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NONSEPARABLE APPROXIMATE EQUIVALENCE

BY

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Dedicated to Sam Holby with admiration and affection

ABSTRACT. This paper extends Voiculescu's theorem on approximate equivalence to the case of nonseparable representations of nonseparable C^*-algebras. The main result states that two representations f and g are approximately equivalent if and only if rank $f(x)$ = rank $g(x)$ for every x. For representations of separable C^*-algebras a multiplicity theory is developed that characterizes approximate equivalence. Thus for a separable C^*-algebra, the space of representations modulo approximate equivalence can be identified with a class of cardinal-valued functions on the primitive ideal space of the algebra. Nonseparable extensions of Voiculescu's reflexivity theorem for subalgebras of the Calkin algebra are also obtained.

1. Introduction. In [V, Theorem 1.5] D. Voiculescu proved a remarkable theorem concerning approximate equivalence of separable unital representations of separable C^*-algebras. For a beautiful account of Voiculescu's theorem and many of its applications see the paper of W. Arveson [**Ar 1**]. This paper proves a version of Voiculescu's theorem for arbitrary unital representations of arbitrary C^*-algebras. Very often nonseparable extensions of theorems tend to be mired in cardinal arithmetic, but in this case the cardinal arithmetic is not too complicated. Of course, the main ingredient of this extension is Voiculescu's theorem itself. Although there have already been many applications of Voiculescu's theorem, the full impact of the theorem is probably yet to come. It is hoped that the results of this paper will aid in future applications of Voiculescu's theorem.

In addition many of the applications of Voiculescu's theorem carry over to nonseparable cases. In particular, Voiculescu's reflexivity theorem [**V**, Theorem 1.8] for unital, norm closed, separable subalgebras of the Calkin algebra is extended to "analogous" quotients for nonseparable Hilbert spaces. Also the results in [**H 4**] on direct integrals are extended to some nonseparable situations; these extensions are used to improve some results of F. J. Thayer [**Th**] on quasidiagonal C^*-algebras. Voiculescu's theorem is also extended to approximate subrepresentations.

Throughout, H denotes a complex Hilbert space and $B(H)$ denotes the set of (bounded linear) *operators* on H. The *dimension* of H, denoted by dim H, is the cardinality of an orthonormal basis for H. If $M \subset H$, then $\bigvee M$ denotes the *span* of M, i.e., the smallest (closed) subspace containing M.

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There are several common operator topologies on $B(H)$: the *norm*, *strong*, *weak*, and $*$-*strong* operator topologies. Of these, the reader may not be familiar with the $*$-strong operator topology. A net $\{T_n\}$ in $B(H)$ converges $*$-strongly to an operator T if and only if $T_n \to T$ strongly and $T_n^* \to T^*$ strongly.

If $T \in B(H)$, then the *rank* of T, denoted by rank T, is the dimension of the closure of the range of T. The range of an arbitrary function F is denoted by ran F; the kernel of a homomorphism G is denoted by ker G.

Since the mapping $T \to \operatorname{rank} T$ is central to this paper, a few remarks are in order. The most important property of this mapping concerns lower semicontinuity. If H is separable, then the mapping $T \to \operatorname{rank} T$ is weakly lower semicontinuous [**PRH 1,** Appendix]; equivalently, if H is separable and m is a cardinal, then $\{T \in B(H)\colon \operatorname{rank} T \leqslant m\}$ is weakly closed. If m is finite and H is nonseparable, then $\{T \in B(H)\colon \operatorname{rank} T \leqslant m\}$ is still weakly closed; however, $\{T \in B(H)\colon \operatorname{rank} T \text{ is finite}\}$ is $*$-strongly dense in $B(H)$. On the other hand, if m and H are arbitrary, then $\{T \in B(H)\colon \operatorname{rank} T \leqslant m\}$ is always closed under limits of weakly convergent *sequences*. (Proof: $T_n \to T$ weakly implies ran $T \subset \bigvee \{\operatorname{ran} T_n\colon n = 1, 2, \ldots\}$.) Thus $\{T \in B(H)\colon \operatorname{rank} T \leqslant m\}$ is norm closed. Therefore, the mapping $T \to \operatorname{rank} T$ is always lower semicontinuous in the norm operator topology.

If m is an infinite cardinal, let $\mathcal{K}_m$ denote the class of all operators that are norm limits of operators with rank less than m, and let $\mathcal{K}_m(H)$ denote $\mathcal{K}_m \cap B(H)$. In the case when $m = \aleph_0$ we shall usually use $\mathcal{K}(H)$ instead of $\mathcal{K}_m(H)$ to denote the set of compact operators on H. It is a part of the folklore of operator theory that $\{\mathcal{K}_m(H)\colon m \text{ an infinite cardinal}\}$ is the set of nonzero, norm closed, two-sided ideals in $B(H)$. The fundamental properties of these ideals vary greatly with the choice of the infinite cardinal m. Since $\{T \in B(H)\colon \operatorname{rank} T < m\}^-$ is determined by limits of sequences, it is not too surprising that one of the most marked differences occurs in the ideals $\mathcal{K}_m(H)$ depending on whether or not m can be approximated by sequences of smaller cardinals (see §4). Call an infinite cardinal m *countably cofinal* if there are countably many cardinals $m_1, m_2, \ldots,$ each less than m, such that $\sup_k m_k = m$. (In [**EEL**] the less suggestive term "$\aleph_0$-irregular" is used instead of countably cofinal.)

Suppose X is a nonempty set and $F, G\colon X \to B(H)$ are functions. The functions F and G are *unitarily equivalent*, denoted by $F \cong G$, if there is a unitary operator U such that $U^*F(x)U = G(x)$ for every x in X.

The functions F and G are *approximately equivalent*, denoted by $F \sim_a G$, if there is a net $\{U_n\}$ of unitary operators for which $\|U_n^*F(x)U_n - G(x)\| \to 0$ for every x in X.

If m is an infinite cardinal, we write $F \sim_a G(\mathcal{K}_m)$ to denote the existence of a net $\{U_n\}$ of unitary operators such that $U_n^*F(x)U_n - G(x) \in \mathcal{K}_m$ for each x and each n, and $\|U_n^*F(x)U_n - G(x)\| \to 0$ for every x. If all of the action takes place on a Hilbert space H, we may write $F \sim_a G(\mathcal{K}_m(H))$.

Throughout, $\mathfrak{A}$ denotes a C^*-algebra. A $*$-homomorphism $\pi\colon \mathfrak{A} \to B(H_\pi)$ for some Hilbert space H_π is a *representation*. If $M = \bigcap \{\ker \pi(a)\colon a \in \mathfrak{A}\}$, then M

reduces ran π, and π followed by restriction to M is called the *zero part* of π; similarly, π followed by restriction to $M^\perp$ is the *nonzero part* of π. If $M = 0$, then π is *nondegenerate*. We shall always assume that $\mathfrak{A}$ has an *identity*, 1, and thus every nondegenerate representation π of $\mathfrak{A}$ is *unital*; i.e., $\pi(1) = 1$. A unital representation π is *irreducible* if no nontrivial subspace of H reduces ran π. Let $\mathrm{Rep}(\mathfrak{A})$ denote the class of unital representations of $\mathfrak{A}$, and let $\mathrm{Irr}(\mathfrak{A})$ denote the subclass of irreducible representations of $\mathfrak{A}$. Also let $\mathrm{Rep}(\mathfrak{A}, H)$ denote the unital representations from $\mathfrak{A}$ into $B(H)$ and let $\mathrm{Irr}(\mathfrak{A}, H)$ denote $\mathrm{Irr}(\mathfrak{A}) \cap \mathrm{Rep}(\mathfrak{A}, H)$. There are two natural topologies on $\mathrm{Rep}(\mathfrak{A}, H)$, the *point-norm* topology (i.e., the topology of pointwise norm convergence) and the *point-weak* topology (i.e., the topology of pointwise weak convergence). We could also define the point-strong and the point-$*$-strong topologies on $\mathrm{Rep}(\mathfrak{A}, H)$, but these coincide with the point-weak topology. (The heart of the proof is that if $T_n \to T$ weakly and $T_n^* T_n \to T^* T$ weakly, then $T_n \to T$ strongly.) For more general mappings, point-weak and point-strong convergence do not coincide. We define dim π as dim H_π.

If $\pi, \rho \in \mathrm{Rep}(\mathfrak{A})$, then π is a *subrepresentation* of ρ, denoted $\pi < \rho$, if π is a summand of ρ.

If m is a cardinal, then $H^{(m)}$ denotes a direct sum of m copies of H, and if $T \in B(H)$, then $T^{(m)}$ denotes a direct sum of m copies of T acting on $H^{(m)}$. Also if $F\colon X \to B(H)$, then $F^{(m)}\colon X \to B(H^{(m)})$ is defined by $F^{(m)}(x) = F(x)^{(m)}$. We often use the symbol ∞ instead of $\aleph_0$, e.g., $T^{(\infty)} = T \oplus T \oplus \cdots$.

In §2 we discuss Voiculescu's theorem (Theorem 2.1) and present an elegant reformulation of this theorem (Theorem 2.5) that is extended to nonseparable cases in §3 (Theorem 3.14). Also §3 contains a characterization of approximate equivalence of representations of separable C^*-algebras that is based on a notion of "approximate multiplicity", which is an extension to representations of the notion of "approximate nullity" used by G. Edgar, J. Ernest and S. G. Lee **[EEL]**.

In §4 we consider quotients of the form $B(H)/\mathcal{K}_m(H)$ where m is an infinite cardinal, $m \leq \dim H$. We investigate the striking difference in the situations when m is, or is not, countably cofinal. We extend some of the compactness results **[V**, Theorem 1.5] related to approximate equivalence in the case when the dimension of the approximately equivalent representations is countably cofinal (Theorem 4.6). Also Voiculescu's reflexivity theorem is extended to separable subalgebras of $B(H)/\mathcal{K}_m(H)$ in the case when m is countably cofinal (Theorem 4.8). It is also shown that certain lifting problems in the quotient $B(H)/\mathcal{K}_m(H)$ with m an uncountable cardinal depend only on whether m is, or is not, countably cofinal (Theorem 4.13). §5 contains an analogue of Voiculescu's theorem for approximate subrepresentations.

§6 contains a result on direct integrals of representations that extends the results in **[H 4]**. Direct integrals of representations are analogues of multiplications by L^∞-functions on L^2-spaces. The problem with the analogy is that the L^∞-functions (direct integrals) are not generally point-norm limits of "simple" functions, or even functions with countable range (such functions correspond to direct sums). The main result (Theorem 6.2) is that direct integrals are approximately equivalent to functions with countable range (i.e., if we are willing to leave the measure-theoretic structure, then we can approximate direct integrals by functions with countable range).

The final section (§7) contains a comparison between approximate equivalence and unitary equivalence from the point of view of spectral multiplicity theory.

Various examples are sprinkled throughout the paper illustrating the limits on further extending these results. The results in §2 were announced in **[H 3]**, and the results in §§3, 4 were announced in **[H 6]**. This paper is part of a preprint of the author entitled *Approximate equivalence and completely positive maps.*

2. Voiculescu's theorem. The aim of this section is to state and reformulate Voiculescu's theorem. The reformulation has the advantage of being easier to state, understand, apply, and remember. The most important advantage is that the reformulation remains true in all of the nonseparable cases. We begin with a statement of Voiculescu's theorem.

THEOREM 2.1 (VOICULESCU [V, THEOREM 1.5]). *Suppose H is separable, $\mathfrak{A}$ is separable, and $\pi, \rho \in \operatorname{Rep}(\mathfrak{A}, H)$. The following are equivalent*:

(1) $\pi \sim_a \rho$,

(2) $\pi \sim_a \rho(\mathcal{K}(H))$,

(3) $\ker \pi = \ker \rho$, $\pi^{-1}(\mathcal{K}(H)) = \rho^{-1}(\mathcal{K}(H))$, *and the nonzero parts of* π, $\rho|\pi^{-1}(\mathcal{K}(H))$ *are unitarily equivalent.*

Our reformulation replaces (2) and (3) by the condition: $\operatorname{rank} \pi(a) = \operatorname{rank} \rho(a)$ for every a in $\mathfrak{A}$.

We first need a few facts concerning representations of C^*-algebras of compact operators. The following lemma is a summary of some of the results in **[Ar 2]**. Essentially the following lemma is a restatement of the facts that a C^*-algebra of compact operators is isomorphic to a direct sum of elementary C^*-algebras, that every representation of such an algebra is a direct sum of irreducible representations, and that every nonzero irreducible representation is unitarily equivalent to a subrepresentation of the identity representation. Note that $\mathcal{J}$ and H need not be separable.

LEMMA 2.2. *Suppose $\mathcal{J}$ is a C^*-subalgebra of $\mathcal{K}(H)$. Then the identity representation is unitarily equivalent to $\pi_0 \oplus \sum_{i \in I}^{\oplus} \pi_i^{(n_i)}$ relative to the decomposition $H = H_0 \oplus \sum_{i\in I}^{\oplus} H_i^{(n_i)}$ such that*

(1) *n_i is a positive integer for each i in I,*

(2) $\pi_0 = 0$,

(3) *π_i is irreducible for each i in I,*

(4) $\mathcal{J} = \{T \in \mathcal{K}(H)\colon T = 0 \oplus \sum_{i\in I}^{\oplus} T_i^{(n_i)}\}$,

(5) $\pi_i \cong \pi_j \Rightarrow i = j$,

(6) *if ρ is a representation of $\mathcal{J}$, then there are cardinals m_i, $i \in I$, such that $\rho \cong \rho_0 \oplus \sum_{i\in I}^{\oplus} \pi_i^{(m_i)}$ where $\rho_0 = 0$.*

Note that if $\pi\colon \mathcal{J} \to B(H)$ is a representation of a C^*-algebra $\mathcal{J}$, then π and the identity representation on $\operatorname{ran} \pi$ have the same reducing subspaces. Therefore the preceding lemma could be restated in terms of representations into $\mathcal{K}(H)$ (see Proposition 2.10).

The following lemma is the main ingredient of our reformulation of Voiculescu's theorem. Note that $\mathcal{G}$ and H need not be separable.

LEMMA 2.3. *Suppose $\mathcal{G}$ is a C^*-algebra and π, ρ: $\mathcal{G} \to \mathcal{K}(H)$ are representations. Then*

(1) *the nonzero parts of π and ρ are unitarily equivalent if and only if* rank $\pi(a) =$ rank $\rho(a)$ *for each a in $\mathcal{G}$;*

(2) *the nonzero part of π is unitarily equivalent to a subrepresentaton of ρ if and only if* rank $\pi(a) \leqslant$ rank $\rho(a)$ *for each a in $\mathcal{G}$.*

PROOF. (1). The "only if" part is obvious. Suppose that rank $\pi(a) =$ rank $\rho(a)$ for each a in $\mathcal{G}$. Then ker $\pi =$ ker ρ (because ker $\pi = \{a \in \mathcal{G}$: rank $\pi(a) = 0\}$). Thus (by considering $\pi(\mathcal{G})$) we can assume that $\mathcal{G} \subseteq \mathcal{K}(H)$ and π is the identity representation on $\mathcal{G}$. Suppose that we have decomposed π, H, and ρ as in Lemma 2.2. We need show only that $n_i = m_i$ for each i in I. If $i \in I$, then it follows from Lemma 2.2(4) that there is an a in $\mathcal{G}$ such that rank $\pi_i(a) = 1$ and rank $\pi_j(a) = 0$ for every j in I with $j \neq i$; whence $n_i =$ rank $\pi(a) =$ rank $\rho(a) = m_i$. This completes the proof of (1).

(2). The proof follows in a fashion similar to that of (1), e.g., it is necessary to show only that $m_i \geqslant n_i$ for every i in I.

The following lemma is used to extend results on approximate equivalence from separable C^*-algebras to nonseparable C^*-algebras. The proof, which is omitted, is obtained by considering nets that are indexed by the finite subsets of X.

LEMMA 2.4. *Suppose X is a nonempty set and F, G: $X \to B(H)$. Then $F \sim_a G$ if and only if $F|Y \sim_a G|Y$ for every finite subset Y of X.*

We are now ready to prove our reformulation of Voiculescu's theorem (Theorem 2.1). Note that the separability assumption on $\mathfrak{A}$ is dropped. We shall later prove (Theorem 3.14) that the separability assumption on H can also be dropped. Our reformulation of Voiculescu's theorem should give the reader an inkling of the depth and power of Voiculescu's theorem; i.e., it is a purely algebraic characterization of approximate equivalence, which is very geometric.

THEOREM 2.5. *Suppose H is separable and π, $\rho \in$* Rep$(\mathfrak{A}, H)$. *Then $\pi \sim_a \rho$ if and only if* rank$(a) =$ rank $\rho(a)$ *for each a in $\mathfrak{A}$.*

PROOF. The "only if" part follows from the lower semicontinuity of the function rank( ) on $B(H)$. To prove the "if" part suppose that rank $\pi(a) =$ rank $\rho(a)$ for each a in $\mathfrak{A}$. We can assume, by Lemma 2.4, that $\mathfrak{A}$ is separable. Since ker $\pi = \{a$: rank $\pi(a) = 0\}$, it follows that ker $\pi =$ ker ρ. Furthermore, it follows from Lemma 2.2(4) that every compact operator in ran π (resp. ran ρ) is a norm limit of finite rank operators in ran π (resp. ran ρ). Hence $\pi^{-1}(\mathcal{K}(H)) = \rho^{-1}(\mathcal{K}(H))$. It follows from Lemma 2.3(1) that the nonzero parts of π, $\rho|\pi^{-1}(\mathcal{K}(H))$ are unitarily equivalent. It now follows from Theorem 2.1 that $\pi \sim_a \rho$.

COROLLARY 2.6. *Suppose H is separable and $\pi, \rho \in \operatorname{Rep}(\mathfrak{A}, H)$. Then $\pi \sim_a \rho$ if and only if there are nets $\{U_n\}$, $\{V_k\}$ of unitary operators such that $U_n^*\pi(a)U_n \to \rho(a)$ weakly and $V_k^*\rho(a)V_k \to \pi(a)$ weakly for each a in $\mathfrak{A}$.*

The preceding corollary was discovered independently by W. Arveson **[Ar 1**, Theorem 5] and the author **[H 1**, Theorem 4.1] in the case when $\mathfrak{A}$ is separable. Note that there is no need for the U_n's and V_k's in the preceding corollary to be unitary; in fact, $\pi \sim_a \rho$ if and only if, for each a in $\mathfrak{A}$, there are nets $\{A_n\}$, $\{B_n\}$, $\{C_k\}$, $\{D_k\}$ of operators (depending on a) such that $A_n\pi(a)B_n \to \rho(a)$ weakly and $C_k\rho(a)D_k \to \pi(a)$ weakly.

Note also that Theorem 2.5 implies the equivalence of (1) and (3) in Theorem 2.1 when $\mathfrak{A}$ is not separable. If $\mathfrak{A}$ is not separable, then the implication (1) $\Rightarrow$ (2) in Theorem 2.1 no longer holds (Proposition 2.7). If H is not separable and $\mathfrak{A}$ is separable, then the implications (1) $\Rightarrow$ (2) and (3) $\Rightarrow$ (1) are no longer true (Proposition 2.8); also, in this case, Corollary 2.6 is no longer true.

The next proposition shows why it is necessary to use nets rather than sequences when defining approximate equivalence for nonseparable C^*-algebras.

PROPOSITION 2.7. *Suppose $\{e_1, e_2, \ldots\}$ is an orthonormal basis for H, let $\mathfrak{A}$ be the C^*-algebra of all operators on H that are diagonal with respect to $\{e_1, e_2, \ldots\}$, and let $\tau\colon \mathfrak{A} \to \mathbf{C}$ be a scalar-valued unital representation that annihilates $\mathfrak{A} \cap \mathcal{K}(H)$. If π is the identity representation on $\mathfrak{A}$ and $\rho = \pi \oplus \tau \oplus \tau \oplus \ldots$, then*

(1) $\pi \sim_a \rho$;

(2) *there is no sequence $\{U_n\}$ of unitary operators such that $U_n^*\pi(a)U_n \to \rho(a)$ weakly for every a in $\mathfrak{A}$;*

(3) *there is no unitary operator U such that $U^*\pi(a)U - \rho(a)$ is compact for every a in $\mathfrak{A}$.*

PROOF. (1) follows from Theorem 2.5. (2) and (3) will both follow once we have proved the following fact: there is no sequence $\{f_n\}$ of unit vectors such that $f_n \to 0$ weakly and $(Af_n, f_n) \to \tau(A)$ for every A in $\mathfrak{A}$. Assume via contradiction that $\{f_n\}$ is such a sequence. Since $f_n \to 0$ weakly, we can find a subsequence $\{f_{n_k}\}$ and an increasing sequence $\{m_k\}$ of positive integers, and an orthonormal sequence $\{g_k\}$ such that $\|f_{n_k} - g_k\| \to 0$ and such that $g_k \in \bigvee\{e_j\colon m_k < j < m_{k+1}\}$ for $k = 1, 2, \ldots$. It follows that $(Ag_k, g_k) \to \tau(A)$ for every A in $\mathfrak{A}$. However, if P is a projection in $\mathfrak{A}$ such that $Pg_k = g_k$ if k is even and $Pg_k = 0$ if k is odd, then $(Pg_k, g_k) \not\to \tau(P)$; this is the desired contradiction. A moment's reflection shows that the existence of a sequence $\{U_n\}$ of unitary operators such that $U_n^*\pi(a)U_n \to \rho(a)$ weakly for every a in $\mathfrak{A}$ or the existence of a unitary operator U such that $U^*\pi(a)U - \rho(a)$ is compact for every a in $\mathfrak{A}$ would imply the existence of a sequence $\{f_n\}$ of unit vectors such that $f_n \to 0$ weakly and $(Af_n, f_n) \to \tau(A)$ for every A in $\mathfrak{A}$. This proves (2) and (3).

PROPOSITION 2.8. *Suppose H is not separable and let P, Q be projections in $B(H)$ such that* rank $P = \aleph_0$ *and* rank $Q = \operatorname{rank}(1 - Q) = \dim H$. *Let $\mathfrak{A} = C^*(P)$, let π*

be the identity representation on $\mathfrak{A}$, and let ρ: $C^*(P) \to C^*(Q)$ be the unital representation with $\pi(P) = Q$. Then:

(1) $\pi \not\sim_a \rho$;

(2) $\ker \pi = \ker \rho$, $\pi^{-1}(\mathcal{K}(H)) = \rho^{-1}(\mathcal{K}(H))$, *and the nonzero parts of* π, $\rho|\pi^{-1}(\mathcal{K}(H))$ *are unitarily equivalent*;

(3) *there are nets* $\{U_n\}$, $\{V_n\}$ *of unitary operators such that* $U_n^*\pi(a)U_n \to \rho(a)$ *weakly and* $V_n^*\rho(a)V_n \to \pi(a)$ *weakly for every a in* $\mathfrak{A}$.

Proof. (1) This follows from Theorem 2.5; i.e., rank $\pi(P) \neq$ rank $\rho(P)$. (2) Clearly $\ker \pi = \ker \rho = \pi^{-1}(\mathcal{K}(H)) = \rho^{-1}(\mathcal{K}(H)) = 0$. (3) Order the finite subsets of H by inclusion. For each finite subset E of H choose unitary operators U_E and V_E so that $U_E^*PU_Ee = Qe$ and $V_E^*QV_Ee = Pe$ for each e in E. It is clear that $\{U_E\}$, $\{V_E\}$ are the required nets.

It should be noted that the equivalence of (2) and (3) in the preceding proposition is true for arbitrary representations. It should also be noted that Theorem 2.5 is false for nonunital representations. To see this suppose $\dim H = \aleph_0$ and P, Q are projections such that rank P = rank Q = rank$(1 - Q) = \aleph_0$ and rank$(1 - P) = 1$. Let $\mathcal{J} = \{\lambda P: \lambda \in \mathbf{C}\}$, let π be the identity representation on $\mathcal{J}$, and let ρ be the representation on $\mathcal{J}$ with $\rho(P) = Q$. Then rank $\pi(a)$ = rank $\rho(a)$ for every a in $\mathcal{J}$, but $\pi \not\sim_a \rho$ (because rank$(1 - P) \neq$ rank$(1 - Q)$).

There is another way of viewing Voiculescu's theorem that will prove useful in the next section. It was proved by R. Gellar and L. Page **[GP]** that two normal operators on a separable Hilbert space are approximately equivalent if and only if they have the same spectrum and their isolated eigenvalues have the same multiplicities. In **[H 1]** the author proved an analogue of this result for arbitrary operators on a separable Hilbert space. We will show how the analogy can be extended to representations. The key ideas are based on Lemma 2.2 and the following lemma (which is contained in **[Ar 2]**).

Lemma 2.9. *Suppose* $\mathfrak{A}$ *is a* C^**-algebra,* $\mathcal{J}$ *is a closed* $*$*-ideal in* $\mathfrak{A}$, *and* $\pi \in$ Rep$(\mathfrak{A}, H)$. *Let* $M = \bigvee \{\operatorname{ran} \pi(a): a \in \mathcal{J}\}$. *Then*:

(1) *M reduces* π;

(2) *a subspace of M reduces π if and only if it reduces* $\pi|\mathcal{J}$;

(3) *if* M_1, M_2 *are subspaces of M that reduce* π, *and if* U: $M_1 \to M_2$ *is a unitary operator such that* $U^*(\pi(a)|M_2)U = \pi(a)|M_1$ *for every a in* $\mathcal{J}$, *then* $U^*(\pi(a)|M_2)U = \pi(a)|M_1$ *for every a in* $\mathfrak{A}$.

The following proposition is easily obtained from Lemma 2.2 and the preceding lemma; it appears in the author's Ph.D. thesis in the case when $\mathfrak{A}$ is separable (see also **[H 1**, Proposition 2.5**]**).

Proposition 2.10. *Suppose* $\pi \in$ Rep$(\mathfrak{A}, H)$ *and let* $\mathcal{J} = \pi^{-1}(\mathcal{K}(H))$. *Then we can write* $\pi \cong \pi_0 \oplus \sum_{i\in I}^{\oplus} \pi_i^{(n_i)}$ *relative to* $H = H_0 \oplus \sum_{i\in I}^{\oplus} H_i^{(n_i)}$ *where*:

(1) n_i *is a positive integer for each i in I*;

(2) $\pi_0|\mathcal{J} = 0$;

(3) $\pi_i(\mathcal{J}) = \mathcal{K}(H_i)$ *for each* i *in* I;

(4) π_i *is irreducible for each* i *in* I;

(5) $\pi_i \cong \pi_j \Rightarrow i = j$;

(6) *if* ρ *is an irreducible representation of* $\mathfrak{A}$ *such that* $\ker \pi \subseteq \ker \rho$ *and* $\mathcal{J} \not\subseteq \ker \rho$, *then* $\rho \cong \pi_i$ *for some* i *in* I.

COROLLARY 2.11. *If* $\mathfrak{A}$ *and* H *are separable*, $\pi \in \operatorname{Rep}(\mathfrak{A}, H)$, *and* ρ *is an irreducible representation of* $\mathfrak{A}$ *such that* $\ker \pi \subseteq \ker \rho$, *then either* ρ *is unitarily equivalent to a subrepresentation of* π *or* $\pi \sim_a \pi \oplus \rho$.

If $\pi \in \operatorname{Rep}(\mathfrak{A}, H)$, then Arveson [**Ar 3**] calls the subrepresentation of π that is complementary to π_0 in the preceding proposition the *essential part* of π. It follows from Lemmas 2.2 and 2.9 that the essential parts of two approximately equivalent representations are unitarily equivalent. Voiculescu's theorem says that if H is separable, then the converse holds; i.e., if the essential parts of two representations are unitarily equivalent and if the representations have the same kernel, then the representations are approximately equivalent (see [**Ar 3**, Theorem 5]).

To get a clearer picture of these ideas let $\mathfrak{A} = C(X)$ where X is a nonempty compact subset of the plane, and let θ be the element of $C(X)$ defined by $\theta(z) = z$. A representation π in $\operatorname{Rep}(\mathfrak{A}, H)$ is completely determined by $T = \pi(\theta)$; the only necessary conditions on T are that T be normal and $\sigma(T) \subseteq X$. Subrepresentations of π correspond to direct summands of T, and irreducible subrepresentations of π correspond to eigenvalues of T. The subrepresentations π_i, $i \in I$, in Proposition 2.10 correspond to the isolated eigenvalues of T that have finite multiplicity; the multiplicity of the eigenvalue corresponding to each π_i is the integer n_i. If $\rho \in \operatorname{Rep}(\mathfrak{A}, H)$ and $\rho(\theta) = S$, then $\ker \rho = \ker \pi$ precisely when S and T have the same spectrum.

Hence we can view the irreducible subrepresentations of a representation π in $\operatorname{Rep}(\mathfrak{A}, H)$ as eigenvalues; let us temporarily use the term *eigen-representation*, and let us call the representations π_i in Proposition 2.10 the *isolated eigen-representations of finite multiplicity*, and call n_i the *multiplicity* of π_i for each i in I. (There is a natural C^*-algebraic setting in which "isolated" has a topological meaning, and, for well-behaved C^*-algebras, the meaning of "isolated" corresponds to its use above. For a brief discussion of these ideas see the last section of [**H 1**].)

PROPOSITION 2.12. *Suppose* $\pi, \rho \in \operatorname{Rep}(\mathfrak{A}, H)$. *Then*:

(1) *if* $\pi \sim_a \rho$, *then* $\ker \pi = \ker \rho$ *and* π, ρ *have the same isolated eigen-representations of finite multiplicity with the same multiplicities*;

(2) *if* H *is separable and* $\ker \pi = \ker \rho$ *and* π, ρ *have the same isolated eigen-representations of finite multiplicity with the same multiplicities, then* $\pi \sim_a \rho$.

3. Nonseparable cases. The main purpose of this section is to extend Theorem 2.5 to the case when H is not separable. We first give a characterization of approximate equivalence when $\mathfrak{A}$ is separable and H is nonseparable that is more in the spirit of Proposition 2.12; the prime ingredient in this characterization is the notion of "approximate multiplicity", which is an analogue for representations of the notion of "approximate nullity" of operators studied in [**EEL**].

The key idea in Voiculescu's proof of Theorem 2.1 is the following lemma. It is clear that this lemma follows from Theorem 2.1 (see Theorem 2.5).

LEMMA 3.1 (VOICULESCU [V, THEOREM 1.3]). *Suppose $\mathfrak{A}$ is separable, H_π, H_ρ are separable Hilbert spaces, $\pi \in \mathrm{Rep}(\mathfrak{A}, H_\pi)$, $\rho \in \mathrm{Rep}(\mathfrak{A}, H_\rho)$, and suppose $\rho|\pi^{-1}(\mathcal{K}(H_\pi)) = 0$. Then $\pi \sim_a \pi \oplus \rho$.*

Our first task is to extend Lemma 3.1.

LEMMA 3.2. *Suppose $\mathfrak{A}$ is separable, H_π, H_ρ are infinite-dimensional Hilbert spaces, $\pi \in \mathrm{Rep}(\mathfrak{A}, H_\pi)$, $\rho \in \mathrm{Rep}(\mathfrak{A}, H_\rho)$, and $m = \dim H_\rho$. If $\rho|\pi^{-1}(\mathcal{K}_m(H_\pi)) = 0$, then $\pi \sim_a \pi \oplus \rho$.*

PROOF. Since $\rho(1) = 1 \neq 0$, it follows that $\dim H_\pi = \mathrm{rank}\, \pi(1) \geqslant m = \dim H_\rho$. In view of Lemma 3.1 we can assume that $\dim H_\pi$ is uncountable. We are going to write H_π as a direct sum of subspaces $\{M_\alpha\colon \alpha \text{ is an ordinal}, \alpha < \dim H\}$ so that for each α:

(1) M_α reduces π,

(2) $\|\pi(a)|M_\alpha\| = \|\pi(a)|(\Sigma^{\oplus}_{\beta<\alpha} M_\beta)^\perp\|$ for every a in $\mathfrak{A}$,

(3) if $a \in \mathfrak{A}$ and $\pi(a)|M_\alpha$ is compact, then $\pi(a)|(\Sigma^{\oplus}_{\beta<\alpha} M_\beta)^\perp$ is compact.

We begin by constructing M_0. Since $\mathfrak{A}$ is separable, it follows that π is a direct sum of separable representations. It therefore follows that for each a in $\mathfrak{A}$ there is a separable subspace N_a of H_π such that N_a reduces π and $\|\pi(a)|N_a\| = \|\pi(a)\|$. Let $\{a_1, a_2, \ldots\}$ be dense in $\mathfrak{A}$ and let $M = \bigvee\{N_{a_k}\colon k = 1, 2, \ldots\}$. Then M is separable, M reduces π, and $\|\pi(a)|M\| = \|\pi(a)\|$ for every a in $\mathfrak{A}$.

Consider the representation $\tau\colon \pi(\mathfrak{A})|M \to \pi(\mathfrak{A})$ defined by $\tau(\pi(a)|M) = \pi(a)$ for every a in $\mathfrak{A}$. Clearly τ is an isomorphism. (We just proved that τ is isometric!) It follows from Lemma 2.1 that there are cardinals $m_1, m_2, \ldots$ and irreducible representations $\tau_1, \tau_2, \ldots$ such that the nonzero part of $\tau|(\pi(\mathfrak{A})|M) \cap \mathcal{K}(M)$ is unitarily equivalent to $\tau_1^{(m_1)} \oplus \tau_2^{(m_2)} \oplus \cdots$. It follows from Lemma 2.9 that we can choose a separable subspace M_0 of H such that $M \subseteq M_0$, M_0 reduces π, and the nonzero part of the representation on $(\pi(\mathfrak{A})|M) \cap \mathcal{K}(M)$ that maps $\pi(a)$ onto $\pi(a)|M_0$ is unitarily equivalent to $\tau_1^{(n_1)} \oplus_2^{(n_2)} \oplus \cdots$, where $n_k = \min(m_k, \aleph_0)$ for $k = 1, 2, \ldots$. It follows that $\|\pi(a)|M_0\| = \|\pi(a)\|$ for every a in $\mathfrak{A}$ and that $(\pi(\mathfrak{A})|M_0) \cap \mathcal{K}(M_0) = [\pi(\mathfrak{A}) \cap \mathcal{K}(H_\pi)]|M_0$. Thus M_0 has the required properties.

Next suppose that α_0 is an ordinal less than $\dim H_\pi$ and that the orthogonal subspaces $\{M_\alpha\colon \alpha < \alpha_0\}$ have been constructed so that (1)–(3) hold for each $\alpha < \alpha_0$. To construct M_{α_0} we just replace H_π by $(\Sigma^{\oplus}_{\alpha<\alpha_0} M_\alpha)^\perp$ and follow the procedure used to construct M_0. Thus all of the M_α's can be constructed by transfinite induction.

To insure that H_π is the direct sum of the M_α's, we can select an orthonormal basis $\{e_\alpha\colon \alpha < \dim H_\pi\}$ and require in our inductive construction that $e_\alpha \in \Sigma^{\oplus}_{\beta\leqslant\alpha} M_\beta$ for each $\alpha < \dim H_\pi$. We can also insist that each M_α be infinite dimensional.

Write $\pi = \Sigma^{\oplus}\{\pi_\alpha\colon \alpha < \dim H_\pi\}$ relative to $H_\pi = \Sigma^{\oplus}\{M_\alpha\colon \alpha < \dim H_\pi\}$. We can also write ρ as a direct sum of separable representations: $\{\rho_\alpha\colon \alpha$ is an ordinal less than $\dim H_\rho = m\}$.

First suppose $m = \aleph_0$. Then $\rho|\pi_0^{-1}(\mathcal{K}(M_0)) = \rho|\pi^{-1}(\mathcal{K}(H_\pi)) = 0$. Thus $\pi_0 \sim_a \pi_0 \oplus \rho$;, whence $\pi \sim_a \pi \oplus \rho$.

Next suppose m is uncountable. Then if $a \in \mathfrak{A}$, $\alpha < m$, and $\pi_\alpha(a)$ is compact, then, since $\sum_{\beta > \alpha}^{\oplus} \pi_\beta(a)$ is compact, it follows that rank $\pi(a) < m$. Thus $\rho|\pi_\alpha^{-1}(\mathcal{K}(M_\alpha)) = 0$ for each $\alpha < m$. It follows from Lemma 3.1 that $\pi_\alpha \sim_a \pi_\alpha \oplus \rho_\alpha$ for each $\alpha < m$, whence $\pi \sim_a \pi \oplus \rho$.

Recall that $\mathrm{Irr}(\mathfrak{A})$ denotes the class of irreducible representations in $\mathrm{Rep}(\mathfrak{A})$, and $\mathrm{Irr}(\mathfrak{A}, H) = \mathrm{Irr}(\mathfrak{A}) \cap \mathrm{Rep}(\mathfrak{A}, H)$.

LEMMA 3.3. *If $\mathfrak{A}$ is separable and H is separable, then* $\mathrm{Rep}(\mathfrak{A}, H)$ *is separable and metrizable in the point-weak topology.*

PROOF. Let D be the closed unit disk in the plane. Choose a (norm) dense sequence $\{a_n\}$ in the unit ball of $\mathfrak{A}$ and a (norm) dense sequence $\{f_n\}$ in the unit ball of H. For each pair (m, n) of positive integers let $D(m, n) = D$, and let Y be the cartesian product $\prod_{(m,n)} D(m, n)$ with the product topology. Define a map Φ: $\mathrm{Rep}(\mathfrak{A}, H) \to Y$ by $\Phi(\pi)(m, n) = (\pi(a_n)f_m, f_m)$. It is easily shown that Φ is an embedding (with the point-weak topology on $\mathrm{Rep}(\mathfrak{A}, H)$) and that Y is separable and metrizable. Hence $\mathrm{Rep}(\mathfrak{A}, H)$ is separable and metrizable in the point-weak topology.

COROLLARY 3.4. *If $\mathfrak{A}$ and H are separable, then every subset of* $\mathrm{Irr}(\mathfrak{A}, H)$ *is Lindelöf in the point-weak topology.*

We are now ready to extend the analogy between eigenvalues of normal operators and irreducible subrepresentations of a representation. Suppose $\mathfrak{A}$ is separable and $\pi \in \mathrm{Rep}(\mathfrak{A})$. For each τ in $\mathrm{Irr}(\mathfrak{A})$ we define the *approximate multiplicity* of τ as a subrepresentation of π, denoted by $\text{Ap-mult}(\tau, \pi)$, as the supremum of the cardinals $m \geq 0$ for which $\tau^{(m)}$ is a subrepresentation of a representation that is approximately equivalent to π. The next proposition contains some of the properties of approximate multiplicity. In particular, the supremum in the definition is shown to actually be a maximum. A more algebraic characterization of approximate multiplicity is given in Lemma 3.13.

PROPOSITION 3.5. *Suppose $\mathfrak{A}$ is separable, $\pi, \rho \in \mathrm{Rep}(\mathfrak{A}, H)$, and $\tau \in \mathrm{Irr}(\mathfrak{A})$. Then*:

(1) $0 < \text{Ap-mult}(\tau, \pi)$ *if and only if* $\ker \pi \subseteq \ker \tau$;

(2) $\text{Ap-mult}(\tau, \pi)$ *is infinite if and only if* $\tau|\pi^{-1}(\mathcal{K}(H)) = 0$;

(3) $\text{Ap-mult}(\tau, \pi)$ *is a positive integer m if and only if τ is an isolated eigen-representation of π with multiplicity m*;

(4) *if $m =$* $\text{Ap-mult}(\tau, \pi)$ *is infinite, then* $\pi \sim_a \pi \oplus \tau^{(m)}$;

(5) *if $\pi \sim_a \rho$ and $\tau \sim_a \sigma$, then* $\text{Ap-mult}(\tau, \pi) = \text{Ap-mult}(\sigma, \rho)$.

PROOF. (1) The "only if" part is obvious. Suppose $\ker \pi \subseteq \ker \tau$. If $\tau|\pi^{-1}(\mathcal{K}(H)) \neq 0$, then, by Proposition 2.10, τ is unitarily equivalent to a subrepresentation of π. If $\tau|\pi^{-1}(\mathcal{K}(H)) = 0$, then it follows from Lemma 3.2 that $\pi \sim_a \pi \oplus \tau$. In either case $\text{Ap-mult}(\tau, \pi) > 0$.

(3) The "if" part follows from Proposition 2.12. Suppose τ is not an isolated eigen-representation of π with finite multiplicity. It follows from Proposition 2.10 (6) that either $\ker \pi \not\subseteq \ker \tau$ or $\tau|\pi^{-1}(\mathcal{K}(H)) = 0$. In the first case, Ap-mult$(\tau, \pi) = 0$, and in the second, Ap-mult(τ, π) is infinite (by Lemma 3.2).

(2) This follows from (1) and (3).

(4) Suppose $m =$ Ap-mult(τ, π) is infinite. Suppose $a \in \mathfrak{A}$ and rank $\pi(a) < m$. Then there is a cardinal k such that rank $\pi(a) < k \leqslant m$ and $\tau^{(k)}$ is a subrepresentation of a representation that is approximately equivalent to π. Thus $k \cdot$ rank $\tau(a) =$ rank $\tau^{(k)}(a) \leqslant$ rank $\pi(a) < k$, whence $\tau(a) = 0$. It follows from Lemma 3.2 that $\pi \sim_a \pi \oplus \tau^{(m)}$.

(5) This is obvious.

Lemma 3.6. *Suppose $\mathfrak{A}$ and H are separable and $\pi \in$ Rep$(\mathfrak{A})$. Then the mapping $\tau \to$ Ap-mult(τ, π) on Irr$(\mathfrak{A}, H)$ is upper semicontinuous in the point-weak topology.*

Proof. Suppose m is a cardinal. We must show that $\mathcal{S} = \{\tau \in \mathrm{Irr}(\mathfrak{A}, H)$: Ap-mult$(\tau, \pi) \geqslant m\}$ is point-weak closed. If $m = 0$, then $\mathcal{S} = \mathrm{Irr}(\mathfrak{A}, H)$. Suppose m is infinite, $\{\tau_n\}$ is a sequence in $\mathcal{S}$ and $\tau_n \to \tau$ in the point-weak topology. Since $\tau_n \in \mathcal{S}$ for each n, we have (by Proposition 3.5(4)) that $\{a\colon \text{rank } \pi(a) < m\} \subseteq \ker \tau_n$ for $n = 1, 2, \ldots$. Thus $\{a\colon \text{rank } \pi(a) < m\} \subseteq \ker \tau$. It follows from Lemma 3.2 that $\pi \sim_a \pi \oplus \tau^{(m)}$, whence $\tau \in \mathcal{S}$. Finally suppose that m is a positive integer. If we have that infinitely many of the τ_n's are unitarily equivalent, then $\ker \tau_n \subset \ker \tau$ for some n. Thus

$$\text{Ap-mult}(\tau, \pi) \geqslant \text{Ap-mult}(\tau, \tau_n) \cdot \text{Ap-mult}(\tau_n, \pi) \geqslant 1 \cdot m = m.$$

Hence we can assume that $\tau_i \cong \tau_j$ only when $i = j$. We can also assume that Ap-mult(τ_n, π) is finite for $n = 1, 2, \ldots$. It follows from Proposition 2.10 that $(\tau_1 \oplus \tau_2 \oplus \cdots)^{(m)}$ is unitarily equivalent to a subrepresentation of π. Also, since $\tau_n \to \tau$ in the point-weak topology, it follows that $\ker(\tau_1 \oplus \tau_2 \oplus \cdots) \subseteq \ker \tau$. Thus Ap-mult$(\tau, \tau_1 \oplus \tau_2 \oplus \cdots) \geqslant 1$, which implies that Ap-mult$(\tau, \pi) \geqslant m$.

The next two propositions are perhaps surprising and very fundamental to the main results of this section. They involve some topology and some cardinal arithmetic.

If $\mathfrak{A}$ is separable and $\pi \in$ Rep$(\mathfrak{A})$, we define the *multiplicity set* of π, denoted by $\mathfrak{M}(\pi)$,to be $\{\text{Ap-mult}(\tau, \pi)\colon \tau \in \mathrm{Irr}(\mathfrak{A})\}$. Also let $\mathfrak{M}_\infty(\pi)$ denote the set of infinite cardinals in $\mathfrak{M}(\pi)$.

Proposition 3.7. *If $\mathfrak{A}$ is separable and $\pi \in$ Rep$(\mathfrak{A})$, then $\mathfrak{M}(\pi)$ is countable.*

Proof. Since $\mathfrak{A}$ is separable, we know that each τ in Irr$(\mathfrak{A})$ is separable. It follows that we need only show that $\mathfrak{M}(\pi, H) = \{\text{Ap-mult}(\tau, \pi)\colon \tau \in \mathrm{Irr}(\mathfrak{A}, H)\}$ is countable for every separable Hilbert space; clearly we need check one Hilbert space of each countable dimension. Suppose H is separable, and assume, via contradiction, that $\mathfrak{M}(\pi, H)$ is uncountable. Hence there is a subset $\mathfrak{M}$ of $\mathfrak{M}(\pi, H)$ consisting only of infinite cardinals such that $\mathfrak{M}$ is order-isomorphic to the first uncountable ordinal Ω. Write $\mathfrak{M} = \{m_\alpha\colon \alpha < \Omega\}$ so that $\alpha < \beta$ implies $m_\alpha < m_\beta$. For each $\alpha < \Omega$ define $\mathcal{O}_\alpha = \{\tau \in \mathrm{Irr}(\mathfrak{A}, H)\colon \text{mult}(\tau, \pi) < m_\alpha\}$, and let

$\mathcal{O}$ be the union of the $\mathcal{O}_\alpha$'s ($\alpha < \Omega$). It follows from Lemma 3.6 that each $\mathcal{O}_\alpha$ is point-weakly open, and it follows from Corollary 3.4 that $\mathcal{O}$ is point-weakly Lindelöf. However, the $\mathcal{O}_\alpha$'s form an open cover with no countable subcover; this is the required contradiction.

PROPOSITION 3.8. *Suppose $\mathfrak{A}$ is separable, $\pi \in \mathrm{Rep}(\mathfrak{A})$, and m is an infinite cardinal. If $\{\tau_i: i \in I\} \subseteq \mathrm{Irr}(\mathfrak{A})$, Ap-mult$(\tau_i, \pi) \leqslant m$ for each i in I, and $\sum_{i\in I}^{\oplus} \tau_i$ is unitarily equivalent to a subrepresentation of π, then* Card $I \leqslant m$.

PROOF. Assume via contradiction that Card $I > m$, and let k be the smallest cardinal greater than m. Choose a sequence $\{a_n\}$ that is dense in $\{a \in \mathfrak{A}: \operatorname{rank} \pi(a) \leqslant m\}$. For each positive integer n, we have

$$\operatorname{rank} \sum_{i\in I}^{\oplus} \tau_i(a_n) \leqslant \operatorname{rank} \pi(a_n) \leqslant m,$$

which implies that there is a subset I_n of I such that Card $I_n \leqslant m$ and $\tau_i(a_n) = 0$ for $i \notin I_n$. Let J be the union of the I_n's. Clearly, card $J \leqslant m$, whence $I - J \neq \varnothing$. Choose $i \in I - J$. Then

$$\pi^{-1}(\mathcal{K}_k(H)) = \{a \in \mathfrak{A}: \operatorname{rank} \pi(a) \leqslant m\} \subseteq \ker \tau_i.$$

Thus, by Lemma 3.2, $\pi \sim_{\mathrm{a}} \pi \oplus \tau_i^{(k)}$, which implies Ap-mult$(\tau_i, \pi) > m$. This is the desired contradiction.

We are now only one lemma away from one of the two main theorems of this section.

LEMMA 3.9. *Suppose $\pi, \pi_1, \pi_2, \ldots \in \mathrm{Rep}(\mathfrak{A})$ and $\pi \sim_{\mathrm{a}} \pi \oplus \pi_n$ for $n = 1, 2, \ldots$. Then $\pi \sim_{\mathrm{a}} \pi \oplus \pi_1 \oplus \pi_2 \oplus \cdots$.*

PROOF. It follows from Lemma 2.4 that we can assume that $\mathfrak{A}$ is finitely generated. For notational convenience, we give the proof only in the case when $\mathfrak{A}$ is singly generated; i.e., $\mathfrak{A} = C^*(a)$. Let $T = \pi(a)$, and let $T_n = \pi_n(a)$ for $n = 1, 2, \ldots$. We can assume that each of the operators $T_1, T_2, \ldots$ appears in the sequence infinitely often. Suppose $\varepsilon > 0$ and suppose $T \in B(H)$. We will construct a sequence $\{M_n\}$ of orthogonal subspaces of H and a sequence $\{S_n\}$ of operators such that for $n = 1, 2, \ldots$ we have:

(1) $M_1, M_2, \ldots, M_n$ each reduces $T + S_1 + \cdots + S_n$;

(2) M_k reduces S_n and $S_n|M_k = 0$ for $1 \leqslant k < n$;

(3) $(T + S_1 + \cdots + S_n)|M_k \cong T_k$ for $1 \leqslant k \leqslant n$;

(4) $(T + S_1 + \cdots + S_n)|(M_1 + \cdots + M_n)^\perp \cong T$;

(5) $\|S_n\| < \varepsilon/4^n$.

Since $T \sim_{\mathrm{a}} T \oplus T_1$, it follows that there is an operator S_1 and a subspace M_1 so that (1)–(5) hold when $n = 1$. Suppose N is a positive integer and $M_1, M_2, \ldots, M_N$ and $S_1, S_2, \ldots, S_N$ have been chosen so that (1)–(5) hold when $n = N$. It follows from (4) and the fact that $T \sim_{\mathrm{a}} T \oplus T_{N+1}$ that there is an operator S_{N+1} and a subspace M_{N+1} such that M_{N+1} is orthogonal to $M_1, M_2, \ldots, M_N$ and such that (1)–(5) hold when $n = N + 1$. Thus, by mathematical induction, we can choose the M_n's and the S_n's so that (1)–(5) hold for each positive integer n. Let $S = S_1 + S_2 + \dots$. Then $\|S\| < \varepsilon/2$ and $T + S$ is reduced by $M_1, M_2, \dots$; moreover, $(T + S)|M_n \cong T_n$ for $n = 1, 2, \dots$. Thus $T_1 \oplus T_2 \oplus \dots$ is unitarily equivalent to a summand of $T + S$. Since each of the operators $T_1, T_2, \dots$ appears infinitely often in the sequence $\{T_n\}$, it follows that

$$(T_1 \oplus T_2 \oplus \dots)^{(2)} \cong T_1 \oplus T_2 \oplus \dots,$$

whence

$$T + S \cong (T + S) \oplus T_1 \oplus T_2 \oplus \cdots = (T \oplus T_1 \oplus \cdots) + (S \oplus 0 \oplus \cdots).$$

Thus there is an operator S' with $\|S'\| < \varepsilon$ such that $T + S' \cong T \oplus T_1 \oplus T_2 \oplus \cdots$. Since $\varepsilon > 0$ was arbitrary, there is a sequence $\{U_n\}$ of unitary operators such that $\|U_n^* T U_n - (T \oplus T_1 \oplus \cdots)\| \to 0$. Since the set of operators A in $B(H)$ for which $\{U_n^* A U_n\}$ is norm convergent is a C^*-algebra, it follows that

$$\|U_n^* \pi(a) U_n - (\pi(a) \oplus \pi_1(a) \oplus \pi_2(a) \oplus \cdots)\| \to 0$$

for every a in $\mathfrak{A}$, whence $\pi \sim_a \pi \oplus \pi_1 \oplus \pi_2 \oplus \cdots$.

We are now ready to prove the first main theorem of this section.

THEOREM 3.10. *Suppose $\mathfrak{A}$ is separable and $\pi, \rho \in$ Rep$(\mathfrak{A}, H)$. Then $\pi \sim_a \rho$ if and only if* Ap-mult$(\tau, \pi) =$ Ap-mult(τ, ρ) *for every τ in* Irr$(\mathfrak{A})$.

PROOF. The "only if" part is obvious. Suppose Ap-mult$(\tau, \pi) =$ Ap-mult(τ, ρ) for every τ in Irr$(\mathfrak{A})$. Let $M_\pi = \bigvee \{\text{ran } \pi(a) : a \in \pi^{-1}(\mathcal{K}(H))\}$ and $M_\rho = \bigvee \{\text{ran } \rho(a) : a \in \rho^{-1}(\mathcal{K}(H))\}$. Write $\pi = \pi_0 \oplus \pi'$ relative to $H = M_\pi \oplus M_\pi^\perp$ and write $\rho = \rho_0 \oplus \rho'$ relative to $H = M_\rho \oplus M_\rho^\perp$. It follows from Proposition 3.5(3) that the isolated eigen-representations of π and ρ of finite multiplicity have the same multiplicities, and it follows from Proposition 2.10 that $\pi_0 \cong \rho_0$.

Since $\mathfrak{A}$ is separable, π' and ρ' can be written as direct sums of separable representations and, by [V, Corollary 1.6], these separable summands are approximately equivalent to direct sums of irreducible representations. Thus π' and ρ' are approximately equivalent to direct sums of irreducible representations, and there is no loss in assuming that π' and ρ' are actually equal to such direct sums. Since $\mathfrak{M}_\infty(\pi) = \mathfrak{M}_\infty(\rho)$, and, by Proposition 3.7, $\mathfrak{M}_\infty(\pi)$ is countable, we can write $\mathfrak{M}_\infty(\pi) = \mathfrak{M}_\infty(\rho) = \{m_1, m_2, \dots\}$. For each positive integer k let π_k (resp. ρ_k) be the direct sum of those irreducible subrepresentations of π' (resp. ρ') whose approximate multiplicity is m_k. Then $\pi' = \pi_1 \oplus \pi_2 \oplus \cdots$ and $\rho' = \rho_1 \oplus \rho_2 \oplus \cdots$. For each positive integer k, it follows from Proposition 3.8 that $\dim \pi_k \leqslant m_k$ and $\dim \rho_k \leqslant m_k$, and it follows from Proposition 3.5(4) that $\pi_k|\rho^{-1}(\mathcal{K}_{m_k}(H)) = 0$ and $\rho_k|\pi^{-1}(\mathcal{K}_{m_k}(H)) = 0$. Thus, by Lemma 3.2, we have $\pi \sim_a \pi \oplus \rho_k$ and $\rho \sim_a \rho \oplus \pi_k$ for $k = 1, 2, \dots$. Hence, by Lemma 3.9, $\pi \sim_a \pi \oplus \rho_1 \oplus \rho_2 \oplus \cdots$ and $\rho \sim_a \rho \oplus \pi_1 \oplus \pi_2 \oplus \cdots$. However, $\pi \oplus \rho_1 \oplus \rho_2 \oplus \cdots$ and $\rho \oplus \pi_1 \oplus \pi_2 \oplus \cdots$ are unitarily equivalent (to $\pi_0 \oplus \pi_1 \oplus \rho_1 \oplus \pi_2 \oplus \rho_2 \oplus \cdots$). Thus $\pi \sim_a \rho$.

COROLLARY 3.11. *Suppose $\mathfrak{A}$ is separable and $\pi \in$ Rep$(\mathfrak{A}, H)$. Then there is a sequence $\{\tau_k\}$ if irreducible representations and a sequence $\{m_k\}$ of cardinals such that $\pi \sim_a \sum_k^\oplus \tau_k^{(m_k)}$.*

PROOF. As in the proof of Theorem 3.10 let π_0 denote the direct sum of the irreducible subrepresentations of π that have finite approximate multiplicity, and write $\mathfrak{M}_\infty(\pi) = \{m_1, m_2, \ldots\}$. For each positive integer k, we can choose a sequence $\tau_{k1}, \tau_{k2}, \ldots$ in $\mathrm{Irr}(\mathfrak{A})$ so that Ap-mult$(\tau_{kn}, \pi) = m_k$ for $n = 1, 2, \ldots$ and so that every τ in $\mathrm{Irr}(\mathfrak{A})$ with Ap-mult$(\tau, \pi) = m_k$ is unitarily equivalent to a point-weak limit of representations in $\{\tau_{k1}, \tau_{k2}, \ldots\}$. (This can be done using Lemma 3.3.) Let $\pi_1 = \sum_k^\oplus \sum_j^\oplus \tau_{kj}^{(m_k)}$, and let $\rho = \pi_0 \oplus \pi_1$. It follows from Proposition 3.5(4) that $\pi \sim_a \pi \oplus \tau_{kj}^{(m_k)}$ for all positive integers j, k. It follows from Lemma 3.9 that $\pi \sim_a \pi \oplus \pi_1$. Since $\rho < \pi \oplus \pi_1$, it follows that Ap-mult$(\tau, \rho) <$ Ap-mult(τ, π) for every τ in $\mathrm{Irr}(\mathfrak{A})$; the reverse inequalities follow from the choice of the τ_{kn}'s and Lemma 3.6. It follows from Theorem 3.10 that $\pi \sim_a \rho$.

COROLLARY 3.12. *If* $\mathfrak{A}$ *is separable and* $\pi \in \mathrm{Rep}(\mathfrak{A}, H)$, *then* $\mathfrak{M}_\infty(\pi) = \{\mathrm{rank}\,\pi(a) : a \notin \pi^{-1}(\mathcal{K}(H))\}$.

To extend Theorem 2.5 to the case when H is not separable we need the following (algebraic) characterization of Ap-mult(τ, π).

LEMMA 3.13. *Suppose* $\mathfrak{A}$ *is separable,* $\pi \in \mathrm{Rep}(\mathfrak{A}, H)$, *and* $\tau \in \mathrm{Irr}(\mathfrak{A})$. *Then* Ap-mult$(\tau, \pi) = \min\{\mathrm{rank}\,\pi(a) : \tau(a) \neq 0\}$.

PROOF. It is clear that Ap-mult$(\tau, \pi) < \min\{\mathrm{rank}\,\pi(a) : \tau(a) \neq 0\}$. If Ap-mult$(\tau, \pi)$ is finite, then it follows from Lemma 2.2(4) and Proposition 2.10(3) that there is an a in $\mathfrak{A}$ such that rank $\tau(a) = 1$ and rank $\pi(a) =$ Ap-mult(τ, π). Thus we can assume that Ap-mult(τ, π) is infinite. It follows from Lemma 3.2 that there is an a in $\mathfrak{A}$ such that $\tau(a) \neq 0$ and rank $\pi(a) =$ Ap-mult(τ, π). Thus Ap-mult$(\tau, \pi) =$ rank $\pi(a)$.

We are now ready to extend Theorem 2.5. Note that there are no separability assumptions on either $\mathfrak{A}$ or H. This theorem had been previously conjectured by the author.

THEOREM 3.14. *Suppose* $\pi, \rho \in \mathrm{Rep}(\mathfrak{A}, H)$. *Then* $\pi \sim_a \rho$ *if and only if* rank $\pi(a)$ = rank $\rho(a)$ *for every* a *in* $\mathfrak{A}$.

PROOF. The "only if" part follows from the (norm) lower semicontinuity of rank( ). Suppose rank $\pi(a) =$ rank $\rho(a)$ for every a in $\mathfrak{A}$. It follows from Lemma 2.4 that we can assume that $\mathfrak{A}$ is separable. It follows from Lemma 3.13 and Theorem 3.10 that $\pi \sim_a \rho$.

4. Operator ideals. So far in our nonseparable extensions of Voiculescu's theorem we have ignored the equivalence of (1) and (2) in Theorem 2.1; i.e., if $\mathfrak{A}$ and H are separable, $\pi, \rho \in \mathrm{Rep}(\mathfrak{A}, H)$, and $\pi \sim_a \rho$, then $\pi \sim_a \rho(\mathcal{K}(H))$. Voiculescu **[V]** viewed this part of his theorem as an extension of the Weyl-von Neumann theorem, which says that every Hermitian operator on a separable Hilbert space is the sum of a diagonal operator and a compact operator. Voiculescu's theorem implies that every representation of a separable C^*-algebra is approximately equivalent to a direct sum of irreducible representations. This result easily includes the extensions of the Weyl-von Neumann theorem to normal operators by I. D. Berg [**B**] and W. Sikonia [**Si**], and to n-normal operators by C. Pearcy and N. Salinas [**PS**].

The Weyl-von Neumann theorem is not true in nonseparable Hilbert spaces; however, G. Edgar, J. Ernest and S. G. Lee [**EEL**] have generalized the Weyl-von Neumann theorem to nonseparable Hilbert spaces by replacing the ideal of compact operators by the ideal $\mathcal{K}_m(H)$ where $m = \dim H$. Even with this replacement, the extension of the Weyl-von Neumann theorem in [**EEL**] works only in the cases when $\dim H$ is countably cofinal. One of the main results of this section extends the nonseparable Weyl-von Neumann theorem in [**EEL**] to representations: if $\mathfrak{A}$ is separable, $m = \dim H$ is countably cofinal, $\pi, \rho \in \operatorname{Rep}(\mathfrak{A}, H)$, and $\pi \sim_a \rho$, then $\pi \sim_a \rho(\mathcal{K}_m(H))$.

Another consequence of Voiculescu's theorem is the theorem (also due to Voiculescu [**V**, Theorem 1.8]) that if $\dim H = \aleph_0$, then every norm closed, separable, unital subalgebra of $B(H)/\mathcal{K}(H)$ is reflexive. We extend this result to quotients of the form $B(H)/\mathcal{K}_m(H)$ where H is nonseparable and $\aleph_0 < m < \dim H$; again the extension is true precisely when m is countably cofinal.

In this section we also examine the quotients of the form $B(H)/\mathcal{K}_m(H)$ and study various lifting problems for these quotients.

The following lemma appears in [**EEL**, Lemma 5.8] (and practically every other paper dealing with these ideals). The proof is included here mainly to give the reader the flavor of the ideas used in proofs involving $\mathcal{K}_m(H)$ when m is not countably cofinal.

LEMMA 4.1. *Suppose $\aleph_0 < m \leqslant \dim H$. Then m is not countably cofinal if and only if $\mathcal{K}_m(H) = \{T \in B(H)\colon \operatorname{rank} T < m\}$.*

PROOF. Suppose m is countably cofinal. Choose a sequence $\{m_k\}$ of cardinals so that $m_1 < m_2 < \cdots < m$ and $m = \sup_k m_k = m_1 + m_2 + \cdots$. Hence we can write $H = H_0 \oplus H_1 \oplus \cdots$ with $\dim H_k = m_k$ for $k = 1, 2, \ldots,$ and relative to this decomposition we can define an operator $T_k = 0 \oplus 1 \oplus \cdots \oplus 1/k \oplus 0 \oplus 0 \oplus \cdots$ for $k = 1, 2, \ldots$. Clearly, $\operatorname{rank} T_k < m$ for $k = 1, 2, \ldots,$ but $\{T_k\}$ is norm convergent to an operator whose rank equals m.

Conversely, suppose m is not countably cofinal and suppose $S \in \mathcal{K}_m(H)$. Then there is a sequence $\{S_k\}$ in $B(H)$ such that $\operatorname{rank} S_k < m$ for $k = 1, 2, \ldots$ and $\|S_k - S\| \to 0$. Thus $\operatorname{rank} S \leqslant \sup_k \operatorname{rank} S_k < m$ (since m is not countably cofinal).

We now turn to some of the results of the preceding section.

PROPOSITION 4.2. *Suppose $\{\pi_i\colon i \in I\}$, $\{\rho_i\colon i \in I\}$ are collections of separable representations of a separable C^*-algebra $\mathfrak{A}$ such that $\operatorname{card} I = m$ is countably cofinal. If $\pi = \sum_{i\in I}^{\oplus} \pi_i$ and $\rho = \sum_{i \in I}^{\oplus} \rho_i$ and $\pi_i \sim_a \rho_i$ for each i in I, then $\pi \sim_a \rho(\mathcal{K}_m(H))$.*

PROOF. Let $a_1, a_2, \ldots$ be dense in $\mathfrak{A}$. Since m is countably cofinal, we can write I as a disjoint union of subsets $I_1, I_2, \ldots$ each with cardinality less than m. For each positive integer n we can choose unitary operators U_{in}, $i \in I$, so that $\| U_{in}^* \pi_i(a_j) U_{in} - \rho_i(a_j) \| < 1/kn$ whenever $k = 1, 2, \ldots$, $i \in I_k$, $1 \leq j \leq k + n$. Let $U_n = \sum_{i \in I}^{\oplus} U_{in}$ for $n = 1, 2, \ldots$. Then

$$\| U_n^* \pi(a_j) U_n - \rho(a_j) \| < 1/n \quad \text{for } 1 \leq j \leq n < \infty.$$

Hence $\| U_n^* \pi(a_j) U_n - \rho(a_j) \| \to 0$ for $j = 1, 2, \ldots$. Since $\{a_1, a_2, \ldots\}$ is dense in $\mathfrak{A}$, it follows that $\| U_n^* \pi(a) U_n - \rho(a) \| \to 0$ for every a in $\mathfrak{A}$. It is also clear that $U_n^* \pi(a_j) U_n - \rho(a_j)$ is in $\mathcal{K}_m(H)$ for $j = 1, 2, \ldots$, and since $\{a_1, a_2, \ldots\}$ is dense in $\mathfrak{A}$, it follows that $U_n^* \pi(a) U_n - \rho(a) \in \mathcal{K}_m(H)$ for every a in $\mathfrak{A}$. Thus $\pi \sim_a \rho(\mathcal{K}_m(H))$.

COROLLARY 4.3. *If $m = \dim H$ is countably cofinal, $\mathfrak{A}$ is separable, and $\pi \in \operatorname{Rep}(\mathfrak{A}, H)$, then there is a ρ in $\operatorname{Rep}(\mathfrak{A}, H)$ such that ρ is a direct sum of irreducible representations and $\pi \sim_a \rho(\mathcal{K}_m(H))$.*

PROOF. Since $\mathfrak{A}$ is separable, we know that π can be written as a direct sum of separable representations, and each of these is approximately equivalent to a direct sum of irreducible representations (Corollary 3.11). Now apply Proposition 4.2.

COROLLARY 4.4. *Suppose $\mathfrak{A}$ is separable, H_π and H_ρ are infinite-dimensional Hilbert spaces, $\pi \in \operatorname{Rep}(\mathfrak{A}, H_\pi)$, $\rho \in \operatorname{Rep}(\mathfrak{A}, H_\rho)$. Suppose $m \geq \dim H_\rho$ and $\rho | \pi^{-1}(\mathcal{K}_m(H_\pi)) = 0$. If m is countably cofinal, then $\pi \sim_a \pi \oplus \rho(\mathcal{K}_m)$.*

PROOF. Review the proof of Lemma 3.2 and apply Proposition 4.2 at the appropriate point.

The preceding corollary is an analogue of Lemma 3.2. The proof of the following lemma is a simple adaptation of Lemma 3.9 and it is omitted.

LEMMA 4.5. *Suppose $\mathfrak{A}$ is separable, $\pi, \pi_1, \pi_2, \ldots \in \operatorname{Rep}(\mathfrak{A})$, and m is an infinite cardinal. If $\pi \sim_a \pi \oplus \pi_n(\mathcal{K}_m)$ for $n = 1, 2, \ldots$, then $\pi \sim_a \pi \oplus \pi_1 \oplus \pi_2 \oplus \cdots (\mathcal{K}_m)$.*

We are now ready to prove the analogue (1) $\Leftrightarrow$ (2) in Theorem 2.1 for nonseparable representations.

THEOREM 4.6. *Suppose $\mathfrak{A}$ is separable, $m = \dim H$ is infinite and countably cofinal, and $\pi, \rho \in \operatorname{Rep}(\mathfrak{A}, H)$. If $\pi \sim_a \rho$, then $\pi \sim_a \rho(\mathcal{K}_m(H))$.*

PROOF. It follows from Corollary 4.3 that we can assume that π, ρ are direct sums of irreducible representations. Let π_0 (resp. ρ_0) be the direct sum of the irreducible subrepresentations of π (resp. ρ) having finite approximate multiplicity. It follows from Lemma 3.5(4) and Proposition 2.12 that $\pi_0 \cong \rho_0$. Write $\mathfrak{M}_\infty(\pi) = \mathfrak{M}_\infty(\rho) = \{m_1, m_2, \ldots\}$, and, for each positive integer k, let π_k (resp. ρ_k) be the direct sum of all of the irreducible subrepresentations of π (resp. ρ) having approximate multiplicity m_k. It follows from Proposition 3.8 that $\dim \pi_k \leq m_k$ and $\dim \rho_k \leq m_k$ for $k = 1, 2, \ldots$. It follows from Corollary 4.4 that $\pi \sim_a \pi \oplus \rho_k(\mathcal{K}_m)$ and $\rho \sim_a \rho \oplus \pi_k(\mathcal{K}_m)$ for $k = 1, 2, \ldots$. Hence, by Lemma 4.6, we have $\pi \sim_a \pi \oplus \rho_1 \oplus \rho_2 \oplus \cdots (\mathcal{K}_m)$ and $\rho \sim_a \rho \oplus \pi_1 \oplus \pi_2 \oplus \cdots (\mathcal{K}_m)$. Since $\pi \oplus \rho_1 \oplus \rho_2 \oplus \cdots$ is unitarily equivalent to $\rho \oplus \pi_1 \oplus \pi_2 \oplus \cdots$, it follows that $\pi \sim_a \rho(\mathcal{K}_m)$.

In contrast to the preceding theorem, the case when $\dim H$ is not countably cofinal is much different.

PROPOSITION 4.7. *Suppose* $\mathfrak{A}$ *is separable*, $\pi, \rho \in \operatorname{Rep}(\mathfrak{A}, H)$, *and* $m = \dim H$ *is not countably cofinal. Then* $\pi \cong \rho(\mathcal{K}_m)$ *if and only if there are representations* π_1, ρ_1, τ *such that* $\pi \cong \pi_1 \oplus \tau$, $\rho \cong \rho_1 \oplus \tau$, $\dim \pi_1 < m$, *and* $\dim \rho_1 < m$.

PROOF. The "if" part is obvious. Suppose $\pi \cong \rho(\mathcal{K}_m)$. We can assume that $\pi(a) - \rho(a) \in \mathcal{K}_m(H)$ for every a in $\mathfrak{A}$. Since $\mathfrak{A}$ is separable, we can write $\pi = \sum_{i\in I}^{\oplus} \pi_i$ and $\rho = \sum_{i\in I}^{\oplus} \rho_i$ relative to $H = \sum_{i\in I}^{\oplus} H_i$, where each H_i is separable. Let $\{a_1, a_2, \dots\}$ be dense in $\mathfrak{A}$, and, for each positive integer k, let $I_k = \{i \in I: \pi_i(a_k) - \rho_i(a_k) \neq 0\}$. Let $I_0 = I_1 \cup I_2 \dots$, and let $J = I - I_0$. Since m is not countably cofinal, it follows that card $I_0 < m$. Thus

$$\pi_1 = \sum_{i\in I_0}^{\oplus} \pi_i, \quad \rho_1 = \sum_{i\in I_0}^{\oplus} \rho_i \quad \text{and} \quad \tau = \sum_{i\in J}^{\oplus} \pi_i$$

are the desired representations.

It is clear that the preceding theorem implies that Theorem 4.6 is not true when $\dim H$ is not countably cofinal (e.g., let π be a representation with no irreducible subrepresentations, and let ρ be a direct sum of irreducible representations such that $\pi \sim_a \rho$).

We now turn our attention to quotients of the form $B(H)/\mathcal{K}_m(H)$. If $\aleph_0 < m < \dim H$, let $\mathcal{C}_m(H)$ denote $B(H)/\mathcal{K}_m(H)$, and let $\nu_m: B(H) \to \mathcal{C}_m(H)$ be the quotient map. If $\mathcal{S} \subseteq \mathcal{C}_m(H)$, let $\operatorname{Lat}_m(\mathcal{S})$ be the set of all projections p in $\mathcal{C}_m(H)$ such that $(1-p)sp = 0$ for every s in $\mathcal{S}$. The set $\mathcal{S}$ is *reflexive* in $\mathcal{C}_m(H)$ if

$$\mathcal{S} = \{a \in \mathcal{C}_m(H): (1-p)ap = 0 \text{ for every } p \text{ in } \operatorname{Lat}_m(\mathcal{S})\}.$$

It is clear that a necessary condition for the reflexivity of $\mathcal{S}$ is that $\mathcal{S}$ be a unital, norm closed algebra. The following theorem is an analogue of Arveson's distance formula [**Ar 1**, Corollary 2] for subalgebras of the Calkin algebra, and the corollaries are analogues of results of Voiculescu [**V**, Theorem 1.8, Corollary 1.9]. The proof of this theorem is only a slight modification of Arveson's proof of Corollary 2 in [**Ar 1**]. Note that we do not assume that $m = \dim H$.

THEOREM 4.8. *Suppose* m *is an infinite, countably cofinal cardinal*, $m < \dim H$, *and* $\mathcal{S}$ *is a separable, unital, norm closed subalgebra of* $\mathcal{C}_m(H)$. *Then for each* t *in* $\mathcal{C}_m(H)$ *there is a* q *in* $\operatorname{Lat}_m(\mathcal{S})$ *such that*

$$\|(1-q)tq\| = \sup\{\|(1-p)tp\|: p \in \operatorname{Lat}_m(\mathcal{S})\} = \operatorname{dist}(t, \mathcal{S}).$$

PROOF. Fix t and choose a separable, unital C^*-subalgebra $\mathfrak{A}$ of $B(H)$ so that $\nu_m(\mathfrak{A})$ is the C^*-algebra generated by t and $\mathcal{S}$. Choose T in $\mathfrak{A}$ so that $\nu_m(T) = t$. According to Arveson's proof of [**Ar 1**, Corollary 2] there is a separable representation $\tau: \mathfrak{A} \to B(H_\tau)$ such that $\mathcal{K}_m(H) \subseteq \ker \tau$ and such that there is a $\tau(\nu_m^{-1}(\mathcal{S}))$-invariant projection P in $B(H_\tau)$ such that $\|(1-P)\tau(T)P\| > \operatorname{dist}(t, \mathcal{S})$. Let π be the identity representation on $\mathfrak{A}$ and let $\rho = \pi \oplus \tau^{(m)}$. It follows from Corollary 4.4 that $\pi \sim_a \rho(\mathcal{K}_m(H))$. Thus there is a unitary operator U such that

$$A - U^*\rho(A)U = \pi(A) - U^*\rho(A)U \in \mathcal{K}_m(H)$$

for every A in $\mathfrak{A}$. Let $q = \nu_m(U^*(0 \oplus P^{(m)})U)$. Then $q \in \mathrm{Lat}_m(\mathbb{S})$ and

$$\|(1-q)tq\| \geqslant \|(1-P)\tau(T)P\| \geqslant \mathrm{dist}(t, \mathbb{S}).$$

On the other hand, $\|(1-q)tq\| = \|(1-q)(t-s)q\| \leqslant \|t-s\|$ for every s in $\mathbb{S}$. Thus $\|(1-q)tq\| = \mathrm{dist}(t, \mathbb{S})$.

Corollary 4.9. *Suppose m is an infinite, countably cofinal cardinal, $m \leqslant \dim H$, and $\mathbb{S}$ is a unital, separable C^*-subalgebra of $\mathcal{C}_m(H)$. Then $\mathbb{S}$ is equal to its own double commutant.*

Corollary 4.10. *Suppose m is an infinite, countably cofinal cardinal, $m \leqslant \dim H$, and $\mathbb{S}$ is a unital, separable, norm closed subalgebra of $\mathcal{C}_m(H)$. Then $\mathbb{S}$ is reflexive.*

The next theorem illustrates the "meta-theorem" that if $\aleph_0 < m \leqslant \dim H$ and m is not countably cofinal, then any separable subset of $\mathcal{C}_m(H)$ having a property that can be "countably" defined can be lifted to a separable subset of $B(H)$ that possesses the same property. It also shows that the countable cofinality of m cannot be dropped in the preceding three results. If $\mathbb{S} \subseteq B(H)$, let $\mathbb{S}'$ denote the commutant of $\mathbb{S}$.

Theorem 4.11. *Suppose m is an infinite cardinal, $m \leqslant \dim H$, and m is not countably cofinal. If $\mathbb{S}$ is a separable subset of $B(H)$, then*:

(1) *there is a separable C^*-subalgebra $\mathfrak{A}$ of $B(H)$ such that $1 \in \mathfrak{A}$, $\nu_m(\mathfrak{A})$ is the C^*-algebra generated by $\nu_m(\mathbb{S})$ and* 1, *and $\nu_m|\mathfrak{A}$ is an isometry*;

(2) $\nu_m(\mathbb{S}') = \nu_m(\mathbb{S})'$;

(3) $\nu_m(\mathrm{Lat}\, \mathbb{S}) = \mathrm{Lat}_m(\nu_m(\mathbb{S}))$.

Proof. (1) Choose $T_1, T_2, \ldots$ in $B(H)$ so that $\nu_m(T_1), \nu_m(T_2), \ldots$ is dense in $C^*(\nu_m(\mathbb{S}))$. For each pair (i, j) of positive integers choose a sequence $\{K_{i,j,n}\}$ in $\mathcal{K}_m(H)$ so that

$$\|T_i - T_j + K_{i,j,n}\| \to \|\nu_m(T_i - T_j)\| = \|\nu_m(T_i) - \nu_m(T_j)\|.$$

Let $M_0 = \bigvee\{\mathrm{ran}\, K_{i,j,n}: i, j, n = 1, 2, \ldots\}$. It follows from Lemma 4.1 that $\mathrm{rank}\, K_{i,j,n} < m$ for $i, j, n = 1, 2, \ldots$. Thus it follows that $\dim M_0 < m$. Let M be smallest subspace of H that contains M_0 and reduces all of the operators $T_1, T_2, \ldots$. Then $\dim M \leqslant \aleph \cdot \dim M_0 < m$. Let P be the orthogonal projection onto $M^\perp$, and let $S_n = PT_n$ for $n = 1, 2, \ldots$. Thus, for each pair (i, j) of positive integers we have

$$\begin{aligned}\|S_i - S_j\| = \|P(T_i - T_j)\| &= \lim_n \|P(T_i - T_j + K_{i,j,n})\| \\ &\leqslant \|\nu_m(T_i) - \nu_m(T_j)\| = \|\nu_m(S_i) - \nu_m(S_j)\|.\end{aligned}$$

Thus $\{\nu_m(S_1), \nu_m(S_2), \ldots\} = \{\nu_m(T_1), \nu_m(T_2), \ldots\}$ is dense in $C^*(\nu_m(\mathbb{S}))$ and $\nu_m|\{S_1, S_2, \ldots\}$ is an isometry. It seems that we are finished. However, the norm closure of $\{S_1, S_2, \ldots\}$ will not generally contain 1. We can remedy this situation by choosing a $*$-isomorphism $\pi: C^*(\nu_m(\mathbb{S})) \to B(M)$ and replacing each S_n by $S_n \oplus \pi(\nu_m(S_n))$. (Note that there is no harm in assuming that M is infinite dimensional.)

(2) Let $S_1, S_2, \ldots$ be dense in $\mathcal{S}$. It is clear that $\nu_m(\mathcal{S}') \subseteq \nu_m(\mathcal{S})'$. Suppose $T \in B(H)$ and $\nu_m(T) \in \nu_m(\mathcal{S})'$. Thus $TS_n - S_nT \in \mathcal{K}_m(H)$ for $n = 1, 2, \ldots$. Let M be the smallest subspace of H that reduces $T, S_1, S_2, \ldots$ and contains $\bigvee\{\operatorname{ran}(TS_n - S_nT): n = 1, 2, \ldots\}$. As in the proof of (1) we conclude that $\dim M < m$. If P is the projection onto $M^\perp$, then $TP \in \mathcal{S}'$ and $T - TP = T(1-P) \in \mathcal{K}_m(H)$. Thus $\nu_m(T) \in \nu_m(\mathcal{S}')$. Therefore $\nu_m(\mathcal{S})' \subseteq \nu_m(\mathcal{S}')$.

(3) The proof of (3) is very similar to the proof of (2) and is omitted.

COROLLARY 4.12. *If m is an infinite cardinal, $m < \dim H$, and m is not countably cofinal, then $\mathcal{C}_m(H)$ contains a separable, unital C^*-algebra that does not equal its own double commutant.*

Note that Theorem 4.11(1) shows that the theory of extensions of Brown, Douglas and Fillmore **[BDF]** is completely trivial in the quotient $\mathcal{C}_m(H)$ when m is not countably cofinal.

In addition, Theorem 4.11(1) shows that any property of an operator that can be defined in terms of the C^*-algebra that it generates can be lifted from the quotient $\mathcal{C}_m(H)$ when m is not countably cofinal. Among such properties are the properties that are simultaneously preserved under direct sums, restrictions to reducing subspaces, and norm limits (see **[H 2**, Theorem 5.1**]**); we will call these latter properties *continuous part properties*. It was proved in **[H 2**, Theorem 5.1**]** that if we are given a continuous part property and a positive number r, then there is a sequence $\{p_n(x, y)\}$ of noncommutative polynomials such that:

(1) $\{p_n(T, T^*)\}$ is uniformly (norm) convergent on every bounded set of operators;

(2) an operator T has $\|T\| \leq r$ and the given property if and only if $p_n(T, T^*) \to 0$.

Define $\varphi(T) = \lim p_n(T, T^*)$ for every operator T; such a function is called a *continuous decomposable function* **[H 2]**. Note that (1) implies that $\varphi(a) = \lim p_n(a, a^*)$ makes sense when a is an element of a unital C^*-algebra. Some of the obvious properties of φ are:

(3) $\varphi(A \oplus B) = \varphi(A) \oplus \varphi(B)$ for all operators A and B;

(4) $\varphi(T) \in C^*(T)$ for every operator T;

(5) if T is an operator and $\pi \in \operatorname{Rep}(C^*(T))$, then $\pi(\varphi(T)) = \varphi(\pi(T))$;

(6) $\varphi | B(H)$ is norm continuous for every Hilbert space H.

Note that the definition $\varphi(T) = \lim p_n(T, T^*)$ where $\{p_n(x, y)\}$ satisfies (1) is not the definition of a continuous decomposable function given in **[H 2]**, but it is equivalent to that definition **[H 2**, Proposition 2.1**]**. We can restate the preceding characterization of continuous part porperties in terms of continuous decomposable functions: given a continuous part property and a positive number r, there is a continuous decomposable function φ such that $\varphi(T) = 0$ precisely when $\|T\| \leq r$ and T has the given property.

Examples of continuous part properties are normality and subnormality. The following theorem shows that if m is an uncountable, countably cofinal cardinal, then many lifting problems in $\mathcal{C}_m(H)$ do not depend upon m. Recall that $A^{(\infty)} = A \oplus A \oplus \cdots$.

THEOREM 4.13. *Suppose m is an uncountable, countably cofinal cardinal, $m <$ $\dim H$, and φ is a continuous decomposable function. The following are equivalent*:

(1) *for every t in $\mathcal{C}_m(H)$ with $\varphi(t) = 0$ there is a T in $B(H)$ such that $\varphi(T) = 0$ and $\nu_m(T) = t$;*

(2) *for every bounded sequence $\{T_n\}$ of operators on a separable Hilbert space with $\|\varphi(T_n)\| \to 0$ there is a sequence $\{S_n\}$ such that $\varphi(S_n) = 0$ for $n = 1, 2, \ldots$ and $\|S_n - (T_n \oplus T_{n+1} \oplus \cdots)^{(\infty)}\| \to 0$.*

PROOF. (2) $\Rightarrow$ (1) Suppose (2) is true, and suppose $t \in \mathcal{C}_m(H)$ with $\varphi(t) = 0$. Choose an operator A in $B(H)$ such that $\nu_m(A) = t$. Since the space $M = \bigvee \{\text{ran } S\colon S \in C^*(A) \cap \mathcal{K}_m(H)\}$ has dimension at most m, and since $\varphi(A|M^\perp) = \varphi(A)|M^\perp = 0$, we can assume that $\dim H = m$. Also, since

$$\dim \bigvee \{\text{ran } S\colon S \in C^*(A) \cap \mathcal{K}(H)\} < \aleph_0,$$

we can assume that $C^*(A) \cap \mathcal{K}(H) = 0$. It follows from Corollary 3.11 that there is a sequence $\{B_k\}$ of irreducible operators and a sequence $\{m_k\}$ of infinite cardinals such that the operator $B = \sum_k^\oplus B_k^{(m_k)}$ is approximately equivalent to A. It follows from Theorem 4.6 that $A \cong B \oplus K$ for some K in $\mathcal{K}_m(H)$. Thus we can assume that $\nu_m(B) = t$. Note that $\varphi(B)$ is the direct sum of the operators $\varphi(B_k^{(m_k)})$. Thus if $m_k = m$ for some k, then $\varphi(B_k) = 0$ (because $\nu_m(\varphi(B)) = \varphi(\nu_m(B)) = \varphi(t) = 0$). Hence we can assume that $m_k < m$ for $k = 1, 2, \ldots$. A moment's reflection shows that there is no harm in assuming $m_1 < m_2 < \cdots$. Since $\varphi(B) \in \mathcal{K}_m(H)$, it follows that $\lim_k \|\varphi(B_k)\| = 0$. Thus, by (2), there is a sequence $\{S_k\}$ of operators such that $\varphi(S_k) = 0$ for $k = 1, 2, \ldots$ and $\|S_k - (B_k \oplus B_{k+1} \oplus \cdots)^{(\infty)}\| \to 0$. We can rewrite B as a direct sum $\sum_k^\oplus((B_k \oplus B_{k+1} \oplus \cdots)^{(\infty)})^{(m_k)}$, and if we let $T = \sum_k S_k^{(m_k)}$, then $B - T \in \mathcal{K}_m(H)$. Thus $\varphi(T) = 0$ and $\nu_m(T) = t$.

(1) $\Rightarrow$ (2) Suppose (1) is true and $\{T_n\}$ is a bounded sequence of operators on a separable Hilbert space with $\|\varphi(T_n)\| \to 0$. Since each T_n is approximately equivalent to a direct sum of (countably many) irreducible operators, there is no harm in assuming that T_n is irreducible for $n = 1, 2, \ldots$. There is also no harm in assuming that $\varphi(T_n) \neq 0$ for $n = 1, 2, \ldots$ and that $T_n \cong T_k$ only when $n = k$. Next we will show that there is an irreducible operator T_0 such that

$$T_0 \oplus T_n \oplus T_{n+1} \oplus \cdots \sim_a T_n \oplus T_{n+1} \oplus \cdots$$

for $n = 1, 2, \ldots$. To do this, consider the C^*-algebra $\mathfrak{A}$ of bounded sequences of operators on the Hilbert space where the T_n's act, and consider the $*$-ideal $\mathfrak{J}$ of sequences that converge (in norm) to 0. If A is the image of the sequence $\{T_n\}$ in $\mathfrak{A}/\mathfrak{J}$, then, for each positive integer n, there is a unital $*$-homomorphism from $C^*(T_n \oplus T_{n+1} \oplus \cdots)$ onto $C^*(A)$ that sends $T_n \oplus T_{n+1} \oplus \cdots$ onto A. Suppose $\pi \in \operatorname{Irr}(C^*(A))$ and let $T_0 = \pi(A)$. It is clear that

$$\|\varphi(T_0)\| < \|\varphi(T_n \oplus T_{n+1} \oplus \cdots)\|$$

for $n = 1, 2, \ldots,$ whence $\varphi(T_0) = 0$. Thus $T_0 \cong T_n$ for $n = 1, 2, \ldots$. It follows from Proposition 3.5 that $T_0 \oplus T_n \oplus T_{n+1} \oplus \cdots \sim_a T_n \oplus T_{n+1} \oplus \cdots$ for $n = 1, 2, \ldots$. Since m is countably cofinal, there is an increasing sequence $\{m_k\}$ of infinite cardinals less than m such that $m = \sup_k m_k$. Let $q = \dim H$, and define

$T = T_0^{(q)} \oplus \sum_k^{\oplus} T_k^{(m_k)}$. It is clear that $\varphi(T) \in \mathcal{K}_m(H)$. Thus if $t = \nu_m(T)$, then $\varphi(t) = \varphi(\nu_m(T)) = \nu_m(\varphi(T)) = 0$. Hence, by (1), there is an S in $B(H)$ such that $\varphi(S) = 0$ and $\nu_m(S) = t = \nu_m(T)$. Thus $S - T \in \mathcal{K}_m(H)$. We can write $H = \sum_{i \in I}^{\oplus} H_i$ where each H_i is separable and reduces both S and T. Suppose $\varepsilon > 0$. Since $T - S \in \mathcal{K}_m(H)$, there is a subset J of I such that $\operatorname{card} J < m$ and $\|(S - T)|H_i\| < \varepsilon$ whenever $i \notin J$. Choose k so that $m_k > \operatorname{Card} J$. Since $m_1 < m_2 < \cdots < m_k$, there is no harm in assuming that $(S - T)|H_i$ has no summand that is unitarily equivalent to one of the operators $T_1, T_2, \ldots, T_{k-1}$ whenever $i \notin J$ (because we need only add to J a set with cardinality less than m_k to obtain this property). Hence there is a countable subset I_k of $I - J$ such that each of the operators $T_0, T_k, T_{k+1}, \ldots$ appears infinitely often as a summand of $T|M_k$ where $M_k = \sum_{i \in I_k}^{\oplus} H_i$. Thus $T|M_k$ is unitarily equivalent to $(T_0 \oplus T_k \oplus T_{k+1} \oplus \cdots)^{(\infty)}$. If $S_k = S|M_k$, then $\|S_k - (T_0 \oplus T_k \oplus T_{k+1} \oplus \cdots)^{(\infty)}\| < \varepsilon$; since this is true for each k with $m_k > \operatorname{Card} J$, we have proved that

$$\|S_k - (T_0 \oplus T_k \oplus T_{k+1} \oplus \cdots)^{(\infty)}\| \to 0 \quad \text{as } k \to \infty.$$

It is clear that $\varphi(S_k) = 0$ for every k. Also $\varphi(U^* S_k U) = U^* \varphi(S_k) U = 0$ for every unitary operator U. Since $(T_0 \oplus T_k \oplus T_{k+1} \oplus \cdots)^{(\infty)}$ is approximately equivalent to $(T_k \oplus T_{k+1} \oplus \cdots)^{(\infty)}$ for every k, we can conclude that there is a sequence $\{U_k\}$ of unitary operators such that $\|U_k^* S_k U_k - (T_k + T_{k+1} + \cdots)^{(\infty)}\| \to 0$. This proves (2).

Note that Theorem 4.13(2) is implied by the following statement: for every bounded sequence $\{T_n\}$ of operators on a separable Hilbert space with $\|\varphi(T_n)\| \to 0$, there is a sequence $\{S_n\}$ of operators such that $\varphi(S_n) = 0$ for $n = 1, 2, \ldots$ and $\|S_n - T_n\| \to 0$. This latter condition was studied by S. Campbell and R. Gellar **[CG]**. In particular, the functions $\varphi(T) = |1 - T^*T| + |1 - TT^*|$ and $\psi(T) = |T^*T - \varepsilon| - (T^*T - \varepsilon)$ satisfy this condition (where $\varepsilon > 0$).

COROLLARY 4.14. *Suppose m is an uncountable, countably cofinal cardinal, $m <$ $\dim H$. Suppose $t \in \mathcal{C}_m(H)$ and t is unitary (resp. invertible); then there is a T in $B(H)$ such that $\nu_m(T) = t$ and T is unitary (resp. invertible).*

The preceding theorem and its corollary show a marked difference between $\aleph_0$ and other countably cofinal cardinals. The fact that unitary elements and invertible elements can be lifted from $\mathcal{C}_m(H)$ when $m > \aleph_0$ suggests that the same might be true for normal elements. However, it is known **[PRH 2]** that there is a bounded sequence of operators $\{T_n\}$ such that $\|T_n^* T_n - T_n T_n^*\| \to 0$ and the distance from the T_n's to the set of normal operators is bounded away from 0. On the other hand, it is not known whether Theorem 4.13(2) is true when φ is defined by $\varphi(T) = T^*T - TT^*$.

It should be noted that the condition in Theorem 4.13(2) is dependent only upon the property defined by $\varphi(T) = 0$ and not by the decomposable function φ used to describe this property. (This is implied by the equivalence of (1) and (2) in Theorem 4.13.) A stronger statement can be proved.

PROPOSITION 4.15. *Suppose φ, ψ are continuous decomposable functions such that, for each operator T, $\varphi(T) = 0$ if and only if $\psi(T) = 0$. Suppose $\{T_n\}$ is a bounded sequence of operators. Then $\|\varphi(T_n)\| \to 0$ if and only if $\|\psi(T_n)\| \to 0$.*

PROOF. Apply the hypothesis to $\mathfrak{A}/\mathfrak{J}$ where $\mathfrak{A}$ is the C^*-algebra of bounded sequences of operators and $\mathfrak{J}$ is the ideal of null sequences.

The following conjecture is weaker than Theorem 4.13(2). It is a converse of the fact that if $\{A_n\}$ is a bounded sequence of operators, φ is a continuous decomposable function with $\varphi(A_n) = 0$ for $n = 1, 2, \ldots,$ and if T_n is "almost" a summand of A_n as $n \to \infty$, then $\|\varphi(T_n)\| \to 0$.

CONJECTURE 4.16. If $\{T_n\}$ is a bounded sequence of operators on a separable Hilbert space, φ is a continuous decomposable function, and $\|\varphi(T_n)\| \to 0$, then there are bounded sequences $\{A_n\}, \{B_n\}$ such that $\varphi(A_n) = 0$ for $n = 1, 2, \ldots$ and $\|A_n - B_n \oplus T_n\| \to 0$.

5. Approximate subrepresentations. In this section we extend some of the results of the preceding sections to approximate subrepresentations. Although many of the results are true for nonseparable C^*-algebras, we restrict ourselves to the separable case. However, the representations are not assumed to be separable. For singly generated C^*-algebras some of these results appear in **[H 1]** and **[BuDe]**.

If $\pi, \rho \in \operatorname{Rep}(\mathfrak{A})$, then π is an *approximate subrepresentation* of ρ, denoted $\pi <_a \rho$, provided there is a net $\{V_n\}$ of isometries such that $\|V_n^*\rho(a)V_n - \pi(a)\| \to 0$ and $\|(V_nV_n^*)\rho(a) - \rho(a)(V_nV_n^*)\| \to 0$ for every a in $\mathfrak{A}$.

THEOREM 5.1. *Suppose $\mathfrak{A}$ is separable and $\pi, \rho \in \operatorname{Rep}(\mathfrak{A})$. The following are equivalent:*

(1) $\pi <_a \rho$;

(2) $\text{Ap-mult}(\tau, \pi) \leq \text{Ap-mult}(\tau, \rho)$ *for every τ in* $\operatorname{Irr}(\mathfrak{A})$;

(3) $\operatorname{rank} \pi(a) \leq \operatorname{rank} \rho(a)$ *for every a in $\mathfrak{A}$;*

(4) *there is a representation ρ' such that $\pi \leq \rho'$ and $\rho \sim_a \rho'$;*

(5) *there is a sequence $\{V_n\}$ of isometries such that $V_n^*\rho(a)V_n \to \pi(a)$ weakly for each a in $\mathfrak{A}$.*

PROOF. The implications (4) $\Rightarrow$ (1) $\Rightarrow$ (5) $\Rightarrow$ (3) are obvious.

(3) $\Rightarrow$ (2) This follows from Lemma 3.13.

(2) $\Rightarrow$ (4) By Corollary 3.11 we can assume that there is a sequence $\{\tau_k\}$ in $\operatorname{Irr}(\mathfrak{A})$ and a sequence $\{m_k\}$ of cardinals such that $\pi = \sum_k^{\oplus} \tau_k^{(m_k)}$. It follows from (2) and Proposition 3.5(3) that we can assume that $\text{Ap-mult}(\tau_k, \pi)$ is infinite for $k = 1, 2, \ldots.$ Since $m_k \leq \text{Ap-mult}(\tau_k, \pi) \leq \text{Ap-mult}(\tau_k, \rho)$ for $k = 1, 2, \ldots,$ it follows that $\rho \sim_a \rho \oplus \tau_k^{(m_k)}$ for $k = 1, 2, \ldots.$ Therefore, by Lemma 3.9, $\rho \sim_a \rho \oplus \pi$. Hence (4) is true.

It should be noted that the implications (1) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4) $\Leftarrow$ (5) remain true when $\mathfrak{A}$ is not separable, while (1) $\Rightarrow$ (5) is generally false (see Proposition 2.7). If $\mathfrak{A}$ is nonseparable and the term "sequence" in (5) is replaced by "net", then (1) $\Rightarrow$ (5) is obviously true, but (5) $\Rightarrow$ (1) no longer true; the problem lies in the fact that the function rank( ) is weakly sequentially lower semicontinuous but not weakly lower semicontinuous on nonseparable Hilbert spaces. It should also be pointed out that it is necessary in (5) to assume that π is a representation because this fact does not follow automatically unless it is known that $V_n^*\rho(a)V_n \to \pi(a)$ strongly for each a in $\mathfrak{A}$.

We conclude this section with an analogue of Theorem 4.6 for approximate subrepresentations.

THEOREM 5.2. *Suppose $\mathfrak{A}$ is separable, $\pi, \rho \in \mathrm{Rep}(\mathfrak{A})$, and $\pi \lessdot_a \rho$. Suppose also that m is an infinite, countably cofinal cardinal with $m \geq \dim \pi$. Then there is a representation ρ' such that $\rho \sim_a \rho'(\mathcal{K}_m(H))$ and $\pi \leq \rho'$.*

PROOF. It follows from Theorem 4.6 and Corollary 3.11 that we can assume that there is a sequence $\{\tau_k\}$ of irreducible representations and a sequence $\{m_k\}$ of cardinals such that $\pi = \sum_k^{\oplus} \tau_k^{(m_k)}$. If Ap-mult$(\tau_k, \rho)$ is finite for some k, then it follows from Proposition 3.5(3) that $\tau_k^{(m_k)}$ is unitarily equivalent to a subrepresentation of ρ. Hence we can assume that Ap-mult(τ_k, ρ) is infinite for $k = 1, 2, \ldots$. Thus, by Corollary 4.4, we have $\rho \sim_a \rho \oplus \tau_k^{(m_k)}(\mathcal{K}_m)$ for $k = 1, 2, \ldots,$ and, by Lemma 4.5, we have $\rho \sim_a \rho \oplus \pi(\mathcal{K}_m)$. This completes the proof.

We conclude this section with a look at what statement (3) in Voiculescu's theorem (Theorem 2.1) means on nonseparable Hilbert spaces.

PROPOSITION 5.3. *Suppose $\pi \in \mathrm{Rep}(\mathfrak{A}, H_\pi)$, $\rho \in \mathrm{Rep}(\mathfrak{A}, H_\rho)$, and $\dim H_\pi \leq \dim H_\rho$. The following are equivalent.*

(1) *There is a net $\{V_n\}$ of isometries such that $V_n^*\rho(a)V_n \to \pi(a)$ weakly for every a in $\mathfrak{A}$.*

(2) $\min(\mathrm{rank}\, \pi(a), \aleph_0) \leq \mathrm{rank}\, \rho(a)$ *for every a in $\mathfrak{A}$.*

(3) *The nonzero part of $\pi|\rho^{-1}(\mathcal{K}(H_\rho))$ is unitarily equivalent to a subrepresentation of $\rho|\rho^{-1}(\mathcal{K}(H_\rho))$.*

PROOF. (1) $\Rightarrow$ (2) This is obvious.

(2) $\Leftrightarrow$ (3) This follows from Lemma 2.3.

(2) $\Rightarrow$ (1). If $\dim H_\rho \leq \aleph_0$, then (2) implies rank $\pi(a) \leq$ rank $\rho(a)$ for every a in $\mathfrak{A}$, whence, by Theorem 5.1 and Lemma 2.4, we have $\pi \lessdot_a \rho$. Thus we can assume that $\dim H_\rho > \aleph_0$. Suppose $\mathbb{S}$ is a finite subset of $\mathfrak{A}$, M is a finite subset of H_π, and $\varepsilon > 0$. Let $N = \bigvee \{\pi(a)f \colon a \in \mathbb{S}, f \in M\}$, and define $\tau\colon C^*(\mathbb{S}) \to B(N)$ by $\tau(a) = \pi(a)|N$. It follows from (2) that

$$\mathrm{rank}\, \tau(a) \leq \min(\mathrm{rank}\, \pi(a), \aleph_0) \leq \mathrm{rank}\, \rho(a)$$

for every a in $C^*(\mathbb{S})$. Thus, by Theorem 5.1, $\tau \lessdot_a \rho|C^*(\mathbb{S})$. Hence there is an isometry $W\colon N \to H_\rho$ such that $\|(W^*\rho(a)W - \tau(a))f\| < \varepsilon$ for every a in $\mathbb{S}$ and every f in M. Since $\dim N \leq \aleph_0 < \dim H_\rho$, it follows that W can be extended to an isometry $V\colon H_\pi \to H_\rho$. Thus $\|(V^*\rho(a)V - \pi(a))f\| < \varepsilon$ for every a in $\mathbb{S}$ and every f in M. Since $\mathbb{S}$, M, and ε were arbitrary, it follows that (1) is true.

PROPOSITION 5.4. *Suppose $\pi, \rho \in \operatorname{Rep}(\mathfrak{A}, H)$. The following are equivalent.*

(1) *There are nets $\{U_n\}$, $\{V_m\}$ of unitary operators such that $U_n^* \pi(a) U_n \to \rho(a)$ weakly and $V_m^* \rho(a) V_m \to \pi(a)$ weakly for every a in $\mathfrak{A}$.*

(2) $\ker \pi = \ker \rho$, $\pi^{-1}(\mathcal{K}(H)) = \rho^{-1}(\mathcal{K}(H))$, *and the nonzero parts of* $\pi | \pi^{-1}(\mathcal{K}(H))$ *and* $\rho | \pi^{-1}(\mathcal{K}(H))$ *are unitarily equivalent.*

(3) $\ker \pi = \ker \rho$, *and the essential parts of π and ρ are unitarily equivalent.*

(4) $\min(\operatorname{rank} \pi(a), \aleph_0) = \min(\operatorname{rank}(\rho(a), \aleph_0))$ *for every a in $\mathfrak{A}$.*

PROOF. The implications $(1) \Leftrightarrow (4)$ follow from Proposition 5.3. The implications $(2) \Leftrightarrow (3)$ follow from Proposition 2.10. The implications $(2) \Leftrightarrow (4)$ follow from Lemmas 2.2 and 2.3.

6. Direct integrals. In **[H 4]** the author used Voiculescu's theorem (Theorem 2.1) to show that every direct integral of unital, separable representations of a separable C^*-algebra is approximately equivalent to a "naturally related" direct sum of representations. In this section we prove a nonseparable version of this theorem. We also extend some of the results of F. J. Thayer **[Th]** on quasidiagonal C^*-algebras. In particular, we show that a separable direct integral of quasidiagonal representations is quasidiagonal.

Throughout this section H is *separable*, $\mathfrak{A}$ is separable, and $(X, \mathfrak{M}, \mu)$ is a sigma-finite measure space. Let $\mathcal{K} = L^2(\mu, H)$ be the Hilbert space of all Borel measurable functions $f: X \to H$ such that $\int_X \|f(x)\|^2 \, d\mu(x) < \infty$, with the inner product defined by

$$(f, g) = \int_X (f(x), g(x)) \, d\mu(x).$$

Let $L^\infty(\mu, B(H))$ denote the set of all essentially (norm) bounded, weakly Borel measurable functions from X into $B(H)$. Each function $x \to T_x$ in $L^\infty(\mu, B(H))$ gives rise to an operator T on $L^2(\mu, H)$ defined by $(Tf)(x) = T_x f(x)$. The operator T is the *direct integral* of the T_x's denoted by $\int_X^\oplus T_x \, d\mu(x)$.

Next consider a mapping $x \to \pi_x$ from X into $\operatorname{Rep}(\mathfrak{A}, H)$ that is Borel measurable in the point-weak topology. Each such mapping defines a representation $\pi: \mathfrak{A} \to B(\mathcal{K})$ defined by

$$\pi(a) = \int_X^\oplus \pi_x(a) \, d\mu(x)$$

for each a in $\mathfrak{A}$. The representation π is the *direct integral* of the π_x's and is denoted by $\int_X^\oplus \pi_x \, d\mu(x)$.

There are more general direct integrals than the ones defined here, but they are unitarily equivalent to direct sums of the ones defined here **[Di 2]**.

Before we get to the main result of this section (Theorem 6.2), we need the following lemma. If $E \subseteq X$, let $\mathcal{K}_E = \{f \in \mathcal{K} : f | X - E = 0\}$.

LEMMA 6.1. *Suppose the mapping $x \to T_x$ is in $L^\infty(\mu, B(H))$ and $T = \int_X^\oplus T_x \, d\mu(x)$, and suppose X has no atoms. If $E = \{x : T_x \neq 0\}$, then* $\operatorname{rank} T = \dim \mathcal{K}_E$.

PROOF. If $T = 0$, the conclusion is obvious. Thus there is no harm in assuming that $E = X$. The proof is based on a series of reductions using the fact that if X is a disjoint union of sets $E_1, E_2, \dots,$ then it suffices to prove the lemma in each of the cases when X is replaced by E_n. We can therefore assume that $\mu(X) < \infty$ (because X is a countable disjoint union of sets with finite measure). Let $e_1, e_2, \dots$ be an orthonormal basis for H. Then, for each x in X there is a smallest positive integer n_x for which $T_x e_{n_x} \neq 0$. Since the mapping $x \to n_x$ is obviously measurable, we can assume that there is a vector e in H such that $T_x e \neq 0$ for every x in X. The mapping $g(x) = \|T_x e\|$ is measurable, and we can assume that g is bounded away from 0 (because X is a countable disjoint union of sets on which this happens). If the vector f in $\mathcal{H}$ is defined by $f(x) = e/g(x)$, and if $\{\varphi_i: i \in I\}$ is an orthonormal basis for $L^2(\mu)$, then $\{T\varphi_i f: i \in I\}$ is an orthonormal subset of ran T. Thus rank $T \geqslant \dim L^2(\mu)$. However, $\{\varphi_i e_n: i \in I,\ n = 1, 2, \dots\}$ is an orthonormal basis for $\mathcal{H}$. Thus (because μ is nonatomic), $\dim \mathcal{H} = \dim L^2(\mu)$ and we are done.

We are now ready for the main result of this section.

THEOREM 6.2. *Suppose $x \to \pi_x$ is a measurable mapping from X into* $\operatorname{Rep}(\mathfrak{A}, H)$ *and $\pi = \int_X^{\oplus} \pi_x \, d\mu(x)$. Then*:

(1) *if X contains no atoms, then $\pi \oplus \pi_x \sim_a \pi$ a.e.*;

(2) *there are points $x_1, x_2, \dots$ in X and cardinals $m_1, m_2, \dots$ such that $\pi \sim_a \sum_k^{\oplus} \pi_{x_k}^{(m_k)}$.*

PROOF. (1) If X contains no atoms, then $\pi(\mathfrak{A}) \cap \mathcal{K}(H) = 0$. Choose a dense sequence $\{a_n\}$ in $\ker \pi$ and let $E = \{x \in X: \pi_x(a_n) = 0 \text{ for } n = 1, 2, \dots\}$. It follows that $\mu(X - E) = 0$. Also, for every x in E, $\ker \pi \subseteq \ker \pi_x$. Thus, by Lemma 3.2, $\pi \sim_a \pi \oplus \pi_x$ for every x in E.

(2) Using the proof of (1) we can assume that $\ker \pi \subseteq \ker \pi_x$ for every x in X. For each nonzero m in $\mathfrak{M}(\pi)$ let m' denote the smallest cardinal greater than m. Choose a countable dense subset $\mathfrak{D}_m$ of $\pi^{-1}(\pi(\mathfrak{A}) \cap \mathcal{K}_{m'}(\mathcal{H}))$, let $E_m = \{x \in X: \pi_x(d) \neq 0 \text{ for some } d \text{ in } \mathfrak{D}_m\}$, and let $F_m = E_m - \bigcup \{E_k: k \in \mathfrak{M}(\pi), k < m\}$ for each cardinal m in $\mathfrak{M}(\pi)$. Since the atoms of μ yield direct summands of π and since X contains only countably many atoms (because μ is sigma-finite), we can assume that X contains no atoms. It follows from Proposition 3.5(3) that all of the nonzero cardinals in $\mathfrak{M}(\pi)$ are infinite. It follows from Lemma 6.1 that $\dim \mathcal{H}_{F_m} \leqslant \dim \mathcal{H}_{E_m} \leqslant m$ for each m in $\mathfrak{M}(\pi)$. Write $\pi = \sum^{\oplus} \pi_m$ relative to $\mathcal{H} = \sum^{\oplus} \mathcal{H}_{F_m}$. Suppose $a \in \mathfrak{A}$ and $k = \operatorname{rank} \pi(a) < m$. It follows from Corollary 3.12 that $k \in \mathfrak{M}(\pi)$. Thus $\mu(\{x \in X: \pi_x(a) \neq 0\} - E_k) = 0$. Hence $\pi_m(a) = 0$. It follows from Lemma 3.2 that $\pi \sim_a \pi \oplus \pi_m^{(m)}$ for each nonzero m in $\mathfrak{M}(\pi)$. Hence, by Lemma 3.9, $\pi \sim_a \sum^{\oplus}\{\pi_m: m \in \mathfrak{M}(\pi)\}$. Also, for each nonzero m in $\mathfrak{M}(\pi)$ we can choose a countable subset I_m of F_m so that $\{\pi_x: x \in I_m\}$ is a point-weak dense subset of $\{\pi_x: x \in F_m, \ker \pi_x \subseteq \ker \pi_m\}$. Let $\rho_m = \sum^{\oplus}\{\pi_x: x \in I_m\}$ for each nonzero m in $\mathfrak{M}(\pi)$. Hence $\ker \pi_m = \ker \rho_m$, and by Theorem 3.14, we can conclude that $\pi_m^{(m)} \sim_a \rho_m^{(m)}$ for each nonzero m in $\mathfrak{M}(\pi)$. Thus

$$\pi \sim_a \sum^{\oplus} \pi_m^{(m)} \sim_a \sum^{\oplus} \rho_m^{(m)} = \sum_m^{\oplus} \sum_{x \in I_m}^{\oplus} \pi_x^{(m)}.$$

Since $\mathfrak{M}(\pi)$ is countable (Proposition 3.7), it follows that $\bigcup\{I_m: m \in \mathfrak{M}(\pi), m \neq 0\}$ is countable; thus (2) is proved.

COROLLARY 6.3. *A property of separable representations of $\mathfrak{A}$ that is preserved under direct sums and approximate equivalence is preserved under direct integrals.*

COROLLARY 6.4. *Suppose $x \to \pi_x$ is a measurable mapping from X into* $\operatorname{Rep}(\mathfrak{A}, H)$ *and $\pi = \int_X^{\oplus} \pi_x \, d\mu(x)$. If π has a property of representations that is preserved under approximate equivalence and subrepresentations, then π_x has the property for almost every x in X.*

We now apply Theorem 6.2 to improve a theorem of F. J. Thayer **[Th]** on quasidiagonal representations of a separable C^*-algebra. A representation $\pi: \mathfrak{A} \to B(H)$ is *quasidiagonal* **[Th]** if there is an orthogonal sequence $\{P_n\}$ of finite-rank projections such that $\sum_n P_n = 1$ and $\pi(a) - \sum_n P_n \pi(a) P_n$ is compact for every a in $\mathfrak{A}$. It is not difficult to show that a countable direct sum of quasidiagonal representations is quasidiagonal **[Th**, Proposition 2**]**. It is also obvious that a representation is quasidiagonal if it is unitarily equivalent modulo the compact operators to a quasidiagonal representation. Therefore, by Theorem 2.1(2), quasidiagonality is preserved under approximate equivalence. The following theorem is therefore a direct consequence of Theorem 6.2. Note that this theorem was proved by F. J. Thayer under some additional measure-theoretic assumptions and the rather severe assumption that there is a separable C^*-subalgebra $\mathfrak{B}$ of $B(H)$ such that ran $\pi_x \subset \mathfrak{B}$ for almost every x in X.

THEOREM 6.5. *Suppose $L^2(\mu)$ is separable and $x \to \pi_x$ is a measurable mapping from X into* $\operatorname{Rep}(\mathfrak{A}, H)$ *such that π_x is quasidiagonal for almost every x in X. Then $\int_X^{\oplus} \pi_x \, d\mu(x)$ is quasidiagonal.*

7. Approximate versus unitary equivalence. This final section gives a brief comparison between the notions of approximate equivalence and unitary equivalence. The purpose of this section is mainly evangelistic; the main theme is that for many purposes approximate equivalence is just as useful as unitary equivalence and is much easier to deal with. In fact, approximate equivalence behaves very much like finite-dimensional unitary equivalence (i.e., unitary equivalence on finite-dimensional Hilbert spaces).

If $\mathfrak{A}$ is finite dimensional, then every representation of $\mathfrak{A}$ is a direct sum of irreducible representations and the problem of unitary equivalence amounts to counting multiplicities of irreducible summands. If $\pi \in \operatorname{Rep}(\mathfrak{A})$ and $\tau \in \operatorname{Irr}(\mathfrak{A})$, then $\text{Ap-mult}(\tau, \pi) = \min\{\operatorname{rank} \pi(a): a \notin \ker \pi\}$ is the number of orthogonal irreducible summands of π that are unitarily equivalent to τ. Since two irreducible representations of $\mathfrak{A}$ with the same kernel are unitarily equivalent, the "multiplicity function" can be defined on $\operatorname{Prim} \mathfrak{A} = \{\ker \tau: \tau \in \operatorname{Irr} \mathfrak{A}\}$.

Even for commutative C^*-algebras the analogous theory involves direct integrals of irreducible representations rather than direct sums and the multiplicity function is defined on measure classes on Prim $\mathfrak{A}$. There is a similar theory for GCR (type I, postliminal) C^*-algebras. However, when $\mathfrak{A}$ is not GCR, irreducible representations with the same kernel need not be unitarily equivalent (e.g., see Example 7.3(2)) [**Di 1**, Theorem 9.1], and Prim $\mathfrak{A}$ no longer plays a central role.

However, if $\mathfrak{A}$ is separable and unital, then every representation on $\mathfrak{A}$ is approximately equivalent to a direct sum of irreducible representations and two irreducible representations with the same kernel are approximately equivalent. Thus each π in $\mathrm{Rep}(\mathfrak{A})$ defines a cardinal-valued function M_π on Prim $\mathfrak{A}$ defined by $M_\pi(\ker \tau) = \text{Ap-mult}(\tau, \pi)$. The function M_π can be defined more directly (Lemma 3.13) by $M_\pi(\mathfrak{J}) = \min\{\text{rank } \pi(a) \colon a \notin \mathfrak{J}\}$ for each $\mathfrak{J}$ in Prim $\mathfrak{A}$. It follows from Lemma 3.6 that M_π is upper semicontinuous relative to the Jacobson (hull-kernel) topology on Prim $\mathfrak{A}$. The proof of Corollary 3.11 contains the key ingredient in showing that every upper semicontinuous cardinal-valued function on Prim $\mathfrak{A}$ is M_π for some π in $\mathrm{Rep}(\mathfrak{A})$. The following is a mere translation of Theorem 3.14.

THEOREM 7.1. *Suppose $\mathfrak{A}$ is separable and unital and $\pi, \rho \in \mathrm{Rep}(\mathfrak{A})$. Then $\pi \sim_a \rho$ if and only if $M_\pi = M_\rho$. Also $\pi \leq_a \rho$ if and only if $M_\pi \leq M_\rho$.*

If one looks at the world through the eyes of approximate equivalence, then all normal operators are diagonalizable, direct integrals are direct sums, and every operator has an eigenvalue; this is a world that should look pleasing to most operator theorists. Because we humans are finitary by nature, any view of operators on an infinite-dimensional space must of necessity be approximate; thus there often is little loss in considering approximate equivalence instead of unitary equivalence.

Perhaps the most compelling reason for considering approximate equivalence is the fact that on a separable Hilbert space approximate equivalence can be determined by finitary methods. For example, suppose H is separable and $S, T \in B(H)$. It follows from [**H 1**, Corollary 4.2] that $S \sim_a T$ if and only if there are sequences $\{U_n\}$, $\{V_n\}$ of unitary operators such that $U_n^* S U_n \to T$ $*$-strongly and $V_n^* T V_n \to S$ $*$-strongly. However, since a sequence of the form $\{U_n^* S U_n\}$ is always bounded, it is only necessary to check for $*$-strong convergence on a spanning set of H (e.g., an orthonormal basis). This leads to the following simple (but useful) conclusion.

PROPOSITION 7.2. *Suppose $S, T \in B(H)$, and $\{e_1, e_2, \ldots\}$ is an orthonormal basis for H. Then there is a sequence $\{U_n\}$ of unitary operators such that $U_n^* S U_n \to T$ $*$-strongly if and only if, for each positive integer m and each positive number ε, there is a unitary operator U such that $\|(U^*SU - T)e_k\| + \|(U^*SU - T)^* e_k\| < \varepsilon$ for $1 \leq k \leq m$.*

This gives us a simple technique for demonstrating the approximate equivalence of two operators. Here are three elementary examples that illustrate this idea.

EXAMPLES 7.3. (1) Let S be a direct sum of finite complex matrices such that $\|S\| \leq 1$ and, for each positive integer n, the $n \times n$ summands of S are dense in the unit ball of $B(\mathbf{C}^{(n)})$. It follows immediately from Proposition 7.2 that $S \sim_a S \oplus T$ for every T in $B(H)$ with $\|T\| \leq 1$. Thus if S' is another operator with the property used to define S, then $S \sim_a S'$.

(2) Suppose S is a weighted unilateral shift operator with postive weights such that $\|S\| \leqslant 1$ and, for each positive integer n, the blocks of weights of S of length n are dense in the Cartesian product of n copies of $[0, 1]$. It is easy to show that $S \sim_{\mathrm{a}} S \oplus T$ whenever T is a weighted (unilateral or bilateral) shift operator and $\|T\| \leqslant 1$. In particular, if S' is any unilateral weighted shift operator with the property used to define S, then $S \sim_{\mathrm{a}} S'$; let $\mathbb{S}$ be the class of all such operators. Since two weighted unilateral shift operators are unitarily equivalent if and only if their weight sequences coincide (assuming the weights are positive), it is clear that there is a family $\{S_i: i \in I\}$ contained in $\mathbb{S}$ such that Card I = Card$[0, 1]$ and $S_i \cong S_j$ only if $i = j$. For each i in I there is a representation π_i of $C^*(S)$ such that $\pi_i(1) = 1$ and $\pi_i(S) = S_i$. Since a weighted unilateral shift operator with positive weights is irreducible, it follows that all of the π_i's are irreducible. Thus the π_i's are irreducible representations of $C^*(S)$ with the same kernel, but no two of them are unitarily equivalent.

(3) Suppose $\{e_1, e_2, \ldots\}$ is an orthonormal basis for H, and let P_n be the projection onto $\bigvee\{e_1, e_2, \ldots, e_n\}$ for $n = 1, 2, \ldots$. For each S in $B(H)$ let $S_n = P_n S|\operatorname{ran} P_n$ for $n = 1, 2, \ldots$, and let $S_0 = S_1 \oplus S_2 \oplus \cdots$. It follows from Proposition 7.2 that $S_0 \sim_{\mathrm{a}} S_0 \oplus S$ for every S in $B(H)$.

The notions of approximate equivalence and approximate summands seem to suggest a general "approximate" structure theory for operators. It would therefore be natural to examine "approximate" analogues of some of the other concepts in operator theory, e.g., similarity, double commutants, reflexivity. One important success in this direction concerns an "approximate" version of reductivity (called *strong reductivity*) introduced by K. Harrison **[Ha]**. C. Apostol, C. Foiaş and D. Voiculescu solved the "approximate" analogue of the reductive algebra problem **[AFV 1]**, **[AFV 2]** with the aid of **[AF]**. An "approximate" version of von Neumann's double commutant theorem as well as an initial study of "approximate" versions of various operator-theoretic concepts is contained in **[H5]**.

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