 and 41 are on the left and, at the opposite end, the less congenial constituencies such as Constituencies 30, 32, 62, and 64.

Figure 5 Here: "Percentage of Constituencies Finding a Project Efficient (ranked)".

Figure 6 Here: "Percentage of Efficient Projects for Each Constituency (ranked)".

One remaining aspect of our analysis of this application of DEA with multiple constituencies is benchmarking. Consider Project 2. This project is distinguished for being inefficient for all constituencies. An interesting question is: How can this project become efficient for a given constituency, say, the most congenial constituency: Constituency 41? When scoring Project 2 with Constituency 41, the optimal solution, although being unity, contains a slack in Attribute 1, "Park", making this entity inefficient. The benchmark for this project is the value of the convex combination of the data set where the coefficients are given by the optimal solution of $(\mathbf{D}^{\ell})$. This yields the "virtual" project $(2.04, 0, 0, 0, 85.1, 0, 85.1, 0, 0)$. Project 2 will be efficient if its value for the "Park" attribute is decreased by 1.96 units of damage to Overton Park.

The obvious conclusion of the application of the methodology developed for this project on the plan to build an interstate through Memphis is that Project 5 emerges as particularly interesting. All constituencies would consider this project efficient and, presumably, would offer no opposition. Also interesting are the inferences about the different constituencies. A public official will be well-served to note the amenability of Constituencies 9 and 41 and his/her dealings with them would be very different than with Constituencies 30, 32, 62, and 64. The data for this study are less than ideal since it involves several categorical attributes. In an ideal model, all attributes have cardinal values but we can expect a less obvious course of action with such data. It is unlikely that a single entity will prevail as Project 5 has in this study. In this case the decision would have to consider not just which entity has the most number of supporting constituencies but also the nature of the constituencies which are opposed. The analysis, nevertheless, provides important insights about the interactions among constituencies and projects that will support decisions made by the decision maker.

Conclusion.

This article treats a broad generalization of standard DEA. We consider what happens when the attributes which define the entities in a DEA study are not fixed to be inputs or outputs as in standard DEA practice. We relax this restriction allowing the classification of attributes to vary according to different constituencies.

Multiple constituencies made it necessary to consider a new LP formulation for scoring entities. Standard formulations turn out to be inadequate. We have introduced a new LP formulation with important features which make it a contribution to general DEA.

The new framework for DEA and the new LP formulation were illustrated and validated by implementing them on an actual application involving a public project. The result was the definition of a methodology for generating and interpreting data with the new framework. Other analysts who wish to perform DEA studies involving constituencies now have a new tool to perform this work.

Appendix 1. Proofs to Theorems

RESULT 1. *Both LPs in the pair* ($\mathbf{P}^\ell$)/($\mathbf{D}^\ell$) *are feasible and bounded.*

PROOF. Let us begin by establishing the feasibility of ($\mathbf{D}^\ell$). We show that the LP is feasible for $\theta = 0$. At this value, the system becomes:

$$\begin{array}{rcll} \sum_{j \neq j^*} a^j \lambda_j & \overset{\ell}{\leftrightarrow} & \alpha^\ell; & \\ \sum_{j \neq j^*} \lambda_j & = & 1 \; ; & \\ \lambda_j & \geq & 0; & \forall j, j \neq j^*. \end{array}$$

Consider desirable attribute $\hat{i}$. Any convex combination of all $\hat{i}$th components $a^j_{\hat{i}}; \forall j, j \neq j^*$ must be such that

$$\sum_{j \neq j^*} a^j_{\hat{i}} \lambda_j \geq \alpha^\ell_{\hat{i}};$$

since, by construction, on a one-on-one basis $a^j_{\hat{i}} \geq \alpha^\ell_{\hat{i}}; \forall j, j \neq j^*$. Similarly for undesirable attributes. So, for $\theta = 0$, *any* convex combination of the data is a feasible solution. This also establishes the boundedness of ($\mathbf{P}^\ell$).

The feasibility of ($\mathbf{P}^\ell$) follows from the following arguments. Consider any supporting hyperplane: $\mathcal{H}(\hat{\pi}, \hat{\beta}) = \langle y, \hat{\pi} \rangle + \hat{\beta} = 0$ for the production possibility set $T^\ell \backslash a^{j^*}$ such that $\hat{\pi}_i \neq 0; \forall i$. Such a hyperplane is always available since there is no restriction on the value of $\hat{\beta}$. Therefore:

$$\begin{array}{rcll} \langle a^j, \hat{\pi} \rangle + \hat{\beta} & \geq & 0; & \forall j \neq j^* \\ \delta^\ell \hat{\pi} & \leq & 0. & \end{array}$$

Now let us take the solution $(\hat{\pi}, \hat{\beta})$ and apply it to the constraint in ($\mathbf{P}^\ell$) involving α^ℓ: $\langle \alpha^\ell - a^{j^*}, \pi \rangle = 1$. By construction, $(\alpha^\ell - a^{j^*}) \overset{\ell}{\leftrightarrow} 0, \hat{\pi} \overset{\ell}{\leftrightarrow} 0$; and since, by assumption $\alpha^\ell \neq a^{j^*}$, $\langle \alpha^\ell - a^{j^*}, \hat{\pi} \rangle = \hat{\gamma} > 0$. Substituting $\tilde{\pi} = \hat{\pi}/\hat{\gamma}$ and recalculating a β provides the feasible solution we started off seeking. This, of course, establishes the boundedness of ($\mathbf{D}^\ell$). ∎

Result 2. *The data point a^{j^*} corresponds to an inefficient entity for constituency δ^ℓ if and only if either*

i) $z^*(\ell) = w^*(\ell) > 1$*; or*

ii) $z^*(\ell) = w^*(\ell) = 1$ *and there exists an optimal solution such that one of the m constraints in* $(\mathbf{D}^\ell)$ *is not binding.*

Proof. Case *i)*: Let $z^*(\ell) = w^*(\ell) > 1$. Consider $(\mathbf{D}^\ell)$ and let (λ^*, θ^*) be its optimal solution. The first m constraints of $(\mathbf{D}^\ell)$ can be rewritten as follows:

$$\sum_{j \neq j^*} a^j \lambda_j^* \overset{\ell}{\leftrightarrow} a^{j^*} + (a^{j^*} - \alpha_i^\ell)(\theta^* - 1).$$

Suppose attribute $\hat{i}$ is desirable. By construction, $\alpha_{\hat{i}}^\ell \leq a_{\hat{i}}^{j^*}$ and since $\theta^* > 1$, then

$$a_{\hat{i}}^{j^*} + \underbrace{(a_{\hat{i}}^{j^*} - \alpha_{\hat{i}}^\ell)}_{\geq 0}\underbrace{(\theta^* - 1)}_{>0} \geq a_{\hat{i}}^{j^*}.$$

Similarly if attribute $\hat{i}$ is undesirable we have that

$$a_{\hat{i}}^{j^*} + (a_{\hat{i}}^{j^*} - \alpha_{\hat{i}}^\ell)(\theta^* - 1) \leq a_{\hat{i}}^{j^*}.$$

However, by assumption, $\alpha^\ell \neq a^{j^*}$ meaning that at least one inequality is strict. Thus, when $\theta^* > 1$, the data point a^{j^*} is dominated by some convex combination of the data making it an inefficient entity for constituency δ^ℓ.

Case *ii)*: $z^*(\ell) = w^*(\ell) = 1$ and slacks occur for, at least, one optimal solution. Let $\hat{a}$ be the 'virtual' entity, i.e., $\hat{a} = \sum_{j \neq j^*} a^j \lambda_j^*$ and $\sum_{j \neq j^*} \lambda_j^* = 1$. So, $\hat{a}_i = a_i^{j^*}$ for all i except at least once, where the slack occurs, and either $\hat{a}_i < a_i^{j^*}$ or $\hat{a}_i > a_i^{j^*}$, depending on whether the attribute is undesirable or desirable. Therefore, $\hat{a}$ dominates a^{j^*} by Definition 2.

To demonstrate the converse, we must show that, if either $z^*(\ell) = w^*(\ell) = 1$ and there are no slacks for any optimal solution or $z^*(\ell) = w^*(\ell) < 1$ then a^{j^*} is efficient.

First let us consider the case where $z^*(\ell) = w^*(\ell) < 1$. This makes $\theta^* < 1$ meaning that $a^{j^*} \notin T^\ell \backslash a^{j^*}$ implying further that a^{j^*} is an extreme point of T^ℓ. As an extreme point, it cannot be expressed as the convex combination of any of the data points or any other collection of points in T^ℓ. Since there are no convex combinations of points in T^ℓ different from a^{j^*} that can express a^{j^*}, there are no convex combinations of points in T^ℓ that can dominate a^{j^*}. In the second case, when all optimal solutions are such that $z^*(\ell) = w^*(\ell) = 1$ and none have slacks, the indication is that all optimal solutions to $(\mathbf{D}^\ell)$ generate virtual entities that are all equal to a^{j^*} and there cannot be other points that can dominate this point without generating scores greater than unity or slacks. ∎

Result 3. *The affinely translated model, $(\widehat{\mathbf{D}^\ell})$, is equivalent to $(\mathbf{D}^\ell)$.*

Proof. Suppose $\hat{a}^j = a^j + u$ and $\hat{\alpha}^\ell = \alpha^\ell + u$. Let $\hat{\theta}^*$, $\hat{\lambda}^*$ be the optimal to $(\widehat{\mathbf{D}^\ell})$.

Then

$$\begin{array}{rcl} \sum_{j\neq j^*} \hat{a}^j \hat{\lambda}_{j^*} + (\hat{\alpha}^\ell - \hat{a}^{j^*})\hat{\theta}^* & \overset{\ell}{\leftrightarrow} & \hat{\alpha}^\ell; \\ \sum_{j\neq j^*} \hat{\lambda}_{j^*} & = & 1. \end{array}$$

Replacing $\hat{a}^j$ and $\hat{\alpha}^\ell$ by their respective translations, we get:

$$\begin{array}{rcl} \sum_{j\neq j^*} (a^j + u)\hat{\lambda}_{j^*} + (\alpha^\ell - a^{j^*})\hat{\theta}^* & \overset{\ell}{\leftrightarrow} & \alpha^\ell + u; \\ \sum_{j\neq j^*} \hat{\lambda}_{j^*} & = & 1. \end{array}$$

$$\begin{array}{rcl} \sum_{j\neq j^*} a^j \hat{\lambda}_{j^*} + (\alpha^\ell - a^{j^*})\hat{\theta}^* & \overset{\ell}{\leftrightarrow} & \alpha^\ell; \\ \sum_{j\neq j^*} \hat{\lambda}_{j^*} & = & 1. \end{array}$$

We conclude that $(\hat{\theta}^*, \hat{\lambda}^*)$ is a feasible solution to $(\mathbf{D}^\ell)$.

Next, we show that $(\hat{\theta}^*, \hat{\lambda}^*)$ is also optimal to $(\mathbf{D}^\ell)$. Let us assume that $(\hat{\theta}^*, \hat{\lambda}^*)$ is not optimal to $(\mathbf{D}^\ell)$. Then there exists $\tilde{\lambda} \geq 0$, $\tilde{\theta} \geq 0$ such that $\tilde{\theta} > \hat{\theta}^*$ and

$$\begin{array}{rcl} \sum_{j\neq j^*} a^j \tilde{\lambda}_j + (\alpha^\ell - a^{j^*})\tilde{\theta} & \overset{\ell}{\leftrightarrow} & \alpha^\ell; \\ \sum_{j\neq j^*} \tilde{\lambda}_j & = & 1. \end{array}$$

$$\begin{array}{lclcl}
\sum_{j \neq j^*} (a^j + u)\tilde{\lambda}_j & + & (\alpha^\ell - a^{j^*})\tilde{\theta} & \stackrel{\ell}{\leftrightarrow} & \alpha^\ell + u; \\
\sum_{j \neq j^*} \tilde{\lambda}_j & & & = & 1.
\end{array}$$

But $(a^j + u) = \hat{a}^j$ and $\alpha^\ell + u = \hat{\alpha}^\ell$. So,

$$\begin{array}{lclcl}
\sum_{j \neq j^*} \hat{a}^j \tilde{\lambda}_j & + & (\hat{\alpha}^\ell - \hat{a}^{j^*})\tilde{\theta} & \stackrel{\ell}{\leftrightarrow} & \hat{\alpha}^\ell; \\
\sum_{j \neq j^*} \tilde{\lambda}_j & & & = & 1.
\end{array}$$

Therefore $(\tilde{\lambda}, \tilde{\theta})$ is feasible to $(\widehat{\mathbf{D}^\ell})$, which contradicts it not being optimal to $(\mathbf{D}^\ell)$ establishing the equivalence between $(\mathbf{D}^\ell)$ and $(\widehat{\mathbf{D}^\ell})$. ∎

Appendix 2. Tables

Table A.2.1. Project description.

Type	No.	Project Name	Park	Midtown East	Midtown West	240 Interchanges	Sam Cooper	Costs ($millions)
Straight thru	1	original plan (@ grade)	bisect	bisect	bisect	base		17.0
	2	cut&cover	under	bisect	bisect	base		85.1
	3	cut & cover stacked	under	bisect	bisect	base		107.5
	4	bored tunnel	under	bisect	bisect	base		280.0
	5	partially depressed	bisect	bisect	bisect	base		16.4
	6	bridge above park	over	bisect	bisect	base		21.2
	7	fully depressed (w.plazas)	under	bisect	bisect	base		25.9
	8	cut & cover under N.Pkwy	around	bisect	bisect	base		157.5
	9	elevated above N.Pkwy	around	bisect	bisect	base		29.8
	10	northern route	around	bisect	bisect	base		31.0
	11	southern route	around	bisect	bisect	base		26.0
	12	L&N route	around	no change	no change	base		40
Around on 240	13	do nothing	no change	bisect	no change	base	partial	0
	14	240 A	no change	bisect	no change	upgraded	partial	34
	15	240 B	no change	bisect	no change	base	full	9.8
	16	240 C	no change	bisect	no change	upgraded	full	43.8
	17	240 D	no change	no change	no change	base	none	-38.8
	18	240 E	no change	no change	no change	upgraded	none	-4.8
Substitute funds	19	split & elevated 1	split around	bisect	bisect	base	full	28.2
	20	split & elevated 2	split around	bisect	bisect	upgraded	full	62.2
	21	S.Cooper to park 1	no change	bisect	no change	base	full	9.8
	22	S.Cooper to park 2	no change	bisect	no change	upgraded	full	43.8
	23	actual	no change	bisect	no change	base	partial	0
	24	actual + interchanges	no change	bisect	no change	upgraded	partial	34
	25	no.S.Cooper 1	no change	no change	no change	base	none	-38.8
	26	no. S.Cooper 2	no change	no change	no change	upgraded	none	-4.8

Table A.2.2. Attribute values.

No.	Park	Midtown East	Midtown West	Funding	Costs	Accidents	Variance	Time (min.)	Work Trip
1	0	0	0	0	17	0	17.0	0	0
2	4	0	0	0	85.1	0	85.1	0	0
3	4	0	0	0	107.5	0	107.5	0	0
4	5	0	0	0	280	0	280	0	0
5	1	0	0	0	16.4	0	16.4	0	0
6	3	0	0	0	21.2	0	21.2	0	0
7	3	0	0	0	25.9	0	25.9	0	0
8	5	0	0	0	157.5	0	157.5	0	0
9	5	0	0	0	29.8	0	29.8	0	0
10	5	0	0	0	31	0	31.0	0	0
11	5	0	0	0	26	0	26.0	0	0
12	5	1	1	0	40	0	40.0	0	0
13	5	0	1	0	0	1997	0.0	15	-2
14	5	0	1	0	34	875	34.0	6	-2
15	5	0	1	0	9.8	1872	9.8	15	-5
16	5	0	1	0	43.8	750	43.8	6	-5
17	5	1	1	0	-38.8	2122	38.8	15	0
18	5	1	1	0	-4.8	1000	4.8	6	0
19	5	0	0	280	28.2	875	28.2	15	-5
20	5	0	0	280	62.2	250	62.2	6	-5
21	5	0	1	280	9.8	1872	9.8	15	-5
22	5	0	1	280	43.8	750	43.8	6	-5
23	5	0	1	280	0	1997	0.0	15	-2
24	5	0	1	280	34	875	34.0	6	-2
25	5	1	1	280	-38.8	2122	38.8	15	0
26	5	1	1	280	-4.8	1000	4.8	6	0

Table A.2.3. DEA with multiple constituencies: project report.

		Projects																									
	Constituency	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	Eff.
1	(1,1,1,1,1,1,-1,-1,1)	–	–	–	1	1	–	–	–	–	–	1	1	–	–	–	–	–	–	–	1	1	1	1	1	1	1	11
2	(1,1,1,1,1,-1,-1,-1,1)	–	–	–	1	1	–	–	–	–	–	1	1	–	–	–	–	–	–	1	1	–	1	1	1	–	1	10
3	(1,1,1,1,-1,1,-1,-1,1)	–	–	–	–	1	–	–	–	–	–	1	1	–	–	–	–	–	–	–	–	–	–	1	–	1	1	6
4	(1,1,1,1,-1,-1,-1,-1,1)	–	–	–	–	1	–	–	–	–	–	1	1	–	–	–	–	–	–	–	1	–	1	1	1	1	1	9
5	(1,1,1,-1,1,1,-1,-1,1)	–	–	–	1	1	–	–	–	–	–	1	1	1	1	1	1	1	1	–	–	–	–	–	–	–	–	10
6	(1,1,1,-1,1,-1,-1,-1,1)	–	–	–	1	1	–	–	–	–	–	1	1	1	–	–	1	–	1	–	–	–	–	–	–	–	–	7
7	(1,1,1,-1,-1,1,-1,-1,1)	–	–	–	–	1	–	–	–	–	–	1	1	1	–	–	–	1	1	–	–	–	–	–	–	–	–	6
8	(1,1,1,-1,-1,-1,-1,-1,1)	–	–	–	–	1	–	–	–	–	–	1	1	1	–	–	–	1	1	–	–	–	–	–	–	–	–	6
9	(1,1,-1,1,1,1,-1,-1,1)	–	–	–	1	1	–	–	–	–	–	1	1	–	–	–	–	–	–	1	1	1	1	1	1	1	1	**12**
10	(1,1,-1,1,1,-1,-1,-1,1)	–	–	–	1	1	–	–	–	–	–	1	1	–	–	–	–	–	–	1	1	–	–	1	1	–	1	9
11	(1,1,-1,1,-1,1,-1,-1,1)	–	–	–	–	1	–	–	–	–	–	1	1	–	–	–	–	–	–	1	1	–	–	1	–	1	1	8
12	(1,1,-1,1,-1,-1,-1,-1,1)	–	–	–	–	1	–	–	–	–	–	1	1	–	–	–	–	–	–	1	1	–	–	1	–	1	1	8
13	(1,1,-1,-1,1,1,-1,-1,1)	–	–	–	1	1	–	–	–	–	–	1	1	1	1	–	–	1	1	1	–	–	–	–	–	–	–	9
14	(1,1,-1,-1,1,-1,-1,-1,1)	–	–	–	1	1	–	–	–	–	–	1	1	1	–	–	–	–	1	–	–	–	–	–	–	–	–	6
15	(1,1,-1,-1,-1,1,-1,-1,1)	–	–	–	–	1	–	–	–	–	–	1	1	1	–	–	–	1	1	1	–	–	–	–	–	–	–	7
16	(1,1,-1,-1,-1,-1,-1,-1,1)	–	–	–	–	1	–	–	–	–	–	1	1	1	–	–	–	1	1	–	–	–	–	–	–	–	–	6
17	(1,-1,1,1,1,1,-1,-1,1)	–	–	–	1	1	–	–	–	–	–	1	1	–	–	–	–	–	–	–	1	1	1	1	1	1	1	11
18	(1,-1,1,1,1,-1,-1,-1,1)	–	–	–	1	1	–	–	–	–	–	1	1	–	–	–	–	–	–	1	1	–	1	1	1	–	1	10
19	(1,-1,1,1,-1,1,-1,-1,1)	–	–	–	–	1	–	–	–	–	–	1	1	–	–	–	–	–	–	–	–	–	–	1	1	1	1	7
20	(1,-1,1,1,-1,-1,-1,-1,1)	–	–	–	–	1	–	–	–	–	–	1	1	–	–	–	–	–	–	1	1	–	1	1	1	1	1	10
21	(1,-1,1,-1,1,1,-1,-1,1)	–	–	–	1	1	–	–	–	–	–	1	1	1	1	1	1	1	1	–	–	–	–	–	–	–	–	10
22	(1,-1,1,-1,1,-1,-1,-1,1)	–	–	–	1	1	–	–	–	–	–	1	1	1	1	–	1	–	1	–	–	–	–	–	–	–	–	8
23	(1,-1,1,-1,-1,1,-1,-1,1)	–	–	–	–	1	–	–	–	–	–	1	1	1	1	–	–	1	1	–	–	–	–	–	–	–	–	7
24	(1,-1,1,-1,-1,-1,-1,-1,1)	–	–	–	–	1	–	–	–	–	1	1	1	1	1	–	1	1	1	–	–	–	–	–	–	–	–	8
25	(1,-1,-1,1,1,1,-1,-1,1)	–	–	–	1	1	–	–	–	–	–	1	–	–	–	–	–	–	–	1	1	1	1	1	1	1	1	11
26	(1,-1,-1,1,1,-1,-1,-1,1)	–	–	–	1	1	–	–	–	–	–	1	–	–	–	–	–	–	–	1	1	–	–	1	1	–	1	8
27	(1,-1,-1,1,-1,1,-1,-1,1)	–	–	–	–	1	–	–	–	–	–	1	–	–	–	–	–	–	–	1	1	–	–	1	1	1	1	8
28	(1,-1,-1,1,-1,-1,-1,-1,1)	–	–	–	–	1	–	–	–	–	–	1	–	–	–	–	–	–	–	1	1	–	–	1	1	1	1	8
29	(1,-1,-1,-1,1,1,-1,-1,1)	–	–	–	1	1	–	–	–	–	–	1	–	1	1	–	–	1	1	1	–	–	–	–	–	–	–	8
30	(1,-1,-1,-1,1,-1,-1,-1,1)	–	–	–	1	1	–	–	–	–	–	1	–	1	–	–	–	–	1	–	–	–	–	–	–	–	–	**5**
31	(1,-1,-1,-1,-1,1,-1,-1,1)	–	–	–	–	1	–	–	–	–	1	–	–	1	1	–	1	1	1	1	–	–	–	–	–	–	–	7
32	(1,-1,-1,-1,-1,-1,-1,-1,1)	–	–	–	–	1	–	–	–	–	–	1	–	1	–	–	–	1	1	–	–	–	–	–	–	–	–	**5**
33	(-1,1,1,1,1,1,-1,-1,1)	1	–	–	1	1	–	–	–	–	–	–	1	–	–	–	–	–	–	–	1	1	1	1	1	1	1	11
34	(-1,1,1,1,1,-1,-1,-1,1)	1	–	–	1	1	–	–	–	–	–	–	1	–	–	–	–	–	–	1	1	–	1	1	1	–	1	10
35	(-1,1,1,1,-1,1,-1,-1,1)	1	–	–	–	1	–	–	–	–	–	–	1	–	–	–	–	–	–	–	–	–	–	1	–	1	1	6
36	(-1,1,1,1,-1,-1,-1,-1,1)	1	–	–	–	1	–	–	–	–	–	–	1	–	–	–	–	–	–	–	1	–	1	1	1	1	1	9
37	(-1,1,1,-1,1,1,-1,-1,1)	1	–	–	1	1	–	–	–	–	–	–	1	1	1	1	1	1	1	–	–	–	–	–	–	–	–	10
38	(-1,1,1,-1,1,-1,-1,-1,1)	1	–	–	1	1	–	–	–	–	–	1	1	1	–	–	–	–	1	–	–	–	–	–	–	–	–	7
39	(-1,1,1,-1,-1,-1,-1,-1,1)	1	–	–	–	1	–	–	–	–	–	–	1	1	–	–	–	1	1	–	–	–	–	–	–	–	–	6
40	(-1,1,-1,1,1,1,-1,-1,1)	1	–	–	–	1	–	–	–	–	–	–	1	1	–	–	–	1	1	–	–	–	–	–	–	–	–	6
41	(-1,1,-1,1,1,-1,-1,-1,1)	1	–	–	1	1	–	–	–	–	–	–	1	–	–	–	–	–	–	1	1	1	1	1	1	1	1	**12**
42	(-1,1,-1,1,1,-1,-1,-1,1)	1	–	–	1	1	–	–	–	–	–	–	1	–	–	–	–	–	–	1	1	–	–	1	1	–	1	9
43	(-1,1,-1,1,-1,1,-1,-1,1)	1	–	–	–	1	–	–	–	–	–	–	1	–	–	–	–	–	–	1	1	–	–	1	–	1	1	8
44	(-1,1,-1,1,-1,-1,-1,-1,1)	1	–	–	–	1	–	–	–	–	–	–	1	–	–	–	–	–	–	1	1	–	–	1	–	1	1	8
45	(-1,1,-1,-1,1,1,-1,-1,1)	1	–	–	1	1	–	–	–	–	–	–	1	1	1	–	–	1	1	1	–	–	–	–	–	–	–	9
46	(-1,1,-1,-1,1,-1,-1,-1,1)	1	–	–	1	1	–	–	–	–	–	–	1	1	–	–	–	–	1	–	–	–	–	–	–	–	–	6
47	(-1,1,-1,-1,1,-1,-1,-1,1)	1	–	–	–	1	–	–	–	–	–	–	1	1	–	–	–	1	1	1	–	–	–	–	–	–	–	7
48	(-1,1,-1,-1,-1,-1,-1,-1,1)	1	–	–	–	1	–	–	–	–	–	–	1	1	–	–	–	1	1	–	–	–	–	–	–	–	–	6
49	(-1,-1,1,1,1,1,-1,-1,1)	1	–	–	1	1	–	–	–	–	–	–	1	–	–	–	–	–	–	–	1	1	1	1	1	1	1	11
50	(-1,-1,1,1,1,-1,-1,-1,1)	1	–	–	1	1	–	–	–	–	–	–	1	–	–	–	–	–	–	1	1	–	1	1	1	–	1	10
51	(-1,-1,1,1,-1,1,-1,-1,1)	1	–	–	–	1	–	–	–	–	–	–	1	–	–	–	–	–	–	–	–	–	–	1	1	1	1	7
52	(-1,-1,1,1,-1,-1,-1,-1,1)	1	–	–	–	1	–	–	–	–	–	–	1	–	–	–	–	–	–	1	1	–	1	1	1	1	1	10
53	(-1,-1,1,-1,1,1,-1,-1,1)	1	–	–	1	1	–	–	–	–	–	–	1	1	1	1	1	1	1	–	–	–	–	–	–	–	–	10
54	(-1,-1,1,-1,1,-1,-1,-1,1)	1	–	–	1	1	–	–	–	–	–	–	1	1	1	–	1	–	1	–	–	–	–	–	–	–	–	8
55	(-1,-1,1,-1,-1,1,-1,-1,1)	1	–	–	–	1	–	–	–	–	–	–	1	1	1	–	–	1	1	–	–	–	–	–	–	–	–	7
56	(-1,-1,1,-1,-1,-1,-1,-1,1)	1	–	–	–	1	–	–	–	–	–	–	1	1	1	–	1	1	1	–	–	–	–	–	–	–	–	8
57	(-1,-1,-1,1,1,1,-1,-1,1)	1	–	–	1	1	–	–	–	–	–	–	–	–	–	–	–	–	–	1	1	1	1	1	1	1	1	11
58	(-1,-1,-1,1,1,-1,-1,-1,1)	1	–	–	1	1	–	–	–	–	–	–	–	–	–	–	–	–	–	1	1	–	–	1	1	–	1	8
59	(-1,-1,-1,1,-1,1,-1,-1,1)	1	–	–	–	1	–	–	–	–	–	–	–	–	–	–	–	–	–	1	1	–	–	1	1	1	1	8
60	(-1,-1,-1,1,-1,-1,-1,-1,1)	1	–	–	–	1	–	–	–	–	–	–	–	–	–	–	–	–	–	1	1	–	–	1	1	1	1	8
61	(-1,-1,-1,-1,1,-1,-1,-1,1)	1	–	–	1	1	–	–	–	–	–	–	–	1	1	–	–	1	1	1	–	–	–	–	–	–	–	8
62	(-1,-1,-1,-1,1,-1,-1,-1,1)	1	–	–	1	1	–	–	–	–	–	–	–	1	–	–	–	–	1	–	–	–	–	–	–	–	–	**5**
63	(-1,-1,-1,-1,-1,1,-1,-1,1)	1	–	–	–	1	–	–	–	–	–	–	–	1	1	–	–	1	1	1	–	–	–	–	–	–	–	7
64	(-1,-1,-1,-1,-1,-1,-1,-1,1)	1	–	–	–	1	–	–	–	–	–	–	–	1	–	–	–	1	1	–	–	–	–	–	–	–	–	**5**
	Total	32	0	0	32	**64**	0	0	0	0	0	32	48	32	16	4	10	24	32	30	28	8	16	32	26	24	32	

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Tables and Figures to Be Inserted in the Body of the Text

Table 1. Attribute Data for Eight Automobiles.

Auto Style	Attribute 1 Size	Attribute 2 Horsepower
1	3	2
2	1	4
3	5	1
4	8	5
5	6	8
6	3	6
7	4	4.5
8	6.5	5.5
9	8	3

Table 2. Efficiency Report in Terms of Entities.

Entity Data Point	Efficient for Constituency ...
a^1	δ^4
a^2	δ^3, δ^4
a^3	δ^2, δ^4
a^4	δ^1
a^5	δ^1, δ^3
a^6	δ^3
a^7	–
a^8	–
a^9	δ^2

Table 3. Efficiency Report in Terms of Constituencies.

Constituency	Efficient Entity
δ^1	a^4, a^5
δ^2	a^3, a^9
δ^3	a^2, a^5, a^6
δ^4	a^1, a^2, a^3

Table 4. Efficiency Matrix.

	Entities									
Constituency	a^1	a^2	a^3	a^4	a^5	a^6	a^7	a^8	a^9	Eff.
δ^1	–	–	–	1	1	–	–	–	–	2
δ^2	–	–	1	–	–	–	–	–	1	2
δ^3	–	1	–	–	1	1	–	–	–	3
δ^4	1	1	1	–	–	–	–	–	–	3
Total	1	2	2	1	2	1	0	0	1	

Figure 1. The Four Production Possibility Sets for the Data in Table 1.

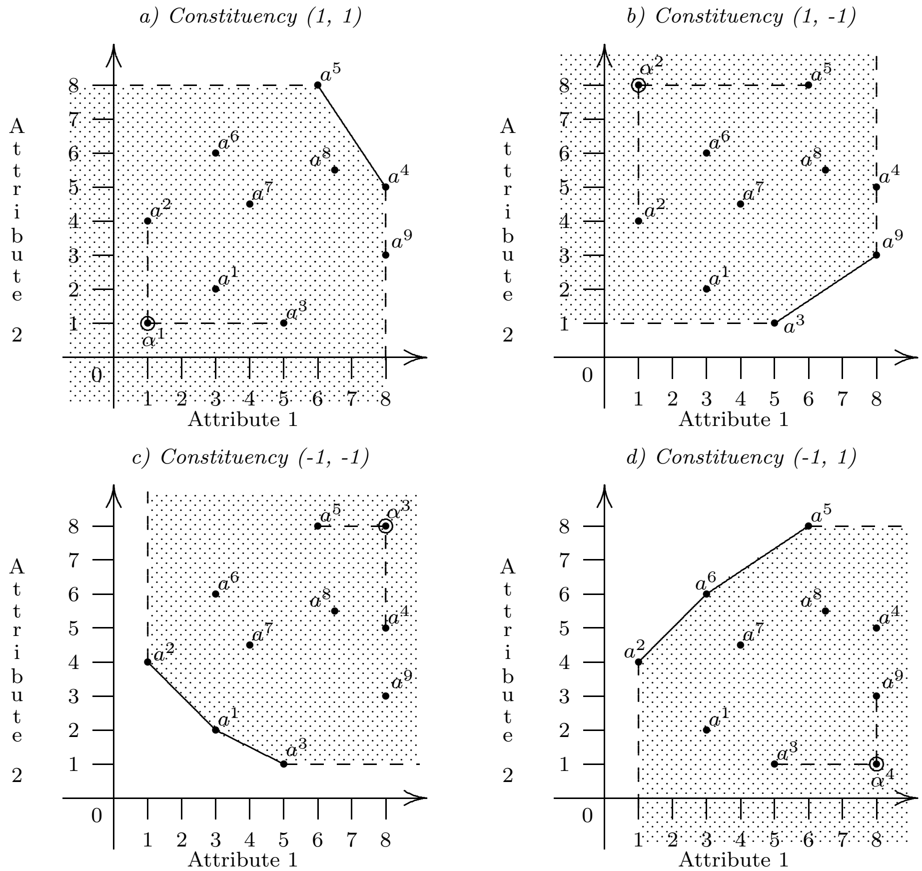

Figure 2. The Four Production Possibility Sets with the Auxiliary Vector.

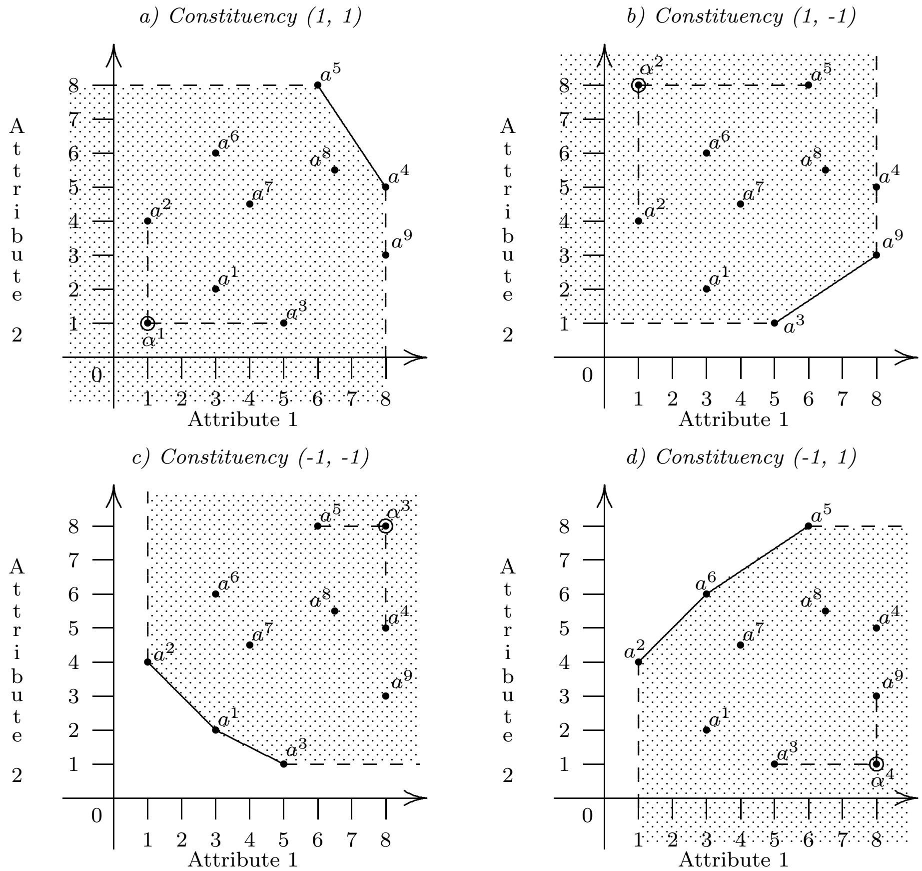

Figure 3. New LP Benchmarks.

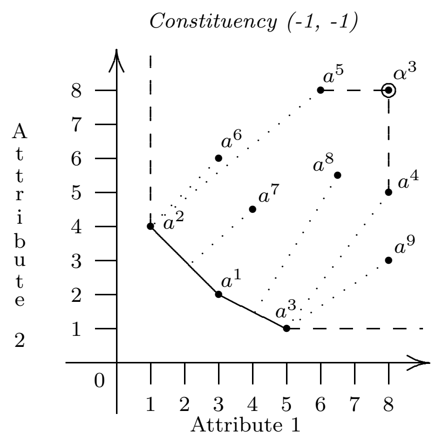

Figure 4. I-40 Project Through Memphis as Planned.

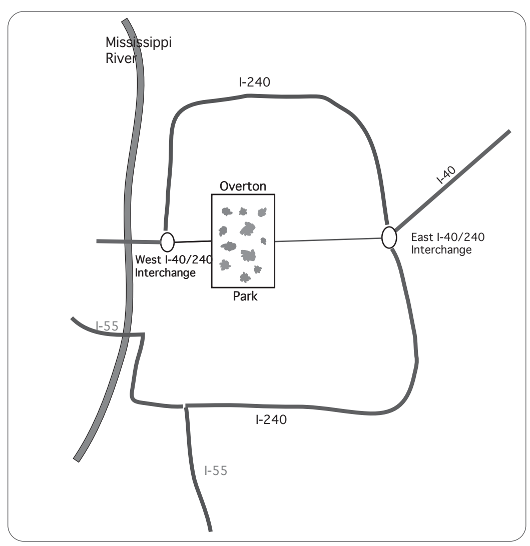

Figure 5. Percentage of Constituencies Finding a Project Efficient (ranked).

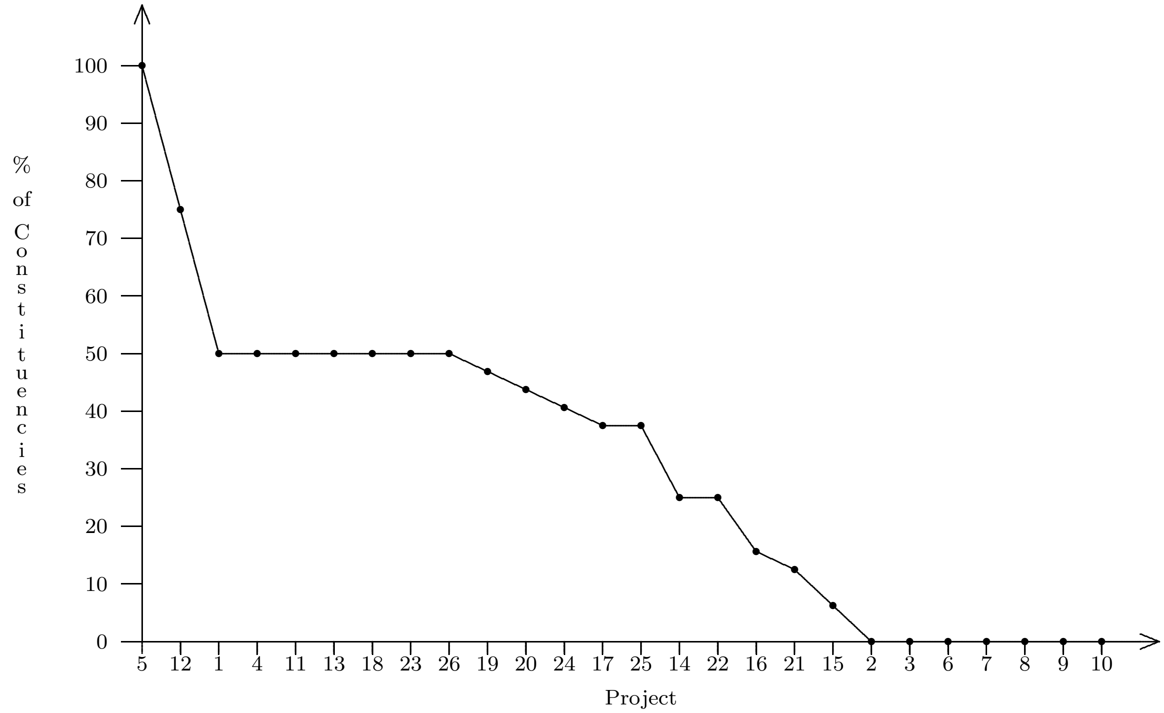

Figure 6. Percentage of Efficient Projects for Each Constituency (ranked).