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Nonregular Triangulations of Products of Simplices*

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Abstract. We exhibit a nonregular triangulation for the product of two tetrahedra. This answers a question by Gel'fand, Kapranov, and Zelevinsky. We also give a complete classification of the symmetry classes of regular triangulations of $\Delta_2 \times \Delta_3$. Our nonregular triangulation of $\Delta_3 \times \Delta_3$ can be extended to a nonregular triangulation of the six-dimensional cube. The four-dimensional cube is the smallest cube with a nonregular triangulation.

1. Preliminaries

Recently, Gel'fand, Kapranov, and Zelevinsky developed the theory of secondary polytopes that originated in the study of generalized hypergeometric functions and discriminants (see [5]–[7]). The vertices of a secondary polytope correspond to the regular triangulations of a point configuration. Other algebraic problems where the regularity of triangulations is important include Viro's construction of real hypersurfaces with prescribed topology [11], Gröbner bases of toric varieties [14], and the elimination theory of sparse polynomial systems [15]. For the discrete geometry community the study of regular subdivisions motivated various developments (see, for example, [4] and [2]). The topic of this note is the construction of nonregular triangulations, which have not been studied profusely and are difficult to obtain. We present solutions to two open problems regarding the existence of nonregular triangulations for the cube and the product of two simplices (see Problem 5.3 in [16] and p. 247 in [7]).

We assume familiarity with basic notions in the theory of convex polytopes (see [16]). Let $\mathcal{A} = \{a_1, a_2, \ldots, a_n\}$ denote a subset of $\mathbb{R}^d$. A *subdivision* of $\mathcal{A}$ is a collection T of subsets of $\mathcal{A}$, called *cells*, whose convex hulls form a polyhedral complex with support

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$Q = \text{conv}(\mathcal{A})$. If each cell in T is a simplex, then T is a *triangulation* of $\mathcal{A}$. Every vector $w = (w_1, \ldots, w_n)$ in $\mathbb{R}^n$ induces a subdivision of $\mathcal{A}$ as follows. Consider the polytope $Q_w = \text{conv}\{(a_1, w_1), \ldots, (a_n, w_n)\})$ which lies in $\mathbb{R}^{d+1}$. Generally, Q_w is a polytope of dimension $\dim(Q) + 1$. The *lower envelope* of Q_w is the collection of faces of the form $\{x \in Q_w \mid cx = c_0\}$ with Q_w contained in the half-space $cx \leq c_0$ and the last coordinate c_{d+1} is negative. The lower envelope of Q_w is a polyhedral complex of dimension $\dim(Q)$. We define T_w to be the subdivision of $\mathcal{A}$ whose cells are the projection of the cells of the lower envelope of Q_w. In other words, $\{a_{i_1}, a_{i_2}, \ldots, a_{i_k}\}$ is a cell of T_w if $\{(a_{i_1}, w_{i_1}), (a_{i_2}, w_{i_2}), \ldots, (a_{i_k}, w_{i_k})\}$ are the vertices of a face in the lower envelope of Q_w. We observe that for a generic choice of the vector w the subdivision T_w is in fact a triangulation of $\mathcal{A}$. A subdivision of $\mathcal{A}$ is *regular* if it is of the form T_w for some vector w. An example of a nonregular triangulation is Rudin's triangulation [12]. We remark that given a regular subdivision T_w, we can find an explicit vector w of lifting heights inducing T_w by solving a linear programming problem.

We denote the d-dimensional simplex by $\Delta_d = \text{conv}(\{e_1, \ldots, e_{d+1}\})$, where e_i is an element of the standard basis of $\mathbb{R}^{d+1}$. In what follows we represent a vertex (e_i, e_j) of $\Delta_r \times \Delta_s$ by the edge (i, j) in the complete bipartite graph $K_{r+1,s+1}$ and label both this vertex and its edge by $a[i, j]$. Under this association maximal dimensional simplices are spanning trees for $K_{r+1,s+1}$. Similarly, minimal affine dependencies among vertices of $\Delta_r \times \Delta_s$ correspond to simple cycles in $K_{r+1,s+1}$. One remarkable result, found by Gel'fand *et al.*, is the existence of a polytope whose face lattice is anti-isomorphic to the lattice of regular subdivisions of $\mathcal{A}$ ordered by refinement. This polytope, $\sum(\mathcal{A})$, is called the *secondary polytope* of $\mathcal{A}$.

2. The Product of Two Simplices

Descriptions are known for all the regular triangulations of $\Delta_1 \times \Delta_s$ with s an arbitrary positive integer and $\Delta_2 \times \Delta_2$ (see p. 246 in [7]). Results characterizing some facets of the secondary polytope $\sum(\Delta_r \times \Delta_s)$ have also been obtained [1]. In the next proposition we describe all regular triangulations of $\Delta_2 \times \Delta_3$.

Proposition 2.1.

- *The secondary polytope of* $\Delta_2 \times \Delta_3$ *has* 4488 *vertices. Up to* $S_3 \times S_4$ *symmetry* $\Delta_2 \times \Delta_3$ *has only* 35 *distinct regular triangulations.*
- *The polytope* $\Delta_2 \times \Delta_4$ *has, up to* $S_3 \times S_5$ *symmetry,* 530 *distinct regular triangulations. Its secondary polytope has* 376,200 *vertices.*

Proof. For brevity we only include the description for the regular triangulations of $\Delta_2 \times \Delta_3$ up to $S_3 \times S_4$ symmetry (for details on $\Delta_2 \times \Delta_4$ see [3]). All its triangulations have 10 maximal simplices of normalized volume one. A simple invariant uniquely distinguishes 33 of the 35 orbits of regular triangulations. This is the sorting of the vector φ_T whose ith component is the number of maximal simplices in the triangulation T that contain the ith vertex of $\Delta_2 \times \Delta_3$. The vector φ_T is the vertex of the secondary polytope $\sum(\Delta_2 \times \Delta_3)$ corresponding to the regular triangulation T. The distinct sorted vectors appear in the second column of Table 1. The fifth column provides, for each $S_3 \times S_4$ orbit, a vector $w = (w[1, 1], w[1, 2], \ldots, w[3, 3], w[3, 4])$ that induces a representative regular triangulation for that particular orbit.

We recall that, for any pair T and T' of regular triangulations of the point configuration $\mathcal{A}$, we can find a finite sequence of geometric bistellar operations that transforms T into T' (see Chapter 7 in [7]). That our classification is complete follows from this fact and the complete list of allowable bistellar operations. The numbers of the third column indicate the possible bistellar operations of a class of regular triangulations. In this way, a regular triangulation belonging to orbit 35 can only be transformed into a triangulation member of orbit 9 or 31. Notice that sometimes a geometric bistellar operation may preserve symmetry and a regular triangulation is transformed into a regular triangulation that lies in the same orbit (this is the case, for example, for orbit 15). Finally, observe that orbits 12 and 14 received the same label. We can distinguish these two types of regular triangulations by looking at the adjacencies among maximal dimensional simplices. Figure 1 shows two graphs whose vertices represent the maximal simplices of a triangulation, with an edge between a pair when the simplices have a common facet. The two graphs are very similar but still not isomorphic. This is clear since all vertices of degree two in the graph associated with orbit 12, are at a distance less than five from the unique vertex of degree one. This is not the case for the graph of orbit 14. □

The question was raised whether a nonregular triangulation could exist for $\Delta_r \times \Delta_s$ (see Chapter 7 in [7]). In the rest of this section we present the answer to this question.

Theorem 2.2. *Nonregular triangulations of $\Delta_r \times \Delta_s$ exist when $s, r \geq 3$.*

Before presenting the proof of Theorem 2.2 we require a useful lemma. Given two spanning trees S_1, S_2 of $K_{r+1,s+1}$ we consider their *superposition graph*, $S_1 * S_2$. This is

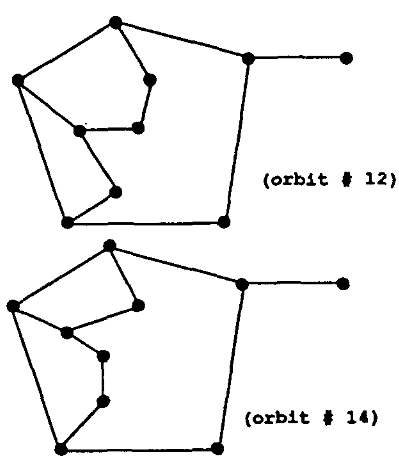

Fig. 1. Distinguishing between two regular triangulations of $\Delta_2 \times \Delta_3$.

Table 1. The classification of regular triangulations of $\Delta_2 \times \Delta_3$.

#	Label of orbit = $sort(\varphi_T)$	Neighboring orbits	Orbit's size	$w = [w[1,1], \ldots, w[3,4]]$
1	[1, 1, 3, 3, 4, 4, 6, 6, 6, 6, 10, 10]	2, 3, 4	72	[4, 3, 0, 0, 0, 0, −2, 0, 0, 1, 0, 3]
2	[1, 2, 2, 3, 3, 4, 6, 6, 6, 7, 10, 10]	1, 4, 6, 11, 27, 28	144	[5, 4, 0, 0, 0, 0, −3, 0, 0, 1, 0, 2]
3	[1, 1, 3, 4, 4, 4, 4, 5, 6, 8, 10, 10]	1, 3, 11, 14, 20, 25	144	[4, 2, 0, 0, 0, 0, −3, 0, 0, 1, 0, 4]
4	[1, 1, 3, 3, 3, 4, 6, 6, 7, 7, 9, 10]	1, 2, 5, 6, 7, 8	144	[0, −1, −5, 0, 0, 0, −2, 2, 0, 1, 0, 6]
5	[1, 1, 3, 3, 4, 4, 5, 5, 6, 9, 9, 10]	4, 9, 10, 11, 12, 13	144	[1, −1, −4, 0, 0, 0, −2, 0, 0, 1, 0, 5]
6	[1, 2, 2, 3, 3, 3, 6, 7, 7, 7, 9, 10]	2, 4, 8, 23, 27, 29	144	[1, 2, −4, 0, 0, 0, −3, 0, 0, 2, 0, 5]
7	[1, 1, 2, 3, 4, 4, 5, 7, 7, 7, 9, 10]	4, 9, 25, 28, 29, 31	144	[2, 1, −3, 0, 0, 0, −1, 0, 0, 2, 0, 4]
8	[1, 1, 3, 3, 3, 3, 7, 7, 7, 7, 9, 9]	4, 6, 32	72	[3, 1, −2, 0, 0, 0, −1, 0, 0, −1, 0, 3]
9	[1, 1, 2, 4, 4, 5, 5, 5, 5, 9, 9, 10]	5, 7, 13, 17, 35	144	[1, −1, −4, 0, 0, 0, −1, 0, 0, 2, 0, 5]
10	[1, 1, 2, 3, 3, 5, 5, 6, 8, 8, 9, 9]	5, 18, 19, 21, 30, 32	144	[0, −3, −5, 0, 0, 0, −1, 1, 0, −1, 0, 6]
11	[1, 2, 2, 3, 4, 4, 4, 5, 6, 9, 10, 10]	2, 3, 5, 14, 15, 16, 17, 18	144	[0, −2, −5, 0, 0, 0, −2, 1, 0, 1, 0, 4]
12	[1, 2, 3, 3, 3, 4, 4, 5, 6, 9, 10, 10]	5, 14, 15, 17, 21, 28	144	[−1, −2, −5, 0, 0, 0, −2, 0, 0, 1, 0, 6]
13	[1, 1, 2, 3, 4, 5, 5, 6, 7, 7, 9, 10]	5, 9, 16, 20, 28, 31	144	[0, −3, −5, 0, 0, 0, −3, 1, 0, 1, 0, 6]
14	[1, 2, 3, 3, 3, 4, 4, 5, 6, 9, 10, 10]	3, 11, 12, 15, 21, 23	144	[1, −3, −6, 0, 0, 0, −2, 0, 0, 1, 0, 3]
15	[2, 2, 2, 3, 3, 3, 4, 5, 6, 10, 10, 10]	11, 12, 14, 15, 22, 26, 34	144	[−1, −2, −5, 0, 0, 0, −2, 0, 0, 1, 0, 4]
16	[1, 2, 2, 2, 4, 4, 5, 6, 7, 8, 9, 10]	11, 13, 19, 20, 21, 22	144	[0, −4, −7, 0, 0, 0, −4, 1, 0, 1, 0, 6]
17	[1, 2, 2, 3, 4, 4, 5, 5, 5, 9, 10, 10]	9, 11, 12, 28, 34	144	[0, −2, −6, 0, 0, 0, −1, 1, 0, 2, 0, 5]
18	[1, 2, 2, 2, 3, 4, 5, 6, 8, 8, 9, 10]	10, 11, 23, 25, 26, 29, 33	144	[0, −3, −5, 0, 0, 0, −1, 1, 0, −1, 0, 3]

19	[1, 1, 2, 2, 4, 5, 5, 7, 7, 8, 9, 9]	10, 16, 24, 29, 31, 33	144	[1, −4, −5, 0, 0, 0, −3, 0, 0, −1, 0, 4]
20	[1, 1, 2, 3, 4, 4, 6, 6, 7, 8, 8, 10]	3, 13, 16, 23, 24, 25	144	[1, −2, −6, 0, 0, 0, −5, 0, 0, 1, 0, 4]
21	[1, 2, 2, 3, 3, 4, 5, 5, 8, 8, 9, 10]	10, 12, 14, 16, 25, 26	144	[1, −4, −6, 0, 0, 0, −3, 0, 0, 1, 0, 4]
22	[2, 2, 2, 2, 3, 3, 5, 6, 7, 8, 10, 10]	15, 16, 23, 26, 27, 28, 29	144	[−1, −3, −6, 0, 0, 0, −4, 0, 0, 1, 0, 5]
23	[1, 2, 2, 3, 3, 3, 6, 6, 7, 8, 9, 10]	6, 14, 18, 20, 22, 32	144	[−1, −3, −8, 0, 0, 0, −6, 0, 0, 1, 0, 5]
24	[1, 1, 1, 3, 4, 5, 6, 7, 7, 8, 8, 9]	19, 20, 30, 31, 32	144	[0, −4, −6, 0, 0, 0, −4, 1, 0, −1, 0, 4]
25	[1, 1, 2, 3, 4, 4, 5, 6, 8, 8, 8, 10]	3, 7, 18, 20, 21, 30	144	[0, −4, −7, 0, 0, 0, −4, 1, 0, 1, 0, 4]
26	[2, 2, 2, 2, 3, 3, 5, 5, 8, 8, 10, 10]	15, 18, 21, 22	72	[−1, −4, −6, 0, 0, 0, −3, 0, 0, 1, 0, 4]
27	[2, 2, 2, 2, 3, 3, 6, 6, 7, 7, 10, 10]	2, 6, 22	72	[−3, −4, −7, 0, 0, 0, −5, 0, 0, 1, 0, 6]
28	[1, 2, 2, 3, 3, 4, 5, 6, 7, 7, 10, 10]	2, 7, 12, 13, 17, 22	144	[−1, −3, −5, 0, 0, 0, −3, 0, 0, 1, 0, 6]
29	[1, 2, 2, 2, 3, 4, 5, 7, 7, 8, 9, 10]	6, 7, 18, 19, 22, 32	144	[−1, −4, −5, 0, 0, 0, −3, 0, 0, −1, 0, 4]
30	[1, 1, 1, 3, 4, 5, 6, 6, 8, 8, 8, 9]	10, 24, 25, 31, 33	144	[1, −3, −5, 0, 0, 0, −3, 0, 0, −2, 0, 2]
31	[1, 1, 1, 3, 5, 5, 5, 7, 7, 7, 9, 9]	7, 13, 19, 24, 30, 35	144	[2, −2, −3, 0, 0, 0, −4, 0, 0, −1, 0, 2]
32	[1, 1, 2, 3, 3, 4, 6, 7, 7, 8, 9, 9]	8, 10, 23, 24, 29, 33	144	[−1, −4, −7, 0, 0, 0, −5, 0, 0, −1, 0, 4]
33	[1, 1, 2, 2, 4, 4, 6, 6, 8, 8, 9, 9]	18, 19, 30, 32	72	[−1, −4, −5, 0, 0, 0, −3, 0, 0, −1, 0, 2]
34	[2, 2, 2, 3, 3, 3, 5, 5, 5, 10, 10, 10]	15, 17	48	[−1, −2, −6, 0, 0, 0, −1, 0, 0, 2, 0, 5]
35	[1, 1, 1, 5, 5, 5, 5, 5, 5, 9, 9, 9]	9, 31	48	[1, −1, −4, 0, 0, 0, 0, 0, 0, 1, −1, 4]

a graph with directed and undirected edges whose vertices are the vertices of $K_{r+1,s+1}$. The vertices of $S_1 * S_2$ are partitioned in two sets A and B (those forming the bipartition of $K_{r+1,s+1}$). If the vertices $i \in A$ and $j \in B$ are adjacent in both S_1 and S_2, then the undirected edge $\{i, j\}$ apppears in $S_1 * S_2$. When the vertices $i \in A$ and $j \in B$ are adjacent in S_1, but not in S_2, then the arrow (i, j) appears in $S_1 * S_2$. Finally, if the vertices $i \in A$ and $j \in B$ are adjacent in S_2, but not in S_1, the arrow (j, i) belongs to $S_1 * S_2$. By a cycle or a path in $S_1 * S_2$ we mean those that respect the direction of the edges (undirected edges can be used in both directions).

Lemma 2.3. *Let σ_1 and σ_2 be two full-dimensional simplices of $\Delta_r \times \Delta_s$. Consider the spanning trees S_1 and S_2 of $K_{r+1,s+1}$ associated with σ_1 and σ_2. The intersection of the two simplices is not a common face if and only if the superposition digraph $S_1 * S_2$ contains a simple cycle.*

Proof. The intersection of two simplices σ_1, σ_2 is improper precisely when a face F_1 of σ_1 and a face F_2 of σ_2 form a minimal Radon partition. Equivalently, there is a minimal affine dependence on the vertices of F_1 and F_2 such that the coefficients of elements of F_1 have positive sign and the coefficients of vertices in F_2 have negative sign. The minimal affine dependence $\sum \lambda_{ij}(e_i, e_j) = 0$ on the vertices of F_1 and F_2 corresponds to a simple cycle in the graph $S_1 * S_2$. The signs of the coefficients of the affine dependency can be interpreted as the direction of edges in the cycle. □

We remark that it can be quickly determined whether $S_1 * S_2$ has a simple cycle. One at a time consider the undirected edges: If i, j are connected in the graph $S_1 * S_2 - \{\{i, j\}\}$, then there is a cycle in $S_1 * S_2$ that contains $\{i, j\}$. Finally, if still necessary, we check the digraph D obtained by removing all the undirected edges. We successively remove vertices with zero indegree or outdegree, until we have a cycle or the empty graph. In this last case no cycle is present in D.

Proof of Theorem 2.2. We present an explicit simplicial complex T and we see that it is a nonregular triangulation of $\Delta_3 \times \Delta_3$. In Fig. 2 we have a collection of 20 six-dimensional simplices presented as spanning trees for $K_{4,4}$. These spanning trees are the vertices of a graph where two spanning trees are neighbors if they share six edges, or, equivalently, if the corresponding six-dimensional simplices have a common facet.

We claim that T is a triangulation of $\Delta_3 \times \Delta_3$. It is well known that any triangulation of $\Delta_r \times \Delta_s$ has $\binom{r+s}{r}$ maximal simplices. The 20 simplices of the simplicial complex T cover the 16 vertices of $\Delta_3 \times \Delta_3$. Lemma 2.3 can be applied to check that the 190 pairs of simplices (spanning trees) in T intersect in a common face. Since no two maximal simplices have a common interior and the number of simplices equals the normalized volume of the whole polytope, T is a triangulation of $\Delta_3 \times \Delta_3$. We also observe that the bistellar transformation that replaces $\{a[1, 4], a[2, 3], a[3, 3], a[4, 1], a[4, 2], a[4, 3], a[4, 4]\}$ (spanning tree 5) and $\{a[1, 4], a[2, 3], a[2, 4], a[3, 3], a[4, 1], a[4, 2], a[4, 4]\}$ (spanning tree 4) with the simplices $\{a[1, 4], a[2, 3], a[2, 4], a[3, 3], a[4, 1], a[4, 2], a[4, 3]\}$ and $\{a[1, 4], a[2, 4], a[3, 3], a[4, 1], a[4, 2], a[4, 3], a[4, 4]\}$ transforms our triangulation into the

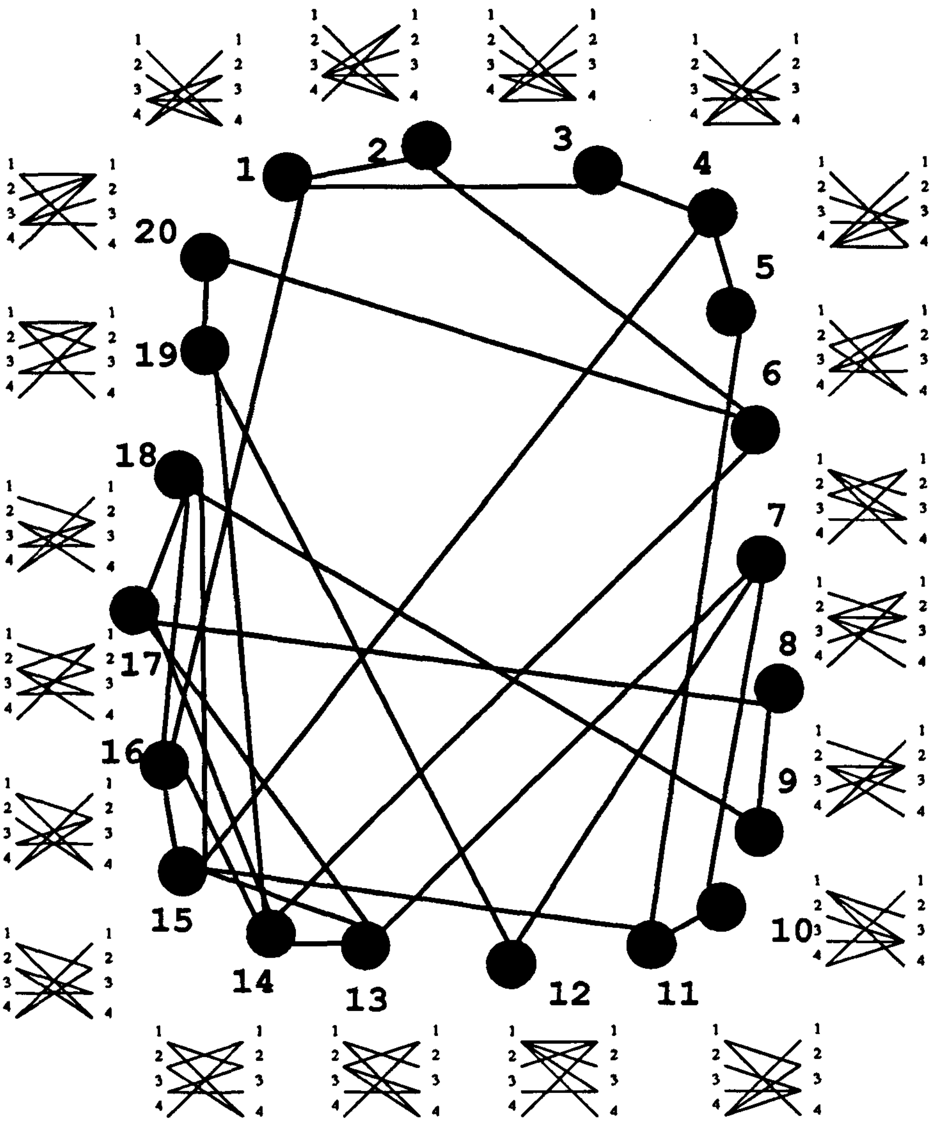

Fig. 2. A nonregular triangulation of $\Delta_3 \times \Delta_3$.

regular triangulation induced by the following lifting vector:

$$
\begin{array}{llll}
W[1,1] = 0, & W[1,2] = -6, & W[1,3] = -5, & W[1,4] = -13, \\
W[2,1] = 0, & W[2,2] = -1, & W[2,3] = -4, & W[2,4] = -10, \\
W[3,1] = 9, & W[3,2] = 2, & W[3,3] = 0, & W[3,4] = 0, \\
W[4,1] = 0, & W[4,2] = 0, & W[4,3] = 0, & W[4,4] = -1,
\end{array}
$$

However, not all the triangulations obtained from T by a bistellar operation are regular.

Now we concentrate on the proof that T is not a regular triangulation and we proceed by contradiction. If T is a regular triangulation, then a certain vector of heights $W = (W[1,1], W[1,2], \dots, W[4,4])$ exists such that the triangulation T equals the triangulation T_w induced by these heights. Observe that the cells $\{a[1,3], a[2,1]\}$, $\{a[1,1], a[3,2]\}$, $\{a[1,2], a[4,3]\}$, $\{a[2,4], a[3,1]\}$, $\{a[2,3], a[4,4]\}$, and $\{a[3,4], a[4,2]\}$ define one-dimensional faces of the triangulation T, while the cells $\{a[1,1], a[2,3]\}$, $\{a[1,2], a[3,1]\}$, $\{a[1,3], a[4,2]\}$, $\{a[2,1], a[3,4]\}$, $\{a[2,4], a[4,3]\}$, and $\{a[3,2], a[4,4]\}$ are not faces of T (they are in fact minimal nonfaces). Moreover, we have the intersection of the following pairs of segments. Each pair of segments intersects in its middle points.

$$\mathrm{conv}(\{a[1,3], a[2,1]\}) \cap \mathrm{conv}(\{a[1,1], a[2,3]\}) \neq \emptyset,$$

$$\mathrm{conv}(\{a[1,1], a[3,2]\}) \cap \mathrm{conv}(\{a[1,2], a[3,1]\}) \neq \emptyset,$$

$$\mathrm{conv}(\{a[1,2], a[4,3]\}) \cap \mathrm{conv}(\{a[1,3], a[4,2]\}) \neq \emptyset,$$

$$\mathrm{conv}(\{a[2,4], a[3,1]\}) \cap \mathrm{conv}(\{a[2,1], a[3,4]\}) \neq \emptyset,$$

$$\mathrm{conv}(\{a[2,3], a[4,4]\}) \cap \mathrm{conv}(\{a[2,4], a[4,3]\}) \neq \emptyset,$$

$$\mathrm{conv}(\{a[3,4], a[4,2]\}) \cap \mathrm{conv}(\{a[3,2], a[4,4]\}) \neq \emptyset.$$

When the points $a[1,1], a[1,2], \dots, a[4,4]$ are lifted the corresponding lifted segments cannot intersect anymore since the nonfaces must lie above the lower envelope of Q_W. This can only happen if the following inequalities are satisfied by the heights assigned to the points:

$$W[1,1] + W[2,3] > W[1,3] + W[2,1], \qquad W[1,2] + W[3,1] > W[1,1] + W[3,2],$$

$$W[1,3] + W[4,2] > W[1,2] + W[4,3], \qquad W[2,1] + W[3,4] > W[2,4] + W[3,1],$$

$$W[2,4] + W[4,3] > W[2,3] + W[4,4], \qquad W[3,2] + W[4,4] > W[3,4] + W[4,2].$$

However, this system of inequalities for the heights cannot be satisfied (the sum of these inequalities gives $0 > 0$). We have reached a contradiction and thus T is indeed a nonregular triangulation.

We observe that the product of tetrahedra $\Delta_3 \times \Delta_3$ is a subpolytope of the product of two simplices of higher dimensions (in fact it appears as a proper face). We can complete T to obtain a triangulation $\hat{T}$ of $\Delta_{(r+3)} \times \Delta_{(s+3)}$ by *placing* (see [9]) the remaining vertices (the extension can be done in many different ways). Since T is a nonregular triangulation, $\hat{T}$ must also be nonregular because any set of heights inducing the triangulation $\hat{T}$ would induce locally a regular triangulation of $\Delta_3 \times \Delta_3$ in place of T. This completes the proof of the theorem. □

3. Triangulations of the Cube

It was observed by Haiman [8] that the product of simplices can be used to produce triangulations of the d-cube using relatively few simplices. Now we use his construction to obtain the following result.

Corollary 3.1. *The six-dimensional cube I^6 has a nonregular triangulation.*

Proof. Given an arbitrary triangulation $T = \{\sigma_1, \ldots, \sigma_k\}$ of I^3. The collection $U = \{\sigma_i \times \sigma_j \mid \sigma_i, \sigma_j \in T\}$ is a polyhedral subdivision of $I^3 \times I^3 = I^6$ into products of tetrahedra. Pick any member of U, say $\sigma_1 \times \sigma_1$, and triangulate it using the nonregular triangulation we presented in Theorem 2.2. We have a new polyhedral subdivision that can be refined to a triangulation K of I^6 (Lemma 1 in [8]). The last arguments of Theorem 2.2 guarantee that K must be a nonregular triangulation of I^6. □

Theorem 3.2. *The four-dimensional cube I^4 is the smallest cube with a nonregular triangulation.*

Proof. We start with a characterization of the distinct triangulations of the 3-cube. We show that the six triangulations of I^3 are lexicographic, in the sense of [9], and hence regular. We say that a facet of a simplex is *exterior* if that facet is on the boundary of I^d. A d-simplex that has d exterior facets is called a *cornered simplex.* A cornered vertex is the vertex of a cornered simplex opposite to the unique interior facet. Mara [10] observed that the d-cube has no more than 2^{d-1} cornered simplices in a triangulation. We can rely on the number of cornered simplices to classify the triangulations of I^3. If a triangulation of I^3 has four cornered simplices, then they can be "sliced off" from the cube to leave a simplex whose vertices are the four noncornered vertices (see Fig. 3(a)). This triangulation is unique up to symmetry of the I^3. We remark that if all the edges of a simplex σ are exterior edges (lie on the boundary of I^3), then σ must be cornered or be adjacent to four cornered simplices. This will be relevant when we consider the cases of three, two, one or zero cornered simplices in a triangulation of I^3, because we know that the noncornered simplices must share a cube's diagonal (they cannot use distinct diagonals, because any two diagonals intersect).

In a triangulation with three cornered simplices the noncornered vertices support a bipyramid. This bipyramid can be triangulated in two ways, with two or three tetrahedra. This first option would produce a fourth cornered simplex and thus the only triangulation with three cornered simplices is shown in Fig. 3(b). Consider now a triangulation with two cornered simplices. Unlike the previous cases, we can distribute two cornered simplices in two ways (up to symmetry). If the two cornered vertices are opposite points of a diagonal of I^3, then the convex hull of the noncornered vertices is an octahedron. This octahedron can be triangulated in three ways (specified by choosing one of the three remaining diagonals of the cube). All three triangulations are the same up to symmetry. See Fig. 3(c) for a representative triangulation. Alternatively, the two cornered vertices may lie in a common facet of I^3. This time the convex hull of the noncornered vertices is a seven facet polytope P. Three triangulations are possible for P and again they are defined by the choice of a diagonal of I^3. Two of them yield isomorphic triangulations of I^3 and the other forces the existence of four cornered simplices. The representative triangulation is shown in Fig. 3(d).

For triangulations with one or zero cornered simplices, we must say how to triangulate the convex hull P of the noncornered vertices. All the simplices used in a triangulation of P must be noncornered and they share one diagonal. Such a triangulation of P is

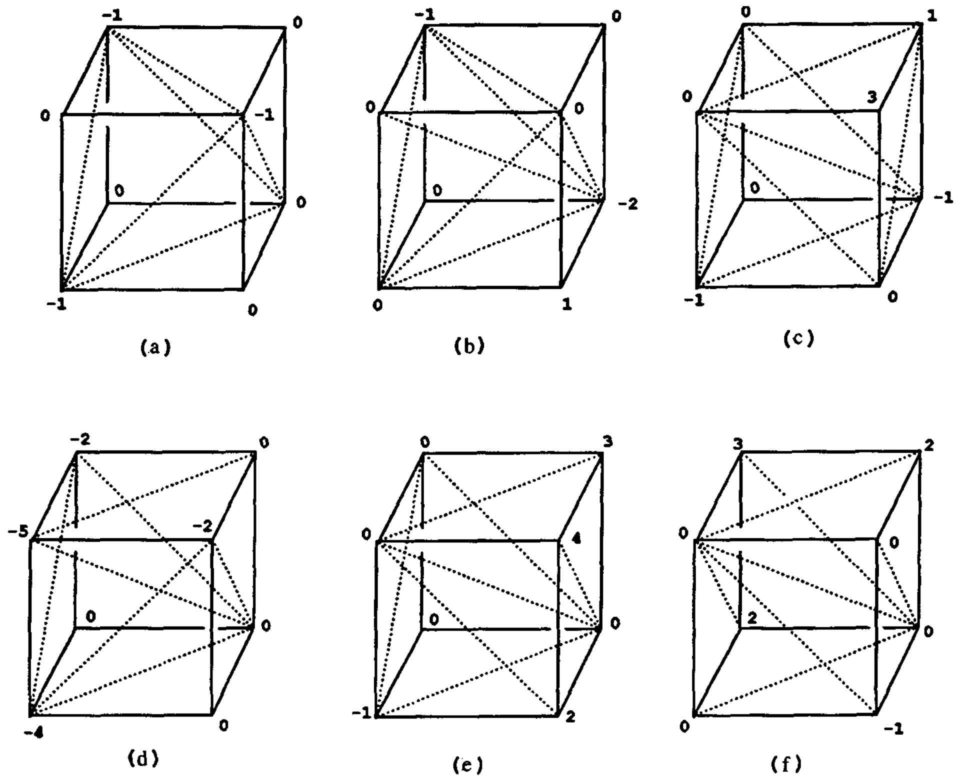

Fig. 3. The triangulations of the 3-cube.

only obtained by pulling (see [9]) the two extreme vertices of the chosen diagonal. See Fig. 3(e) and (f) for representative triangulations. For completeness we show, on top of the vertices of Fig. 3, explicit lifting vectors inducing the six triangulations of I^3.

Finally we present a nonregular triangulation of I^4 into 24 simplices. For the presentation we label the vertices of I^4 by their positions as columns of the following matrix:

$$\begin{bmatrix} v_1 & v_2 & v_3 & v_4 & v_5 & v_6 & v_7 & v_8 & v_9 & v_{10} & v_{11} & v_{12} & v_{13} & v_{14} & v_{15} & v_{16} \\ 1 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

The explicit triangulation T is given by the following maximal simplices:

$$\{7, 9, 10, 11, 13\}, \{7, 10, 11, 13, 14\}, \{10, 11, 13, 14, 16\}, \{2, 10, 13, 15, 16\},$$

$$\{2, 10, 13, 14, 16\}, \{9, 10, 11, 13, 16\}, \{8, 10, 13, 15, 16\}, \{8, 10, 12, 15, 16\},$$

$$\{1, 5, 6, 7, 10\}, \{1, 3, 5, 7, 10\}, \{5, 7, 9, 10, 13\}, \{5, 6, 7, 10, 11\},$$

$$\{5, 7, 9, 10, 11\}, \{4, 5, 7, 9, 11\}, \{5, 6, 10, 11, 12\}, \{1, 5, 6, 7, 11\},$$

$$\{1,4,5,7,11\}, \{5,10,11,12,16\}, \{5,8,10,12,16\}, \{5,9,10,11,16\},$$
$$\{5,9,10,13,16\}, \{5,8,10,13,16\}, \{3,5,8,10,13\}, \{3,5,7,10,13\}.$$

We remark that the geometric bistellar operation that replaces the simplices {5, 6, 10, 11, 12}, {5, 10, 11, 12, 16} with the simplices {5, 6, 10, 11, 16}, {5, 6, 10, 12, 16} tranforms our simplicial complex T into the regular triangulation induced by the following vector:

$$W := (0, 0, 12, -17, 0, -11, -8, 19, -16, 0, -29, 0, -2, -14, 20, -22).$$

This shows that our proposed simplicial complex T is indeed a triangulation of I^4.

We can give a proof of nonregularity of the above triangulation by looking, as we did in Theorem 2.2, at the way the faces and the nonfaces of the simplicial complex intersect. Observe that the cells {6, 7}, {11, 12}, {8, 16}, and {3, 13} define one-dimensional faces of the triangulation T, while the cells {3, 11}, {6, 16}, {12, 13}, {7, 8} are not faces of T. We have intersection of the following pairs of segments in their middle points: $\text{conv}(\{3,11\}) \cap \text{conv}(\{6,7\}) \neq \emptyset$, $\text{conv}(\{6,16\}) \cap \text{conv}(\{11,12\}) \neq \emptyset$, $\text{conv}(\{12,13\}) \cap \text{conv}(\{8,16\}) \neq \emptyset$, and $\text{conv}(\{7,8\}) \cap \text{conv}(\{3,13\}) \neq \emptyset$. One condition for the existence of lifting heights $W_1, \ldots, W_{16}$ that might induce T is that the pairs of intersecting segments above do not intersect anymore. The given nonfaces must lie above the lower envelope of the lifted points. This translates into the following infeasible system of inequalities (the sum of these inequalities implies $0 > 0$):

$$\begin{aligned} -W_3 + W_6 + W_7 - W_{11} &< 0, & -W_{16} - W_6 + W_{12} + W_{11} &< 0,\\ -W_{12} + W_{13} + W_8 + W_{16} &< 0, & -W_7 + W_3 - W_8 + W_{13} &< 0. \end{aligned}$$

This completes the proof of our theorem. We remark that we have found over 20 nonisomorphic, nonregular triangulations of I^4. The smallest of these nonregular triangulations has 22 simplices. □

Finally, we comment that recently Sturmfels has extended Theorem 2.2. Using the algebraic techniques of Gröbner bases and $\mathcal{A}$-graded monomial ideals he derived a nonregular triangulation of $\Delta_2 \times \Delta_5$, and showed that every triangulation of $\Delta_2 \times \Delta_3$ and $\Delta_2 \times \Delta_4$ is regular.

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Note added in proof. The classification of the triangulations of the three-dimensional cube was done previously by F. Bigdeli in her Ph.D. thesis (University of Kentucky, 1991).