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EINDHOVEN UNIVERSITY OF TECHNOLOGY

Department of Mathematics

Probability Theory, Statistics and Operations Research Group

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Nonsingular factors of polynomial matrices and (A,B)-invariant subspaces

by

E. Emre

Eindhoven, May 1978

The Netherlands

NONSINGULAR FACTORS OF POLYNOMIAL MATRICES AND (A,B)-INVARIANT SUBSPACES

by

E. Emre [†]

ABSTRACT

Given a polynomial matrix $B(s)$, we consider the class of nonsingular polynomial matrices $L(s)$ such that $B(s) = R(s)L(s)$ for some polynomial matrix $R(s)$. It is shown that finding such factorizations is equivalent to finding (A,B)-invariant subspaces in the kernel of C where A,B,C are linear maps determined by $B(s)$. In particular, the results yield, as a corollary, a method to determine simultaneously a row proper greatest right divisor of a left invertible polynomial matrix as well as the resulting polynomial matrix whose greatest right divisors are unimodular.

The results also relate, the same way, such subspaces of constant systems $(\bar{C},\bar{A},\bar{B})$ where $(\bar{C},\bar{A})$ is observable and $(\bar{A},\bar{B})$ is reachable, to the nonsingular right factors of the numerator polynomial matrices in coprime factorizations of the form $D^{-1}(s)B(s)$ of their transfer matrices.

May 1978

[†] The author's address after June 1978 is with the Center for Mathematical System Theory, University of Florida, Gainesville.

1. INTRODUCTION

Factorization of a polynomial matrix B(s) has been a subject of several authors both in mathematics and system theory literature [1 - 3], [8 - 14]. In [12 - 14], B(s) has been assumed to be square and monic (i.e. highest degree coefficient matrix is unit matrix), and only monic factors of B(s) have been considered.

In [1 - 3], [8 - 11], B(s) has been taken to be a left invertible polynomial matrix [1 - 3] and the main purpose has been the extraction of a greatest right divisor and obtaining the remaining factor as a polynomial matrix with all unity invariant factors [1 - 3].

In this paper, we consider a general polynomial matrix B(s) with coefficients of its entries in a field, and its nonsingular right polynomial divisors (NRD) L(s) (i.e., factorizations of the form, B(s) = R(s)L(s) for some polynomial matrix R(s) such that $\det(L(s)) \not\equiv 0$). Motivated by the results of [4] on exact matching,it is shown in section 2 that such factorizations are equivalent to finding (A,B) invariant subspaces in the kernel of C[5 - 7], where A,B,C are linear maps determined by B(s). Every NRD yields such a subspace and, once such a subspace is found, it is shown that corresponding L(s) can be found(in row proper form [1 - 3]). In particular, the results of the paper yield a method to determine a row proper greatest right divisor of a left invertible polynomial matrix as well as a resulting polynomial matrix which is a left factor whose invariant factors are all unity. Here we consider only the case of right factors because the case of nonsigular left factors can be approached by duality.

Finally, it is shown that if $(\bar{A},\bar{B},\bar{C})$ is any observable and reachable system, then the NRD's of B(s) in any coprime factorization [1 - 3] of $\bar{C}(sI - \bar{A})^{-1}\bar{B}$ as $D^{-1}(s)B(s)$ are related in the same way to $(\bar{A},\bar{B})$-invariant subspaces in the kernel of $\bar{C}$.

The notation is such that the maps and their matrix representations are denoted by the same symbols and for a matrix R, {R} denotes the span of the columns of R. If A is a linear map and ψ is an A-invariant subspace, $A \mid \psi: \psi \to \psi$ denotes the restriction of A to ψ. By a basis matrix for a subspace ψ we mean a matrix R whose columns are a basis for ψ.

KerC denotes the kernel of the mapping C.

2. NONSINGULAR RIGHT FACTORS AND (A,B)-INVARIANT SUBSPACES

Let B(s) be an f × r polynomial matrix.

Definition 1: An r × r polynomial matrix L(s) is said to be a *nonsingular right divisor* of B(s) (NRD) iff

1) det(L(s)) is nonzero, and
2) there exists an f × r polynomial matrix R(s) such that

$$B(s) = R(s)L(s) .$$

Proposition 1 [1 - 3] If L(s) is an r × r nonsingular polynomial matrix, then there exists a unimodular polynomial M(s) and a row proper matrix $\bar{L}(s)$ such that

$$M(s)L(s) = \bar{L}(s) \tag{1}$$

In general the polynomial matrices M(s) and $\bar{L}(s)$ satisfying (1) are not necessarily unique.
However, if v_i is the degree of the i-th row of an $\bar{L}(s)$ as in (1), then the set $\{v_1,\dots,v_r\}$ is the same (modulo the ordering of v_i's) for every $\bar{L}(s)$ as in (1). □

It follows from Definition 1 and Proposition 1 that, if L(s) is a NRD of B(s), then the elements of the set

$$S_L = \{M(s)L(s) \mid M(s) \text{ is a unimodular polynomial matrix}\}$$

are all NRD's of B(s). Further each S_L contains at least one element whose highest degree row coefficient matrix is nonsingular.

Another result that we use is the following:

Lemma 1 [1 - 3]. If L(s) is an r × r row proper matrix with the i-th row degree v_i, then $L^{-1}_{(s)}$ is a proper rational matrix. If $v_i \geq 1$, i = 1,...,r, then $L^{-1}(s)$ is strictly proper. □

Now motivated by the approach in [4] to the exact model matching, we have the following theorems characterizing NRD's of a polynomial matrix B(s), where we assume, without loss of generality, that B(s) has no zero rows. Let the i-th row of B(s) be

$$b_i(s) = \sum_{j=0}^{\lambda_i} b_j^i s^j \ , \qquad i = 1,\dots,f \ ,$$

where b_j^i's are constant row vectors, $\lambda_i \geq 1$, and $b_{\lambda_i}^i \neq 0$, $i = 1,\dots,f$.
Let

$$\bar{B}_i = \begin{bmatrix} b_{\lambda_i}^i \\ \vdots \\ b_0^i \end{bmatrix} , \qquad B = \begin{bmatrix} \bar{B}_1 \\ \hdashline \vdots \\ \hdashline \bar{B}_f \end{bmatrix} ,$$

$$P_i = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ \vdots & & \ddots & \ddots & 1 \\ 0 & \dots & \dots & \dots & 0 \end{bmatrix} \quad \text{is } (\lambda_i + 1) \times (\lambda_i + 1) \text{ if } \lambda_i \geq 1 \text{ and } P_i = 0 \text{ if } \lambda_i = 0$$

$$A = \begin{bmatrix} P_1 & & 0 \\ & \ddots & \\ 0 & & P_f \end{bmatrix} , \quad \bar{C}_i = [1 \quad 0 \quad \dots \quad 0] \ \text{ is } 1 \times (\lambda_i + 1) \ ,$$

$$C = \begin{bmatrix} \bar{C}_1 & & 0 \\ & \ddots & \\ 0 & & \bar{C}_f \end{bmatrix} .$$

__Theorem 1.__ Let $L(s)$ be a row proper NRD of $B(s)$ with the i-th row degree V_i and let (A_1, B_1, C_1, D_1) be a minimal realization of $L^{-1}(s)$, i.e.

$$L^{-1}(s) = C_1(sI - A_1)^{-1}B_1 + D_1 \ . \tag{2}$$

Then the following holds:

1) There exists a subspace ψ, of dimension less than or equal to

$$\bar{n} = \sum_{i=1}^{r} v_i$$

satisfying

$$\begin{aligned} A\psi &\subset \psi + \{B\} \\ \psi &\subset \operatorname{Ker} C \ . \end{aligned} \tag{3}$$

2) There exists a matrix X such that $\psi = \{X\}$, and the matrices X, A_1, C_1 satisfy

$$AX = XA_1 + BC_1 \tag{4}$$

(Thus in case dim $\psi = \bar{n}$ and X is a basis matrix, there exists a feedback map F such that $(A + BF)\psi \subset \psi$, and $(A + BF)|\psi$ is represented by A_1 [5 - 6].)

<u>Proof</u>. If L(s) is as in hypothesis, then (1) holds for some f × r polynomial matrix R(s), or

$$B(s)C_1(sI - A_1)^{-1}B_1 = R(s) - B(s)D_1 \tag{5}$$

Then considering the formal power series expansion of $L_1^{-1}(s)$ and equating the coefficients in (5), row by row, we obtain

$$\underbrace{\begin{bmatrix} b^i_{\lambda_i} & :\ldots: & b^i_0 \end{bmatrix}}_{\bar{B}_i} \begin{bmatrix} C_1A_1^{\lambda_i}B_1 & : & C_1A_1^{\lambda_i+1}B_1 & : & \ldots \\ \vdots & & \vdots & & \\ C_1B_1 & : & C_1A_1B_1 & : & \ldots \end{bmatrix} = [0 :\ldots 0 :\ldots]$$

or

$$\bar{B}_i \begin{bmatrix} C_1A_1^{\lambda_i} \\ \vdots \\ C_1 \end{bmatrix} [B_1 : A_1B_1 :\ldots] = [0:\ldots:0:\ldots] \tag{6}$$

But, since (A_1, B_1) is reachable,

$$\bar{B}_i \begin{bmatrix} C_1A_1^{i} \\ \vdots \\ C_1 \end{bmatrix} = 0 , \qquad i = 1,\ldots,f . \tag{7}$$

But (7) shows that the polynomial matrix $b_i(s)C_1$ is right divisible by $(sI - A_1)$ [15], i.e. there exists a $1 \times \bar{n}$ polynomial matrix $\psi_i(s)$ such that

$$b_i(s)C_1 = \psi_i(s)(sI - A_1) \tag{8}$$

or, letting

$$\psi(s) = \begin{bmatrix} \psi_1(s) \\ \vdots \\ \psi_f(s) \end{bmatrix} ,$$

$$B(s)C_1 = \psi(s)(sI - A_1) . \tag{9}$$

From (8) it follows that degree $(\psi_i(s)) < \lambda_i$. Now let

$$\psi_i(s) = \sum_{j=0}^{\lambda_i-1} \psi^i_j s^j ,$$

$$\bar{\psi}_i = \begin{bmatrix} 0_{1,\bar{n}} \\ \psi^i_{\lambda_i-1} \\ \vdots \\ \psi^i_0 \end{bmatrix} , \qquad X = \left[\begin{array}{c} \bar{\psi}_1 \\ \hline \vdots \\ \hline \bar{\psi}_f \end{array}\right] . \tag{10}$$

Then (9) yields (4), i.e., if $\psi = \{X\}$, we have

$$P\psi \subset \psi + \{B\} .$$

$CX = 0$ is clear and hence $\psi \subset \mathrm{Ker}C$. □

Remark 1. Note that we have

$$R(s) = \psi(s)B_1 + B(s)D_1 .$$

Also note that if $v_i \geq 1$, $i = 1,\ldots,r$, D_1 is zero.

Remark 2. In case $B(s)$ is left invertible then from (9) it is seen that X in (10) has full column rank in which case $\dim \psi = \bar{n}$ and A_1 always $A + BF|\psi$ where F is such that $(A + BF)\psi \subset \psi$.

Theorem 2. Let ψ be a subspace satisfying (3). Let X be a basis matrix for ψ. Let A_1, C_1 be matrices satisfying (4). (In this case A_1 represents $(A + BF)|\psi$ for some feedback map F such that $(A + BF)\psi \subset \psi$).

Also, suppose that C_1 has full row rank. Then the following holds:

1) (A_1,C_1) is observable,

2) there exists a unique matrix B_1 such that (A_1,B_1) is reachable and such that

$$L(s) = [C_1(sI - A_1)^{-1}B_1]^{-1}$$

is a NRD which is row proper with the highest row coefficient matrix being I_r, and i-th row degree, v_i, being ≥ 1, $i = 1,2,\ldots,r$.

Proof. With the same notation before, defining $\psi(s)$ as in (10) we see that (9) holds.

Now since

$$B(s)C_1(sI - A_1)^{-1} = \psi(s) \quad ,$$

and X has full column rank (C_1,A_1) is observable. Then since C_1 has full row rank, the observability indices v_i of (C_1,A_1) are ≥ 1. Then there exists a nonsingular constant matrix $\tilde{T}$ such that

$$C_1(sI - A_1)^{-1} = L^{-1}(s)W(s)\tilde{T}$$

where L(s) is an r × r row proper polynomial matrix with row degrees being equal to v_i and the highest coefficient row matrix being I_r [1 - 3], and

$$W(s) = \begin{bmatrix} W_1(s) & & 0 \\ & \ddots & \\ 0 & & W_r(s) \end{bmatrix} ,$$

where

$$W_i(s) = \begin{bmatrix} s^{v_i-1} \ldots 1 \end{bmatrix} .$$

Thus if we let

$$\tilde{T}B_1 = \left[\begin{array}{ccc} v_1\left\{\begin{matrix}0\\ \vdots\\ 0\\ 1\end{matrix}\right. & & 0 \\ & \ddots & \\ 0 & & \left.\begin{matrix}0\\ \vdots\\ 0\\ 1\end{matrix}\right\}v_r \end{array}\right] ,$$

we have

$$C_1(sI - A_1)^{-1}B_1 = L^{-1}(s) .$$

Since $W(s)\tilde{T}B_1 = I_r$ is coprime with $L(s)$, (A_1,B_1) is reachable [1 - 3]. Then

$$B(s)L^{-1}(s) = \psi(s)B_1 = R(s)$$

and

$$B(s) = R(s)L(s) .$$

□

Remark 3. In Theorem 2, if C_1 does not have full row rank, let $\hat{T}$ be any nonsingular matrix such that

$$\hat{T}C_1 = \begin{bmatrix}\bar{C}_1\\ \\ 0\end{bmatrix}$$

where $\bar{C}_1$ has full row rank. Then we again have

$$B(s)C_1 = \psi(s)(sI - A_1)$$

with (C_1,A_1) observable, and

$$B(s)\hat{T}^{-1}\begin{bmatrix}\bar{C}_1\\ \\ 0\end{bmatrix} = \psi(s)(sI - A_1)$$

with $(\bar{C}_1,A_1)$ observable.

Let $B(s)\hat{T}^{-1} = [B_1(s) \; \vdots \; B_2(s)]$.

Then if we choose B_1 as in Theorem 2 for $(\bar{C}_1,A_1)$, the resulting $L(s)$ will satisfy

$$B_1(s)L^{-1}(s) = \psi(s)B_1 = \bar{R}(s) \ .$$

Then

$$B(s)\hat{T}^{-1}\begin{bmatrix} L(s) & 0 \\ 0 & I \end{bmatrix}^{-1} = [\bar{R}(s) : B_2(s)] = R(s)$$

or

$$B(s) = R(s)\begin{bmatrix} L(s) & 0 \\ 0 & I \end{bmatrix}\hat{T}$$

yields

$$L_1(s) = \begin{bmatrix} L(s) & 0 \\ 0 & I \end{bmatrix}\hat{T}$$

as a row proper NRD of B(s).

Now we have the following corollary which yields a method to find a greatest common right divisor [1 - 3] of two polynomial matrices V(s), T(s), where T(s) is nonsingular, as well as the resulting coprime pair simultaneously.

It is clear that this is equivalent to finding a greatest right divisor [1 - 3] of

$$B(s) = \begin{bmatrix} T(s) \\ V(s) \end{bmatrix} .$$

Corollary 1. Let B(s) be an $f \times r$ polynomial matrix with $f \geq r$, which is left invertible (i.e. no zeros among the diagonal entries of its Smith form).

Let ψ_{max} be the maximal dimensional subspace satisfying (3). Let X, A_1, C_1 be as in Theorem 2. If C_1 has full row rank let B_1 be as in Theorem 2 and if C_1 does not have full row rank let $\bar{C}_1$ and B_1 be as in Remark 3. Then the resulting NRD, L(s) is a row proper greatest right factor of B(s).

Proof. Suppose that L(s) is not a greatest right divisor. Let $\bar{L}(s)$ be a greatest row proper right divisor. Then by Theorem 1 and Remark 2, there exists a subspace $\bar{\psi}$ satisfying (3) with dim $\bar{\psi}$ = degree(det $\bar{L}(s)$).

But then degree(det $\bar{L}(s)$) > degree(det $L(s)$). However, degree(det $L(s)$) is dimension of ψ_{max} by Theorem 2 and Remark 3. This is a contradiction. Thus $L(s)$ is a row proper greatest right divisor of $B(s)$. □

Remark 4. There are several methods to find a maximal (A,B)-invariant subspace ψ_{max} in KerC [5 - 7].
Once we have found a basis matrix, X_{max}, for ψ_{max}, the corresponding $\psi(s)$ is already available.
Then applying Theorem 2 and Remark 3, we have both a row proper greatest right divisor as well as the resulting polynomial matrix whose only polynomial right divisors are unimodular polynomial matrices. Now the following corollary is immediate:

Corollary 2. An $f \times r$ $(f \geq r)$ left invertible polynomial matrix $B(s)$ has only unity invariant factors iff

$$\psi_{max} = \{0\} .$$

Based on Theorems 1 and 2 we also have the following result.

Theorem 3. Let $(\bar{A},\bar{B},\bar{C})$ be a system such that $(\bar{C},\bar{A})$ is observable and $(\bar{A},\bar{B})$ is reachable. Let

$$G(s) = \bar{C}(sI - \bar{A})^{-1}\bar{B} = D^{-1}(s)B(s)$$

be a coprime factorization of $G(s)$ such that $D(s)$ is row proper with the highest degree row coefficient matrix being the unit matrix and $B(s)$ has no zero rows.
Then Theorems 1 and 2 hold with $(\bar{A},\bar{B},\bar{C})$ replacing (A,B,C) as defined previously.

Proof. Let the observability indices of $(\bar{C},\bar{A})$ be $\bar{v}_i$, $i = 1,\dots,f$.
Since $G(s)$ is strictly proper and $B(s)$ has no zero rows, $\bar{v}_i \geq 1$, $i = 1,\dots,f$, [1 - 3]. Since $(\bar{C},\bar{A})$ is observable, there exist matrices K, $\tilde{T}$, $\tilde{T}$ being nonsingular, such that

$$\begin{aligned} \tilde{T}(\bar{A} + K\bar{C})\tilde{T}^{-1} &= \tilde{A} \\ \bar{C}\tilde{T}^{-1} &= \tilde{C} \end{aligned} \qquad (11)$$

[1 - 3],

where

$$\tilde{A} = \begin{bmatrix} \tilde{A}_1 & & 0 \\ & \ddots & \\ 0 & & \tilde{A}_f \end{bmatrix} ,$$

$\tilde{A}_i$ is a $\bar{V}_i \times \bar{V}_i$ matrix given as

$$\tilde{A}_i = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ & & \ddots & & \vdots \\ & & & \ddots & 0 \\ & 0 & & & 1 \\ 0 & \cdots & \cdots & \cdots & 0 \end{bmatrix}$$

if $\bar{V}_i > 1$ and $\tilde{A}_i = 0$ if $\bar{V}_i = 1$.

$$\tilde{C} = \begin{bmatrix} \tilde{C}_1 & & 0 \\ & \ddots & \\ 0 & & \tilde{C}_f \end{bmatrix} ;$$

$\tilde{C}_i$ is the $1 \times \bar{V}_i$ matrix given as

$$\tilde{C}_i = [1 \quad 0 \ \ldots \ 0] .$$

Then, since $D^{-1}(s)B(s)$ is strictly proper and $D(s)$ is row proper, $\lambda_i < \bar{V}_i$, $i = 1,\ldots,f$.

Now let

$$\tilde{B} = \tilde{T}\bar{B} = \begin{bmatrix} \hat{B}_1 \\ \hline \vdots \\ \hline \hat{B}_f \end{bmatrix}$$

Then since $(\bar{A},\bar{B})$ is reachable, $\hat{B}_i$ is a $\bar{V}_i \times r$ matrix given as

$$\hat{B}_i = \begin{bmatrix} 0_{\bar{V}_i - \lambda_i - 1,\ r} \\ \hline \bar{B}_i \end{bmatrix} , \quad i = 1,\ldots,f . \quad [1 - 3] .$$

Since the subspaces $\bar{\psi}$ satisfying

$$\bar{A}\bar{\psi} \subset \bar{\psi} + \{\bar{B}\} , \quad \bar{\psi} \subset \mathrm{Ker}\bar{C} \tag{12}$$

are independent of the type of transformations occurring in (11) which are invertible, the subspaces $\bar{\psi}$ satisfying (12) are the same as the subspaces $\tilde{\psi}$ satisfying

$$\tilde{A}\psi \subset \tilde{\psi} + \{\tilde{B}\} \ , \qquad \tilde{\psi} \subset \mathrm{Ker}\tilde{C} \tag{13}$$

But the subspaces $\tilde{\psi}$ satisfying (13) are the same as the subspaces ψ satisfying

$$A\psi \subset \psi + \{B\} \ , \qquad \psi \subset \mathrm{Ker}C$$

imbedded into a larger dimensional vector space. Also the matrices A_1, C_1 satisfying

$$\bar{A}\bar{X} = \bar{X}A_1 + \bar{B}C_1$$

where $\bar{\psi} = \{\bar{X}\}$, satisfy

$$\widetilde{AT}\bar{X} = \tilde{T}\bar{X}A_1 + \tilde{B}C_1 \ ,$$

and thus they satisfy

$$AX = XA_1 + BC_1$$

for some X such that $\{X\} = \psi$ which is the same as $\bar{\psi}$ (modulo embedding ψ into a larger vector space). Then, by Theorems 1 and 2 the proof follows. □

Remark 5. If $\bar{D}^{-1}(s)\bar{B}(s)$ is any coprime factorization of G(s), there exists a unimodular polynomial matrix M(s) such that

$$M(s)\bar{B}(s) = B(s) \quad [1-3] \ .$$

Hence $\bar{B}(s)$ and B(s) have the same set of NRD's. Thus Theorem 3 is valid for any $\bar{B}(s)$ in any coprime factorization of G(s) as $\bar{D}^{-1}(s)\bar{B}(s)$.

Remark 6. Theorem 3 shows that given any NRD, L(s), of B(s), in $G(s) = D^{-1}(s)B(s)$ which is a coprime factorization of $\bar{C}(sI - \bar{A})^{-1}\bar{B}$ with $(\bar{C},\bar{A})$ being observable (equivalently the set S_L), there corresponds a unique $(\bar{A},\bar{B})$-invariant subspace in Ker$\bar{C}$. Also, given any such subspace, there corresponds at least one NRD of B(s).

3. CONCLUSION

We have given a characterization of NRD's of a polynomial matrix in terms of (A,B)-invariant subspaces in KerC. The results in particular yield a method to obtain simultaneously a row proper greatest right divisor of a left invertible polynomials matrix as well as the resulting polynomial matrix whose greatest right divisors are unimodular polynomial matrices. The results also yield a characterization of the NRD's of the numerator polynomial matrix in a coprime factorization of a transfer matrix in terms of $(\bar{A},\bar{B})$ invariant subspaces in $\mathrm{Ker}\bar{C}$ where $(\bar{A},\bar{B},\bar{C})$ is an observable and reachable realization of the transfer matrix.

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