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Non-Perturbative Methods and Extended Hadron Models in Field Theory

II. Two-Dimensional Models and Extended Hadrons

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ABSTRACT

We discuss the application of semiclassical quantization methods to two-dimensional model field theories for which exact non trivial classical solutions are known analytically. This yields results which cannot be reached by ordinary perturbation methods. In particular, we obtain extended objects which can be considered as prototypes for hadrons. We study their quantum corrections and renormalization. We also develop a method for including fermions.

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SECTION I. - INTRODUCTION

For some time now, it has been known that certain 2-dimensional field theories have particular classical solutions which look like extended particles.[1] There has however, not been much progress on the question of whether these objects would appear as true particle states in the corresponding quantum field theory. In this paper we address this problem using the semi-classical functional quantization scheme developed in the previous paper.[2]

The usual way of doing perturbation theory has built into it the assumption that the asymptotic states of a field theory are free fields. In a functional language, free field modes are just the solution to the extreme linearization of the Schwinger equations for the generating functional. This is reflected in the Feynman path integral language through the instruction to integrate over all possible field histories after expanding the interaction functional, either in ascending powers of the coupling constant, or topologically, in terms of loop functionals, around the free field modes.

This assumption selects only a sector of admissible solutions to the full interacting problem. We will be concerned with those solutions that pass through the usual functional sieve and are not asymptotically free fields.

In particular, we want to take as fundamental an exact solution to the classical full-nonlinear interacting equations.

The bulk of this paper is devoted to the study of a particular model that is a remarkable one since all relevant equations can be solved analytically. Furthermore it has many interesting properties even for weak coupling, a regime where we believe we have control over our approximations. In particular we will display an extended particle solution which has many properties reminiscent of hadrons.

A peculiar feature of the model extended particle is that it involves a classical field configuration which has a topology different than that of the classical vacuum. This feature serves to stabilize the state and we conjecture that it is a general characteristic of interesting extended objects. The possibility of topologically unusual field configurations appears to be related to spontaneously broken symmetry, a discrete one in the present case. In a sense the field theory becomes a model for a superconductor.

Our model extended particle also serves as a well which can trap and confine fermions. In fact, by turning the solution in one space dimension into a (locally one dimensional) thin spherical shell, a group at SLAC has independently been able to construct an interesting and perhaps realistic model of hadrons with confined quarks.[3] The methods we have developed for including fermions, trapped or otherwise, in semi-classical calculations are presented in Section 4 of this paper. Included are a set of self-consistent field equations, which while written for two dimensions, can be trivially generalized to four.

We will compute the first quantum corrections to the masses of our extended particles. In doing so we will meet ultraviolet divergences and in the process of removing them, illustrate some renormalization techniques. Generalizing these methods to arbitrary renormalizable interactions is in principle straightforward. In more complex models however, the calculations would be extremely difficult.

There is a persistent conceptual problem associated with the identification of a classical particle-like field configuration with a quantum particle. It is that the classical extended object has to be localized at some point in space. In the previous paper we show how this apparent difficulty goes away when semi-classical quantization methods are applied consistently. The reader who is troubled by this point is referred to Section 5 of the preceeding paper.

The following section contains a review of those aspects of our semi-classical functional methods which are needed for the present work. We then proceed to the model and the fermion techniques described above.

Section 2. Review of the Semi-Classical Method

In this section we wish to review those results of the previous paper which are directly relevant for the present work.

Suppose we have a classical field theory described by a field φ (more generally a set of fields) and a Lagrangian $\mathcal{L}(\varphi)$. Suppose further that we can find a time-independent solution $\varphi_0(x)$ to the time-independent Euler-Lagrange equation

$$-\frac{\partial}{\partial x}\left(\frac{\delta \mathcal{L}}{\delta(\frac{\partial}{\partial x}\varphi)}\right) + \frac{\delta \mathcal{L}}{\delta \varphi} = 0 \tag{2.1}$$

We require that the solution be localized in space so that the classical energy, given by

$$E_{cl} = -\int \mathcal{L}(\varphi_0)\, dx \tag{2.2}$$

for a time independent field, is finite. Note that we are not looking for a new vacuum state which would be a constant φ with a constant energy density and hence a divergent energy.

The classical solution φ_0 is also supposed to be stable in the following sense. In the Lagrangian set $\varphi(x,t) = \varphi_0(x) + \eta(x,t)$ and expand in powers of η keeping only the quadratic terms. (The linear terms will be absent because φ_0 satisfies the classical equations of motion.) The resulting quadratic Lagrangian should then be reducible to a set of independent harmonic oscillators with real frequencies ω_k $(k = 1 \ldots \infty)$.

Intuitively, one would expect this localized, stable classical solution to correspond in some sense to a particle at rest. In first

approximation its mass should be E_{cl} with the first quantum correction coming from the zero-point energy $\sum_k \frac{1}{2}\omega_k$ of the small oscillations around φ_0. Of course, we will have to subtract the zero point energy of the vacuum to make $\sum_k \frac{1}{2}\omega_k$ finite and perhaps have to make further renormalizations. According to the previous paper, this is a valid procedure for finding new kinds of particles in field theory provided that the coupling constants are small. For strong coupling, semi-classical approximation methods take a more complicated form akin to the usual WKB method in ordinary quantum mechanics. In the next two sections we will restrict ourselves to weak coupling.

If $\varphi_0(x)$ is any particle like classical solution so, of course, is $\varphi_0(x+a)$ for any spacial translation a. Also φ_0 can be Lorentz transformed to obtain moving solutions. It was shown in the previous paper that when these additional degrees of freedom are taken into account one obtains a quantum mechanical particle which has the proper energy-momentum relation $E = p^2 + M^2$ where M can be computed as outlined above. The quantized state of zero momentum does not correspond to any one of the particular solutions $\varphi_0(x+a)$ but rather to the whole set of classical solutions obtained by letting a vary. Furthermore, it was shown that states with many of these particles can exist and that they will obey Bose statistics if φ is a Bose field.

In the previous paper, it was shown that one can systematically improve on the weak coupling semi-classical approximation as follows.

In the Lagrangian $\mathcal{L}$ write the quantized field φ as $\varphi(x,t) = \varphi_c(x) + \eta(x,t)$ where $\varphi_c(x)$ is a time-independent c-number field and $\eta(x,t)$ is a new quantized field. Separating out the pieces of $\mathcal{L}$ which are linear and quadratic in η gives a new free lagrangian $\mathcal{L}_0$. The quadratic terms in $\mathcal{L}_0$ define a propagator $\mathcal{D}(x, x', t-t')$ which depends on x and x' separately, but because φ_c is time independent the time dependence is the usual one $t-t'$. We denote the fourier transform of $\mathcal{D}$ with respect to $t-t'$ as $\tilde{\mathcal{D}}(x, x', \omega)$. The part of $\mathcal{L}$ which is trilinear or higher in η defines a interaction Lagrangian $\mathcal{L}_I$. In terms of $\mathcal{D}$ and $\mathcal{L}_I$ one can clearly define a Feynman diagram expansion which has exactly the same topological and combinatoric properties as the usual one. We now introduce a functional $\mathcal{E}(\varphi_c)$ which is defined to be the sum of all connected one-particle irreducible diagrams with no external lines. Specifically $\mathcal{E}(\varphi_c)$ is

$$\frac{\mathcal{E}(\varphi_c)}{\hbar} = -\int \frac{dx\,\mathcal{L}(\varphi_c)}{\hbar} + \frac{1}{4\pi i}\int_{-i\infty}^{+i\infty} d\omega \ \mathrm{tr}\,\log\tilde{\mathcal{D}}(\omega) + \ldots \tag{2.3}$$

where tr $\log\tilde{\mathcal{D}}(\omega)$ is the log of the determinant of $\tilde{\mathcal{D}}(x, x', \omega)$ with respect to the variables x and x' and the omitted terms come from diagrams with two or more closed loops and are of order $(\hbar)^n$ with $n \geq 1$.

Note that the first term in the expansion (2.3) for $\mathcal{E}(\varphi_c)$ is just the classical energy. Any stable solution to the classical field

equation (2.1) is a local minimum of the classical energy. In the previous paper, it was shown that a way to improve on the weak coupling semi-classical approximation is to look for a local minimum of $\mathcal{E}$ instead of the classical energy. The equations are

$$\left.\frac{\delta \mathcal{E}(\varphi_c)}{\delta \varphi_c(x)}\right|_{\varphi_c = \overline{\varphi}_c} = 0$$

$$M = \mathcal{E}(\overline{\varphi}_c) \tag{2.4}$$

where we denote by $\overline{\varphi}_c$ the localized field which is supposed to be a relative minimum of $\mathcal{E}$. We note that the particle mass M is just $\mathcal{E}$ evaluated at $\overline{\varphi}_c$.

For weak coupling Eqs. (2.4) were justified in the previous paper. Except for special cases, they do <u>not</u> provide a useful variational method for finding particle states. The reason is that a particle-like, localized field can at best be a local minimum of $\mathcal{E}$. The reader who is familiar with functional methods will immediately recognize that the absolute minimum of $\mathcal{E}$ comes at the constant field $<0|\varphi|0>$ equal to the vacuum expectation value of the quantum field φ. Nevertheless, Eqs. (2.4) are useful. For example, they tell us how to renormalize. In a renormalizable theory, for every subtraction needed to define the second term on the right of Eq. (2.3) or any of the omitted higher terms there is a corresponding counter term which can be added to the Lagrangian in the first term.

For weak coupling where this recipe is useful, the higher order terms in (2.3) are small and since they are prohibitively difficult to

compute, in practice one will work with the two terms shown explicitly in Eq. (2.3). If one looks for a minimum of $\mathcal{E}(\varphi_c)$ computed in this approximation, the resulting equations lead to an interesting self-consistent-field Hartree-like approximation. An example of this procedure is given in Section (4). For the present, let us content ourselves with an iterative solution to the Eqs. (2.4). Let $\bar{\varphi}_c$ be expanded as $\bar{\varphi}_c = \varphi_0 + \hbar\varphi_1 + \hbar^2\varphi_2 + \ldots$. Clearly the first term φ_0 is just the classical field which is a local minimum of the classical energy. To compute the corresponding particle mass through order $\hbar$, we need only insert φ_0 into Eq. (2.3). The error incurred by neglecting $\hbar\varphi_1$ can easily be seen to be of order $\hbar^2$. The particle mass is then

$$\begin{aligned} M &= -\int dx \mathcal{L}(\varphi_0) + \frac{\hbar}{4\pi i}\int_{-i\infty}^{+i\infty} d\omega \, \mathrm{tr}\log\mathcal{D}(\omega,\varphi_0) \\ &= E_{cl} + \frac{\hbar}{4\pi i}\int_{-i\infty}^{+i\infty}\sum_k \log(\omega-\omega_k) \end{aligned} \tag{2.5}$$

where in the second line E_{cl} is the classical energy as before and the ω_k in the second term are the oscillator frequencies defined above. The ω integral is divergent and in need of regularization and renormalization, but an integration by parts shows that it is formally equivalent to the sum of zero point energies $\sum_k \frac{1}{2}\omega_k$.

SECTION 3.

BOUND STATES IN A TWO DIMENSIONAL FIELD THEORY MODEL

We now discuss a simple, soluble example, of the application of these ideas to field theory. Our example is the quantization of a classical kink-like solution of the field theory described by the Lagrangian density

$$\tilde{\mathcal{L}} = -\tfrac{1}{2}(\partial_\mu \tilde{\varphi})^2 + \frac{m^2}{2}\tilde{\varphi}^2 - \frac{\lambda}{4}\tilde{\varphi}^4 \qquad (3.1)$$

where $\tilde{\varphi}(\tilde{x}, \tilde{t})$ is a real scalar field. The sign of the mass term generates spontaneous symmetry breaking (the symmetry being $\varphi \to -\varphi$). As will be clear, this is necessary for the existence of our solution.

By making the scaling

$$\varphi = \sqrt{\frac{\lambda}{m}}\,\tilde{\varphi}$$

$$x = m\tilde{x} \qquad (3.2)$$

the Lagrangian becomes

$$\tilde{\mathcal{L}} = \frac{m^4}{\lambda} \mathcal{L}$$

with

$$\mathcal{L} = -\tfrac{1}{2}(\partial_\mu \varphi)^2 + \tfrac{1}{2}\varphi^2 - \tfrac{1}{4}\varphi^4 \qquad (3.3)$$

After this rescaling, the limit $\hbar \to 0$ is equivalent to the limit $\lambda/m^2 \to 0$, which is the weak coupling limit. Hence, we expect our results to be at least valid in the range of validity of ordinary perturbation theory.

The classical equation of motion is

$$(-\partial_t^2 + \partial_x^2)\varphi + \varphi - \varphi^3 = 0 \qquad (3.4)$$

The boundary conditions are $|\varphi| = 1$ at infinity. We look for stationary solutions. Multiplying (3.4) by φ' and integrating, one obtains, after using the boundary condition

$$\varphi'^2 + \varphi^2 - \tfrac{1}{2}\varphi^4 = \tfrac{1}{2} \qquad (3.5)$$

(We use the notation $\dot{\varphi} = \partial_t \varphi$ and $\varphi' = \partial_z \varphi$.) Besides the solution $\varphi = \pm 1$, which represents the ordinary vacuum, we find the solutions

$$\varphi = \pm \tanh \frac{1}{\sqrt{2}} (x - x_0) \qquad (3.6)$$

These solutions represent "kinks" where the energy density vanishes exponentially away from x_0. In the following, we shall restrict ourselves to $x_0 = 0$ and the + sign. We also notice that $\varphi(x)$ approaches

two different values as $x \to \pm \infty$, which correspond to the two possible vacuum states. Because it is the lowest energy state which connects these two vacua, it can be translated, boosted or excited, but never decay. Its decay would require flipping the vacuum over an infinite range of space, which requires infinite energy. This persists after quantization. It is of course essential that the vacua have the same numerical magnitude.

It is possible to have two (or many) solutions of the type (3.6) located around different values of x_0 and with alternating signs. There is an additive conserved kink number, which can only take the values +1, 0, or -1. Such a system is actually a well-known model for a one dimensional superconductor, φ being the order parameter. This relationship to superconductivity is no accident and will also be found in higher dimensional models.

We now turn to the calculation of the quantum mechanical energy of such a state, in our W. K. B. approximation. There are two contributions: the classical energy and the quantum mechanical fluctuations around the fundamental solution. We shall have to subtract the ordinary zero point vacuum energy. It will also turn out that another subtraction, corresponding to mass renormalization will be needed.

The harmonic oscillator frequencies are the square root of the eigenvalues of the differential operator $(\frac{d}{dx})^2 + 1 - 3\tanh^2 \frac{x}{\sqrt{2}}$.

It turns out that this problem is soluble in terms of elementary functions.[4]

The equation to be solved is

$$\frac{d^2y}{dx^2} + (1 + \omega^2)y - 3y \tanh^2 \frac{x}{\sqrt{2}} = 0 \qquad (3.7)$$

This is an ordinary one-dimensional Schrodinger equation for the potential $1 - 3 \tanh^2 \frac{x}{\sqrt{2}}$. There are both bound states and scattering states. By the change of variables,

$$z = \frac{x}{\sqrt{2}}$$

$$\epsilon = 2\omega^2 + 2$$

we recognize a particular case of Eq. (12.3.22) of ref. (4). The values of the bound-state energies are $\omega^2 = 0$ and $\omega^2 = \frac{3}{2}$. The continuum begins at $\omega^2 = 2$.

The bound state at $\omega = 0$ is recognized as a translation mode of the kink. Indeed, its wave function is

$$\frac{1}{\cosh^2 \frac{x}{\sqrt{2}}} = -\frac{d}{dx_0} \tanh \frac{x - x_0}{\sqrt{2}} \Bigg|_{x_0 = 0}$$

This is the zero frequency mode mentioned in Sec. (4) of the previous paper. It does not contribute to the mass of our kink. Its presence is an indication of the fact that if center of mass motion were taken into

account the energy momentum relation would be the correct $E = \sqrt{p^2 + m^2}$.

The bound state at $\omega = \sqrt{\frac{3}{2}}$ corresponds to some eigenvibration of the kink, which dies off exponentially as one goes away from the center of the kink. This vibration, if of small enough amplitude, is a harmonic oscillator with an excitable spectrum. Its ground state energy contributes $\frac{1}{2}\hbar\omega$ to the ground state energy of the kink.

We now come to the continuum states. They are defined by a wave vector $k = \sqrt{2\omega^2 - 4}$ (Eq. 12.3.28 in ref (4)). One finds that there is no reflection: all of the incoming wave is transmitted through the potential well with a phase shift δ which turns out to be

$$\delta = 2\pi - 2 \text{ Arctg } k - 2 \text{ Arctg } \tfrac{1}{2} k. \qquad (3.8)$$

To compute the total contribution to the energy, we need the density of states as a function of k. Putting the system in a very large box of length L with periodic boundary conditions makes the modes discrete, the n^{th} mode being given by

$$Lk_n + \delta = 2n\pi$$

Hence the contribution $E_{cont.}$ of the continuum to the zero-point energy of the kink:

$$E_{cont} = \sum_m \frac{1}{2\sqrt{2}} \sqrt{k_m^2 + 4} \underset{L\to\infty}{\to} \int_{-\infty}^{+\infty} \frac{1}{2\sqrt{2}} \sqrt{k^2 + 4} \frac{L}{2\pi} dk - \int_{-\infty}^{+\infty} \frac{1}{2\sqrt{2}} \sqrt{k^2 + 4} \frac{d\delta}{dk} \frac{dk}{2\pi} \tag{3.9}$$

The first term in (3.9), which is proportional to the volume of the box is cancelled by subtracting out the vacuum energy. From (3.8):

$$\frac{d\delta}{dk} = 6 \frac{(2 + k^2)}{(1 + k^2)(4 + k^2)}$$

The remaining term in (3.9) is then still logarithmically divergent. This divergence is exactly cancelled by the usual mass renormalization counterterm. The actual calculation requires a lot of care, in particular, because the vacuum energy is linearly divergent, and finite parts can easily be missed. The regularization scheme which is best suited for this problem is to define the field theory on a lattice in a very large box: the number of degrees of freedom becomes finite; one performs the calculations and then lets the lattice spacing go to zero and the size of the box to infinity. These points are discussed in the Appendix. The final result for the energy of the kink is then

$$E = \frac{2}{3}\sqrt{2}\,\frac{m^3}{\lambda} + m\left[-\frac{3}{\pi\sqrt{2}} + \frac{1}{2\sqrt{6}}\right] + O(\lambda) \tag{3.10}$$

where the first term is the classical energy, and the second one is the first quantum mechanical correction.

It is interesting to note that the kink has excited states. As noted above, the $\omega = \frac{3}{2}$ bound state is a local oscillation of the kink. The state with n quanta in this oscillator will have an energy

$$E_n = E_{kink} + \sqrt{\frac{3}{2}}\, n \qquad (3.11)$$

The $n = 1$ state is below the continuum and is stable. The higher states can decay into an unexcited kink and one of the ordinary quanta of the theory. They would appear as resonances.

The kink is vaguely reminicent of a hadron. It is an extended object with a ground state and a tower of excited resonances above it.

SECTION 4.

ADDING FERMIONS

We can generalize our model by adding to the Lagrangian the terms

$$i\bar{\psi}\partial\!\!\!/\psi + g\,\tilde{\varphi}\bar{\psi}\psi \tag{4.1}$$

which describe a fermion coupled to the scalar field $\tilde{\varphi}$. Note that our fermion has no bare mass; its physical mass comes from the vacuum expectation value of $\tilde{\varphi} = \pm \frac{m}{\sqrt{\lambda}}$. Hence, denoting the fermion mass by M, we have

$$M = \pm \frac{gm}{\sqrt{\lambda}} \tag{4.2}$$

The transformation $\psi \to \gamma_5\psi$ will reverse the sign of a fermion mass so that M can be taken as positive for either sign of $< \varphi >$.

The rescaling

$$\begin{aligned} x &\to \frac{x}{m} \\ \tilde{\varphi} &\to \frac{m}{\sqrt{\lambda}}\,\varphi \end{aligned} \tag{4.3}$$

leads to the equation

$$i\partial\!\!\!/\psi + G\,\varphi\,\psi = 0 \tag{4.4}$$

where G is the dimensionless quantity

$$G = \frac{g}{\sqrt{\lambda}} = \frac{M}{m} . \tag{4.5}$$

Finally, we observe that the "charge"

$$Q = \int \bar{\psi} \gamma^0 \psi \tag{4.6}$$

is a conserved quantum number and that if we take the representation

$$\gamma^0 = \sigma_y \quad \gamma^1 = i \sigma_z \tag{4.7}$$

the charge conjugation is simply hermitian conjugation, i.e.

$$C \psi C^{-1} = \psi^+ \tag{4.8}$$

It is of interest to study the solutions of (4.4) in the static kink field of Eq. (3.6). Giving ψ a time dependence

$$\psi = e^{i\omega t} U, \tag{4.9}$$

multiplying (4.4) on the left by $i \not{\partial} - G \varphi \psi$ and using the representation (4.7) yields

$$\omega^2 U + U'' - G^2 \varphi^2 U + G \sigma_z (1 - \varphi^2) U = 0 \tag{4.10}$$

where primes denote derivatives with respect to x and we have used the equation $\varphi' = 1 - \varphi^2$, satisfied by the kink. Eq. (4.10), like (3.7), is then a Schroedinger equation with a hyperbolic tangent potential and can be solved

analytically. The frequencies ω_n of the bound states are

$$\omega_n^2 = 2nG - n^2 \; ; \quad n = 0, 1, \ldots < G \tag{4.11}$$

Note that ω_0 is zero. It is a non-degenerage eigenvalue of (4.4). The corresponding wave function U_0 is real in the representation (4.7) and is therefore a self charge conjugate fermion state which carries no charge Q.[5] Also, because U_0 turns out to be an eigenstate of $\gamma^1 = i\sigma_z$, the density $\bar{U}_0 U_0$ vanishes, i.e.

$$\bar{U}_0 U_0 = 0 \tag{4.12}$$

The non-zero frequencies come in pairs, $\omega_n = \pm\sqrt{2nG - n^2}$. The positive frequencies are fermions with Q = + 1 while the negative frequencies are anti fermions with Q = -1. The proper normalization for these states is

$$\int U_n^+ U_n = 1 \; ; \quad n = 1, 2, \ldots < G \tag{4.13}$$

There is also a continuous spectrum to (4.10).

So far, all we have done is to introduce some notation and discuss the solutions (4.4) in the static kink field (3.6). Now we would like to see how the Fermi field can be incorporated into our general program of looking for semi-classical particle-like solutions in field theory. A simple device which will be sufficient for our purposes is to observe that the Fermi field enters only

bilinearly in the Lagrangian and can be integrated out of a functional integral. Doing this will yield, in the standard manner, an effective action for the field φ itself;

$$S = \int \left(-\tfrac{1}{2}(\partial_\mu \tilde{\varphi})^2 + \frac{m^2}{2} \tilde{\varphi} - \frac{\lambda \tilde{\varphi}^4}{4} \right) dx + \frac{i}{2} \text{ tr log } (i\partial\!\!\!/ + g\tilde{\varphi}) \tag{4.14}$$

where

$$\text{tr log } (i\partial\!\!\!/ + g\tilde{\varphi}) = \log \det (i\partial\!\!\!/ + g\tilde{\varphi}) \tag{4.15}$$

is the log of the Fredholm determinant of the differential operator $(i\partial\!\!\!/ + g\tilde{\varphi})$. The determinant is, of course, divergent and must be renormalized. We will return to this later.

At this point it should be understood that $\tilde{\varphi}$ is now a general classical scalar field not the kink field. When we wish to refer to the kink field we will say so. If we restrict ourselves to fields $\tilde{\varphi}$ which are independent of time, i.e. $\tilde{\varphi} = \tilde{\varphi}(x)$ where x is the spacial variable, the time part of the trace in (4.15) can be done in frequency space yielding

$$\tfrac{1}{2} \text{ tr log } (i\partial\!\!\!/ + g\tilde{\varphi}) = \frac{1}{2\pi i} \int d\omega \, \text{tr}_x \log \left(\gamma^0 \omega + i\gamma^1 \frac{d}{dx} + g\tilde{\varphi} \right) \times \frac{\delta(0)}{2\pi} \tag{4.16}$$

where the symbol tr_x log stands for the log of the Fredholm determinant of the one-dimensional differential operator $\left(\gamma^0 \omega + i\gamma^1 \frac{d}{dx} + g\tilde{\varphi}\right)$. The factor $\delta(0)/2\pi$ corresponds to the integration over time in the definition of the action S. It will be dropped from now on.

An integration by parts in (Eq. 16) yields

$$\tfrac{1}{2}\,\mathrm{tr}\,\log\,(i\partial\!\!\!/ + g\,\tilde{\varphi}) = \frac{1}{2\pi i}\int_C d\omega\;\omega\;\mathrm{tr}_x\left(\frac{\gamma^0}{-\gamma^0\omega + i\gamma^1\frac{d}{dx} + g\tilde{\varphi}}\right) \qquad (4.17)$$

where the contour C of the ω integration remains to be specified. The integrand has poles at the bound states of the one-dimensional operator $i\gamma^0\gamma^1\frac{d}{dx} + g\gamma^0\tilde{\varphi}$. The lie on the real axis and (by charge conjugation) are symmetric around $\omega = 0$. If we assume that $|\tilde{\varphi}(x)|$ goes to a constant $|\tilde{\varphi}(\infty)|$ as $|x| \to \infty$, then a free fermion has mass $M = g|\tilde{\varphi}(\infty)|$ and the integrand in (4.17) has cuts running from $+M$ to $+\infty$, and $-M$ to $-\infty$. If we are interested in a $Q = 0$ sector of the theory then by charge conjugation, the contour C must cross the axis at $\omega = 0$ as shown in Fig. (1). To obtain the contour for a $Q \neq 0$ sector, one proceeds as follows. Add to the Lagrangian a Lagrange multiplier $\mu\bar{\psi}\gamma^0\psi$, compute as before and then adjust μ to obtain the desired value of Q. Clearly, the effect of the Lagrange multiplier is simply to replace ω by $\omega + \mu$ in the integrand of Eq. (4.16). This simply shifts the contour to the right or left depending on the sign of μ. If we shift to the right passing N poles of the integrand in the process, we will be in a sector with $Q = N$. Conversely, shifting to the left and picking up N poles at negative ω will put us in a sector with $Q = -N$. The contour for the $Q = 2$ sector is shown in Fig. (2a). Fig. (2b) shows this contour deformed back into the $Q = 0$ contour C_0 plus a loop around the two lowest positive energy bound states of the Dirac equation. Let us call the poles in the loop occupied states. For the $Q = N$ sector, the N lowest positive energy

states will be occupied and for the Q = -N sector, the N lowest negative energy states will be occupied. Having now determined the contour C for the general case, the Lagrange multiplier no ionger plays any role and will be ignored.

Let us now look for time independent (static) solution to the variational equation for $\tilde{\varphi}$. Varying the effective action (4.14) with respect to the time independent, but spacially varying field $\tilde{\varphi}(x)$, yields

$$\tilde{\varphi}''(x) + m^2\tilde{\varphi}(x) - \lambda\tilde{\varphi}^3(x) + \frac{g}{2\pi i}\int_C d\omega \,\langle x|\, \frac{1}{-\gamma^0\omega + i\gamma^1\frac{d}{dx} + g\tilde{\varphi}}\,|x\rangle = 0 \qquad (4.18)$$

where the integrand is the diagonal x-space matrix element of the indicated inverse differential operator. This matrix element depends functionally on φ. Deforming the contour to the standard Q = 0 contour C_0 and picking up any occupied states, we have

$$\tilde{\varphi}''(x) + m^2\tilde{\varphi}(x) - \lambda\tilde{\varphi}^3(x) \pm g\sum_{k\in \text{occ. states}} U_k(\tilde{\varphi},x)\,U_k(\tilde{\varphi},x) + \text{"Fermion Loop"} \qquad (4.19)$$

where

$$\text{"Fermion Loop"} = \frac{1}{2\pi i}\int_{C_0} d\omega \,\langle x|\, \frac{1}{-\gamma^0\omega + i\gamma^1\frac{d}{dx} + g\tilde{\varphi}}\,|x\rangle \qquad (4.20)$$

and we have indicated that the U_k depend functionally on φ; that is they are solutions to the equation $-\omega_k\gamma^0 U_k(\tilde{\varphi},x) + i\gamma U'_k(\tilde{\varphi},x) + g\tilde{\varphi}(x)\,U_k(\tilde{\varphi},x) = 0$. The ± sign in Eq. (4.19) takes the value + for positive energy states and - for negative energy states. The product of wave functions $\bar{U}U$

also changes sign as so that this term is actually the same for $Q > 0$ and $Q < 0$ as it should be from charge conjugation considerations.

In Eq. (4.19) we have separated the fermion loop from the occupied states since the former is a true quantum mechanical correction of order $\hbar$ relative to the terms explicitly shown in Eq. (4.19). Concentrating on the explicit terms in Eq. (4.19) it is clear that they define a self consistent field, Hartree-like approximation.[6] One could imagine guessing a value of $\tilde{\varphi}$, computing the occupied states U_k, solving Eq. (4.19) to obtain a new field $\tilde{\varphi}$ and repeating the cycle until a self consistent solution is obtained. One could even contemplate doing this with the fermion loop included, but such a problem would be intractable unless some clever approximation could be devised.

While the general problem of solving an equation like (4.19) is very difficult, there is a special case which is extremely simple. Recall, that with our static kink solution for $\tilde{\varphi}$, the Dirac equation has one solution with $\omega = 0$. As mentioned above, this state carries no charge and has $\bar{U}_0(\text{kink},x)$ $U_0(\text{kink}, x) = 0$. Now, if we occupy only this state, there will be no reaction of the Fermi field back on the scalar field $\tilde{\varphi}$, and the kink will remain an exact self-consistent solution. (Here, we are ignoring the fermion loop.) The particle thus obtained by taking the basic scalar kink and occupying the $\omega = 0$ fermion state will be a new object in the theory which has $Q = 0$ but otherwise behaves like a fermion, i.e., it will have spin $\frac{1}{2}$ and obey Fermi statistics. To zeroth order in $\hbar$ the mass of this new fermion state will be

the same as the mass of the kink. This must be so, since it costs no energy to put in a fermion with $\omega = 0$. Of course if g is small, we can find further approximate solutions to Eq. (4.19). If we take small g in Eq. (4.19), then in first approximation the $U_k(\tilde{\varphi}, x)$ will simply be $U_k(\text{kink}, x)$. The solutions of the Dirac equation in the kink potential were discussed above. For small g, the mass of a $Q = N$ state composed of kink plus N trapped fermions will be

$$M(Q = N) = M(\text{kink}) + m \sum_{n=1}^{N} \sqrt{2nG - n^2} \qquad (4.21)$$
$$\text{(small g)}$$

where $G = g\sqrt{2/\lambda}$ as before and we must have $N < G$. We leave it to the reader to convince himself of this result.

It remains to tackle the fermion loop. We assume that we are talking about a particle with $Q = 0$, either the original kink or the kink with the $\omega = 0$ fermion state occupied. We wish to evaluate the lowest order contribution of the fermion loop to the mass of particle. From Eq. (4.17), this is simply

$$\Delta M = -\frac{1}{2\pi} \int_{C_0} d\omega\, \omega\, \text{tr}_x \left(\frac{\gamma^0}{-\gamma^0 \omega + i\gamma^1 \frac{\partial}{\partial x} + g\,\tilde{\varphi}(\text{kink})} \right) \qquad (4.22)$$

Deforming the contour C_0 to encircle the negative energy pole and cut as shown in Fig. (3), yields

$$\Delta M = \sum_{\substack{\omega_k \\ \text{neg. energy states}}} \qquad (4.23)$$

where we have imagined the cut to be a series of closely spaced poles. The interpretation of (4.23) is simple. It is the sum of the energies of all the negative energy states which, according to Dirac hole theory, should be occupied. Recall that for a boson loop we found that ΔM was given by $\sum_{\frac{1}{2}}^{\infty} \omega_k$ for positive ω_k which has the interpretation of the zero point energy of a set of oscillators. A fermion loop has the opposite sign (the ω_k in (4.23) are all negative) and a factor of two difference in magnitude. This change of sign and factor of two are familiar from perturbation theory. Here we see that this is a reflection of some rather different physics.

To make ΔM finite we first have to subtract the (infinite) energy of the states which would be filled in the vacuum, i.e., in the absence of a kink. Then,

$$\Delta M = \sum_{\substack{k \\ \text{neg. energy} \\ \text{states}}} \left[\omega_k(\text{kink}) - \omega_k(\text{vacuum})\right] \tag{4.24}$$

in an obvious notation. The expression is still logrithmically divergent but is made finite by a simple mass renormalization, as was the closed boson loop.

The actual computation of ΔM can be carried out along the lines of the Appendix. As the result is rather complicated and not particularly illuminating we will not give it here.

One of the authors (B. H.) would like to express his appreciation to Dr. Carl Kaysen for the hospitality extended to him by the Institute for Advanced Study.

APPENDIX

In this appendix, we compute the first quantum mechanical correction to the energy of the kink of Section 3. The classical energy is

$$E_{cl} = \frac{m^3}{\lambda} \int_{-\infty}^{+\infty} (\tfrac{1}{2}\varphi'^2 - \tfrac{1}{2}\varphi^2 + \frac{1}{4}\varphi^4 + \frac{1}{4})\, dx \qquad (A..1)$$

with $\varphi = \tanh x/\sqrt{2}$.

Hence $$E_{cl} = \frac{2}{3}\sqrt{2}\,\frac{m^3}{\lambda} \qquad (A..2)$$

For the quantum mechanical corrections, one must be very careful to appropriately count the modes, and subtract the vacuum energy for each mode. The regularization of the theory, both in the infrared and in the ultraviolet, is obtained most naturally by putting the system on a finite lattice in a box, with periodic boundary conditions. This makes the number of degrees of freedom finite. We compute the difference between the quantum corrections to the ground state energy of the ordinary vacuum and to the ground state energy of the kink. We shall neglect any quantity which goes to zero fast enough, when the length L of the box goes to infinity. One must follow what happens to the eigenmodes of that system when the kink is introduced into the box.

In the absence of the kink, the energy of the vacuum comes only from continuum states (travelling waves). When the kink is introduced, the first two continuum states disappear to become bound states with $\omega = 0$ and $\omega = \sqrt{\frac{3}{2}}$. The contribution of these two states to the energy of the kink

will then be $\frac{1}{2}(0 - \sqrt{2}) + \frac{1}{2}\left(\sqrt{\frac{3}{2}} - \sqrt{2}\right)$. The contribution from the other states, which remain in the continuum in the presence of the kink will be

$$E_{cont} = \sum_n \frac{1}{2\sqrt{2}} \left[\left(k_n^2 + 4\right)^{\frac{1}{2}} - \left(k'^2_n + 4\right)^{\frac{1}{2}}\right] \qquad (A.3)$$

where k_n is the wave number of the n^{th} mode in the continuum in the presence of the kink, and k'_n the wave number in the vacuum. They are related by the periodic boundary condition:

$$Lk_n + \delta = Lk'_n = 2n\pi$$

with δ given by eq. (3.8). In the limit $L \to \infty$, the discrete sum of (A.3) becomes the integral

$$E = \frac{1}{\sqrt{2}} \int_0^{\Lambda} \delta \frac{d\omega}{dk} dk$$

where Λ is the ultraviolet cutoff, given by the lattice spacing, Using eq. (3.8) and restoring the dimensional units, one finds

$$E_{cont} = m \left[\frac{4\pi - 6}{2\pi\sqrt{2}} - \frac{3}{\pi\sqrt{2}} \int_0^{\infty} \frac{m^2 dk}{(m^2 + 2k^2)\sqrt{k^2 + 2m^2}}\right.$$

$$\left. - \frac{3}{\pi\sqrt{2}} \int_0^{\Lambda} \frac{dk}{\sqrt{k^2 + 2m^2}}\right] \qquad (A..4)$$

This expression is now logarithmically divergent only. This remaining divergence is cancelled by the ordinary mass renormalization counterterms as follows:

By computing all the one loop graphs in the ordinary fashion for the Lagrangian (3.1), one finds that they become finite if the Lagrangian (3.1) is

replaced by

$$\tilde{\mathcal{L}}_1 = -\tfrac{1}{2}\,(\partial_\mu \tilde{\varphi})^2 + \tfrac{1}{2}\,(m^2 + \delta m^2)\,\tilde{\varphi}^2 - \frac{\lambda}{4}\,\tilde{\varphi}^4 \qquad (A.5)$$

where $$\delta m^2 = \frac{3\lambda}{2\pi}\int_0^{\Lambda}\frac{dk}{\sqrt{k^2+2m^2}}$$

and in which one performs the ordinary shift $\tilde{\varphi} \to \tilde{\varphi} + \frac{m}{\sqrt{\lambda}}$. The fact that this shift remains unchanged generates the tadpole counterterm $\tilde{\varphi}\ \frac{m}{\sqrt{\lambda}}\,\delta m^2$ which is precisely needed to cancel the one loop tadpole. For the kink, the shift is $\tilde{\varphi} \to \tilde{\varphi} + \frac{m}{\sqrt{\lambda}}\ \tanh\ \frac{x\,m}{\sqrt{2}}$.

Hence the mass counterterm contributes to the energy of the kink by the amount

$$-\tfrac{1}{2}\,\delta m^2\ \frac{m}{\lambda}\int_{-\infty}^{+\infty} dx\ \left(\tanh^2\frac{x}{\sqrt{2}} - 1\right),$$

which exactly cancels the divergent term of eq. (A..4). Collecting all the finite terms, one then arrives at eq. (3.10).

REFERENCES

1) See, for example, T. Skyrme Nuclear Physics 31 556 (1962) or D. Finkelstein and J. Rubenstein J. Math. Phys. 9, 1762 (1968) and references cited therein.

2) R. Dashen, B. Hasslacher, and A. Neveu IAS Preprint.

3) Report at London Conference by T. M. Yan.

4) Morse and Feshbach, "Methods of Theoretical Physics", McGaw-Hill (New York), 1953.

5) Although Q defined by (4.6) looks positive definite, one must remember that ψ is really an anti-commuting object so that a real field can carry no charge.

6) Similar equations have been considered by J. Cornwall, R. Jackiw and E. Tomboulis, MIT Center for Theoretical Physics Preprint No. 419.

7) This state is a self charge-conjugate object like K_1^0 . It has a partner (K_2^0) built on the kink with the negative sign in (3.6).

FIGURE CAPTIONS

Fig. 1. The singularities of the integrand of Eq. (4.17) and the integration contour C_0 for a charge zero sector of the theory.

Fig. 2. The contour for the charge two sector (a) and its shift to C_0 with two states occupied (b).

Fig. 3. Contour for evaluating the fermion loop.

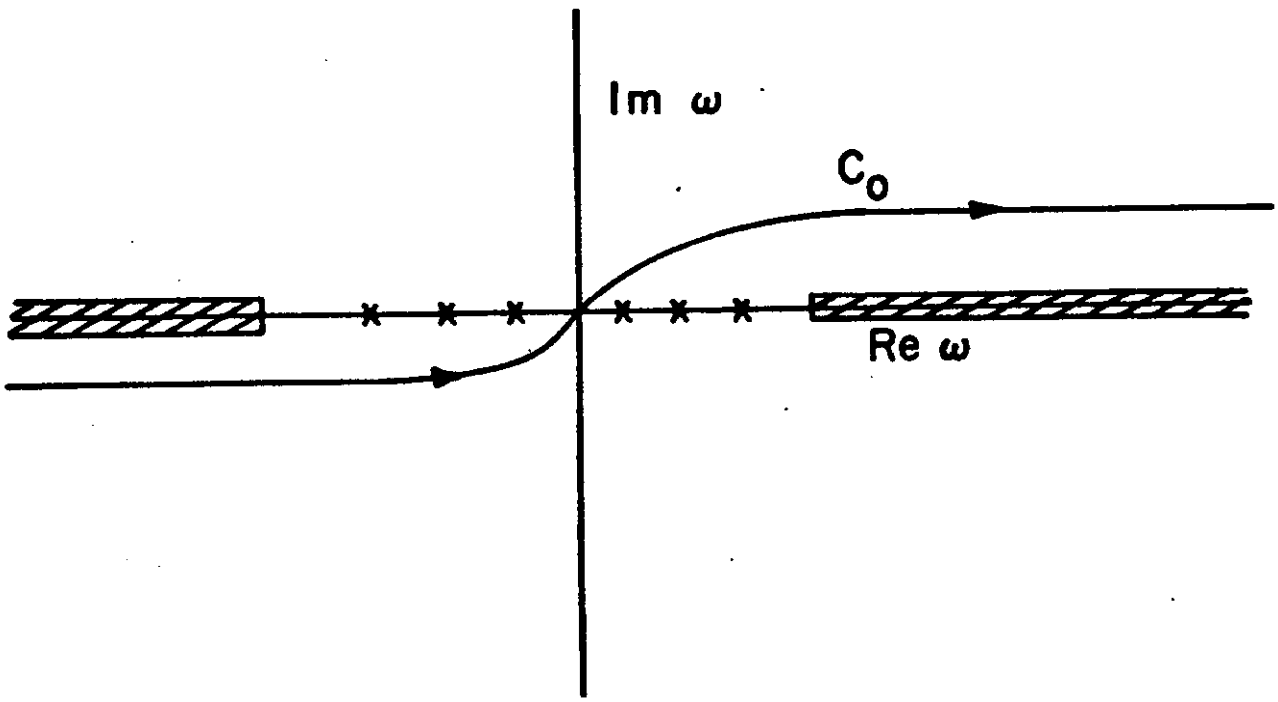

FIGURE 1.

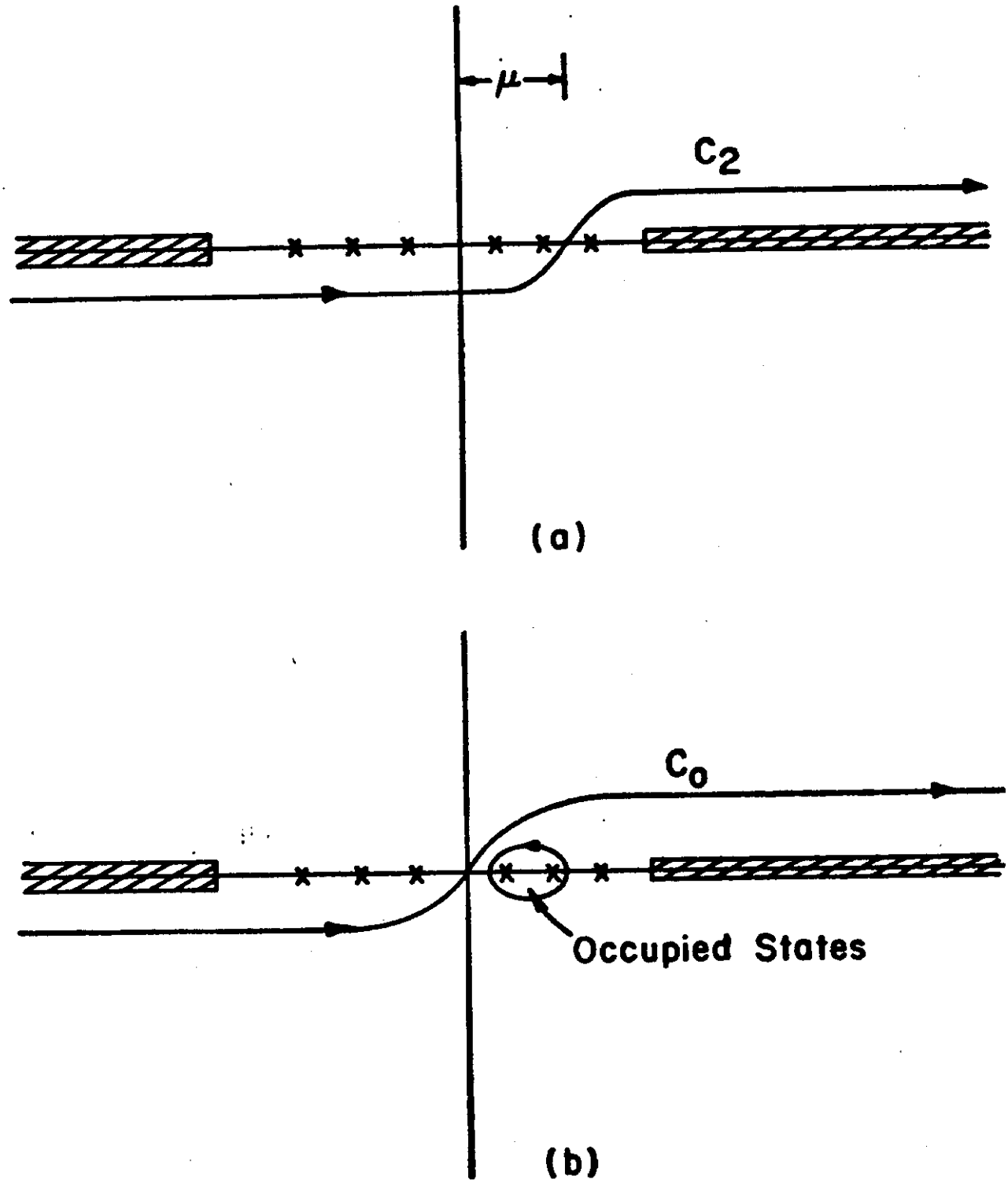

FIGURE 2.

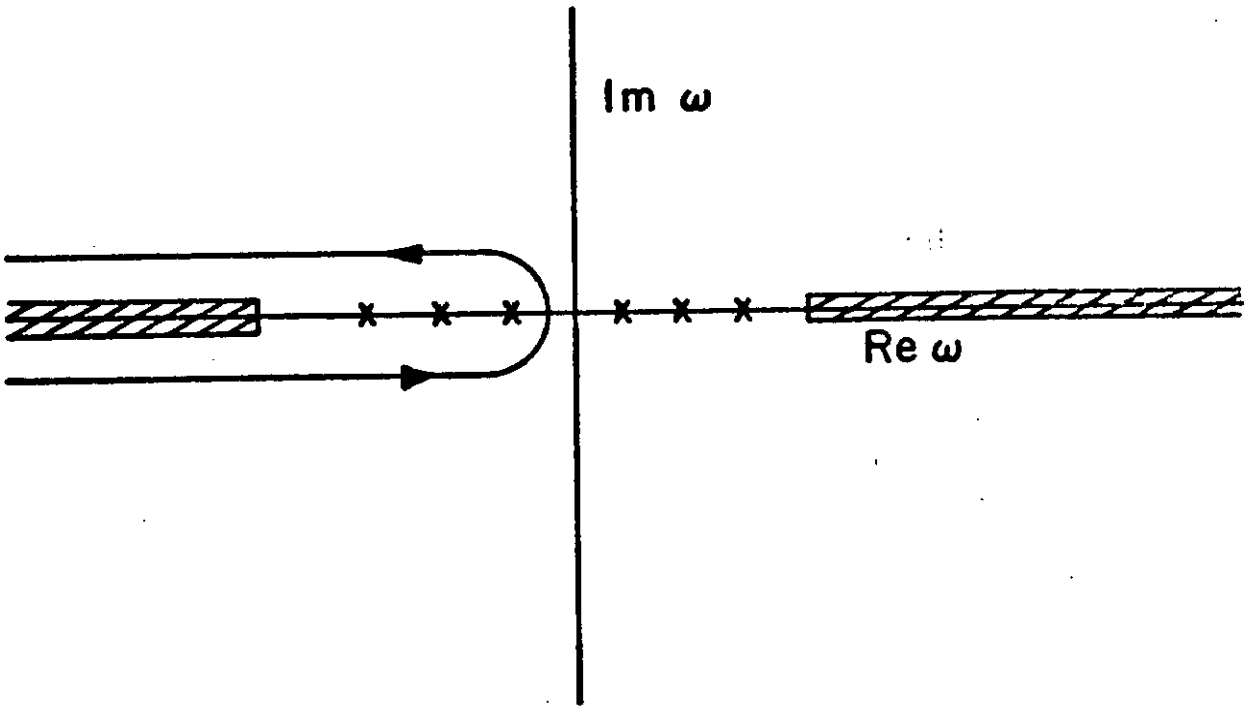

FIGURE 3.