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Non-Perturbative Methods and Extended Hadron Models in Field Theory*

I. Semi-Classical Functional Methods

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ABSTRACT

This is the first of a series of papers on the use of semi-classical approximations to find particle states in field theory. The meaning of the WKB approximation is examined from a functional integral approach. Special emphasis is placed on the distinction between a true WKB or semi classical approach and the weak coupling approximation to it. Other topics include the center of mass motion of particle states and some problems special to field theory such as multiple particle states, statistics and infinite volume systems. Ultraviolet divergences are touched on but dealt with more thoroughly in the following paper where specific models are examined. The central result of this series is that certain kinds of non-linear field theories have extended particle solutions which survive quantization. The most interesting of these objects, which are reminiscent of hadrons, come from theories with spontaneous symmetry breaking.

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I. INTRODUCTION

This is the first of a series of papers that will present methods for finding solutions to field theories which are inaccessible to perturbation techniques. We approach this problem through the quantum action principle in the Feynman path integral representation, since this provides the most natural connection between the classical problem and its second quantized analog.

In particular, it is possible to find solutions to the full non-linear interacting classical equations of motion of various models, which behave like bound, stable field configurations in space-time, with particle properties. The question arises as to whether these solutions survive the process of second quantization. In this paper we give a method for answering that question, the accuracy of which depends both on how much one knows about the classical problem, and the strength of the coupling constant, in direct proportion.

Our methods are based on the works of Gutzwiller[1] and Maslov,[2] who developed a general semi-classical formalism for use in atomic physics. These techniques are directed toward the computation of energy levels, or particle masses in field theory. They work in such a way that one never has to construct any wave functions. In field theory this is a great advantage, since a field theoretic wave function, or more properly, a state functional, is an exceedingly complicated object. Also, since we start from a Lagrangian formalism, any divergences that emerge can be handled by more or less standard renormalization techniques.

This paper and the two following it are organized as follows: In Section 2 and 3, we develop a functional formulation of the W. K. B. approximation via path integrals, and cast it in a form suitable for generalization to field theory. The strategy is basically that of Gutzwiller[1] and Maslov,[2] whose ideas we follow closely, making only small technical improvements. We take particular care to distinguish between general semi-classical methods and their weak coupling approximation. The general methods are very complex, but work for strong coupling as well as weak.

In Section 4 and 5 we clear up two conceptual problems which have long plagued the idea of identifying particle-like solutions to classical field theories with quantum particles. The first is how to take the center of mass motion into account and in the process dispose of the annoyance that classical particle-like solutions always appear to be fixed arbitrarily at some point in space. The other problem is how to handle multiparticle states and some related formal difficulties which arise in spacially infinite systems.

Section 6 connects weak coupling semi-classical methods to the more familiar loop expansion of what is essentially the vacuum generating functional or effective action. This alternative but equivalent point of view is especially useful when renormalization is necessary.

The second paper is devoted to the study of some explicit two dimensional models. We discovered a model which has some very interesting properties, yet is simple enough so that all calculations can be done analytically. The model has a solution which looks like a two dimensional,

stable, extended hadron, which has a spectrum of excited states and serves as a well which traps and contains fermions.

To follow this work, it is not necessary to have grasped the present paper in any detail. Some readers may prefer to proceed directly to the models. Also included in the second paper are methods for handling fermions. In particular, we develop a set of self-consistent field equations for coupled scalar and fermi fields which can be trivially generalized to four dimensions.

Finally, in the third paper, we treat models in four dimensions, which require some discussion of analogs in field theory to Type II superconducting metals, and the associated vortex line solutions to the Landau-Ginsberg equations. In particular we display a remarkable classical solution to the system of a non-abelian gauge field coupled to a scalar field with a broken Higgs vacuum. This is extended to include fermion fields. The solution is essentially a closed ball outside of which quanta get a mass through a Higgs mechanism. Particles are effectively trapped since in the ball they have zero mass and get heavy if they try to leave it. This model can be looked at as a non-abelian superconductor for a color quantum number, which in 4-space solves the endpoint problem of the abelian vortex case. The analog to a metal is the Higgs vacuum itself.

2A. BOUND STATES IN SYSTEMS WITH ONE DEGREE OF FREEDOM[1]

We compute the bound state energies of a one-dimensional potential well using path integral methods. To do so, consider the trace of the propagator

$$G(E) = \mathrm{tr}\,\frac{1}{H-E} = \sum_n \frac{1}{E_n - E} \tag{1.1}$$

where E_n is the energy of the n^{th} bound state. We write

$$G(E) = i\,\mathrm{tr}\int_0^\infty \frac{dT}{\hbar}\,\exp\left[\,i\,\frac{(E-H)T}{\hbar}\,\right]. \tag{1.2}$$

Now e^{-iHT} is the propagation kernel, which can be expressed directly as a Feynman path integral, over periodic paths,[3]

$$\mathrm{tr}\,e^{-iHT} = \int \mathcal{D}x(\tau)\,e^{i\,S/\hbar} \tag{1.3}$$

where S stands for the classical action computed along the path $x(\tau)$:

$$S = \int_0^T \left[\tfrac{1}{2}\dot{x}^2 - V(x)\right] d\tau\,. \tag{1.4}$$

The paths along which one integrates in (1.3) are all periodic paths with fixed period T. The periodicity condition is the translation into path space of the trace operation of Eq. (1.2).

We now evaluate Eq. (1.3) to leading order in $\hbar$. This is done by a stationary phase method: the dominant contribution to the functional integral comes from those paths which are close to the classical periodic orbits $x_{cl}(\tau)$, with period T. Expanding the action around these classical orbits, the integral becomes:

$$\mathrm{tr}\,e^{-iHT} \approx e^{iS_{cl}(T)} \int \mathcal{D}x(\tau)\,dx_{cl}(0)\,e^{i\tilde{S}/\hbar} \tag{1.5}$$

where

$$\tilde{S} = \int_0^T \{ \tfrac{1}{2}\dot{x}^2 - \tfrac{1}{2}x^2 V''(x_{cl}(\tau)) \}\, d\tau \ . \tag{1.6}$$

The functional integral is taken over the paths $x(\tau)$ which satisfy $x(0) = x(\tau) = 0$. The extra measure $dx_{cl}(0)$, the starting point factor, which is done along the classical orbit, represents the freedom we have to begin the expansion anywhere along the classical orbit.

The functional integral (1.5) is evaluated by well-known shifting methods.[4] We introduce the mapping

$$y(\tau) = x(\tau) - \int_0^\tau \frac{\dot{N}(\mu)}{N(\mu)}\, x(\mu)\, d\mu \tag{1.7}$$

and its inverse

$$x(\tau) = y(\tau) + N(\tau) \int_0^\tau \frac{\dot{N}(\mu)}{N^2(\mu)}\, y(\mu)\, d\mu \tag{1.8}$$

where N is defined by the equation

$$\ddot{N} = -V'' N \ . \tag{1.9}$$

Then the expression for $\mathrm{tr}\, e^{-iHT}$ becomes

$$\mathrm{tr}\, e^{-iHT} \approx e^{iS_{cl}(T)/\hbar} \int \mathcal{D}y(\tau)\, d\alpha\, dx_{cl}(0) \left| \frac{\mathcal{D}x}{\mathcal{D}y} \right| \times \exp\left(\frac{i}{\hbar}\right) \left[\tfrac{1}{2} \int_0^T \dot{y}^2\, d\tau + \alpha \left(y(T) + N(T) \int_0^T \frac{\dot{N}(\tau)}{N^2(\tau)}\, y(\tau)\, d\tau \right) \right] . \tag{1.10}$$

In Eq. (1.10), α is a Lagrange multiplier which inserts the constraint on $y(\tau)$ induced by the end point conditions on $x(\tau)$. The range of integration of $y(\tau)$ is then unrestricted. The end point $y(0)$ is still fixed at zero. The functional Jacobian $\left| \frac{\mathcal{D}x}{\mathcal{D}y} \right|$ is just $[N(T)/N(0)]^{\frac{1}{2}}$ since (1.7) is a Volterra integral equation.[4] The integrations over y and α are direct and the result is:

$$\mathrm{tr}\, e^{-iHT} \approx \sqrt{\frac{i}{2\pi\hbar}}\, e^{iS_{cl}(T)} \left| \frac{N(T)}{N(0)} \right|^{\frac{1}{2}} \left[\int_0^T \frac{N^2(T)}{N^2(\tau)}\, d\tau \right]^{-\frac{1}{2}} \int dx_{cl}(0). \tag{1.11}$$

We prove in Appendix A that this reduces to

$$\operatorname{tr} e^{-iHT} \approx \sqrt{\frac{i}{2\pi\hbar}}\, T \left| \frac{d E_{c\ell}}{dT} \right|^{\frac{1}{2}} e^{i[S_{c\ell}(T)/\hbar - \pi]} \tag{1.12}$$

where $E_{c\ell} = d S_{c\ell}/dT$ is the energy of the classical trajectory.

To complete the sum over paths, we insert (1.12) into (1.2). However, we must take into account that each classical orbit can be traversed n times, so that one has

$$G(E) = \sqrt{\frac{-i}{2\pi\hbar}} \sum_{n=1}^{\infty} \int_0^{\infty} \frac{dT}{\hbar}\, T \left| n \frac{d E_{c\ell}}{dT} \right|^{\frac{1}{2}} \exp i n \left[\frac{S_{c\ell}(T)+ET}{\hbar} - \pi \right]. \tag{1.13}$$

The leading term in $\hbar^{-1}$ is again given by a stationary phase approximation: the stationary phase point is fixed by

$$\frac{d S_{c\ell}}{dT} = -E_{c\ell} = -E \tag{1.14}$$

which determines T as a function of $T(E)$ of the energy E. Defining,

$$W(E) = S_{c\ell}(T(E)) + ET(E) \tag{1.15}$$

the stationary phase approximation to (1.13) is

$$\begin{aligned} G(E) &\approx \frac{i\, T(E)}{\hbar} \sum_{n=1}^{\infty} \exp[\, i\, n(W(E)/\hbar - \pi)\,] \\ &= \frac{-i\, T(E)}{\hbar}\, \frac{e^{i\, W(E)/\hbar}}{1 + e^{iW(E)/\hbar}} \,. \end{aligned} \tag{1.16}$$

Now if E_m is a root

$$W(E_m) = (2m+1)\pi\hbar \tag{1.17}$$

then near $E = E_m$, G goes like

$$G(E) \sim (E_m - E)^{-1} \tag{1.18}$$

where we have used the relation $dW(E)/dE = T(E)$. Finally, noting that

$$W(E) = 2 \int_{x_1}^{x_2} \sqrt{2(E-V)}\, dx \tag{1.19}$$

where x_1 and x_2 are the turning points, one sees that (1.17) is the usual WKB condition.

Summarizing, there are three ingredients to this derivation of the WKB energies. First the approximate evaluation of $\mathrm{tr}\, e^{-iHT}$ to give (1.12). Secondly, the stationary phase integral over T which converts (1.12) into a factor times e^{iW} and finally the sum over multiple traverses of the basic orbit which produces a geometric series in e^{iW} and poles in G. In systems of many degrees of freedom the same steps will appear. The problem is to find the orbits.

Having shown how $\hbar$ enters the calculation we will henceforth set $\hbar = 1$, except occasionally when we wish to emphasize a point.

2B. AN EXAMPLE

It is instructive to see how the general method works in a particular example. In particular, the following example will illustrate the difference between weak and strong coupling.

Consider the anharmonic oscillator whose Lagrangian is

$$\mathcal{L} = \frac{\dot{x}^2}{2} - \frac{x^2}{2} - \lambda \frac{x^4}{4} \tag{1.1}$$

where λ is an adjustable coupling constant. The solutions to the classical equations of motion are all periodic, with the period T and energy E being related by

$$T = 2 \int_{-x_0}^{+x_0} \frac{dx}{\sqrt{2\left(E - \frac{x^2}{2} - \frac{\lambda}{4} x^4\right)}} \tag{1.1}$$

where $\pm x_0$ are the turning points; i.e. the places where the expression under the square root sign vanishes. The change of variables $x = \sqrt{E}y$ gives

$$T = 2 \int_{-y_0}^{+y_0} \frac{dv}{\sqrt{2\left(1 - \frac{y^2}{2} - \frac{E\lambda}{4} y^4\right)}} \tag{1.19}$$

from which one sees that T depends only on the product $E\lambda$. For small $E\lambda$, the weak coupling limit, T approaches the harmonic oscillator period $T = 2\pi$. As $E\lambda$ increases T decreases steadily approaching zero like $(E\lambda)^{-1/4}$ in the extreme strong coupling limit. The qualitative behavior of T as a function of $E\lambda$ is shown by the bottom curve in Fig. (1). The higher curves in Fig. (1) show the periods which are integral multiples of this basic period, corresponding to orbits in which the oscillator runs around the basic loop 2, 3, 4 etc. times.

To apply the semi-classical method to this example, we first compute $\mathrm{tr}\, e^{-iHT}$. To do so we have to find all periodic orbits with period T. As can easily be seen from Fig. (1), there is a discrete infinity of such orbits. For example, if T is less than 2π, there is the motion whose basic period is T and then the multiple traverses of motions whose basic period is an integral fraction of T. Each of these infinitely many orbits is a stationary phase point in the path-integral for $\mathrm{tr}\, e^{-iHT}$. In the semi-classical approximation we get a contribution from each such stationary phase point and we have

$$\mathrm{tr}\, e^{-iHT} \sim \sum_n C_n e^{iS_n} \tag{1.20}$$

where n runs over the orbits indicated in Fig. (1), S_n is the action for the n^{th} orbit and C_n is a constant evaluated in Sec. 2A.

The next step to make a Laplace transform on $\mathrm{tr}\, e^{-iHT}$ to get $G(E)$. Doing the transform by stationary phase we get contributions from each orbit of energy E. This infinite sum produces poles in $G(E)$ at

$$W(E) = 2E \int_{-y_0}^{+y_0} \sqrt{2\left(1 - \frac{y^2}{2} - \frac{\lambda E y^4}{4}\right)} = (2n+1)\,\pi \ . \qquad (1.21)$$

Let us now consider weak couplings starting with a precise definition. If for a given value of $n = n_0$, Eq. (1.21) can be satisfied for a value of E such that $\lambda E y_0^4/4$ is small compared to unity, then we are in a weak coupling regime. In this case, the energy levels for $n < n_0$ are clearly given by the harmonic oscillator levels $E_n = (n + \frac{1}{2})$. This is no surprise, since for weak coupling one is operating on the left hand edge of Fig. (1), where the periods are almost equally spaced horizontal lines, i.e. where the system is acting like a harmonic oscillator.

Since the WKB method is exact for a harmonic oscillator, it will therefore give good results for weak coupling. We wish to emphasize, however, that the semi-classical approximation is not basically a weak-coupling scheme. Indeed, if the potential is not too wiggly, WKB gives qualitatively correct answers even for very strong coupling. To use WKB for strong coupling, however one does have to get a handle on the strong coupling classical problem. In terms of Fig. (1), this means that we would really have to compute the curves for T as a function of E rather than approximating them by straight lines as can be done for weak coupling.

The difficulty in handling the classical theory will be the stumbling block in the application of semi-classical methods to strong coupling field theory.

2C. WEAK COUPLING

In the case of zero coupling one does not need all the formalism because the functional integral for the harmonic oscillator can be done exactly, yielding

$$\text{tr } e^{-iHT} = (2 - 2\cos T)^{-\frac{1}{2}} . \tag{1.22}$$

For weak coupling we can get the same result by expanding the path integral around the trivial orbit $x \equiv 0$ and keeping only quadratic terms in the Lagrangian which of course, reduces the problem to a harmonic oscillator.

So far we have been ignoring this trivial orbit. We wish now to see how this orbit and the resulting harmonic oscillator approximation fits into our general scheme. First it should be understood that $x \equiv 0$ is an exact periodic solution to the classical anharmonic oscillator for any period T. It is therefore a stationary phase point in the functional integral for $\text{tr } e^{-iHT}$. Why then have we been ignoring it? To answer this we will have to distinguish between the cases of weak and strong coupling.

A stationary phase point in the functional integral corresponding to a non-trivial orbit makes a contribution proportional to (see Eq. (1.12)).

$$T\left(\frac{dE}{dT}\frac{1}{\hbar}\right)^{\frac{1}{2}} e^{iS(T)/\hbar} \tag{1.23}$$

where we have inserted factors of $\hbar$ to see the classical limit. One sees immediately that (1.23) is a factor of $(\hbar)^{-\frac{1}{2}}$ bigger than the corresponding

term (1.22) from the trivial orbit, except for the special case where T is an integer times 2π and (1.22) blows up. Referring now to Fig. (1) we see that for strong coupling one is not interested in orbits with these particular periods. Therefore, for strong coupling the contribution of the trivial orbit is negligible compared to that of the non-trivial ones.

For weak coupling something else happens. Referring again to Fig. (1) we see that in this case we are interested in only those orbits whose periods are close to an integral multiple of 2π. These orbits are, however, very small excursions away from the trivial one at $x = 0$. Consequently the stationary phase points corresponding to the trivial and non-trivial orbits are no longer well separated and do not make additive contributions to the functional integral. That is to say, for weak coupling the Gaussian integral around the trivial orbit includes the contribution of the non-trivial orbit and visa-versa. Therefore, for weak coupling one calculates around one orbit or the other but not both.

To summarize:

1) It is consistent to ignore the trivial (time independent) orbit for either weak or strong coupling.

2) For weak coupling one has the alternative option of expanding the path integral around the trivial orbit and treating the system as a harmonic oscillator.

Finally, we note that if there are two time independent solutions to the classical equations of motion, $x(t) = x_1$ and $x(t) = x_2$, say, they will generally represent separate stationary phase points in the path integral

and their contributions (if relevant at all) must be added. This is illustrated by the potential with two minima shown in Fig. (2). If near the two minima the potential is sufficiently well approximated by parabolas then weak coupling applies and we can evaluate the path integral by expanding around the two trivial orbits $x = x_1$ and $x = x_2$. Their contributions will add in G(E) yielding poles at $E_n = V(x_1) + (n + \frac{1}{2})(V''(x_1))^{\frac{1}{2}}$ and at $E_n = V(x_2) + (n+\frac{1}{2})(V''(x_2))^{\frac{1}{2}}$ for $n = 1, 2, \ldots$. We argue in Appendix B that under the stated conditions of weak coupling this will be a good approximation to the low lying energy levels. We will see that this phenomenon of multiple, but simple, stationary points occurs in field theory.

2D. SEPARABLE SYSTEMS

In the next section we will discuss the semi-classical method for systems with more than one degree of freedom. This is in general a difficult problem simply because the classical mechanics is hard. Separable systems are simple, however. At this point it will be instructive to see how the method works in a separable case.

Consider the system defined by the Lagrangian

$$L = \frac{\dot{x}^2}{2} - \omega_1^2 \frac{x^2}{2} - \lambda_1 \frac{x^4}{4} + \frac{\dot{y}^2}{2} - \omega_2^2 \frac{y^2}{2} - \lambda_2 \frac{y^4}{4} \tag{1.24}$$

where the harmonic oscillator frequencies ω_1 and ω_2 as well as the couplings λ_1 and λ_2 are assumed to be different to avoid degeneracies. In general the motion of this system is multiply periodic with the x-variable having a basic period

$$T_1 = 2\int_{-x_0}^{x_0} \frac{dx}{\sqrt{2(\epsilon_1 - \omega_1^2 \frac{x^2}{2} - \lambda_1 \frac{x^4}{4})}} \tag{1.25}$$

and y having the basic period

$$T_2 = 2\int_{-y_0}^{+y_0} \frac{dy}{\sqrt{2(\epsilon_2 - \omega_2^2 \frac{y^2}{2} - \lambda_2 \frac{y^4}{4})}}$$

where ϵ_1 and ϵ_2 are parameters whose sum is the total energy E. The truly periodic motions occur for the special values of T_1 and T_2 such that

$$MT_1 = NT_2 \tag{1.26}$$

where M and N are integers. There will in general be one such orbit for each value of the energy E. Note that in the case of one degree of freedom the periodic orbits (for given E) could be labelled by one integer (see Fig. (1)); here we need two integers.

To compute $\mathrm{tr}\, e^{-iHT}$ we have to pick out all truly periodic orbits of period T. The computation is then a straightforward extension of what we have already done and one finds that $\mathrm{tr}\, e^{-iHT}$ can be represented as

$$\mathrm{tr}\, e^{-iHT} \sim \sum_{M,N} \int_0^{2\pi/\omega_1} dT_1 \int_0^{2\pi/\omega_2} dT_2 \left[NT_1 \left|\frac{d\epsilon_1}{dT_1}\right|^{\frac{1}{2}} e^{(iS_1(T_1)N)}\right] \times \left[MT_2 \left|\frac{d\epsilon_2}{dT_2}\right|^{\frac{1}{2}} e^{(iS_2(T_2)M)}\right] \times \delta\left(\frac{NT_1 + MT_2}{2} - T\right) \delta(NT_1 - MT_2) \tag{1.27}$$

where $S_1(T_1)$ and $S_2(T_2)$ are the actions for the individual x and y variables. To compute G(E) we take the Laplace transform and make the substitution

$$\delta(NT_1 - MT_2) = \frac{1}{2\pi}\int d\alpha\, e^{i\alpha(NT_1 - MT_2)} \tag{1.28}$$

which gives after some algebra and collecting constants omitted in (1.27)

$$G(E) = \frac{i}{2\pi}\int d\alpha\, G_1(\frac{E}{2}+\alpha)\, G_2(\frac{E}{2}-\alpha) \tag{1.29}$$

$$G_i(z) = -\frac{iT_i\, e^{iW_i(z)}}{1+e^{iW_i(z)}}$$

$$W_i(z) = 2\int_{-x_0}^{x_0}[\,2(z-\omega_i^2\frac{x^2}{2}-\lambda_i\frac{x^4}{4})]\,dx.$$

The separate G_i have poles at $z = \epsilon_{n,i}$ where $W_i(\epsilon_{n,i}) = (n+\frac{1}{2})\pi$. It is then straightforward to verify that G has poles at $E = \epsilon_{n,1}+\epsilon_{m,2}$ $n, m = 1, 2\ldots$ as it should.

We did not learn anything very surprising in this calculation. The point of it was the following. In the one dimensional case all classical orbits are periodic and it is not surprising that one can find the energy levels by concentrating on the periodic orbits. The separable two dimensional system is generally multiply periodic and classical periodic orbits are the exception rather than the rule. Nevertheless, we can get the energy levels by concentrating on the periodic orbits.

Our previous discussion of weak coupling and trivial (time independent) orbits carries over to the present case in an obvious way. We wish to add only one remark.

To get the correct energy levels one has to be sure to find <u>all</u> relevant periodic orbits. Suppose we had noticed only the special orbits $y = 0$ with x oscillating with period an integer times T_1. One can convince himself that summing only over these orbits would have yielded energy levels $E = \epsilon_{n,1}+\frac{1}{2}\omega_2$ where as before $\epsilon_{n,1}$ is the WKB energy for

the x-system. These levels are, in fact, correct if the coupling λ_2 is small. However, if λ_2 is large the contribution of these orbits is down by a factor of $(\hbar)^{\frac{1}{2}}$ (as discussed above) and are not the leading terms; hence they give the wrong energies.

3. MANY DEGREES OF FREEDOM

A. Preliminary Remarks

As has been mentioned before, the difficulty in applying a WKB approach to a general system is one's lack of ability to make any progress with the classical problem. There are only two types of multi-dimensional systems which are analytically tractable. One is a weakly coupled (harmonic oscillator) system and the other is a separable one. Both of these cases have already been discussed. Weakly coupled systems will come up again in Sec. 6. Fortunately there are model field theories which display new and interesting phenomena in a weak coupling regime.

There is a conceptual tool that is useful in thinking about the information that a properly treated WKB requires. The generalization of the one mode case to systems with N modes is not straightforward because of the topological complexity of the allowed types of classical motions. In a one dimensional example, the total energy is a time invariant of the system, which restricts the kinds of paths the system evolves along; i.e., they must be at least consistent with energy conservation. The energy eventually gets quantized with the WKB method.

For a separable system with N degrees of freedom, the N invariants fix an N dimensional manifold in a 2N dimensional phase space. This manifold is called an invariant torus in the literature and is topologically a complex object.[2] Quantization conditions emerge because the torus is multiply connected. Each time one encircles one of the holes in the torus, the phase of the wave function is constrained to change by a multiple of 2π. In general one should expect as many quantum numbers as there are degrees of freedom. Weak coupling systems are equivalent to sets of coupled harmonic oscillators. They are separable by transforming to normal co-ordinates.

In the non-separable case, it has been shown that invariant tori also exist, and one can in principle give a constructive algorithm for approximating it.[2,5] Unfortunately, knowledge of the invariant torus is equivalent to solving for all the classical motions of the system by quadratures, which for most systems is a hopeless task in practice, and for field theory hopeless in principle. There is however, one case where such complete information is available and because of its importance, it bears mentioning. There are classes of non-linear wave equations that support solitary wave solutions, that have the property of emerging unchanged in shape and velocity from a complete non-linear scattering of two of them. These solutions are called solitons in the literature[6] and if a wave equation admits them then there is available an infinite number of conserved integrals of the motion. In such a case the invariant torus is completely known and a complete semi-classical calculation

becomes possible. It remains to be seen whether interesting quantum field theories support soliton modes.

In what follows we continue to follow a Lagrangian-Functional Integral approach. These methods, which form a natural bridge to field theory were pioneered by Gutzwiller. As in the previous section, the method is based on summing over periodic orbits of the classical system. For one degree of freedom there is generally a single periodic orbit for a given energy, where one does not count multiple traverses of a single orbit as a new orbit. In a system with many degrees of freedom there will generally be a discrete infinity of periodic orbits for each given energy E. With the methods we are using, one has to know all of these orbits to properly generalize WKB. For separable systems the orbits are known and the summation can be carried out by the methods of Sec. 2. In the general case one will have to resort to approximations.

B. The Functional Integral

For a system with N co-ordinates x_i i = 1, 2...N, we define the Feynman propagation kernel by the functional integral

$$K(\underset{\sim}{x}'', \underset{\sim}{x}'; T) = \int \mathcal{D}\underset{\sim}{x}(\tau)\, e^{iS}$$

$$S = \int_0^T \left[\frac{\dot{\underset{\sim}{x}}^2(\tau)}{2} - V(\underset{\sim}{x}(\tau))\right] d\tau \qquad (2.1)$$

where the integration is over all paths satisfying

$$\underset{\sim}{x}(0) = \underset{\sim}{x}' \qquad \underset{\sim}{x}(T) = \underset{\sim}{x}'' \ . \qquad (2.2)$$

The semi-classical approximation to K is known to be

Note: we use $\underset{\sim}{x}$ to denote the space piece of a vector.

$$K(\underset{\sim}{x}', \underset{\sim}{x}''; T) = C_N \left| \frac{\partial^2 S_{c\ell}(\underset{\sim}{x}', \underset{\sim}{x}'':T)}{\partial \underset{\sim}{x}' \, \partial \underset{\sim}{x}''} \right|^{\frac{1}{2}} e^{iS_{c\ell}(\underset{\sim}{x}', x'';T)}$$

$$C_N = (\sqrt{-2\pi i})^N . \tag{2.3}$$

This result, which was obtained by Pauli[7] from an approximate integration of the Schrödinger equation and by Morette[8] from a completeness argument, can be gotten from a stationary phase approximation to the functional integral. We sketch the essential ingredients of the computation which is algebraically complex.

After shifting by the classical solution we get

$$K = \exp[\, i\, S_{c\ell}(t) \,] \int \mathcal{D}x(\tau) \exp[\, i\, \tilde{S}/\hbar] \tag{2.4}$$

where the functional integral is now taken over paths such that

$$\underset{\sim}{x}(0) = 0 \quad \text{and} \quad \underset{\sim}{x}(t) = 0 \quad \text{and}$$
$$\tilde{S} = \int_0^t d\tau \left[\tfrac{1}{2} \dot{x}_i^2 - \tfrac{1}{2} x_i \, x_j \frac{\partial^2 V(x(\tau))}{\partial x_i \, \partial x_j} \right]. \tag{2.5}$$

The calculation proceeds in essentially the same way as in Sec. 2. Define a shift and its inverse by

$$y_i(\tau) = x_i(\tau) - \int_0^\tau \dot{N}_{ik} \, N^{-1}_{kj} \, x_j \, du \tag{2.6}$$

$$x_i(\tau) = N_{ij}(\tau) \int_0^\tau N^{-1}_{jk} \, \dot{y}_k \, du \tag{2.7}$$

where N satisfies:

$$\ddot{N}_{ij} = V''_{ik} \, N_{kj} . \tag{2.8}$$

After substituting into (2.3) and some algebra

$$K(x_i(0), \, x_i(t)) = \exp(i\, S_{c\ell}(t)) \int \mathcal{D}y_i \exp\left(\frac{i}{2} \int_0^\tau du \, \dot{y}_i^2(u)\right)$$
$$\times \int_{-\infty}^{+\infty} d\alpha_j \exp\left(i\, \alpha_j \, N_{ji}(\tau) \int_0^\tau N^{-1}_{ik} \, \dot{y}_k(u) \, du\right) \left| \frac{\mathcal{D}x_i}{\mathcal{D}y_j} \right| \tag{2.9}$$

where the path space is fixed by $y(0) = 0$ and $y(t)$ arbitrary. The α integration is just the incorporation of constraints via a Lagrange multiplier, as in the one mode case.

The functional integral is now trivial and we are left with

$$K(x(0),\ x(t)) = \exp(i\ S_{c\ell}(t)) \int_{-\infty}^{+\infty} d\alpha_j \left|\frac{\det N(t)}{\det N(0)}\right|^{\frac{1}{2}} \exp\left(-\frac{i}{2}\int_0^t \alpha_j N_{jk}(t)\ N^{-1}_{k_i}(u)\ N^{-1}_{\ell_i}(u)\ N_{m\ell}(t)\ \alpha_m\ du\right). \tag{2.10}$$

Performing the α integration gives the final form:

$$K(x(0),\ x(t)) = \exp(i\ S_{c\ell}(t))\ \ |N(t)N(0)|^{-\frac{1}{2}}\ \left|\int_0^t N^{-1}_{ki}(u)\ N^{-1}_{\ell i}(u)du\right|^{-\frac{1}{2}}. \quad (2$$

Now, let $x_i = x_i(\tau);\ 0 \leq \tau \leq T$

$$\bar{x}_i = x_i(\bar{\tau}), \bar{\tau} < 0\ . \tag{2.12}$$

Then, using the Hamilton-Jacobi equations, one checks that

$$N^{-1}_{ij} = \frac{\partial^2 S_{c\ell}(\bar{x}, x)}{\partial \bar{x}_i\ \partial x_j} \tag{2.13}$$

satisfies Eq. (2.8). Choosing the particular solution $\underset{\sim}{\bar{x}} \rightarrow \underset{\sim}{x}(0) = \underset{\sim}{x}'$ simplifies the final calculation and noting that $\underset{\sim}{x}(T) = \underset{\sim}{x}''$ one obtains (2.3). When $|\partial^2 S/\partial \underset{\sim}{x}\partial \underset{\sim}{\bar{x}}|$ vanishes between 0 and T, additional phases in Eq. (2.12) are needed, in direct analogy to those introduced by the turning point in one dimension.[1, 2] The advantage of the functional integral derivation lies in the physical interpretation of the correction terms generated by the determinant in K, Eq. (2.12). As one can see from Eq. (2.3) and Eq. (2.4), they represent the effect of small, quantum mechanical fluctuations around the classical orbit. The functional approach also makes the transition to field theory quite natural.

C. Periodic Orbits

Proceeding as before, we have

$$\mathrm{tr}\, e^{-iHT} = \int d\underset{\sim}{x}\, K(\underset{\sim}{x}, \underset{\sim}{x}; T) \ . \tag{2.14}$$

The $\underset{\sim}{x}$ integration may be done by stationary phase: the stationary phase point is determined by

$$\frac{\partial S}{\partial \underset{\sim}{x}'} + \frac{\partial S}{\partial \underset{\sim}{x}''}\Bigg|_{\underset{\sim}{x}'=\underset{\sim}{x}''=\underset{\sim}{x}} = \underset{\sim}{p}'' - \underset{\sim}{p}' = 0 \ . \tag{2.15}$$

As pointed out by Gutzwiller,[1] this stationary phase condition selects classical periodic orbits. If we label the (discrete) set of orbits with period T by an integer n, then one has

$$\mathrm{tr}\, e^{-iHT} \approx \sum_{\substack{\text{orbits} \\ \text{with period } T}} D_n(T)\, e^{iS_n(T)} \tag{2.16}$$

where S_n is the action for the orbit in question and D_n is the product of the determinant in (2.3) and a determinant coming from the integration around the stationary phase point.

In principle, we can multiply (2.16) by e^{iET} and do the T integration by stationary phase to obtain the semiclassical approximation to $G(E)$. The problem is to find all the periodic orbits.

When all the orbits are not available, a possible approximation to (2.16) is the following. Suppose that we have access to one family of such orbits, i.e., one orbit at some fixed energy E_0 and the set of orbits swept out as we vary the energy continuously away from E_0. The sum in (2.16) might then be approximated by summing over this one parameter family of orbits, including of course multiple traverses of the basic orbits.

The calculation can then be carried out exactly as in Sec. 2. The details are given by Gutzwiller.[1] To give an example, in the case of a stable classical orbit of period T of a particle in a two-dimensional potential well, G(E) computed this way is

$$G(E) = T \sum_{n=1}^{\infty} \frac{1}{2\sin \frac{1}{2} n\nu} \exp(i\, n\, W) \qquad (2.17)$$

where ν is the stability angle. This in turn gives the quantization condition

$$W = (2m\pi + \tfrac{1}{2}(2p+1)\nu) \qquad (2.18)$$

where m and p are non-negative integers. One finds, as expected that the bound states of the 2-dimensional system are labelled by the two quantum numbers m and p. Since $\sin \frac{1}{2} n\nu$ in Eq. (2.17) represents the effect of small deviations around the orbit, one expects that the approximation (2.18) is valid only if p is not too large. Of course, the approximation of keeping one family of orbits in (2.16) makes sense only if the coupling to the transverse degree of freedom (represented by the oscillator energies $(p+\frac{1}{2})\nu$) is weak. It is not clear whether or not such an approximation could be useful in field theory.

D. Weak Coupling

According to Sec. 2C, for weak coupling, we have the option of expanding around a trivial orbit $\underset{\sim}{x}(\tau) \equiv \underset{\sim}{x}_0$ where $\underset{\sim}{x}_0$ is a (local) minimum of the potential V. The functional integral in (2.4) is then simple to do and the energy levels are

$$E = V(\underset{\sim}{x}_0) + \sum_{\alpha=1}^{N} (P_\alpha + \tfrac{1}{2})\, \omega_\alpha \qquad (2.1$$

where $[P_\alpha] = [P_1 \ldots P_N]$ is any set of positive integers and the ω_α are the eigenfrequencies of the classical oscillator system defined by the action

$$\int d\tau \left[\tfrac{1}{2} \dot{x}_i \dot{x}_i - \tfrac{1}{2} x_i x_j \frac{\partial^2 V(x_0)}{\partial x_i \partial x_j} \right] . \tag{2.20}$$

E. Field Theory

Letting the number of degrees of freedom become infinite, the results of this section carry over directly to field theory. What do we mean by a periodic orbit in field theory? Consider a theory containing fields $\varphi^i(\underset{\sim}{x}, t)$. [Note that $\underset{\sim}{x}$ now labels space points: the φ^i are the coordinates.] A periodic orbit is clearly a solution φ^i_{cl} to the classical field equations which has the property that

$$\varphi^i_{cl}(\underset{\sim}{x}, t + T) = \varphi^i_{cl}(\underset{\sim}{x}, t) . \tag{2.21}$$

We will be particularly interested in particle-like solutions which have a finite (classical) energy relative to the vacuum. Such solutions must satisfy

$$\lim_{|\underset{\sim}{x}| \to \infty} \varphi^i_{cl}(\underset{\sim}{x}, t) \cong \varphi^i_{vac} \tag{2.22}$$

where φ^i_{vac} is the vacuum expectation of the field in question and by $\cong$ we mean equal up to a symmetry operation. In the simplest theories, φ^i_{vac} vanishes and (2.20) reduces to

$$\lim_{|\underset{\sim}{x}| \to \infty} \varphi^i_{cl}(\underset{\sim}{x}, t) = 0 \tag{2.23}$$

In field theory, a trivial orbit is a time independent solution $\varphi_0(\underset{\sim}{x})$ to the classical field equations. To have a particle interpretation, $\varphi_0(\underset{\sim}{x})$ should be a nonconstant field satisfying the appropriate one of (2.22) or (2.23). The weak coupling approximation in field theory leads to formulas like

(2.18) where the number of oscillators is infinite.

So far, we have been ignoring translational invariance and infinite volume questions which must be faced up to if one is to seriously talk about applying semi-classical methods in field theory. These are the topics which will concern us in the next two sections.

In Sec. 6 we will give a more systematic account of the weak coupling approximation in field theory.

4. CENTER OF MASS MOTION

Continuing to work towards our goal of finding particle like solutions in field theories, we note that a conceptional problem arises. Any classical particle-like solution to a field theory will be both localized in space and have a definite momentum. This is impossible in quantum mechanics. Another indication that our formal developments have so far been incomplete is that in cases of real interest the spectrum of H is continuous due to translational invariance. In this section we will fill in these holes in our formalism.

The same basic problems appear in the simpler case of non-relativistic systems. Let us first consider the non-relativistic free particle. Working in one space dimension for simplicity, we discretize the energy levels by making space finite and imposing periodic boundry conditions. Classically, this may be done by imagining that the (one space dimensional) world is a large closed loop with a perimeter of length L. The motion of a free particle with velocity v is then periodic

with period

$$T = \frac{L}{v} \tag{3.1}$$

i.e. the time it takes to go once around the loop. The action for a free particle of mass M is $\frac{Mv^2T}{2} = \frac{ML^2}{2T}$. Inserting this into the general formulas of Sec. 2 gives energy levels

$$E_n = \frac{P_n^2}{2M} \quad P_n = \frac{2\pi n}{L} \tag{3.2}$$
$$n = 0, 1, 2.....$$

which is of course the correct answer for a particle in a periodic box.

We can see already how our conceptional difficulties will be resolved. The energy levels in Eq. (3.2) correspond to the quantization of complete orbits. For these energy eigenstates only the complete orbit has a meaning; the position of the particle along the orbit is meaningless quantum-mechanically. Also one sees that as $L \to \infty$, Eq. (3.2) produces the correct continuous spectrum for H.

For multiparticle non-relativistic systems everything generalizes as expected. Because of Galilean invariance, the center of mass motion separates from the internal motion. Thus we have a separable system of the type treated in Sec. 2. The energy levels are the sums of internal energy levels computed by ignoring the center of mass motion plus the kinetic energy of the center of mass as given by Eq. (3.2).

Moving on now to relativistic problems let us first consider the free particle. With the same periodic boundary conditions, the action for a relativistic particle with mass M is

$$S = -M(1 - v^2)^{\frac{1}{2}} T \tag{3.3}$$

where v is the velocity measured in units of the velocity of light. For future reference, we wish now to compute the trace of the free propagator, $G_0(E)$. Eq. (3.1) gives the period for one trip around the loop; for n trips the period is nL/v. Summing over these multiple passes to get $\mathrm{Tr}\, e^{-iHT}$ we have

$$\sqrt{-2\pi i}\ \mathrm{tr}\, e^{-iHT} \approx \sum_n L\left[\frac{M}{T}\frac{1}{(1-v_n^2)^{3/2}}\right]^{\frac{1}{2}} e^{-iM\sqrt{1-v_n^2}\,T} \qquad (3.4)$$

$$v_n = \frac{nL}{T}$$

where the factor in front of the exponential is just the $\frac{T}{n}\left(\frac{dE}{dT}\right)^{1/2}$ from our general semi-classical formula (1.12). Transforming to $G(E)$, we obtain

$$G_0(E) = \sqrt{\frac{-i}{2\pi}} \sum_n \int dT\ L\left[\frac{M}{T}\frac{1}{(1-v_n^2)^{3/2}}\right]^{\frac{1}{2}} \exp i\, T(E - M\sqrt{1-v_n^2}) \qquad (3.5)$$

$$= \left[\frac{i\, d\, W_0(E)}{dE}\right] \frac{e^{iW_0(E)}}{1 - e^{iW_0(E)}} \qquad (3.6)$$

where we have done the integral in (3.5) by stationary phase and

$$W_0(E) = L(E^2 - M^2)^{\frac{1}{2}} \ . \qquad (3.7)$$

Following our general formalism, we see that Eq. (3.6) has poles of unit residue at

$$E_n = \sqrt{M^2 + P_n^2} \ , \quad P_n = \frac{2\pi n}{L} \qquad (3.8)$$

$$n = 0, 1, 2, \ldots$$

as expected.

Armed with the formulas of the preceeding paragraph we can now tackle the general problem of a composite relativistic system. Suppose that we have found a classical particle-like solution which is at rest.

In a field theory, an example would be a solution $\varphi(t, x)$ to the classical field equations which is bounded in space, i.e. $\varphi(t, \underset{\sim}{x}) \to 0$ as $|\underset{\sim}{x}| \to \infty$, and periodic in time, i.e. $\varphi(t + \tau, \underset{\sim}{x}) = \varphi(t, \underset{\sim}{x})$. Space and time translational invariance will allow us to find new solutions by taking $\underset{\sim}{x} \to \underset{\sim}{x} + \underset{\sim}{a}$ and $t \to t + b$. Also Lorentz invariance allows us to boost the solution to obtain a particle like object moving with any velocity $|v| < 1$. What happens when we quantize the system?

For simplicity we again consider one space dimension and impose periodic boundary conditions with length L. The periods of translational *motion for a moving object are then* $T = nL/v$ *where* n *is an integer.* Because of time dilation the periods of internal motion for the particle moving with velocity v are $T = \frac{m\tau}{\sqrt{1 - v^2}}$ where m is an integer and τ which has the same meaning as before, is the period in the rest frame. Following closely the methods used to discuss separable systems in Sec. 2, we note that the truly periodic motions satisfy

$$T = \frac{m\tau}{\sqrt{1-v^2}} = \frac{nL}{v} \qquad (3.9)$$

$$m, n = 0, 1, 2 \ldots\ldots \infty.$$

We assume that classical solutions are available for a range of values of τ and since v is an arbitrary number less than one, it is clear that for each T there is a two fold discrete infinity of orbits satisfying Eq. (3.9). Now let the action in the rest frame be $\bar{S}$ i.e.

$$\bar{S}(\tau) = \int_0^{\tau} \mathcal{L}(\tau')\, d\tau' \Big|_{v=0} \qquad (3.10)$$

and $v_n = nL/T$ as before so that (3.9) becomes

$$m\tau = \sqrt{1 - v_n^2}\, T \tag{3.11}$$

Then the sum over all periodic orbits which satisfy (3.9) can be shown to be

$$\sqrt{-2\pi i}\ \mathrm{tr}\, e^{-iHT} \approx \sum_{m,n,i} \int d\tau\, L \left| \frac{d\bar{S}_i(\tau)/d\tau}{T(1-v_n^2)^{3/2}} \right|^{\frac{1}{2}} [\, m\Delta_i(\tau)]^{\frac{1}{2}} \tag{3.12}$$

$$\times\, e^{im\bar{S}_i(\tau)}\, \delta\left(m\tau - \sqrt{1 - v_n^2}\, T\right)$$

where $\Delta_i(\tau)$ is a determinant independent of m and n and the index i runs over the set of all distinct periodic orbits which have a period τ in the rest frame. The explanation of the factors in (3.12) is as follows. The delta function clearly enforces (3.11). Since the action is a Lorentz invariant the total action for motion through m internal periods will be $m\bar{S}(\tau)$ in any frame; hence this quantity appears in the exponent. The remaining factors come from the factor D_n in (2.16). Noting that $-d\bar{S}/d\tau$ is the classical mass one sees that apart from the Lorentz invariant $|m\Delta|^{\frac{1}{2}}$, these factors are the same as in (3.5). This is what one expects: we leave the detailed verification of (3.12) to the reader.

Now inserting

$$\delta\left(m\tau - \sqrt{1-v_n^2}\, T\right) = \frac{1}{2\pi} \int dM\, \exp\left[\, iM\left(m\tau - \sqrt{1-v_n^2}\, T\right)\right] \tag{3.13}$$

we see that doing the τ integration by stationary phase will lead to

$$M = -\frac{d\bar{S}}{d\tau}\ . \tag{3.14}$$

Hence (3.12) can be written as

$$\sqrt{-2\pi i}\ \mathrm{tr}\, e^{-iHT} \approx \sum_n \int \frac{dM}{2\pi i}\, L \left| \frac{M}{T(1-v_n)^{3/2}} \right| \bar{G}(M) \tag{3.15}$$

$$\exp\left[\, -i\, M\sqrt{1-v_n^2}\, T\right]$$

where

$$\bar{G}(M) = i\sum_{m,i} \int d\tau [\, m\Delta_i(\tau)]^{\frac{1}{2}} \exp[\, i\, m(\bar{S}_i(\tau) - M\tau)\,] \ . \tag{3.16}$$

The quantity $\bar{G}$ is, of course, just the thing which we would have computed if we had ignored translational invariance and had worked with a set of periodic solutions at rest and arbitrarily localized in space. Note that $\bar{G}$ does not depend on the length of space L. We may assume that $\bar{G}$ has poles, i.e.

$$\bar{G}(M) \sim \frac{1}{M_k - M} \qquad k = 1, 2.... \tag{3.17}$$

which corresponds to the mass spectrum of the theory. Finally, comparing with (3.5) and (3.6) one easily sees that

$$G(E) = \frac{1}{2\pi i} \int G_0(M, E)\, \bar{G}(M)\, dM \tag{3.18}$$

where G_0 is given by (3.6) and we have explicitly indicated that it depends on M. The M integration picks up the poles of (3.17) and G(E) consequently has poles at

$$E = \sqrt{P_j^2 + M_k^2}\ , \quad k = 1, 2.... \quad P_j = \frac{2\pi}{L}\, j, \ j = 1, 2.... \tag{3.19}$$

which is the desired result.

Note that the energy momentum relation did not have to be put in by hand, but appeared kinematically as a consequence of the delta function in (3.12). Evidently, we have solved the problem of center of mass motion in the general, relativistic case. Our remarks about the complete lack of localization of a particle in a definite energy state apply as before.

We have worked in one spacial dimension, but it should be obvious

that the result generalizes trivially to three space dimensions.

In the following paper we will look for time-independent solutions to field theories which have a behavior at spacial infinity, consistent with a particle interpretation. In particular they will have a finite classical energy which we interpret as a first approximation to the mass of a quantum mechanical state. These time independent solutions are like the trivial orbits discussed in connection with weak coupling approximation. The trivial internal motion of these solutions is periodic for any period T. Furthermore they can be boosted to obtain solutions which move. Taking the weak coupling approximation to the general results of Eq. (3.19) tells that if the energy of the time independent solution is M, then when translational invariance is taken into account the energies will be the expected $\sqrt{M^2 + P^2}$.

Recall that in weak coupling the energy is the classical energy plus the energy of zero point oscillations around the trivial (time independent) orbit. For a translationally invariant system there are always zero frequency oscillations corresponding to a translation of the spacial origin. These oscillations are to be interpreted as small motions of the center of mass. According to the general results of this section, such oscillations can be ignored when computing the mass. Their role is to give the proper mass energy relation when translational invariance is properly taken into account.

5. INFINITE VOLUME SYSTEMS AND STATISTICS

By analogy with statistical mechanics we expect that in field theory $\mathrm{tr}\, e^{-iHT}$ will be of the form $\exp(-i\,t\,F(T)\,\Omega)$ where Ω is the volume of space and F is a real time analog of the free energy per unit volume. If we expand the exponential, it is evident that $G(E)$ will contain terms proportional to any power of Ω. This might appear to be a serious difficulty in the limit of infinite volume. It is, however, really just a formal problem which when understood will not cause trouble in practice. In the process of resolving this formal difficulty, we will see how particle statistics and scattering states fall out of our general formalism.

To get a feeling for how the volume independence of G goes, let us consider a very simple example. Consider a particle in one dimension with Lagrangian $L = \dot{x}^2/2 - V(x)$ where V is an attractive potential of finite range. As in the previous section, we make space finite and of length L by imposing periodic boundary conditions. There are now two kinds of periodic orbits which contribute to $\mathrm{tr}\, e^{-iHT}$. First, there are negative energy bound orbits which exist only in the region where V is non-zero. Secondly, there are positive energy orbits (scattering orbits) in which the particle covers the entire periodic space. The contribution of the bound orbits is obviously independent of the length of space L: they clearly give the bound states poles of G in the usual way.

Turning now to the scattering orbits, we note that on passing through the potential a particle with energy E suffers a time delay $\Delta(E)$ of

$$\Delta(E) = -\int\left[\frac{1}{\sqrt{2(E-V)}}\right] - \frac{1}{\sqrt{2E}}\, dx \tag{4.1}$$

where by "time delay" we mean the additional time required to pass through the potential over what a free particle would take to traverse the same distance. Since a free particle with energy E has velocity $\sqrt{2E}$, the periods of the periodic orbits are

$$T(E) = \frac{L}{\sqrt{2E}} + \Delta(E) \quad . \tag{4.2}$$

The quantization condition is

$$W(E) = S(T(E)) + ET(E) = 2\pi n \tag{4.3}$$

$$n = 0, 1, 2, \ldots$$

Now for large L we are interested only in the density of scattering states $\frac{dn}{dE}$. Treating n as a continuous variable in Eq. (4.3) and differentiating yields[9]

$$\frac{dn}{dE} = \frac{T(E)}{2\pi} = \frac{L}{2\pi\sqrt{2E}} + \frac{\Delta(E)}{2\pi} = \frac{dn_0}{dE} + \frac{\Delta(E)}{2\pi} \tag{4.4}$$

where dn_0/dE is the L-dependent free particle density of states and $\Delta(E)$ contains the effect of the potential. The latter term is L independent as it should be. Evidently $G(E)$ is given by

$$G(E) = \sum_k \frac{1}{E-E_k} + \frac{1}{2\pi}\int_0^\infty \frac{\Delta(E')}{E-E'}\, dE' + G_0(E) \tag{4.5}$$

where E_k are the energies of bound states and G_0 is the free particle term and is proportional to L. From Eq. (4.5) we learn two things. First that, in this case, the terms containing interesting physics are independent of L and secondly that our formalism provides us with some information about the scattering process, namely the density of states.

We could make the above problem translationally invariant by imagining that we really have two particles with the potential depending on their separation. In this case the uninteresting term G_0 will be proportional to L^2. According to what was said in Sec. 4, the bound states will acquire a kinetic energy and their contribution to G will go like L. The same thing will happen to the term containing $\Delta(E)$ in Eq. (4.5). Thus we might guess that for translationally invariant systems the interesting parts of G(E) will contain a single power of the volume. This, as we shall see, is indeed what happens.

Rather than proceeding directly to field theory, let us consider a non-relativistic system which has essentially the same volume-of-space properties. We define a non-relativistic system with an indefinite number of particles by summing $\mathrm{tr}\, e^{-iHT}$ over a one particle space, a two particle space, a three particle space and so on. This, of course, is just what one does to obtain the grand canonical ensemble in statistical mechanics. Indeed in what follows we will simply be describing the real time version of the cluster expansion in statistical mechanics.

The one particle term thus obtained for G(E) is proportional to the volume Ω and is just that of a free particle. The two particle term contains a piece proportional to Ω^2 corresponding to two free particles and a nontrivial term proportional to Ω which contains the bound states and the density of scattering states as in Eq. (4.5). The three particle term has a Ω^3 piece equal to the free G for three particles and a Ω^2 term coming from processes where two particles interact while the third

propagates freely. In particular, this Ω^2 term contains the free propagation of particle number 3, say, and a bound states of numbers 1 and 2. In addition there will be a term proportional to Ω which, unlike the Ω^3 and Ω^2 terms, yields new information. It will contain the three particle bound states and information about the three body scattering process.[9] Going on to four particles, there will clearly be a Ω^4 term where all four propagate free, a Ω^3 term where one pair interacts, and two kinds of Ω^2 terms where three particles interact while the fourth propagates freely or where two separate pairs interact. Finally, the term proportional to Ω will again be nontrivial giving four particle bound states and scattering information.

Evidently, the terms in G going like Ω^n for $n > 1$ are not interesting since they simply repeat information already known from the term proportional to Ω. Thus, if we write $\text{tr } e^{-iHT} = e^{-iF(T)\Omega}$ for our model many particle system, the interesting pieces of G would be gotten by Laplace transforming $-i\ F(T)\ V$ rather than the exponential. That $\log \text{tr } e^{-iHT}$ will actually be proportional to Ω in the model is direct consequence of the cluster expansion in statistical mechanics. The reader who is not familiar with this line of argument may find it helpful to convince himself that if there were no interactions in the model then $F(T)$ would be proportional to $e^{iS_0(T)}$ not the free action S_0 itself. In this case, only the one particle term would have a piece going like Ω.

Almost by definition, field theory is a many body problem where the number of particles is not fixed. The proper thing to do in field theory

is the same as in the above model; work with $\log \operatorname{tr} e^{-iHT}$. Let us see how this would work in the sort of applications which we have in mind. For the moment, we will ignore some points related to Bose and Fermi statistics. They will be treated later. Suppose that we have found some particle like solutions to a classical field theory, i.e., fields satisfying $\varphi_{c\ell}(t + \tau, \underline{x}) = \varphi_{c\ell}(t, \underline{x})$ for some period τ and $\varphi_{c\ell}(t, \underline{x}) \to 0$ as $|\underline{x}| \to \infty$. In the previous section, we saw how such a solution becomes a quantized propagating particle when the center of mass motion is taken into account. The contribution of such a state to $\operatorname{tr} e^{-iHT}$ or $G(E)$ is proportional to Ω. Now since our particle like solutions are localized in space we can find approximate solutions representing two well separated particles, for example let the field be, $\varphi_{c\ell}(t, \underline{x}) + \varphi_{c\ell}(t, \underline{x} + \underline{a})$ where $|\underline{a}|$ is large. Putting the system in a periodic box we could look for the exact periodic orbits which take into account the interaction of the two particles. For large Ω we would then get two terms in $G(E)$, one going like Ω^2 and the other like Ω. As before, the Ω^2 term will just count the states of two free particles and is not of interest. The term proportional to Ω will yield information about the scattering of two particles. Any "bound" two particle orbits would be added to our original list of single particle solutions. Similarly, we could in principle construct solutions which display the scattering[10] of three particles, four particles and so on. Each of these is a stationary phase point in the functional integral for $\operatorname{tr} e^{-iHT}$ and in principle must be kept. The sum over all these classical orbits will be of the form $\exp(-iF(T)\Omega)$ where $-iF(T)\,\Omega$ contains two things. First it

is the sum over propagating particle-like solutions. Secondly, there will be a residual piece from the scattering orbits.[10] Since it is unlikely that these scattering orbits will be available in cases of interest we can concentrate on the particle-like solutions. As far as they are concerned, we are back where we started. Transforming $-iF(T)\Omega$ to get the interesting part of G is just the same thing as simply examining the localized one particle solutions and forgetting the whole business of higher powers Ω. This is indeed what one is supposed to do. The point of all this has only been to assure ourselves that this physically obvious ansatz is formally correct.

We conclude this section with a brief discussion of how Bose and Fermi statistics work in the semi-classical approximation. In particular, we will indicate how particle-like solutions to classical field theory end up, when quantized, with the proper statistics. We will treat only Bose statistics explicitly, the generalization to Fermi statistics is straightforward. Consider a non-relativistic system of two identical bosons. When computing $\mathrm{tr}\, e^{-iHT}$ we have to include an exchange term, i.e.,

$$\mathrm{tr}\, e^{-iHT} = \int (\langle x_1 x_2 | e^{-iHT} | x_1 x_2 \rangle + \langle x_2 x_1 | e^{-iHT} | x_1 x_2 \rangle)\, dx_1 dx_2 . \quad (4.6)$$

The first term in Eq. (4.6) is the usual one which comes from periodic orbits in the semi-classical approximation. The same stationary phase approximation applied to the exchange term in Eq. (4.6) will pick out exchange orbits where if the coordinate and momentum of particle 1 at time zero are x_1, p_1 and those of particle 2 are x_2, p_2 then after time T the coordinate and momenta of particle 1 are x_2, p_2 and for particle 2

x_1, p_1. The presence of these exchange orbits will give rise to a new term in G(E) proportional to one power of the volume of space Ω. This correction to the density of states is the signature of bosons in our formalism. We leave to the reader the instructive exercise of verifying that the classical exchange orbits do indeed give the correct change in G.

Suppose now that we have a particle like solution to field theory. For simplicity we will assume that the solution is time independent in its rest frame, i.e., $\varphi = \varphi_0(x)$ where $\varphi_0 \to 0$ as $|x| \to \infty$, and take space to be one dimensional. Boosting, gives the moving solutions $\varphi_0(\gamma(x+vt))$ where $\gamma^{-1} = \sqrt{1 - v^2}$. An approximate two particle solution is given by

$$\varphi = \varphi_0(\gamma(x+vt)) + \varphi_0(\gamma'(x+a+v't)) \tag{4.7}$$

where v' is in general a different velocity and $|a|$ is large. For a periodic space of length L, this solution is periodic with period T if $T = nL/v = mL/v'$ where m and n are integers. In the special case where $v = v'$ and $a = L/2$ there are further periodic orbits with period $(n + 1/2)\, L/v$ for any integer n. For these orbits the two terms on the right of Eq. (4.7) go into each other. They are the analog of the exchange orbits in the above example and we will call them by that name. The difference here is that in field theory the exchange orbits do not have to be added by hand but already exist among the periodic solutions of the classical problem. For the exchange orbits the value of a is fixed. Therefore their contribution to G(E) will contain a single power of L (Ω in the three dimensional case). This change in the density of states is, as noted

above, a characteristic of bosons. We leave the details of the calculation as well as the generalizations to N-particle exchanges, fermions, etc. to the reader.

6. USING AN EFFECTIVE ACTION

The semi-classical method discussed in the previous sections requires an actual knowledge of the dynamics, i.e. orbits, of the classical system. In this section we turn to a different but not unrelated method which will be seen to be a systematic scheme for improving on the weak coupling approximations discussed in Sec. 2.

First we will obtain a method for finding the ground state of a quantum system. In our later applications to field theory, we will not be interested in the ground state (vacuum) but rather excited states (particles). How the method works for excited states will be discussed later. For the ground state we will be discussing a method which is well known in the literature.[11,12]

Let us start with the simple system with one degree of freedom defined by the Lagrangian

$$\mathcal{L} = \frac{\dot{x}^2}{2} - V(x) \quad . \tag{5.1}$$

We assume that V has a unique minimum at $x = x_0$. The classical "ground state" of this system is simply the particle sitting still at the point $x = x_0$; its energy is $V(x_0)$. Because of zero point motion, the quantum mechanical ground state energy must be higher.

Rather than work with $\mathrm{tr}\, e^{-iHT}$ as we did in previous sections,

let us set $T = -i\beta$ and consider the limit of large β which gives

$$\text{tr } e^{-\beta H} \underset{\beta\to\infty}{\rightarrow} e^{-\beta E_0} \tag{5.2}$$

where E_0 is the ground state energy. We will compute $\text{tr } e^{-\beta H}$ via the functional integral in the following way. Define

$$L = L_0 + L_I - V(x_0)$$

where

$$L_0 = \frac{\dot{x}^2}{2} - \frac{m^2}{2}(x - x_0)^2 \tag{5.3}$$

$$m^2 = V''(x_0)$$

$$L_I = -\sum_{n=3}^{\infty} V^{(n)}(x_0)\,(x-x_0)^n/n!$$

Then we have

$$\text{tr } e^{-\beta H} = e^{\beta V(x_0)} \int \mathcal{D}x(T)\, e^{-\int_0^\beta \frac{\dot{x}^2}{2} + \frac{m^2}{2}(x-x_0)^2} \sum_{n=0}^{\infty} \frac{1}{n!}\left(-\int_0^\beta L_I\right)^n$$

$$= e^{-\beta\left(V(x_0) + \frac{m}{2}\right)} + \cdots \tag{5.4}$$

where we have explicitly done the $n = 0$ term which is just the weak coupling result. The higher order terms can be done by standard means and yield the perturbation expansion for $\text{tr } e^{-\beta H}$ in Feynman diagram form. As is well known, the expansion in Eq. (5.4) contains both connected and disconnected diagrams. The series can be partially summed to give the exponential of the sum of connected diagrams, so we have

$$\text{tr } e^{-\beta H} = \exp\left(-\beta\left[V(x_0) + \frac{m}{2} + \Delta E\right]\right) \tag{5.5}$$

where $-\beta\Delta E$ is the sum of all connected diagrams with two or more closed loops.

Some examples of these diagrams are shown in Fig. (3). The single diagram with one closed loop gives the $m/2$ term in (5.5). It is well known from statistical mechanics that in the limit of large β, ΔE is independent of β and the terms in its expansion are simply ordinary Feynman diagrams in Wick rotated form. The details of the diagrams, e.g. combination factors, can be straightforwardly deduced from functional integral (5.4).

The diagrams in Fig. (3a) and Fig. (3b) are one particle reducible, i.e. they contain a single particle line which, if cut, would make the diagram disconnected. The one particle reducible diagrams can be summed in the following way.

The sum of all tadpole diagrams shown in Fig. 4 where a single line disappears is equal to the difference between the expectation value $\langle x \rangle$ of the operator x in the ground state and its classical value x_0, i.e. "complete tadpole" $= \langle x \rangle - x_0$. If we were to expand the Lagrangian around $\langle x \rangle$ rather than x_0, obtaining in the process a new mass $m^2(\langle x \rangle) = V''(\langle x \rangle)$ and new n-point vertices $V^{(n)}(\langle x \rangle)$, then there would be no tadpoles or, equivalently, no one particle reducible diagrams in the perturbation series for ΔE. Of course, we do not know the a priori value of $\langle x \rangle$. Nevertheless, let us define a function $\Gamma(y)$ by

$$-\beta\Gamma(y) = -\beta\left(V(y) + \frac{m(y)}{2}\right) - \beta\sum(y) \tag{5.6}$$

where $\sum(y)$ is the sum of all connected, one particle irreducible diagrams computed with the Feynman rules derived from (5.4) using the

interaction $L_1 = \sum_n \frac{1}{n!} V^{(n)}(y)(x-y)^n$ and the mass $m^2(y) = V''(y)$ in the free Lagrangian. It is obvious that $\Gamma(\langle x \rangle) = E_0$, the ground state energy. To compute $\langle x \rangle$ we note that $\Gamma'(y) = \frac{d}{dy}\Gamma(y)$ is the sum of all one particle irreducible tadpole diagrams, including a "bare" tadpole $V'(y)$. Now it is obvious that the complete tadpole vanishes if and only if the one particle irreducible tadpole vanishes. Hence the vanishing of the complete tadpole is equivalent to the vanishing of Γ' and we can therefore find $\langle x \rangle$ by looking for a zero of Γ'. In fact

$$E_0 = \min_y \Gamma(y) \tag{5.7}$$

which is the quantum analog of looking for the minimum of V.

The extension to a system with n degrees of freedom is straightforward: Γ simply becomes a function of n variables $y_1 \cdots y_n$. In field theory Γ is a <u>functional</u> of the field φ considered as a function of space co-ordinates at a fixed time.

Because Γ is difficult to compute, Eq. (5.7) is only useful for weak coupling. In this case, it is easy to see the equivalence of the present method and the semi-classical method. For the latter, the weak coupling approximation amounts to expanding the functional integral around the trivial orbit $x \equiv x_0$. Previously we kept only the quadratic terms in this expansion but the higher order terms could be computed perturbatively as was done here. In fact, the result would simply be an analytic continuation to real time of the diagrams for ΔE. One could sum the one particle reducible diagrams by introducing a real time Γ and it is easy to convince oneself that, to any finite order in L_I, the $G(E)$

computed in this manner would have a pole at the energy given by (5.7).

What if we tried the same trick but chose to expand around a trivial orbit which does not correspond to the classical ground state. For example, one could try expanding around the higher minimum of the potential shown in Fig. (2).

In Appendix (B) we argue that this procedure will give good results for an excited state provided that the coupling around the point in question is reasonably weak and that the perturbation expansion for Γ is truncated at a low order. It will not give a convergent expansion for the energy, but this is not likely to be of importance in practice.

Actually for the field theoretic problems which we have in mind, there are rigorous methods for using Γ to find excited (particle) states. One has to arrange things in such a way that the vacuum state cannot enter the problem. One can do this in two ways.

1) In some field theories (see the following paper) there are field configurations whose topology is inequivalent to the field configurations which enter into the vacuum wave function. The part of functional integral for $\mathrm{tr}\, e^{-\beta H}$ which runs over these topologically abnormal field configurations contributes only to excited (particle) states. It follows that one can find particle states by minimising Γ over fields with an abnormal topology.

2) Another method is to insist that the state that one is looking for have non-vacuum quantum numbers or a non-

zero three momentum. This may be accomplished by using a Lagrange multiplier μ and computing $\mathrm{tr}\, e^{-\beta(H+\mu Q)}$ where Q is, say, baryon number or charge. Considering $H+\mu Q$ as an effective Hamiltonian, $\mathrm{tr}\, e^{-\beta(H+\mu Q)}$ can be computed by a path integral and one proceeds exactly as before. Terms in the effective Γ which are independent of μ have to be dropped to avoid the vacuum. In the end μ is adjusted to get the correct quantum numbers. There is a simple example of this procedure in the following paper.

In situations where we can use Eq. (5.9) or some variant thereof to find particle states there is an obvious, but perhaps very useful approximation scheme available. That is to compute Γ to some finite order and look for a minimum. In field theory this turns out to yield a self-consistent field, Hartree type of approximation. In the following paper we show how this works in a specific example.

The identification of the weak coupling semi-classical method with the properties of the effective action Γ serves another very useful function in field theory. It is a good guide to how to renormalize when divergences appear.

Finally, the diagramatic methods discussed here can be used to compute quantum corrections to the general semi-classical method where we expand around periodic orbits. This is discussed in Appendix C. -

Using the effective action to improve on weak coupling approximations has one drawback. The role played by translational invariance, statistics and field-theoretic infinite volume difficulties is not transparent. A way to handle these problems, which lie at the heart of any particle interpretation of a c-number field, was discussed in Sections 4 and 5. By remembering how the effective action relates to the weak coupling approximation to WKB, one can reinterpret the results of Sections 4 and 5 in the context of an effective action. Order by order in perturbation theory this is a relatively straightforward problem. In paper II we will see how this goes in the lowest non-trivial order.

One of the authors (B. H.) would like to express his appreciation to Dr. Carl Kaysen for the hospitality extended to him by the Institute for Advanced Study.

APPENDIX A

The general solution of Eq. (1.9) is

$$N(T) = (\alpha \int \dot{x}_{c\ell}^{-2}\, d\mu + \beta)\, \dot{x}_{c\ell} \tag{A.1}$$

where α and β are the two integration constants. One can then check that

$$\frac{N(T)}{N(0)} \left[\int_0^T \frac{N^2(T)}{N^2(\mu)}\, d\mu \right]^{-1} = \frac{\partial^2 S_{c\ell}}{\partial x(0) \partial x(T)} \tag{A.2}$$

for any α and β, such that $N^2(u)$ does not vanish for $0 \leq u \leq \tau$. One then uses the formulas

$$\frac{\partial^2 S_{c\ell}}{\partial x(0)\, \partial x(T)} = - \frac{\partial^2 W_{c\ell}}{\partial E_{c\ell}\, \partial x(0)} \frac{\partial^2 W_{c\ell}}{\partial E_{c\ell} \partial x(T)} \frac{1}{\frac{\partial^2 W_{c\ell}}{\partial E_{c\ell}^2}}$$

with

$$W_{c\ell} = \int_{x(0)}^{x(T)} \sqrt{2(E-V)}\, dx = S_{c\ell} + E_{c\ell} T$$

and

$$\frac{\partial^2 W}{\partial E_{c\ell}\, \partial x(T)} = \frac{1}{\dot{x}(T)}$$

to arrive at Eq. (1.12). The phase $e^{-i\pi}$ comes from the fact that $\frac{\partial^2 S_{c\ell}}{\partial x(0)\, \partial x(\tau)}$ vanishes each time one goes through a turning point: each turning point thus introduces the phase $e^{i\frac{\pi}{2}}$.

APPENDIX B

Consider the two-minimum potential shown in Fig. (2). Suppose that each dip in V could, ignoring the other dip, support a bound state of energy well below the barrier separating the two. From elementary considerations the coupling of these two states will then be of order of the exponential of $-\int \sqrt{V-E}\ dx$ integrated across the barrier. For reasonable barrier thickness this will be very small so that the two states are almost decoupled. Therefore the low lying energy levels are almost entirely determined by the individual shapes of the two dips. For weak coupling we replace the wells by parabolas and obtain harmonic oscillator levels. Low orders of perturbation theory would correct for deviations from a parabolic shape. It follows that a few orders of perturbation theory would give good results. Similarly, minimizing Γ computed to a few orders could be a good approximation. This sort of perturbation expansion cannot, however, be carried to far since it ignores the coupling between the states. For a calculation based on minimizing Γ, one can see this as follows. In lowest order Γ is just V and has two well-separated minima. On the other hand, the exact $\Gamma(y)$ can be shown to be the minimum of $\langle \psi | H | \psi \rangle$ over normalized wave functions such that $\langle \psi | x | \psi \rangle = y$. We can approximately compute Γ by setting $\psi = \cos\theta\, \psi_1 + \sin\theta\, \psi_2$ where ψ_1 and ψ_2 are the lowest states in wells one and two respectively. If $\langle \psi_i | H | \psi_i \rangle = \epsilon_i$ and $\langle \psi_i | x | \psi_i \rangle = x_i$ for $i = 1, 2$ then (since $\langle \psi_1 | H | \psi_2 \rangle \approx 0$ and $\langle \psi_1 | x | \psi_2 \rangle \approx 0$) $\Gamma(y) \approx \cos^2\theta(y)\epsilon_1 + \sin^2\theta(y)\epsilon_2$ when $\theta(y)$ is determined by $y = \cos^2\theta(y)\, x_1 + \sin^2\theta(y)\, x_2$.

Now for weak coupling $\epsilon_1 \approx V(x_1)$ and $\epsilon_2 \approx V(x_2)$, from which one sees that the true Γ does not have the hump between x_1 and x_2, but rather connects the minima with a straight line. This effect which does not occur to low order in perturbation theory need not keep one from using Γ to obtain a useful estimate of the energies.

APPENDIX C

When we compute $\mathrm{tr}\, e^{-iHT}$ by stationary phase, we expand the Lagrangian around the periodic classical orbit $x_{cl}(T)$. Separating the Lagrangian into terms which are quadratic in $(x-x_{cl})$ and those which are cubic or higher defines a split into an L_0 and L_1 which is analogous to that in Eq. (5.3). One could expand in powers of L_1 obtaining, in each order of $(L_1)^n$, functional integrals which can be computed by a simple extension of the methods of Sec. (3) and Appendix (A). This will lead to a Feynman diagramatic perturbation expansion which has the same topological and combinational properties as that discussed in Sec. 6. For example, the set of all diagrams will sum to an exponential of the sum of all connected diagrams and there will be both one particle reducible and irreducible diagrams. The one particle reducible diagrams can be summed as before. Let $y(\tau)$ be any periodic function. Define a functional $\Gamma(y)$ where y is any periodic function as $\Gamma(y) = S(y)$ + (the sum of all connected one particle reducible diagrams obtained by expanding the functional integral for $\mathrm{tr}\, e^{-iHT}$ around the periodic path $y(t)$). The same argument as was used in Sec. 6 then implies that $\mathrm{tr}\, e^{-iHT} = e^{i\Gamma(<x>)}$ where the periodic function $<x(\tau)>$ is determined by $\frac{\delta \Gamma}{\delta y(\tau)} = 0$ at $y(\tau) = <x(\tau)>$. This result, while elegant, is very unlikely to be of any use. For strong coupling where one has to use the full semi-classical method, the real problem is to find all the classical orbits which are stationary phase points in the functional integral.

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FIGURE CAPTIONS

Fig. 1. Periods of the anharmonic oscillator as a function of $E\lambda$, the energy times the coupling constant. The higher curves are integral multiples of the bottom curve.

Fig. 2. A potential with two minima leading to two classes of orbits in the weak coupling approximation.

Fig. 3. Some typical connected diagrams in the expansion of Eq. (5.4). Diagrams a and b are one particle reducible while c and d are one particle reducible.

Fig. 4. Some tadpole diagrams.

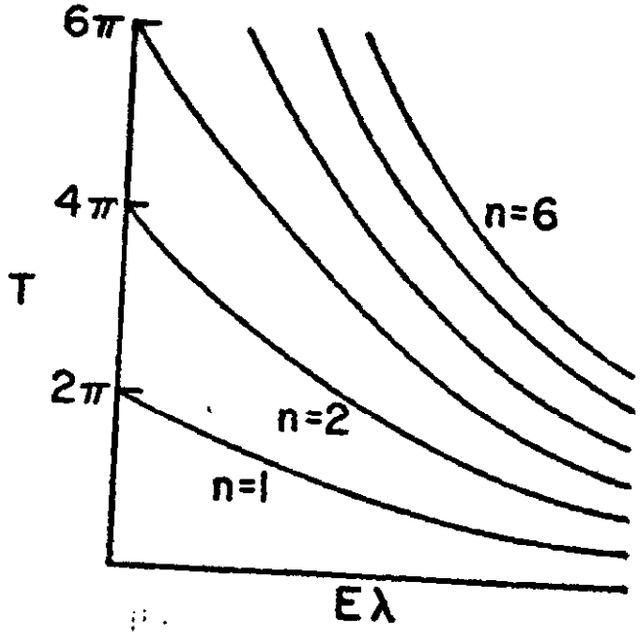

FIGURE 1.

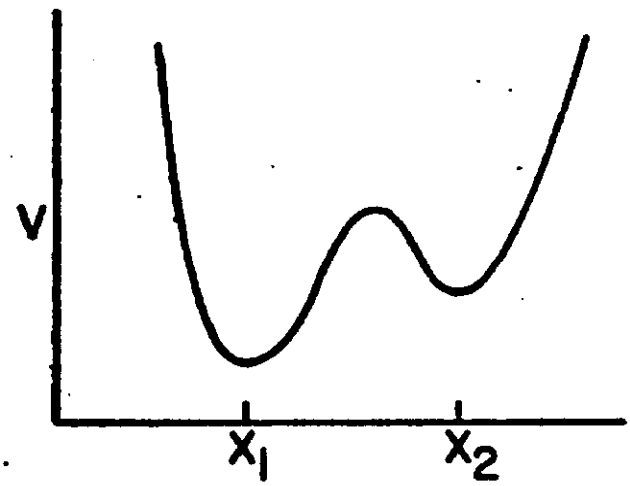

FIGURE 2.

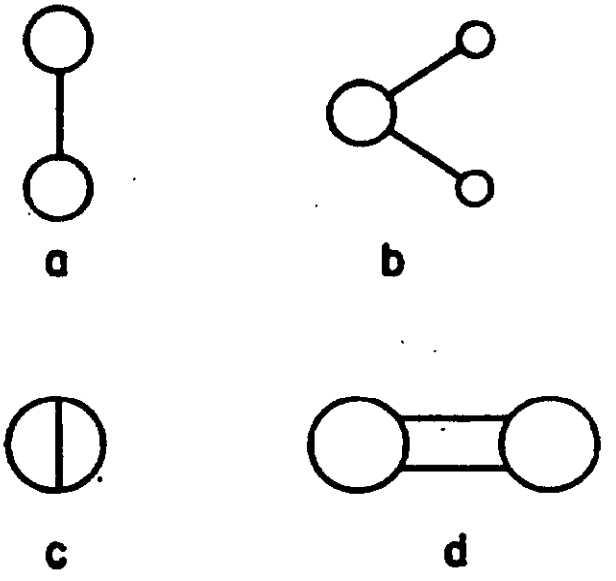

FIGURE 3.

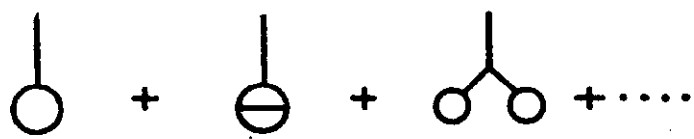

FIGURE 4.