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NONREGULAR CONTACT STRUCTURES ON BRIESKORN MANIFOLDS[1]

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1. **Introduction.** In 1958, Boothby and Wang [1] showed that if a compact, $(2n - 1)$-dimensional differentiable manifold M^{2n-1} admits a regular contact structure, then M^{2n-1} is the total space of a principal S^1 bundle over a symplectic manifold. To the authors' knowledge no examples of nonregular contact structures on compact manifolds have appeared in the literature. The purpose of this announcement is to exhibit a large class of compact manifolds that admit a contact structure; this class of manifolds includes the Brieskorn and generalized Brieskorn manifolds, and, in particular, those exotic spheres that arise as Brieskorn manifolds. Furthermore, when $n = 2$, these contact structures are often nonregular.

In the next section we recall the relevant definitions, state our main theorems, and indicate their proofs. The details will appear elsewhere.

2. Let M^{2n-1} be a $(2n - 1)$-dimensional differentiable manifold. A contact structure on M^{2n-1} is a 1-form ω that satisfies

$$\omega \wedge (d\omega)^{n-1} \neq 0 \quad \text{everywhere.} \tag{1}$$

A distribution V is associated with ω as follows. Let

$$V_p = \{X \in T_p(M^{2n-1}) \,|\, d\omega(X, Y) = 0 \;\forall\; Y \in T_p(M^{2n-1})\}.$$

Because of (1), dim $V_p = 1$. Thus, V is integrable and determines a one-dimensional foliation of M^{2n-1}. The contact structure is called regular if this foliation is regular in the sense of foliations [6]; otherwise, it is called nonregular. Recall that a foliation is called regular if for each $p \in M^{2n-1}$ there exists Froebenius coordinates around p such that different slices belong to different leaves.

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Our main theorems are the following.

THEOREM 1 . *Let N be a complex submanifold of* C^{n+1}. *Let* S^{2n+1} *be a standard sphere in* C^{n+1}. *Suppose that N intersects* S^{2n+1} *transversally in a differentiable manifold M* $(=N \cap S^{2n+1})$. *Then the restriction of the standard contact form on* S^{2n+1} *to M is a contact form on M.*

The contact form above is the same contact form considered indepenently by Hsu and Sasaki [3] when M is a Brieskorn manifold.

For example, let N be an algebraic variety minus its singular points defined by a single polynomial with the origin as an isolated singular point or a regular point. Let S^{2n+1} be a sphere of small radius ϵ with center at the origin. For ϵ sufficiently small, it is well known [4] that N intersects S^{2n+1} transversally. If the polynomial is a Brieskorn polynomial [2], [4] or weighted homogenous, we obtain the Brieskorn or generalized Brieskorn manifolds, which includes those exotic spheres which bound parallelizable manifolds. Incidentally, the above transversality holds for any complex algebraic set in C^{n+1} near an isolated singular point or a regular point.

LEMMA. *If* M^3 *is a compact three-dimensional manifold with a regular contact form, then either* $\pi_1(M^3)$ *contains an isomorphic copy of Z or* $\pi_1(M^3)$ *is cyclic.*

THEOREM 2. *There exists infinitely many three-dimensional compact manifolds with a nonregular contact structure.*

OUTLINE OF PROOFS. Let us assume that the sphere S^{2n+1} has its center at the origin of C^{n+1}. The standard contact form $\tilde{\omega}$ on S^{2n+1} is given by $\tilde{\omega}(X) = \langle X, J\vec{x} \rangle$, where $\vec{x}$ is the position vector in C^{n+1}, J the complex structure on C^{n+1}, and $\langle\ ,\ \rangle$ is the standard inner product on C^{n+1}. Let ω be the restriction of $\tilde{\omega}$ to M. The proof of Theorem 1 depends upon the fact that transversality implies that the tangent space $T_p(M)$ at each $p \in M$ is spanned by a complex subspace of $T_p(C^{n+1})$ and the projection of $J\vec{x}(p)$ onto $T_p(M)$. Using this and the intrinsic definitions of the exterior derivative and the exterior product, it is not difficult to show that $\omega \wedge (d\omega)^k \neq 0$, where $2k + 1 = \dim M$. One simply evaluates $\omega \wedge (d\omega)^k$ on an orthonormal basis $X_1, X_2, \ldots, X_{2k+1}$, where $X_1, \ldots, X_{2k}$ spans the complex subspace of $T_p(M)$.

The lemma is an easy consequence of the Boothby-Wang fibration theorem [1] and the homotopy exact sequence of a fibration.

Theorem 2 is proved by noting that the generalized Brieskorn manifold associated to the polynomial $z_1^2 + z_2^2 z_3 + z_3^{k+1}$ has fundamental group equal to the binary dihedral group with $4k$ elements [4], [5]; and this group is a finite, noncommutative group. Therefore, by the lemma, the contact structure guaranteed by Theorem 1 is nonregular.

The other examples known to the authors of compact manifolds with a nonregular contact structure are the Brieskorn and generalized Brieskorn manifolds with fundamental groups the binary icosohedral group, the binary tetrahedral group, the binary octahedral group, and the quaternion group [4].

ADDED IN PROOF.

REMARK 1. It has been pointed out to the authors that S. Tanno, *Sasakian manifolds with constant ϕ-holomorphic sectional curvature*, Tôhoku Math. J. 21 (1969), 501–507 has shown that certain $(4n + 3)$-dimensional compact manifolds of constant positive curvature admit a nonregular contact structure.

REMARK 2. Recently, Montgomery and Yang, *Differentiable actions on homotopy seven spheres*. II, Proceedings of the Conference on transformation groups, New Orleans, 1967, Springer-Verlag, New York (1968), pp. 125–134 showed that there are certain exotic 7-spheres which do not admit any free S^1-*action*. Combining that result with the Boothby-Wang fibration theorem, the contact structures exhibited above are nonregular on those exotic spheres.

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