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Nonrelativistic Geodesic Motion

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We show that any second-order dynamic equation on a configuration space $X \to \mathbf{R}$ of nonrelativistic time-dependent mechanics can be seen as a geodesic equation with respect to some (nonlinear) connection on the tangent bundle $TX \to X$ of relativistic velocities. We compare relativistic and nonrelativistic geodesic equations, and study the Jacobi vector fields along nonrelativistic geodesics.

1. INTRODUCTION

To provide a geometric formulation of nonrelativistic mechanics, one usually tries to introduce a metric on a configuration space. Following Cartan's idea, we show that any second-order dynamic equation of nonrelativistic mechanics is equivalent to a particular geodesic equation on a phase space of relativistic 4-velocities. The key point is that relativistic and nonrelativistic geodesic equations are defined on different subspaces of the same 4-velocity phase space. One can perform a relativization of nonrelativistic dynamic equations by means of their extension onto the relativistic subspace of the 4-velocity phase space. However, such an extension fails to be unique. In Section 4, we will consider the most important examples. Treating nonrelativistic dynamic equations as the geodesic ones, we can study them by means of well-known differential geometric methods. In particular, Jacobi vector fields along nonrelativistic geodesics can be introduced in a natural way, and conjugate points of these geodesics can be investigated (see Section 5).

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Let X be a 4-dimensional world manifold of a relativistic theory, coordinated by (x^λ). Then the tangent bundle TX of X plays the role of a 4-velocity phase space. By a relativistic equation of motion usually is meant a geodesic equation

$$\ddot{x}^\mu = K^\mu_\lambda(x^\nu, \dot{x}^\nu)\dot{x}^\lambda \tag{1}$$

with respect to a (nonlinear) connection

$$K = dx^\lambda \otimes (\partial_\lambda + K^\mu_\lambda \dot{\partial}_\mu) \tag{2}$$

on the tangent bundle $TX \to X$. It is supposed additionally that there is a pseudo-Riemannian metric g of signature $(+, ---)$ on X, and that a geodesic vector field does not leave the subbundle of relativistic hyperboloids

$$W_g = \{\dot{x}^\lambda \in TX | g_{\lambda\mu}\dot{x}^\lambda x^\mu = 1\} \tag{3}$$

in TX. It suffices to require that the condition

$$(\partial_\lambda g_{\mu\nu}\dot{x}^\mu + 2g_{\mu\nu}K^\mu_\lambda)\dot{x}^\lambda\dot{x}^\nu = 0 \tag{4}$$

holds for all tangent vectors which belong to W_g, (3). Of course, the Levi-Civita connection $\{_\lambda{}^\mu{}_\nu\}$ of the metric g fulfills the condition (4). Any connection K on the tangent bundle $TX \to X$ can be written as

$$K^\mu_\lambda = \{_\lambda{}^\mu{}_\nu\}\dot{x}^\nu + \sigma^\mu_\lambda(x^\lambda, \dot{x}^\lambda)$$

where the soldering form $\sigma = \sigma^\mu_\lambda\, dx^\lambda \otimes \dot{\partial}_\mu$ plays the role of an external force. Then the condition (4) takes the form

$$g_{\mu\nu}\sigma^\mu_\lambda\dot{x}^\lambda\dot{x}^\nu = 0 \tag{5}$$

Let now a world manifold X admit a projection $X \to \mathbf{R}$, where $\mathbf{R}$ is a time axis. One can think of the bundle $X \to \mathbf{R}$ as being the configuration space of nonrelativistic mechanics (Massa and Pagani, 1994; Mangiarotti and Sardanashvily, 1998; Sardanashvily, 1998). It is provided with the adapted bundle coordinates (x^0, x^i), where the transition functions of the temporal one are $x'^0 = x^0 + \text{const}$. The corresponding velocity phase space is the first-order jet manifold J^1X of $X \to \mathbf{R}$, coordinated by (x^λ, x^i_0). There is the canonical imbedding of J^1X onto the affine subbundle of the tangent bundle TX [see (10) below], given by the coordinate conditions

$$\dot{x}^0 = 1, \qquad \dot{x}^i = x^i_0 \tag{6}$$

Then one can regard (6) as the 4-velocities of a nonrelativistic system. The relation (6) differs from the familiar relation between 4- and 3-velocities of a relativistic system. It follows that the 4-velocities of relativistic and nonrelativistic systems occupy different subbundles of the tangent bundle TX. The key point of our consideration is the following.

Proposition 1. Let J^2X be the second-order jet manifold of $X \to \mathbf{R}$, coordinated by $(x^\lambda, x^i_0, x^i_{00})$. Any second-order dynamic equation

$$x^i_{00} = \xi^i(x^0, x^j, x^j_0) \tag{7}$$

of nonrelativistic mechanics on $X \to \mathbf{R}$ is equivalent to the geodesic equation

$$\begin{aligned} \ddot{x}^0 &= 0, \qquad \dot{x}^0 = 1 \\ \ddot{x}^i &= \overline{K}^i_0 \dot{x}^0 + \overline{K}^i_j \dot{x}^j \end{aligned} \tag{8}$$

with respect to a connection $\overline{K}$ on $TX \to X$ which fulfills the conditions

$$\overline{K}^0_\lambda = 0, \qquad \xi^i = \overline{K}^i_0 + x^j_0 \overline{K}^i_j \big|_{\dot{x}^0 = 1, \dot{x}^i = x^i_0} \tag{9}$$

Thus, we observe that both relativistic and nonrelativistic equations of motion can be seen as the geodesic equations on the same tangent bundle TX. The difference between them lies in the fact that their solutions live in the different subbundles (3) and (6) of TX. At the same time, relativistic equations, expressed via the 3-velocities $x^i_0 = \dot{x}^i/\dot{x}^0$, tend exactly to the nonrelativistic equations on the subbundle (6) when $\dot{x}^0 \to 1$, $g_{00} \to 1$, i.e., only when nonrelativistic mechanics and the nonrelativistic approximation of a relativistic theory coincide.

2. GEOMETRY OF NONRELATIVISTIC MECHANICS

This section is devoted to the proof of Proposition 1. Let a fiber bundle $X \to \mathbf{R}$, coordinated by (x^0, x^i) be a configuration space of nonrelativistic mechanics. Its velocity phase space J^1X is provided with the adapted coordinates (x^0, x^i, x^i_0). Recall that J^1X comprises the equivalence classes $j^1_{x^0}c$ of sections of $X \to \mathbf{R}$ which are identified by their values $c^i(x^0)$ and the values of their derivatives $\partial_0 c^i(x^0)$ at points $x^0 \in \mathbf{R}$, i.e., $x^i_0(j^1c) = \partial_0 c^i(x^0)$. There is the canonical imbedding

$$\lambda: \quad J^1X \hookrightarrow TX, \qquad \lambda = \partial_0 + x^i_0 \partial_i \tag{10}$$

over X. From now on, we will identify J^1X with its image in TX. It is an affine bundle modeled over the vertical tangent bundle VX of $X \to \mathbf{R}$.

In particular, every connection on a bundle $X \to \mathbf{R}$ is given by the nowhere-vanishing vector field

$$\Gamma: \quad X \to J^1X \subset TX, \qquad \Gamma = \partial_0 + \Gamma^i \partial_i \tag{11}$$

on X. It can be treated as a reference frame in nonrelativistic mechanics.

Every connection Γ^i (11) defines an atlas of local constant trivializations of the bundle $X \to \mathbf{R}$ and the associated coordinates (x^0, x^i) on X such that the transition functions $x^i \to x'^i$ are independent of x^0 and vice versa (Mangiarotti and Sardanashvily, 1998; Mangiarotti *et al.*, 1999). We find $\Gamma^i = 0$ with respect to this coordinate atlas, also called a reference frame. In particular, there is one-to-one correspondence between the complete connections Γ (11) and the trivializations $X \cong \mathbf{R} \times M$ of the configuration bundle X.

A nonrelativistic second-order dynamic equation on a configuration bundle $X \to \mathbf{R}$ is defined as the geodesic equation

$$x^i_{00} = \xi^i(x^\mu, x^j_0)$$

for a holonomic connection

$$\xi = \partial_0 + x^i_0\,\partial_i + \xi^i(x^\mu, x^i_0)\,\partial_i^0 \tag{12}$$

on the jet bundle $J^1X \to \mathbf{R}$, which takes its values into the second-order jet manifold J^2X. It has the transformation law

$$\xi'^i = (\xi^j\partial_j + x^j_0 x^k_0\,\partial_j\,\partial_k + 2x^j_0\,\partial_j\,\partial_0 + \partial_0^2)x'^i$$

Let us consider the relationship between the holonomic connections ξ (12) on the jet bundle $J^1X \to \mathbf{R}$ and the connections

$$\gamma = dx_\lambda \otimes (\partial_\lambda + \gamma^i_\lambda\,\partial_i^0) \tag{13}$$

on the affine jet bundle $J^1X \to X$. The connections γ have the transformation law

$$\gamma'^i_\lambda = (\partial_j x'^i \gamma^j_\mu + \partial_\mu\, x'^i_0)\,\frac{\partial x^\mu}{\partial x'^\lambda} \tag{14}$$

Proposition 2. Any connection γ (13) on the affine jet bundle $J^1X \to X$ defines the holonomic connection

$$\xi_\gamma = \partial_0 + x^i_0\,\partial_i + (\gamma^i_0 + x^j_0\,\gamma^i_j)\,\partial_i^0 \tag{15}$$

on the jet bundle $J^1X \to \mathbf{R}$ (De León and Rodrigues, 1989; Mangiarotti and Sardanashvily, 1998; Mangiarotti *et al.*, 1999).

It follows that every connection γ (13) on the affine jet bundle $J^1X \to X$ yields the dynamic equation

$$x^i_{00} = \gamma^i_0 + x^j_0\,\gamma^i_j \tag{16}$$

on the configuration space X. Of course, different dynamic connections may lead to the same dynamic equation (16). The converse assertion is the following (Crampin *et al.*, 1996; Mangiarotti and Sardanashvily, 1998; Mangiarotti *et al.*, 1999).

Proposition 3. Any holonomic connection ξ (12) on the jet bundle $J^1X \to \mathbf{R}$ defines a connection

$$\gamma = dx^0 \otimes [\partial_0 + (\xi^i - \tfrac{1}{2}x^j_0\partial^0_j\xi^i)\,\partial^0_i] + dx^j \otimes [\partial_j + \tfrac{1}{2}\partial^0_j\xi^i\,\partial^0_i] \tag{17}$$

on the affine jet bundle $J^1X \to X$.

The connection γ (17), associated with a dynamic equation, possesses the property

$$\gamma^k_i = \partial^0_i\,\gamma^k_0 + x^j_0\,\partial^0_i\,\gamma^k_j$$

which implies the relation $\partial^0_j\,\gamma^k_i = \partial^0_i\,\gamma^k_j$. Such a connection γ is called symmetric.

Let γ be a connection (13) and ξ_γ the corresponding dynamic equation (15). Then the connection (17) associated with ξ_γ takes the form

$$\gamma_{\xi_\gamma}{}^k_i = \tfrac{1}{2}(\gamma^k_i + \partial^0_i\,\gamma^k_0 + x^j_0\,\partial^0_i\,\gamma^k_j), \qquad \gamma_{\xi_\gamma}{}^k_0 = \xi^k - x^i_0\gamma_{\xi_\gamma}{}^k_i$$

It is readily observed that $\gamma = \gamma_{\xi_\gamma}$ if and only if γ is symmetric.

Now let us prove Proposition 1. We start from the relation between the connections γ on the affine jet bundle $J^1X \to X$ and the connections K, (2) on the tangent bundle $TX \to X$ of the configuration space X. Let us consider the diagram

$$\begin{tikzcd} J^1_XJ^1X \arrow[r, "J^1\lambda"] & J^1_XTX \\ J^1X \arrow[u, "\gamma"] \arrow[r, "\lambda"] & TX \arrow[u, "K"'] \end{tikzcd} \tag{18}$$

where $J^1_XJ^1X$ is the first-order jet manifold of the affine jet bundle $J^1X \to X$ with coordinates $(x^\lambda, x^i_0, x^i_{\mu 0})$ and $J^1_X TX$ is the first-order jet manifold of the tangent bundle $TX \to X$, coordinated by $(x^\lambda, \dot{x}^\lambda, \dot{x}^\lambda_\mu)$. The jet prolongation over X of the canonical imbedding λ (10) reads

$$J^1\lambda\colon\quad (x^\lambda, x^i_0, x^i_{\mu 0}) \mapsto (x^\lambda, \dot{x}^0 = 1, \dot{x}^i = x^i_0, \dot{x}^0_\mu = 0, \dot{x}^i_\mu = x^i_{\mu 0})$$

We have

$$J^1\lambda \circ \gamma\colon\quad (x^\lambda, x^i_0) \mapsto (x^\lambda, \dot{x}^0 = 1, \dot{x}^i = x^i_0, \dot{x}^0_\mu = 0, \dot{x}^i_\mu = \gamma^i_\mu)$$

$$K \circ \lambda\colon\quad (x^\lambda, x^i_0) \mapsto (x^\lambda, \dot{x}^0 = 1, \dot{x}^i = x^i_0, \dot{x}^0_\mu = K^0_\mu, \dot{x}^i_\mu = K^i_\mu)$$

It follows that the diagram (18) can be commutative only if the components K^0_μ of the connection K on $TX \to X$ vanish. Since the transition functions $x^0 \to x'^0$ are independent of x^i, a connection K with the components $K^0_\mu =$

0 can exist on the tangent bundle $TX \to X$. In particular, let (x^0, x^i) be a reference frame. Given an arbitrary connection K (2) on $TX \to X$, one can put $K^0_\mu = 0$ in order to obtain a desired connection

$$\overline{K} = dx^\lambda \otimes (\partial_\lambda + K^i_\lambda \dot{\partial}_i) \tag{19}$$

obeying the transformation law

$$K'^i_\lambda = (\partial_j x'^i K^j_\mu + \partial_\mu \dot{x}'^i) \frac{\partial x^\mu}{\partial x'^\lambda} \tag{20}$$

Now the diagram (18) becomes commutative if the connections γ and $\overline{K}$ fulfill the relation

$$\gamma^i_\mu = K^i_\mu(x^\lambda, \dot{x}^0 = 1, \dot{x}^i = x^i_0) \tag{21}$$

It is easily seen that this relation holds globally because the substitution of $\dot{x}^i = x^i_0$ into (20) restates the transformation law (14). In accordance with the relation (21), a desired connection $\overline{K}$ is an extension of the section $J^1\lambda \circ \gamma$ of the affine bundle $J^1_X TX \to TX$ over the closed submanifold $J^1X \subset TX$ to a global section. Such an extension always exists, but it is not unique. Thus, we state the following.

Proposition 4. In accordance with the relation (21), every dynamic equation on the configuration space X can be written in the form

$$x^i_{00} = K^i_0 \circ \lambda + x^j_0 K^i_j \circ \lambda \tag{22}$$

where $\overline{K}$ is a connection (19) on the tangent bundle $TX \to X$.

Let us consider the geodesic equation (8) on TX with respect to the connection $\overline{K}$. Its solution is a geodesic curve $c(t)$ also satisfying the dynamic equation (7) and vice versa. It states Proposition 1.

The above proof also leads to the following converse of Proposition 1.

Proposition 5. Given a reference frame, any connection K (2) on the tangent bundle $TX \to X$ defines a connection γ on the affine jet bundle $J^1X \to X$ and the dynamic equation (22) on the configuration space X.

Remark. Note that any second-order dynamic equation on $Q \to \mathbf{R}$ also defines a linear connection on the tangent bundle $TJ^1Q \to J^1Q$ (Massa and Pagani, 1994; Crampin *et al.*, 1996; Mangiarotti and Sardanashvily, 1998). A conservative second-order dynamic equation on a manifold Z also defines a connection on $TZ \to Z$, but it is a geodesic equation with respect to this connection if and only if this connection is a spray (Marmo *et al.*, 1990; Morandi *et al.*, 1990; Mangiarotti and Sardanashvily, 1998).

3. QUADRATIC DYNAMIC EQUATIONS

From the physical viewpoint, the most interesting dynamic equations are the quadratic ones, i.e.,

$$\xi^i = a^i_{jk}(x^\mu)x^j_0 x^k_0 + b^i_j(x^\mu)\, x^j_0 + f^i(x^\mu) \tag{23}$$

This property is coordinate independent due to the affine transformation law of coordinates x^i_0. Then, it is readily observed that the corresponding connection γ (17) is affine:

$$\gamma = dx^\lambda \otimes [\partial_\lambda + (\gamma^i_{\lambda 0}(x^\nu) + \gamma^i_{\lambda j}(x^\nu)x^j_0)\, \partial^0_i]$$

and vice versa. This connection is symmetric if and only if $\gamma^i_{\lambda\mu} = \gamma^i_{\mu\lambda}$.

Lemma 6. There is one-to-one correspondence between the affine connections γ on the affine jet bundle $J^1X \to X$ and the linear connections $\overline{K}$ (19) on the tangent bundle $TX \to X$. This correspondence is given by the relation (21), which takes the form

$$\gamma^i_\mu = \gamma^i_{\mu 0} + \gamma^i_{\mu j}\, x^j_0, \qquad \gamma^i_{\mu\lambda} = K^i_{\mu\lambda}$$

In particular, if an affine connection γ is symmetric, so is the corresponding linear connection $\overline{K}$. Then we come to the following corollaries of Propositions 1 and 5.

Proposition 7. Any quadratic dynamic equation

$$x^i_{00} = a^i_{jk}(x^\mu)\, x^j_0 x^k_0 + b^i_j(x^\mu)\, x^j_0 + f^i(x^\mu) \tag{24}$$

is equivalent to the geodesic equation

$$\begin{aligned} &\ddot{x}^0 = 0, \qquad \dot{x}^0 = 1 \\ &\ddot{x}^i = a^i_{jk}(x^\mu)\, \dot{x}^k \dot{x}^j + b^i_j(x^\mu)\dot{x}^j\dot{x}^0 + f^i(x^\mu)\dot{x}^0\dot{x}^0 \end{aligned} \tag{25}$$

for the symmetric linear connection

$$\overline{K} = dx^\lambda \otimes (\partial_\lambda + K^\mu_{\lambda\nu}(x^\alpha)\dot{x}^\nu\, \dot{\partial}_\mu)$$

on $TX \to X$, given by the components

$$K^0_{\lambda\nu} = 0, \qquad K^i_{0_j} = f^i, \qquad K^i_{0_0} = K^i_{j_0} = \frac{1}{2}\, b^i_j, \qquad K^i_{j_k} = a^i_{jk} \tag{26}$$

Proposition 8. Conversely, any linear connection K on the tangent bundle $TX \to X$ defines the quadratic dynamic equation

$$x^i_{00} = K^i_{0_0} + (K^i_{0_j} + K^i_{j_0})\, x^j_0 + K^i_{j_k}\, x^j_0,\, x^k_0$$

written with respect to a given reference frame (x^0, x^i)

The geodesic equation (25), however, is not unique for the dynamic equation (24).

Proposition 9. Any quadratic dynamic equation (24), being equivalent to the geodesic equation with respect to the linear connection K (26), is also equivalent to the geodesic equation with respect to an affine connection K' on $TX \to X$ which differs from K (26) in a soldering form σ on $TX \to X$ with the components

$$\sigma^0_\lambda = 0, \qquad \sigma^i_k = h^i_k + (s - 1)h^i_k \dot{x}^0, \qquad \sigma^i_0 = -sh^i_k \dot{x}^k - h^i_0 \dot{x}^0 + h^i_0$$

where s and h^i_λ are local functions on X.

In particular, it follows that, if there is no topological obstruction and the Minkowski metric η on TX exists, a nonrelativistic dynamic equation

$$x^i_{00} = b^i_j(x^\mu)x^j_0 + f^i(x^\mu) \tag{27}$$

gives rise to the geodesic equation

$$\ddot{x}^0 = 0, \qquad \dot{x}^0 = 1$$
$$\ddot{x}^i = b^i_j(x^\mu)\, \dot{x}^j + f^i(x^\mu)\dot{x}^0 \tag{28}$$

The above-mentioned ambiguity often occurs. The nonrelativistic dynamic equations (27) can be represented as both the geodesic equation (28) and the one (25), where $a = 0$. The first is the case for external forces, e.g., an electromagnetic theory, while the latter is that for a gravitation theory.

4. EXAMPLES

In order to compare relativistic and nonrelativistic dynamics, one should consider a pseudo-Riemannian metric on TX, compatible with the fibration $X \to \mathbf{R}$. Note that $\mathbf{R}$ is a time of nonrelativistic mechanics. It is one for all nonrelativistic observers. In the framework of a relativistic theory, this time can be seen as a cosmological time. Given a fibration $X \to \mathbf{R}$, a pseudo-Riemannian metric on the tangent bundle TX is said to be admissible if it is defined by a pair (g^R, Γ) of a Riemannian metric g^R on X and a nonrelativistic reference frame Γ (11), i.e.,

$$g = \frac{2\Gamma \otimes \Gamma}{|\Gamma|^2} - g^R, \qquad |\Gamma|^2 = g^R_{\mu\nu}\Gamma^\mu\Gamma^\nu = g_{\mu\nu}\Gamma^\mu\Gamma^\nu \tag{29}$$

in accordance with the well-known theorem (Hawking and Ellis, 1973). The vector field Γ is timelike relative to the pseudo-Riemannian metric g (29), but not with respect to other admissible pseudo-Riemannian metrics in general.

As we have shown above, given a reference frame (x^0, x^i), any connection $K(x^\lambda, \dot{x}^\lambda)$ (2) on the tangent bundle $TX \to X$ defines the connection $\overline{K}$ on $TX \to X$ with the components

$$\overline{K}^0_\lambda = 0, \qquad \overline{K}^i_\lambda = K^i_\lambda \tag{30}$$

It follows that, given a fibration $X \to \mathbf{R}$, every relativistic equation of motion (1) yields the geodesic equation (8) and, consequently, has the counterpart

$$x^i_{00} = K^i_0(x^\lambda, 1, x^k_0) + K^i_j(x^\lambda, 1, x^k_0)x^i_0$$

(7) in nonrelativistic mechanics. Note that, written with respect to a reference frame (x^0, x^i), the connection $\overline{K}$ (9) and the corresponding geodesic equation (8) are well defined relative to any coordinates on X, while the dynamic equation (1) is done relative to arbitrary coordinates on X, compatible with the fibration $X \to \mathbf{R}$. The key point is that, for another reference frame (x^0, x'^i) with time-dependent transition functions $x^i \to x'^i$, the same connection K (2) on TX sets another connection $\overline{K}'$ on $TX \to X$ with the components

$$K'^0_\lambda = 0, \qquad K'^i_\lambda = \left(\frac{\partial x'^i}{\partial x^j} K^j_\mu + \frac{\partial x'^i}{\partial x^\mu}\right)\frac{\partial x^\mu}{\partial x'^\lambda} + \frac{\partial x'^i}{\partial x^0}\frac{\partial x^\mu}{\partial x'^\lambda} K^0_\mu$$

while the connection $\overline{K}$ (30) has the components

$$\overline{K}'^0_\lambda = 0, \qquad \overline{K}'^i_\lambda = \left(\frac{\partial x'^i}{\partial x^j} K^j_\mu + \frac{\partial x'^i}{\partial x^\mu}\right)\frac{\partial x^\mu}{\partial x'^\lambda}$$

relative to the same reference frame. This illustrates the obvious fact that a nonrelativistic approximation is not relativistic invariant (see, e.g., Lévy-Leblond, 1967).

The converse procedure is more intricate. First, a nonrelativistic dynamic equation (7) is brought into the geodesic equation (8) with respect to the connection $\overline{K}$ (9). A solution is not unique in general. Then, one finds a pair (g, K) of a pseudo-Riemannian metric g and a connection K on $TX \to X$ such $K^i_\lambda = \overline{K}^i_\lambda$ and the condition (4) is fulfilled.

Given a coordinate system (x^0, x^i) compatible with the fibration $X \to \mathbf{R}$, let us consider a nondegenerate quadratic Lagrangian

$$L = \frac{1}{2} m_{ij}(x^\mu)x^i_0 x^j_0 + k_i(x^\mu)x^i_0 + f(x^\mu) \tag{31}$$

where m_{ij} is a Riemannian mass tensor. Similarly to Lemma 6, one can show that any quadratic polynomial on $J^1X \subset TX$ is extended to a bilinear form on TX. Then the Lagrangian L (31) can be written as

$$L = -\frac{1}{2} g_{\alpha\mu} x_0^\alpha x_0^\mu, \qquad x_0^0 = 1 \tag{32}$$

where g is the metric

$$g_{00} = -2f, \qquad g_{0i} = -k_i, \qquad g_{ij} = -m_{ij} \tag{33}$$

on X. The corresponding Lagrange equation takes the form

$$x_{00}^i = -(m^{-1})^{ik}\{_{\lambda k\nu}\} x_0'^\lambda x_0^\nu, \qquad x_0^0 = 1 \tag{34}$$

where

$$\{_{\lambda\mu\nu}\} = -\frac{1}{2}(\partial_\lambda g_{\mu\nu} + \partial_\nu g_{\mu\lambda} - \partial_\mu g_{\lambda\nu})$$

are the Christoffel symbols of the metric (33). Let us assume that this metric is nondegenerate. By virtue of Proposition 7, the dynamic equation (34) can be brought into the geodesic equation (25) on TX, which reads

$$\ddot{x}^0 = 0, \qquad \dot{x}^0 = 1$$

$$\ddot{x}^i = (\{_\lambda{}^i{}_\nu\} - \frac{g^{i0}}{g^{00}}\{_\lambda{}^0{}_\nu\})\dot{x}^\lambda \dot{x}^\nu \tag{35}$$

Let us now bring the Lagrangian (31) into the form

$$L = \frac{1}{2} m_{ij}(x^\mu)(x_0^i - \Gamma^i)(x_0^j - \Gamma^j) + f'(x^\mu) \tag{36}$$

where Γ is a Lagrangian connection on $X \to \mathbf{R}$. This connection Γ defines an atlas of local constant trivializations of the bundle $X \to \mathbf{R}$ and the corresponding coordinates $(x^0, \overline{x}^i)$ on X such that the transition functions $\overline{x}^i \to \overline{x}'^i$ are independent of x^0, and $\Gamma^i = 0$ with respect to $(x^0, \overline{x}^i)$. In this coordinates, the Lagrangian L (36) reads

$$L = \frac{1}{2} \overline{m}_{ij} \overline{x}_0^i \overline{x}_0^j + f'(x^\mu)$$

One can think of its first term as the kinetic energy of a nonrelativistic system with the mass tensor $\overline{m}_{ij}$ relative to the reference frame Γ, while $(-f')$ is a potential. Let us assume that f' is a nowhere-vanishing function on X, i.e., the metric (33) is nondegenerate. Then the Lagrange equation (34) takes the form

$$\overline{x}_{00}^i = \{_\lambda{}^i{}_\nu\} \overline{x}_0^\lambda \overline{x}_0^\nu, \qquad \overline{x}_0^0 = 1$$

where $\{_\lambda{}^i{}_\nu\}$ are the Christoffel symbols of the metric (33) whose components with respect to the coordinates $(x^0, \overline{x}^i)$ read

$$g_{ij} = -\overline{m}_{ij}, \qquad g_{0i} = 0, \qquad g_{00} = -2f' \tag{37}$$

This metric is Riemannian if $f' > 0$ and pseudo-Riemannian if $f' < 0$. Then the spatial part of the corresponding geodesic equation

$$\begin{aligned} \ddot{\overline{x}}^0 &= 0, \qquad \dot{\overline{x}}^0 = 1 \\ \ddot{\overline{x}}^i &= \{_\lambda{}^i{}_\nu\}\dot{\overline{x}}^\lambda \dot{\overline{x}}^\nu \end{aligned}$$

is exactly the spatial part of the geodesic equation with respect to the Levi-Civita connection of the metric (37) on TX. It follows that, as declared above, the nonrelativistic dynamic equation (37) describes the nonrelativistic approximation of the geodesic motion in the Riemannian or pseudo-Riemannian space with the metric (37). Note that the spatial part of this metric is the mass tensor, which may be treated as a variable (Mangiarotti and Sardanashvily, 1998).

Conversely, let us consider a geodesic motion

$$\ddot{x}^\mu = \{_\lambda{}^\mu{}_\nu\}\dot{x}^\lambda \dot{x}^\nu \tag{38}$$

in the presence of a pseudo-Riemannian metric g on a world manifold X. Let $(x^0, \overline{x}^i)$ be local hyperbolic coordinates such that $g_{00} = 1$, $g_{0i} = 0$. These coordinates define a nonrelativistic reference frame for a local fibration $X \to \mathbf{R}$. Then Eq. (38) has the nonrelativistic limit

$$\begin{aligned} \ddot{\overline{x}}^0 &= 0, \qquad \dot{\overline{x}}^0 = 1 \\ \ddot{\overline{x}}^i &= \{_\lambda{}^i{}_\nu\}\dot{\overline{x}}^\lambda \dot{\overline{x}}^\nu \end{aligned} \tag{39}$$

which is the Lagrange equation for the Lagrangian

$$L = \frac{1}{2}\,\overline{m}_{ij}\overline{x}^i_0\overline{x}^j_0$$

describing a free nonrelativistic mechanical system with the mass tensor $\overline{m}_{ij} = -g_{ij}$. Relative to another frame $(x^0, x^i, (x^0, \overline{x}^j))$ associated with the same local splitting $X \to \mathbf{R}$, the nonrelativistic limit of Eq. (38) keeps the form (39), whereas the nonrelativistic equation (39) is brought into the Lagrange equation (35) for the Lagrangian

$$L = \frac{1}{2}\,m_{ij}(x^\mu)(x^i_0 - \Gamma^i)(x^j_0 - \Gamma^j) \tag{40}$$

This Lagrangian describes a mechanical system in the presence of the inertial force associated with the reference frame Γ. The difference between (35) and (39) shows that a gravitational force cannot model an inertial force in general. Nevertheless, if the mass tensor in the Lagrangian L (40) is independent of time, the corresponding Lagrange equation is a spatial part of the geodesic equation in a pseudo-Riemannian space.

In view of the ambiguity that we have mentioned, the relativization (32) of an arbitrary nonrelativistic quadratic Lagrangian (31) may lead to confusion. In particular, it can be applied to a gravitational Lagrangian (36) where f' is a gravitational potential. An arbitrary quadratic dynamic equation can be written in the form

$$x^i_{00} = -(m^{-1})^{ik}\{_{\lambda k\mu}\}x^\lambda_0\, x^\mu_0 + b^i_\mu\,(x^\nu)x^\mu_0, \qquad x^0_0 = 1$$

where $\{_{\lambda k\mu}\}$ are the Christoffel symbols of some admissible pseudo-Riemannian metric g, whose spatial part is the mass tensor $(-m_{ik})$, while

$$b^i_k(x^\mu)x^k_0 + b^i_0(x^\mu) \tag{41}$$

is an external force. With respect to the coordinates where $g_{0i} = 0$, one may construct the relativistic equation

$$\ddot{x}^\mu = \{_\lambda{}^\mu{}_\nu\}\dot{x}^\lambda\dot{x}^\nu + \sigma^\mu_\lambda\dot{x}^\lambda \tag{42}$$

where the soldering form σ must fulfill the condition (5). It holds only if

$$g_{ik}b^i_j + g_{ij}b^i_k = 0,$$

i.e., the external force (41) is the Lorentz-type force plus some potential one. Then we have

$$\sigma^0_0 = 0, \qquad \sigma^0_k = -g^{00}g_{kj}\,b^j_0, \qquad \sigma^i_0 = b^i_0$$

The relativization (42) exhausts almost all familiar examples. It means that a wide class of mechanical systems can be represented as a geodesic motion with respect to some affine connection in the spirit of the above-mentioned idea of Cartan.

To complete our exposition, we point out also another "relativization" procedure. Let a force $\xi^i(x^\mu)$ in the nonrelativistic dynamic equation (7) be a spatial part of a 4-vector ξ^λ in the Minkowski space (X, η). Then one can write the relativistic equation

$$\ddot{x}^\lambda = \xi^\lambda - \eta_{\alpha\beta}\zeta^\beta\dot{x}^\alpha\,\dot{x}^\lambda \tag{43}$$

This is the case, e.g., for a relativistic hydrodynamics that we meet usually in the literature on a gravitation theory. However, this is not a geodesic equation, and the nonrelativistic limit $\dot{x}^0 = 1$ of Eq. (43) does not coincide with the initial nonrelativistic equation. There are also other variants of relativistic hydrodynamic equations (Kupershmidt, 1992).

5. NONRELATIVISTIC JACOBI FIELDS

Let us consider the quadratic dynamic equation (23) and the equivalent geodesic equation (8) with respect to the symmetric linear connection $\overline{K}$ (26). Its curvature

$$R_{\lambda\mu}{}^{\alpha}{}_{\beta} = \partial_\lambda K_\mu{}^\alpha{}_\beta - \partial_\mu K_\lambda{}^\alpha{}_\beta + K_\lambda{}^\gamma{}_\beta K_\mu{}^\alpha{}_\gamma - K_\mu{}^\gamma{}_\beta K_\lambda{}^\alpha{}_\gamma$$

has the temporal component

$$R_{\lambda\mu}{}^{0}{}_{\beta} = 0 \tag{44}$$

It should be emphasized that our expressions for connections and the curvature differ in a minus sign from those usually used. Then the equation for a Jacobi vector field u along a geodesic reads

$$\dot{x}^\beta \dot{x}^\mu \, (\nabla_\beta(\nabla_\mu u^\alpha) - R_{\lambda\mu}{}^{\alpha}{}_{\beta} u^\lambda) = 0, \qquad \nabla_\beta \dot{x}^\alpha = 0 \tag{45}$$

where ∇ denotes covariant derivatives relative to the connection $\overline{K}$ (Kobayashi and Nomizu, 1969). Due to the relation (44), Eq. (45) for the temporal component u^0 of a Jacobi field takes the form

$$\dot{x}^\beta \dot{x}^\mu (\partial_\mu \partial_\beta u^0 + K_\mu{}^\gamma{}_\beta \partial_\gamma u^0) = 0$$

We chose its solution

$$u^0 = 0 \tag{46}$$

because all nonrelativistic geodesics obey the constraint $\dot{x}^0 = 0$.

In the case of a quadratic Lagrangian L, Eq. (45) coincides with the Jacobi equation

$$u^j \, d_0(\partial_j \dot{\partial}_i L) + d_0(\dot{u}^j \dot{\partial}_i \dot{\partial}_j L) - u^j \partial_i \partial_j \, L = 0$$

for a Jacobi field on solutions of the Lagrange equations for L. This equation is the Lagrange equation for the vertical extension L_V of the Lagrangian L (Mangiarotti and Sardanashvily, 1998; see also Dittrich and Reuter, 1992).

Let us consider the quadratic Lagrangian (31) with a Riemannian mass tensor m_{ij}. The corresponding Lagrange equations are equivalent to the geodesic equation (8) for the linear connection

$$\overline{K}_\lambda{}^0{}_\mu = 0, \qquad \overline{K}_\lambda{}^i{}_\mu = (-m^{-1})^{ik} \{_{\lambda k \mu}\} \tag{47}$$

where $\{_{\lambda k \mu}\}$ are the Christoffel symbols of the metric (33). This metric is not necessarily Riemannian. Therefore, given a reference frame (x^0, x^λ), let us consider another metric

$$\overline{g}_{00} = -1, \qquad \overline{g}_{0i} = 0, \qquad \overline{g}_{ij} = -m_{ij} \tag{48}$$

which is always Riemannian. However, its covariant derivative with respect to the connection K (47) does not vanish. We have

$$\nabla_\lambda \overline{g}_{00} = \nabla_\lambda \overline{g}_{ik} = 0, \qquad \nabla_\lambda \overline{g}_{0k} = \{_{\lambda k 0}\} \neq 0$$

Nevertheless, due to the condition (46), the well-known formula

$$\int_a^b (\overline{g}_{\lambda\mu}\dot{x}^\alpha \nabla_\alpha u^\lambda \dot{x}^\beta \nabla_\beta u^\mu + R_{\lambda\mu\alpha\nu}\, u^\lambda u^\alpha \dot{x}^\mu \dot{x}^\nu)\, dt$$
$$+ \overline{g}_{\lambda\mu}\dot{x}^\alpha \nabla_\alpha u^\lambda u'^\mu\Big|_{t=a} - \overline{g}_{\lambda\mu}\dot{x}^\alpha \nabla_\alpha u^\lambda u'^\mu\Big|_{t=b} = 0 \tag{49}$$

holds for a Jacobi vector field u along a geodesic c. Accordingly, the following assertions also hold (Kobayashi and Nomizu, 1969).

Proposition 10. If the sectional curvature $R_{\lambda\mu\alpha\nu}u^\lambda u^\alpha \dot{x}^\mu \dot{x}^\nu$ is nonpositive, a geodesic motion has no conjugate points.

Proposition 11. If the sectional curvature $R_{\lambda\mu\alpha\nu}u^\lambda u^\alpha v^\mu v^\nu$, where u, v are arbitrary unit vectors on a Riemannian manifold X, exceeds $k > 0$, then for every geodesic the distance between two consecutive conjugate points is at most $\pi/\sqrt{k}$.

For instance, let us consider a one-dimensional motion described by the Lagrangian

$$L = \frac{1}{2}(\dot{x}^1)^2 - \phi(x^1)$$

where ϕ is a potential. The corresponding Lagrange equations are equivalent to the geodesic one on the 2-dimensional space $\mathbf{R}^2$ with respect to the connection K whose nonzero component is $\{_0{}^1{}_0\} = -\partial_1\phi$. The curvature of K has the nonzero component

$$R_{10}{}^1{}_0 = \partial_1\{_0{}^1{}_0\} = -\partial_1^2\phi$$

Choosing the Riemannian metric (48)

$$\overline{g}_{11} = -1, \qquad \overline{g}_{01} = 0, \qquad \overline{g}_{00} = -1$$

we come to the formula (49)

$$\int_a^b [(\dot{x}^\mu \partial_\mu u^1)^2 - \partial_1^2\phi(u^1)^2]\, dt = 0$$

for a Jacobi vector field u which vanishes at points a and b. Then we obtain from Proposition 10 that, if $\partial_1^2\phi < 0$ at points of c, this motion has no conjugate points. In particular, let us consider the oscillator $\phi = k(x^1)^2/2$. In this case, the sectional curvature is $R_{0101} = k$, while the half-period of this oscillator is exactly $\pi/\sqrt{k}$ in accordance with Proposition 11.

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