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NONSELFADJOINT SUBALGEBRAS ASSOCIATED WITH COMPACT ABELIAN GROUP ACTIONS ON FINITE VON NEUMANN ALGEBRAS

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1. Introduction. Let G be a compact abelian group whose dual Γ has a total order. Suppose that M is a von Neumann algebra with a faithful normal tracial state τ and $\{\alpha_g\}_{g\in G}$ is a σ-weakly continuous representation of G as *-automorphisms of M such that $\tau\circ\alpha_g=\tau$, $g\in G$. Put $\Gamma_+=\{\gamma\in\Gamma:\gamma\geqq 0\}$ and let $H^\infty(\alpha)$ be the set of $x\in M$ such that $Sp_\alpha(x)\subset\Gamma_+$. Recently, the structure of $H^\infty(\alpha)$ has been investigated by several authors (cf. [7], [8], [9], [10], [12], [13], [15]). It is well-known that $H^\infty(\alpha)$ is a finite maximal subdiagonal algebra of M (cf. [8]). However, $H^\infty(\alpha)$ is not necessarily maximal as a σ-weakly closed subalgebra of M. McAsey, Muhly and the author in [9], [10] and [15] studied the maximality of typical examples of $H^\infty(\alpha)$ which are called nonselfadjoint crossed products.

Our aim in this paper is to investigate the maximality of $H^\infty(\alpha)$ as a σ-weakly closed subalgebra of M. Our method is based on a characterization of spectral subspaces and the invariant subspace structure of the noncommutative Lebesgue space $L^2(M,\tau)$ associated with M and τ in the sense of Segal [16]. In §2, we give a characterization of spectral subspaces. For every $\gamma\in\Gamma$, we put $M_\gamma=\{x\in M:\alpha_g(x)=\langle g,\gamma\rangle x, g\in G\}$. Suppose that the center $\mathfrak{Z}(M_0)$ of M_0 is contained in the center $\mathfrak{Z}(M)$ of M. If $M_\gamma\neq\{0\}$, then there is a partial isometry u_γ in M_γ and a projection e_γ in $\mathfrak{Z}(M_0)$ such that $M_\gamma=M_0u_\gamma$ and $u_\gamma^*u_\gamma=u_\gamma u_\gamma^*=e_\gamma$. In particular, if M_0 is a factor, then we may choose a unitary element u_γ in M_γ such that $M_\gamma=M_0u_\gamma$. In §3, we first define the cocycles of canonical left-invariant subspaces of $L^2(M,\tau)$. If M_0 is a factor, then every two-sided invariant subspace is left-pure and left-full. As the main result in this paper, we show that, if $\mathfrak{Z}(M_0)\subset\mathfrak{Z}(M)$ and if there is no nonzero projection p of $\mathfrak{Z}(M_0)$ with $Mp=M_0p$, then $H^\infty(\alpha)$ is a maximal σ-weakly closed subalgebra of M if and only if M_0 is a factor and $Sp\alpha$ is a subgroup (of Γ) with an archimedean order.

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2. A characterization of spectral subspaces. Suppose that M is a finite von Neumann algebra acting on a Hilbert space H and that $\{\alpha_g\}_{g\in G}$ is a σ-weakly continuous representation of a compact abelian group G as a group of *-automorphisms of M. For simplicity, such an $\{\alpha_g\}_{g\in G}$ is called a compact abelian group action on M in this paper. Following Arveson [3] and Loebl-Muhly [8], we define a representatin $\alpha(\cdot)$ of $L^1(G)$ into the algebra of bounded operators on M by

$$\alpha(f)x = \int_G f(g)\alpha_g(x)d\mu(g) ,$$

where $f \in L^1(G)$ and μ is the normalized Haar measure on G. Let Γ be the dual group of G. The pairing between G and Γ will be written as $\langle g, \gamma\rangle$, $g \in G$, $\gamma \in \Gamma$, hence the Fourier transform will take this form: $\hat{f}(\gamma) = \int_G \langle g, \gamma\rangle f(g)d\mu(g)$, $f \in L^1(G)$. If $f \in L^1(G)$, we let $Z(f) = \{\gamma \in \Gamma: \hat{f}(\gamma) = 0\}$. We let $Sp\alpha$ be $\bigcap Z(f)$, where f runs through the set of functions in $L^1(G)$ such that $\alpha(f) = 0$. If $x \in M$, we let $Sp_\alpha(x) = \bigcap Z(f)$, where $\alpha(f)x = 0$, $f \in L^1(G)$. If S is a subset of Γ, we denote by $M^\alpha(S)$ the set of $x \in M$ such that $Sp_\alpha(x) \subset S$. For every $\gamma \in \Gamma$ we define a σ-weakly continuous linear map ε_γ on M by the integration

$$\varepsilon_\gamma(x) = \int_G \overline{\langle g, \gamma\rangle}\alpha_g(x)d\mu(g) , \qquad x \in M .$$

Put $\varepsilon_\gamma(M) = M_\gamma$. Then it is clear that

$$M_\gamma = \{x \in M: \alpha_g(x) = \langle g, \gamma\rangle x, g \in G\} .$$

The following lemma is well-known and easy to prove.

LEMMA 2.1 (cf. [12], [4]). *Keep the notations as above. Then*

(1) $M_\gamma = M^\alpha(\{\gamma\})$.

(2) $M_\gamma M_\lambda \subset M_{\gamma+\lambda}$ *and* $M_\gamma^* = M_{-\gamma}$ *for every* $\gamma, \lambda \in \Gamma$.

(3) *Let* $x, y \in M$. *If* $\varepsilon_\gamma(x) = \varepsilon_\gamma(y)$ *for each* $\gamma \in \Gamma$, *then* $x = y$.

(4) $Sp_\alpha(x) = \{\gamma \in \Gamma: \varepsilon_\gamma(x) \neq 0\}$ *for* $x \in M$.

(5) $Sp\alpha = \{\gamma \in \Gamma: M_\gamma \neq \{0\}\}$.

(6) *Let* $x \in M_\gamma$ *and let* $x = v|x|$ *be the polar decomposition of* x. *Then* $v \in M_\gamma$ *and* $|x| \in M_0$.

By a result of Connes [4, Théorème 2.2.4], if M_0 is a factor, then $Sp\alpha$ is a subgroup of Γ. Thus we have the following analogue of Størmer [17, Theorem 3.2].

LEMMA 2.2. *Keep the notations as above. If M_0 is a factor, then the dual $(Sp\alpha)\hat{}$ of $Sp\alpha$ is canonically isomorphic to G/N, where N is the kernel* $\ker\alpha$ *of* α *in* G.

Our goal in this section is the following theorem whose proof is inspired by Araki [1].

THEOREM 2.3. *In the notations above, suppose that the center* $\mathfrak{Z}(M_0)$ *of* M_0 *is contained in the center* $\mathfrak{Z}(M)$ *of* M. *Then for every* $\gamma \in Sp\alpha$, *there exist a partial isometry* u_γ *in* M_γ *and a projection* e_γ *in* $\mathfrak{Z}(M_0)$ *such that* $M_\gamma = M_0 u_\gamma$ *and* $u_\gamma^* u_\gamma = u_\gamma u_\gamma^* = e_\gamma$.

PROOF. Let $\gamma \in Sp\alpha$. By Lemma 2.1 (2), it is clear that the linear span S of $M_\gamma^* M_\gamma$ is a two-sided ideal of M_0. Then there exists a nonzero projection e_γ in $\mathfrak{Z}(M_0)$ such that the σ-weak closure $\bar{S}$ of S equals $M_0 e_\gamma$. Further, since $4y^*x = (x+y)^*(x+y) - (x-y)^*(x-y) + i(x+iy)^*(x+iy) - i(x-iy)^*(x-iy)$, $x, y \in M$, we have

$$S = \left\{\sum_{n=1}^{m} \alpha_n x_n^* x_n : x_n \in M, \alpha_n \in \boldsymbol{C}\right\} ,$$

where $\boldsymbol{C}$ is the complex field. Hence there exists a sequence $\{y_\lambda\}_{\lambda \in \Lambda}$ in S such that $e_\gamma = \sigma$-weak limit y_λ. Put $p = \sup\{u^*u : u$ is a partial isometry of $M_\gamma\}$. By Lemma 2.1 (6), $e_\gamma - p = (e_\gamma - p)e_\gamma = \sigma$-weak limit $(e_\gamma - p)y_\lambda = 0$ and so $e_\gamma = p$. Since e_γ is a central projection of M, we have $uu^* \leqq e_\gamma$ for every partial isometry u in M_γ. Thus we similarly have $e_\gamma = \sup\{uu^* : u$ is a partial isometry of $M_\gamma\}$.

Next we show that there is a partial isometry u_γ of M_γ such that $u_\gamma^* u_\gamma = u_\gamma u_\gamma^* = e_\gamma$. Consider a maximal family $\{u_\lambda\}_{\lambda \in \Lambda}$ of partial isometries of M_γ such that $u_\lambda u_\lambda^*$ are mutually orthogonal and $u_\lambda^* u_\lambda$ are mutually orthogonal. Put $u_\gamma = \sum_{\lambda \in \Lambda} u_\lambda$. Then u_γ is a partial isometry of M_γ. Suppose that $e_\gamma - u_\gamma^* u_\gamma \neq 0$. Since $e_\gamma = \sup\{u^*u : u$ is a partial isometry of $M_\gamma\}$, there exists a partial isometry v in M_γ such that $v^*v(e_\gamma - u_\gamma^* u_\gamma) \neq 0$. By the comparability theorem, there are a central projection z in M_0 and partial isometries u_1 and u_2 in M_0 such that $u_1^* u_1 = z(e_\gamma - u_\gamma^* u_\gamma)$, $u_1 u_1^* \leqq z v^* v$, $u_2^* u_2 = (1-z)v^*v$ and $u_2 u_2^* \leqq (1-z)(e_\gamma - u_\gamma^* u_\gamma)$. Then we have either $u_1 \neq 0$ or $u_2 \neq 0$. If $u_1 \neq 0$, then we set $v_1 = zvu_1$. Thus $v_1^* v_1 = u_1^* z v^* v u_1 = u_1^* u_1 u_1^* u_1 = u_1^* u_1 = z(e_\gamma - u_\gamma^*) \leqq e_\gamma - u_\gamma^* u_\gamma$ and v_1 is a nonzero partial isometry in M_γ. If $u_2 \neq 0$, then we set $v_1 = (1-z)vu_2^*$. Thus $v_1^* v_1 = u_2 u_2^* \leqq e_\gamma - u_\gamma^* u_\gamma$ and v_1 is a nonzero partial isometry in M_γ. Let T (resp. T_0) be the center valued trace of M (resp. M_0). Since $\mathfrak{Z}(M_0) \subset \mathfrak{Z}(M)$, the restriction of T to M_0 equals T_0. Hence we have

$$\begin{aligned} T_0(e_\gamma - u_\gamma u_\gamma^*) &= T(e_\gamma - u_\gamma u_\gamma^*) = T(e_\gamma - u_\gamma^* u_\gamma) \\ &\geqq T(v_1^* v_1) = T(v_1 v_1^*) = T_0(v_1 v_1^*) . \end{aligned}$$

By [18, p. 314, Corollary 2.8], $v_1 v_1^* \precsim e_\gamma - u_\gamma u_\gamma^*$. Thus there is a partial isometry u in M_0 such that $u^*u = v_1 v_1^*$ and $uu^* \leqq e_\gamma - u_\gamma u_\gamma^*$. Put $v_2 =$

uv_1. Then

$$v_2^* v_2 = v_1 u^* u v_1 = v_1^* v_1 \leqq e_\gamma - u_\gamma^* u_\gamma$$

and

$$v_2 v_2^* = u v_1 v_1^* u^* = u u^* \leqq e_\gamma - u_\gamma u_\gamma^* .$$

Since v_2 is a nonzero partial isometry in M_γ, this contradicts the maximality of $\{u_\lambda\}_{\lambda\in\Lambda}$. It is clear that $M_\gamma = M_0 u_\gamma$. Hence we are done.

COROLLARY 2.4. *If M_0 is a factor, then there exists a unitary element u_γ of M_γ such that $M_\gamma = M_0 u_\gamma$ for every $\gamma \in Sp\alpha$.*

3. Invariant subspaces and maximality of $H^\infty(\alpha)$. Let M be a von Neumann algebra with a faithful normal tracial state τ. Let $\{\alpha_g\}_{g\in G}$ be a compact abelian group action on M such that $\tau\circ\alpha_g = \tau$, $g \in G$. We suppose that the dual group Γ of G has a total order. Set $\Gamma_+ = \{\gamma \in \Gamma: \gamma \geqq 0\}$ and $\Gamma_{+0} = \{\gamma \in \Gamma: \gamma > 0\}$, respectively. Let $L^2(M, \tau)$ be the noncommutative Lebesgue space associated with M and τ (cf. [16]). For every $x \in M$, we define operators L_x and R_x on $L^2(M, \tau)$ by the formulae $L_x y = xy$ and $R_x y = yx$, $y \in L^2(M, \tau)$. For a subset S of M, we write $L(S) = \{L_x: x \in S\}$ and $R(S) = \{R_x: x \in S\}$, respectively. For a subset S of $L^2(M, \tau)$, we denote by $[S]_2$ the closed linear span of S in $L^2(M, \tau)$. Further, we define $H^\infty(\alpha) = M^\alpha(\Gamma_+)$, which is called the noncommutative Hardy space with respect to $\{\alpha_g\}_{g\in G}$. We also define $H_0^\infty(\alpha) = M^\alpha(\Gamma_{+0})$, $H^2(\alpha) = [H^\infty(\alpha)]_2$ and $H_0^2(\alpha) = [H_0^\infty(\alpha)]_2$. Since $\tau\circ\alpha_g = \tau$, there is a unitary group $\{W_g\}_{g\in G}$ on $L^2(M, \tau)$ such that $W_g L_x W_g^* = L_{\alpha_g(x)}$ and $W_g R_x W_g^* = R_{\alpha_g(x)}$, $g \in G$, $x \in M$. By Lemma 2.1 and [8], we have the following:

PROPOSITION 3.1. (1) *$H^\infty(\alpha)$ is a finite maximal subdiagonal algebra of M with respect to ε_0 and τ.*

(2) $H^\infty(\alpha) = \{x \in M: \varepsilon_\gamma(x) = 0, \gamma \in \Gamma, \gamma < 0\}$.

(3) $H_0^\infty(\alpha) = \{x \in H^\infty(\alpha): \varepsilon_0(x) = 0\}$.

We first define invariant subspaces of $L^2(M, \tau)$ according to [9], [10] and [15].

DEFINITION 3.2. Let $\mathfrak{M}$ be a closed subspace of $L^2(M, \tau)$. We say that $\mathfrak{M}$ is left-invariant, if $L(H^\infty(\alpha))\mathfrak{M} \subset \mathfrak{M}$; left-reducing, if $L(M)\mathfrak{M} \subset \mathfrak{M}$; left-pure, if $\mathfrak{M}$ contains no left-reducing subspace; and left-full, if the smallest left-reducing subspace containing $\mathfrak{M}$ is all of $L^2(M, \tau)$. The right-hand versions of these concepts are defined similarly. A closed subspace which is both left- and right- invariant will be called two-sided invariant.

Throughout this section, we suppose that M_0 is a factor. By Corollary 2.4, there exists a family $\{u_\gamma\}_{\gamma \in Sp\alpha}$ of unitary operators in M such that $M_\gamma = M_0 u_\gamma$, $\gamma \in Sp\alpha$.

PROPOSITION 3.3 (cf. [15, Proposition 3.2]). *Let $\mathfrak{M}$ be a left-invariant subspace of $L^2(M, \tau)$. Then we have the following*:

(1) *$\mathfrak{M}$ is left-reducing if and only if $u_\gamma \mathfrak{M} \subset \mathfrak{M}$ for every $\gamma \in Sp\alpha$.*

(2) *$\mathfrak{M}$ is left-pure if and only if $\bigwedge_{\gamma \in Sp\alpha} u_\gamma \mathfrak{M} = \{0\}$.*

(3) *$\mathfrak{M}$ is left-full if and only if $\bigvee_{\gamma \in Sp\alpha} u_\gamma \mathfrak{M} = L^2(M, \tau)$.*

Throughout this section, suppose that $Sp\alpha$ has an Archimedean order, that is, $Sp\alpha$ may be regarded as a subgroup of $\boldsymbol{R}$ with the discrete topology ([19, Theorem 8.1.2]). Thus $Sp\alpha$ is order isomorphic onto $\boldsymbol{Z}$ or a dense subgroup of $\boldsymbol{R}$ with the discrete topology.

Let $\mathfrak{M}$ be a left-invariant subspace of $L^2(M, \tau)$. Put $\mathfrak{M}_\gamma = u_\gamma \mathfrak{M}$, $\gamma \in Sp\alpha$. The family of subspaces $\mathfrak{M}_\gamma$ decreases as γ increases in $Sp\alpha$. If $Sp\alpha$ is a dense subgroup of $\boldsymbol{R}$ with the discrete topology, then we have

$$\mathfrak{M}_{(+)} = \bigwedge \{\mathfrak{M}_{-\gamma} : \gamma \in Sp\alpha \cap \Gamma_{+0}\} \quad \text{and} \quad \mathfrak{M}_{(-)} = \bigvee \{\mathfrak{M}_\gamma : \gamma \in Sp\alpha \cap \Gamma_{+0}\} .$$

DEFINITION 3.4. Let $\mathfrak{M}$ be a left-invariant subspace of $L^2(M, \tau)$. If $Sp\alpha$ is a dense subgroup of $\boldsymbol{R}$ with the discrete topology, then $\mathfrak{M}$ is said to be left- (resp. right-) normalized in case $\mathfrak{M} = \mathfrak{M}_{(+)}$ (resp. $\mathfrak{M} = \mathfrak{M}_{(-)}$). If $\mathfrak{M}$ is both left- and right-normalized, then $\mathfrak{M}$ is said to be completely normalized. Further, if $Sp\alpha$ is a dense subgroup of $\boldsymbol{R}$ (resp. $Sp\alpha$ is order-isomorphic onto $\boldsymbol{Z}$), then a left-invariant subspace $\mathfrak{M}$ of $L^2(M, \tau)$ is said to be canonical in case $\mathfrak{M}$ is left-pure, left-full and left-normalized (resp. left-pure and left-full).

Next we define cocycles of canonical left-invariant subspaces of $L^2(M, \tau)$. We now fix such a subspace $\mathfrak{M}$ of $L^2(M, \tau)$. For $\gamma \in Sp\alpha$, we denote by P_γ tne projection of $L^2(M, \tau)$ onto $\mathfrak{M}_\gamma$. As γ increases in $Sp\alpha$, P_γ decreases from the identity 1 to 0, by Proposition 3.3. For each real number λ not in $Sp\alpha$, we define P_λ so that the family $\{P_\lambda\}_{\lambda \in \boldsymbol{R}}$ is continuous from the left. Then $1 - P_\lambda$ is a resolution of the identity in $L^2(M, \tau)$, to which by Stone's theorem is associated the unitary group $\{V_t\}_{t \in \boldsymbol{R}}$ defined by

$$V_t = -\int_{-\infty}^{\infty} e^{it\lambda} dP_\lambda . \tag{3.1}$$

Since $L(M_0)\mathfrak{M}_\lambda \subset \mathfrak{M}_\lambda$, it is clear that P_t and V_t are in $L(M_0)'$ for $t \in \boldsymbol{R}$. Hence we have $P_{\lambda+\gamma} = L_{u_\gamma} P_\lambda L_{u_\gamma^*}$ and

$$L_{u_\gamma^*}V_tL_{u_\gamma} = -\int_{-\infty}^{\infty} e^{it\lambda}d(L_{u_\gamma^*}P_\lambda L_{u_\gamma}) = -\int_{-\infty}^{\infty} e^{it\lambda}dP_{\lambda-\gamma} = e^{it\gamma}V_t\ .$$

PROPOSITION 3.5 (cf. [15, Theorem 4.1]). *Keep the notations and the assumptions as above. The families* $\{P_t\}_{t\in\boldsymbol{R}}$ *and* $\{V_t\}_{t\in\boldsymbol{R}}$ *associated with a canonical left-invariant subspace* $\mathfrak{M}$ *satisfy*

$$(3.2)\qquad \begin{cases} P_{\lambda+\gamma} = L_{u_\gamma}P_\lambda L_{u_\gamma^*}\ , \\ V_tL_u = e^{it\gamma}L_{u_\gamma}V_t\ , \\ P_t,\ V_t \in L(M_0)'\ ,\quad t, \lambda\in\boldsymbol{R}\ ,\quad \gamma\in Sp\alpha\ . \end{cases}$$

Conversely, every left-continuous family $\{P_t\}_{t\in\boldsymbol{R}}$ *of projections and every continuous unitary group* $\{V_t\}_{t\in\boldsymbol{R}}$ *satisfying* (3.2) *are obtained from a unique, canonical left-invariant subspace of* $L^2(M, \tau)$.

Put $N=\ker\alpha$. Since $Sp\alpha$ is a subgroup of Γ, the dual $(Sp\alpha)^\wedge$ of $Sp\alpha$ is canonically isomorphic to G/N by Lemma 2.2. Since $Sp\alpha$ is also a subgroup of $\boldsymbol{R}$, let e_t for each real number t be the element of G/N defined by $e_t(\lambda)=e^{it\lambda}$, $\lambda\in Sp\alpha$. It is easy to verify that the mapping ω defined by $\omega(t)=e_t$ is a continuous homomorphism of $\boldsymbol{R}$ into G/N and the image $\omega(\boldsymbol{R})$ is a dense subgroup of G/N. Now $\{\alpha_g\}_{g\in G}$ (resp. $\{W_g\}_{g\in G}$) induces a σ-weakly continuous representation of $\{\tilde{\alpha}_{[g]}\}_{[g]\in G/N}$ (resp. $\{\tilde{W}_{[g]}\}_{[g]\in G/N}$) of *-automorphisms of M (resp. unitary operators on $L^2(M,\tau)$), where $\tilde{\alpha}_{[g]}=\alpha_g$ (resp. $\tilde{W}_{[g]}=W_g$), with the coset $[g]$ of g in G/N. It is clear that $L_{\tilde{\alpha}_{[g]}}(x)=\tilde{W}_{[g]}L_x\tilde{W}^*_{[g]}$, $[g]\in G/N$. Put $S_t=\tilde{W}_{\omega(t)}$, $t\in R$. Then $\{S_t\}_{t\in\boldsymbol{R}}$ is a continuous unitary group on $L^2(M,\tau)$ and we have the following:

THEOREM 3.6. *Keep the notations and the assumptions as above. Then each continuous unitary group* $\{V_t\}_{t\in\boldsymbol{R}}$ *on* $L^2(M,\tau)$ *satisfying* (3.2) *has the form* $V_t=R_{a_t}S_t$, *where* $\{a_t\}_{t\in\boldsymbol{R}}$ *is a continuous unitary family of* M *such that*

$$(3.3)\qquad a_{t+u}=\tilde{\alpha}_{\omega(t)}(a_u)a_t\ ,\quad t, u\in\boldsymbol{R}\ .$$

Conversely, if $\{a_t\}_{t\in\boldsymbol{R}}$ *is any such unitary family of* M, *then* $V_t=R_{a_t}S_t$ *defines a continuous unitary group on* $L^2(M,\tau)$ *which satisfies* (3.2).

PROOF. Put $A_t=V_tS_t^*$. Since $(Sp\alpha)^\wedge$ is canonically isomorphic to G/N, $Sp\alpha$ is the annihilator of N, that is, $Sp\alpha=\{\gamma\in\Gamma\colon \langle g,\gamma\rangle=1$ for all $g\in N\}$. Thus we have

$$\begin{aligned} S_tL_{u_\gamma}S_t^* &= \tilde{W}_{\omega(t)}L_{u_\gamma}\tilde{W}^*_{\omega(t)} = L_{\tilde{\alpha}_{\omega(t)}(u_\gamma)} = L_{\alpha_g(u_\gamma)} \\ &= \langle g,\gamma\rangle L_{u_\gamma} = \langle\omega(t),\gamma\rangle L_{u_\gamma} = e^{it\gamma}L_{u_\gamma}\ , \end{aligned}$$

where $t\in\boldsymbol{R}$, $\gamma\in Sp\alpha$ and $g\in\omega(t)$. Thus

$$A_t^* L_{u_\gamma} A_t = (V_t S_t^*)^* L_{u_\gamma} (V_t S_t^*) = S_t V_t^* L_{u_\gamma} V_t S_t^* = e^{-it\gamma} S_t L_{u_\gamma} S_t^* = L_{u_\gamma} .$$

Since V_t and S_t are elements in $L(M_0)'$ and $L(M)$ is generated by $L(M_0)$ and $\{L_{u_\gamma}\}_{\gamma \in Sp\alpha}$, we have $A_t \in L(M)' = R(M)$. Thus there is a unitary family $\{a_t\}_{t \in R}$ of M such that $A_t = R_{a_t}$. Further, we have

$$\begin{aligned} A_{t+u} &= V_{t+u} S_{t+u}^* = V_t S_t^* S_t V_u S_u^* S_t^* = A_t S_t A_u S_t^* \\ &= R_{a_t} S_t R_{a_u} S_t^* = R_{a_t} R_{\tilde{\alpha}_{\omega(t)}(a_u)} = R_{\tilde{\alpha}_{\omega(t)}(a_u) a_t} . \end{aligned}$$

Thus $a_{t+u} = \tilde{\alpha}_{\omega(t)}(a_u) a_t$.

Conversely, put $V_t = R_{a_t} S_t$. By (3.3), $\{V_t\}_{t \in R}$ is a continuous unitary group of $L(M_0)'$. By Stone's Theorem, there is a left-continuous family $\{P_t\}_{t \in R}$ of projections of $L(M_0)'$ such that $V_t = -\int_{-\infty}^{\infty} e^{it\lambda} dP_\lambda$. Now, for $\gamma \in Sp\alpha$ and $t \in \boldsymbol{R}$, we have

$$\begin{aligned} L_{u_\gamma} V_t L_{u_\gamma}^* &= L_{u_\gamma} R_{a_t} S_t L_{u_\gamma}^* = R_{a_t} S_t S_t^* L_{u_\gamma} S_t L_{u_\gamma}^* \\ &= R_{a_t} S_t L_{\tilde{\alpha}_{\omega(-t)}(u_\gamma)} L_{u_\gamma}^* = e^{-it\gamma} R_{a_t} S_t = e^{-it\gamma} V_t . \end{aligned}$$

Therefore $\{P_t\}_{t \in R}$ and $\{V_t\}_{t \in R}$ satisfy (3.2). This completes the proof.

DEFINITION 3.7. A unitary family $\{a_t\}_{t \in R}$ of M satisfying the conditions of Theorem 3.6 is called a cocycle determined by a canonical left-invariant subspace of $L^2(M, \tau)$.

Next we show that, if M_0 is a factor, then every two-sided invariant subspace of $L^2(M, \tau)$ which is not left-reducing is left-pure and left-full. To prove this, we need the following lemmas.

LEMMA 3.8. *Suppose that M_0 is a factor and $Sp\alpha$ has an Archimedean order. If B is an $\{\alpha_g\}_{g \in G}$-invariant σ-weakly closed subalgebra of M containing $H^\infty(\alpha)$, then either $B = H^\infty(\alpha)$ or $B = M$.*

PROOF. Since B is $\{\alpha_g\}_{g \in G}$-invariant and σ-weakly closed, $\varepsilon_\gamma(x)$ lies in B for all $x \in B$. Hence, if $H^\infty(\alpha) \neq B$, then there is an $x \in B$ and a γ $(<0) \in Sp\alpha$ such that $\varepsilon_\gamma(x) \neq 0$. For this x, we may write $\varepsilon_\gamma(x) = a u_\gamma$ for some $a \in M_0$. But, since $M_0 \subset H^\infty(\alpha) \subset B$, we have $M_0 a M_0 u_\gamma = M_0 a u_\gamma M_0 \subset B$. Since finite factors are algebraically simple ([3, p. 257]), $M_0 a M_0 = M_0$, and $u_\gamma \in B$. For every γ' $(<0) \in Sp\alpha$, if $\gamma' > \gamma$, then $M_0 u_{\gamma'} = M_0 u_{\gamma' - \gamma} u_\gamma \subset B$. On the other hand, if $\gamma' < \gamma$, then there exists an $n > 0$ such that $n\gamma \leqq \gamma'$. Thus $M_0 u_{\gamma'} = M_0 u_{\gamma' - n\gamma} u_\gamma^n \subset B$ and $B = M$. This completes the proof.

LEMMA 3.9. *Suppose that M_0 is a factor, M is not a factor and $Sp\alpha$ has an Archimedean order. Then $\mathfrak{Z}(M) \cap H^\infty(\alpha)$ is a maximal σ-weakly closed subalgebra of $\mathfrak{Z}(M)$.*

PROOF. Set $\mathfrak{Z}(M)\cap H^{\infty}(\alpha)=\mathfrak{A}$ and $[\mathfrak{Z}(M)]_2=K$. Let x be a nonzero element in $\mathfrak{A}$. We now consider the closed subspace $[\mathfrak{A}x]_2$ $(=\mathfrak{M})$ of $[\mathfrak{A}]_2$. Since $\tilde{\alpha}_{[g]}(\mathfrak{Z}(M))=\mathfrak{Z}(M)$, we put $\beta_{[g]}=\tilde{\alpha}_{[g]}|_{\mathfrak{Z}(M)}$, $[g]\in G/N$. Since $\{\beta_{[g]}\}_{[g]\in G/N}$ acts ergodically on $\mathfrak{Z}(M)$, $Sp\beta$ is a subgroup of $Sp\alpha$ by Lemma 2.1. Let E be the support projection of x. As in the proof of [15, Proposition 5.2], we have $\beta_{\omega(t)}(E)=E$. Since $\omega(\boldsymbol{R})$ is dense in G/N, we have $\beta_{[g]}(E)=E$ for every $[g]\in G/N$, hence $E=1$. By [11, Theorem], $\mathfrak{A}$ is a maximal σ-weakly closed subalgebra of $\mathfrak{Z}(M)$ and the proof is completed.

Since M is generated by M_0 and $\{u_\gamma\}_{\gamma\in Sp\alpha}$, we have the following theorem by Lemmas 3.8 and 3.9 as in the proof of [15, Theorem 5.3].

THEOREM 3.10. *Suppose that M_0 is a factor and $Sp\alpha$ has an Archimedean order. Then every-sided invariant subspace of $L^2(M,\tau)$ which is not left-reducing is left-pure and left-full.*

Finally we study the maximality of $H^{\infty}(\alpha)$ as a σ-weakly closed subalgebra of M.

THEOREM 3.11. *Suppose that M_0 is a factor and $Sp\alpha$ has an Archimedean order. Let $\mathfrak{M}$ be a canonical left-invariant subspace of $L^2(M,\tau)$. If $B=\{x\in M: L_x\mathfrak{M}\subset\mathfrak{M}\}$, then $B=H^{\infty}(\alpha)$.*

PROOF. Let $\{V_t\}_{t\in\boldsymbol{R}}$ be a continuous unitary group associated with $\mathfrak{M}$. Since $L_{\tilde{\alpha}_{\omega(t)}(x)}=S_tL_xS_t^*=V_tL_xV_t^*$ by Theorem 3.6, we have

$$L_{\tilde{\alpha}_{\omega(t)}(x)}\mathfrak{M}=V_tL_xV_t^*\mathfrak{M}\subset V_tL_x\mathfrak{M}\subset V_t\mathfrak{M}\subset\mathfrak{M}$$

for $x\in B$. Thus $\tilde{\alpha}_{\omega(t)}(x)\in B$. Since $\omega(\boldsymbol{R})$ is dense in G/N, we have $\tilde{\alpha}_{[g]}(x)\in B$ for every $[g]\in G/N$ and so $\alpha_g(x)\in B$, $g\in G$. Therefore B is $\{\alpha_g\}_{g\in G}$-invariant. Since B is a σ-weakly closed subalgebra of M containing $H^{\infty}(\alpha)$, we have $B=H^{\infty}(\alpha)$ by Lemma 3.8. This completes the proof.

THEOREM 3.12. *Suppose that M_0 is a factor and $Sp\alpha$ has an Archimedean order. Then $H^{\infty}(\alpha)$ is a maximal σ-weakly closed subalgebra of M.*

To prove this theorem, we need the following lemma as in the proof of [15, Theorem 6.3] if $Sp\alpha$ is a dense subgroup of $\boldsymbol{R}$.

LEMMA 3.13. *Suppose that M_0 is a factor and $Sp\alpha$ is a dense subgroup of $\boldsymbol{R}$ with the discrete topology. Let $\mathfrak{M}$ be a left-invariant subspace of $L^2(M,\tau)$. If $\mathfrak{M}$ is not left-reducing, then so is $\mathfrak{M}_{(+)}$.*

PROOF. Suppose that $\mathfrak{M}_{(+)}$ is left-reducing. For every $x\in\mathfrak{M}$, we have $u_{-2\rho}x\in\mathfrak{M}_{(+)}$ for each $\rho\in Sp\alpha\cap\Gamma_{+0}$. Hence $u_\gamma u_{-2\rho}x\in\mathfrak{M}$ for each $\gamma\in Sp\alpha\cap\Gamma_{+0}$. Since there is an element $\gamma\in Sp\alpha\cap\Gamma_{+0}$ such that $\gamma<\rho$, we see that $M_0 u_{-\rho} x = M_0 u_{\rho-\gamma} u_\gamma u_{-2\rho} x \subset \mathfrak{M}$. Thus $u_{-\rho} x \in \mathfrak{M}$ and so $\mathfrak{M}$ is left-reducing. This is a contradiction and completes the proof.

PROOF OF THEOREM 3.12. Let B be a proper σ-weakly closed subalgebra of M containing $H^\infty(\alpha)$. Let $[B]_2$ be the closed linear span of B in $L^2(M, \tau)$. By [9, Corollary 1.5], we have $[B]_2 \neq L^2(M, \tau)$. It is clear that $[B]_2$ is a two-sided invariant subspace of $L^2(M, \tau)$ which is not left-reducing. If $Sp\alpha$ is a dense subgroup of $\boldsymbol{R}$ (resp. isomrphic onto $\boldsymbol{Z}$), let $\mathfrak{M}$ be the two-sided invariant subspace $([B]_2)_{(+)}$ (resp. $[B]_2$) of $L^2(M, \tau)$. By Lemma 3.11, $\mathfrak{M}$ is not left-reducing. Hence, by Theorem 3.10, $\mathfrak{M}$ is left-full and left-pure and so $\mathfrak{M}$ is canonical. As in the proof of [15, Theorem 6.3], we have Theorem 3.12 by Theorem 3.11. This completes the proof.

It is attractive to conjecture that the converse of Theorem 3.12 is true. As a partial answer, we have the following:

THEOREM 3.14. *Suppose that $\mathfrak{Z}(M_0) \subset \mathfrak{Z}(M)$ and there is no nonzero projection $p \in \mathfrak{Z}(M_0)$ such that $M_0 p = Mp$. Then $H^\infty(\alpha)$ is a maximal σ-weakly closed subalgebra of M if and only if M_0 is a factor and $Sp\alpha$ is a subgroup (of Γ) with an Archimedean order.*

PROOF. ($\Leftarrow$) is trivial by Theorem 3.12.

($\Rightarrow$). First we suppose that M_0 is not a factor. Then there exists a nonzero projection $p \in \mathfrak{Z}(M_0)$ such that $M_0 p \neq Mp$. Considering a σ-weakly closed subalgebra B generated by $H^\infty(\alpha)p$ and $M(1-p)$, this is clearly a contradiction. Therefore M_0 is a factor. Hence $Sp\alpha$ is a subgroup of Γ. Next we suppose that $Sp\alpha$ does not have an Archimedean order. Then there are $\lambda, \gamma \in Sp\alpha \cap \Gamma_{+0}$ such that $n\lambda \leqq \gamma$, $n = 1, 2, 3, \cdots$. Let B be the σ-weakly closed subalgebra of M generated by u_λ^* and $H^\infty(\alpha)$. Then $B \neq H^\infty(\alpha)$. Since $u_\lambda^{*n} u_\lambda \in H_0^\infty(\alpha)$, $n = 1, 2, 3, \cdots$, we have $\tau(x u_\lambda^{*n} u_\lambda) = 0$ for every $x \in H^\infty(\alpha)$. Hence it is clear that $\tau(y u_\lambda) = 0$ for every $y \in B$. This implies that $B \neq M$, a contradiction.

REMARK 3.15. Suppose that $\mathfrak{Z}(M_0) \subset \mathfrak{Z}(M)$. By Theorem 2.3, for every $\gamma \in Sp\alpha$ there are a partial isometry u_γ in M_γ and a projection e_γ in $\mathfrak{Z}(M_0)$ such that $M_\gamma = M_0 u_\gamma$ and $u_\gamma^* u_\gamma = u_\gamma u_\gamma^* = e_\gamma$. Put $e = \sup\{e_\gamma : \gamma \in Sp\alpha \cap \Gamma_{+0}\}$. Then $M_0(1-e) = M(1-e)$ and $M_0 p \neq Mp$ for every projection $p \in \mathfrak{Z}(M_0)$ such that $0 < p \leqq e$. Thus $H^\infty(\alpha) = H^\infty(\alpha)e \oplus M_0(1-e)$. To prove the maximality of $H^\infty(\alpha)$, it is sufficient to consider the part of $H^\infty(\alpha)e$. Therefore, by Theorem 3.14, $H^\infty(\alpha)$ is a maximal σ-weakly closed subalgebra of M if and only if $M_0 e$ is a factor and $Sp\alpha$ has an Archimedean order.

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