Nonsingularity and group invertibility of linear combinations of two k-potent matrices*

Julio Benítez[a†] Xiaoji Liu[b‡] Tongping Zhu[c]

[a] Departamento de Matemática Aplicada, Instituto de Matemática Multidisciplinar, Universidad Politécnica de Valencia, Camino de Vera s/n, 46022, Valencia, Spain.

[b, c] College of Mathematics and Computer Science, Guangxi University for Nationalities, Nanning 530006, P.R. of China

Abstract

A square matrix A is said to be k-potent when $A^k = A$. Let T_1 and T_2 be two $n \times n$ complex matrices and c_1, c_2 be two nonzero complex numbers. We study the range space and the null space of a linear combination of the form $T = c_1T_1 + c_2T_2$. Also the problems of when T is nonsingular and group invertible are considered. We derive explicit formulae of the inverse and the group inverse of T under some conditions.

AMS classification: 15A03, 15A09, 15A24, 15A57

Key words: k-potent matrix; linear combination; nonsingularity; rank; group inverse.

1 Introduction

The symbols $\mathbb{C}$, $\mathbb{C}^*$, and $\mathbb{C}^{m\times n}$ will denote the sets of complex numbers, nonzero complex numbers, and $m \times n$ complex matrices, respectively. Moreover, A^*, $\mathrm{rk}(A)$, $\mathcal{R}(A)$, and $\mathcal{N}(A)$ will stand for the conjugate transpose, rank, range space, and null space of $A \in \mathbb{C}^{m\times n}$. From the point of view of the present paper, the key role is played by matrices belonging to the set of k-potent matrices, being k a positive integer greater than 1. This set of matrices is defined as $\{T \in \mathbb{C}^{n\times n} : T^k = T\}$. In [1] this set has been extensively studied, in where the authors proved (among other things) the following: Let $T \in \mathbb{C}^{n\times n}$ and k be a natural number greater than 1, then T satisfies $T^k = T$ if and only if T is diagonalizable and the spectrum of T is contained in $\sqrt[k-1]{1} \cup \{0\}$. In [2] this research was extended to the setting of bounded operators in a Hilbert space. It is customary to say that a matrix A is idempotent (or oblique projector) when $A^2 = A$.

*The second and the third author are supported by Guangxi Science Foundation (0640016, 0832084).

†Email: jbenitez@mat.upv.es

‡Email: xiaojiliu72@yahoo.com.cn

If $\mathcal{X}$ is a subspace of $\mathbb{C}^{n\times 1}$, then $\mathcal{X}^{\perp}$ will mean its orthogonal complement. It will be used along this paper that $\mathcal{R}(A^*) = \mathcal{N}(A)^{\perp}$ and $\mathcal{R}(A)^{\perp} = \mathcal{N}(A^*)$ for any matrix A and $\mathcal{X}^{\perp} \cap \mathcal{Y}^{\perp} = (\mathcal{X} + \mathcal{Y})^{\perp}$ for any pair of subspaces $\mathcal{X}, \mathcal{Y}$ of $\mathbb{C}^{n\times 1}$.

Recently, there have been several papers devoted to the invertibility of a linear combination of two idempotents (matrices, operators in a Hilbert space or even elements in a C^*-algebra). In [3] and using rank theory, Groß and Trenkler gave characterizations of the nonsingularity of the difference $P_1 - P_2$ and the sum $P_1 + P_2$, where $P_1, P_2 \in \mathbb{C}^{n\times n}$ are two idempotents in terms of the range and null spaces of either P_1 and P_2 directly or other expressions involving P_1 and P_2. In [4, 5] Buckholtz studied the idempotency of the difference of two operators in a Hilbert space. In [6], the invertibility of the difference of two self-adjoint idempotents in a C^*-algebra is studied. In [7] Koliha, Rakočević, and Straškaba gave characterizations and explicit formulae of the inverse of the difference and the sum of two idempotent matrices. In [8] Baksalary and Baksalary discussed the invertibility of a linear combination of idempotent matrices. This last paper was improved in [9] by Koliha and Rakočević by showing that the rank of a linear combination of two idempotents is constant. Du, Yao, and Deng extended the conclusion on an infinite dimensional Hilbert space (see [10]).

In this paper, we study the range space, the null space, the invertibility, and the group invertibility of a linear combination $T = c_1T_1 + c_2T_2$, where T_1, T_2 are k-potent matrices and c_1, c_2 are two nonzero complex numbers. Also, we find some formulae for T^{-1} and the group inverse of T under some conditions. The results obtained here are more general than the aforementioned. Special types of matrices, as idempotent, tripotent, etc... are very useful in many contexts and they have been extensively studied in literature. For example, it is well known that quadratic forms with idempotent matrices are used extensively in statistical theory (see for example [11, Th. 5.1.1] or [12, Lemma 9.1.2]). So it is worth to stress and spread these kind of results.

2 Invertibility of a linear combination of two k-potent matrices

As we already pointed out, some of the main results of this paper, which are similar to the ones obtained by Baksalary and Baksalary (see [8]) and by Koliha and Rakočević (see [9]) for idempotent matrices, deal with the nonsingularity and the rank of linear combinations of k-potent matrices. Since an idempotent matrix is also a k-potent matrix, we think that it is suitable to start by repeating the observation in [8]:

If we define the following two idempotent,

$$T_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \text{and} \qquad T_2 = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix},$$

then

$$c_1T_1 + c_2T_2 = \begin{bmatrix} c_1 + c_2 & c_2 \\ 0 & c_1 \end{bmatrix}$$

is nonsingular for all $c_1, c_2 \in \mathbb{C}^*$ such that $c_1 + c_2 \neq 0$, but $T_1 - T_2$ is clearly a singular matrix.

Again, as in [8], it is interesting to ask if there is any relationship between members of the family $\{c_1T_1 + c_2T_2 : c_1, c_2 \in \mathbb{C}^*\}$ and the subfamily of that being $\{c_1T_1 + c_2T_2 : c_1, c_2 \in \mathbb{C}^*$ and $c_1 + c_2 \neq 0\}$. This is the context of the theorems below.

Theorem 2.1. *Let $T_1, T_2 \in \mathbb{C}^{n\times n}$ be two k-potent matrices for some natural $k > 1$. If there exist $a_1, a_2, a_3, a_4 \in \mathbb{C}$ such that*

$$a_1T_1 + a_2T_2 + a_3T_1^{k-1}T_2 + a_4T_2^{k-1}T_1 = 0, \tag{2.1}$$

then

$$\mathcal{N}(c_1T_1 + c_2T_2) = \mathcal{N}(T_1) \cap \mathcal{N}(T_2)$$

for all $c_1, c_2 \in \mathbb{C}^*$ satisfying

$$c_1(a_2 + a_3) \neq c_2(a_1 + a_4). \tag{2.2}$$

Proof. The inclusion $\mathcal{N}(T_1) \cap \mathcal{N}(T_2) \subseteq \mathcal{N}(c_1T_1 + c_2T_2)$ is obvious and does not need any assumption on $c_1, c_2 \in \mathbb{C}^*$. Let us prove the opposite inclusion: Pick any $x \in \mathcal{N}(c_1T_1 + c_2T_2)$ and $c_1, c_2 \in \mathbb{C}^*$ satisfying (2.2). We have

$$c_1T_1x + c_2T_2x = 0. \tag{2.3}$$

Premultiplying (2.3) by T_1^{k-1} and by T_2^{k-1} we get, respectively,

$$c_1T_1x + c_2T_1^{k-1}T_2x = 0 \qquad \text{and} \qquad c_1T_2^{k-1}T_1x + c_2T_2x = 0 \tag{2.4}$$

beacuse $T_1^k = T_1$ and $T_2^k = T_2$. Postmultiplying (2.1) by x we obtain

$$a_1T_1x + a_2T_2x + a_3T_1^{k-1}T_2x + a_4T_2^{k-1}T_1x = 0. \tag{2.5}$$

Equations (2.3), (2.4), and (2.5) can be written together:

$$\begin{bmatrix} T_1x & T_2x & T_1^{k-1}T_2x & T_2^{k-1}T_1x \end{bmatrix} \begin{bmatrix} c_1 & c_1 & 0 & a_1 \\ c_2 & 0 & c_2 & a_2 \\ 0 & c_2 & 0 & a_3 \\ 0 & 0 & c_1 & a_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix}. \tag{2.6}$$

Since

$$\det\left(\begin{bmatrix} c_1 & c_1 & 0 & a_1 \\ c_2 & 0 & c_2 & a_3 \\ 0 & c_2 & 0 & a_3 \\ 0 & 0 & c_1 & a_4 \end{bmatrix}\right) = c_1c_2[c_1(a_2 + a_3) - c_2(a_1 + a_4)] \neq 0,$$

from (2.6) we get $T_1x = T_2x = 0$. This finishes the proof. □

Theorem 2.1 permits prove the stability of the rank of a linear combination of two k-potent matrices that satisfy conditions (2.1) and (2.2).

Theorem 2.2. *Let $T_1, T_2 \in \mathbb{C}^{n\times n}$ be two k-potent matrices for some natural $k > 1$. If there exist $a_1, a_2, a_3, a_4 \in \mathbb{C}$ such that $a_1T_1 + a_2T_2 + a_3T_1^{k-1}T_2 + a_4T_2^{k-1}T_1 = 0$, then*

$$\mathrm{rk}(c_1T_1 + c_2T_2) = \mathrm{rk}(d_1T_1 + d_2T_2)$$

for all $c_1, c_2, d_1, d_2 \in \mathbb{C}^*$ satisfying $c_1(a_2 + a_3) \neq c_2(a_1 + a_4)$ and $d_1(a_2 + a_3) \neq d_2(a_1 + a_4)$.

In particular, if $d_1T_1 + d_2T_2$ is nonsingular for some $d_1, d_2 \in \mathbb{C}^*$ with $d_1(a_2 + a_3) \neq d_2(a_1 + a_4)$, then $c_1T_1 + c_2T_2$ is nonsingular for all $c_1, c_2 \in \mathbb{C}^*$ with $c_1(a_2 + a_3) \neq c_2(a_1 + a_4)$.

Proof. It follows from the following elementary facts: if $X \in \mathbb{C}^{n\times n}$, then $\mathrm{rk}(X) + \dim \mathcal{N}(X) = n$ and X is nonsingular if and only if $\mathrm{rk}(X) = n$. $\square$

We can obtain similar results than Theorems 2.1 and 2.2 if we slightly change the condition (2.1). A related result is the following (see [13]): for every pair of orthogonal projectors $P, Q \in \mathbb{C}^{n\times n}$ (i.e. $P^2 = P = P^*$ and $Q^2 = Q = Q^*$) one has $\mathcal{N}(P + Q) = \mathcal{N}(P) \cap \mathcal{N}(Q)$ and $\mathcal{R}(P + Q) = \mathcal{R}(P) + \mathcal{R}(Q)$.

Theorem 2.3. *Let $T_1, T_2 \in \mathbb{C}^{n\times n}$ be two k-potent matrices for some natural $k > 1$. If there exist $b_1, b_2, b_3, b_4 \in \mathbb{C}$ such that*

$$b_1T_1 + b_2T_2 + b_3T_1T_2^{k-1} + b_4T_2T_1^{k-1} = 0, \tag{2.7}$$

then

$$\mathcal{R}(c_1T_1 + c_2T_2) = \mathcal{R}(T_1) + \mathcal{R}(T_2)$$

for all $c_1, c_2 \in \mathbb{C}^*$ satisfying

$$c_1(b_2 + b_4) \neq c_2(b_1 + b_3) \tag{2.8}$$

Proof. Denote $S_1 = T_1^*$ and $S_2 = T_2^*$. Relation (2.7) can be written as

$$\overline{b_1}S_1 + \overline{b_2}S_2 + \overline{b_3}S_2^{k-1}S_1 + \overline{b_4}S_1^{k-1}S_2 = 0.$$

Moreover, it is evident that S_1 and S_2 are two k-potent matrices. Pick any $c_1, c_2 \in \mathbb{C}^*$ satisfying (2.8). Hence $\overline{c_1}(\overline{b_2} + \overline{b_4}) \neq \overline{c_2}(\overline{b_1} + \overline{b_3})$. By Theorem 2.1 we get $\mathcal{N}(\overline{c_1}S_1 + \overline{c_2}S_2) = \mathcal{N}(S_1) \cap \mathcal{N}(S_2)$. Therefore

$$\begin{aligned}\mathcal{R}(c_1T_1 + c_2T_2)^\perp &= \mathcal{N}\left((c_1T_1 + c_2T_2)^*\right) = \mathcal{N}(\overline{c_1}S_1 + \overline{c_2}S_2) = \\ &= \mathcal{N}(S_1) \cap \mathcal{N}(S_2) = \mathcal{N}(T_1^*) \cap \mathcal{N}(T_2^*) = \mathcal{R}(T_1)^\perp \cap \mathcal{R}(T_2)^\perp = \left(\mathcal{R}(T_1) + \mathcal{R}(T_2)\right)^\perp.\end{aligned}$$

By taking perp on both sides, the proof is finished. $\square$

The following result follows immediately from Theorem 2.3 and we do not give its proof.

Theorem 2.4. *Let $T_1, T_2 \in \mathbb{C}^{n\times n}$ be two k-potent matrices for some natural $k > 1$. If there exist $b_1, b_2, b_3, b_4 \in \mathbb{C}$ such that $b_1T_1 + b_2T_2 + b_3T_1T_2^{k-1} + b_4T_2T_1^{k-1} = 0$, then*

$$\mathrm{rk}(c_1T_1 + c_2T_2) = \mathrm{rk}(d_1T_1 + d_2T_2)$$

for all $c_1, c_2, d_1, d_2 \in \mathbb{C}^*$ satisfying $c_1(b_2 + b_4) \neq c_2(b_1 + b_3)$ and $d_1(b_2 + b_4) \neq d_2(b_1 + b_3)$.

In particular, if $d_1T_1 + d_2T_2$ is nonsingular for some $d_1, d_2 \in \mathbb{C}^*$ with $d_1(b_2 + b_4) \neq d_2(b_1 + b_3)$, then $c_1T_1 + c_2T_2$ is nonsingular for all $c_1, c_2 \in \mathbb{C}^*$ with $c_1(b_2 + b_4) \neq c_2(b_1 + b_3)$.

Observe that when matrices T_1, T_2 satisfy $T_1^{k-1}T_2 = T_2^{k-1}T_1$ or $T_2T_1^{k-1} = T_1T_2^{k-1}$, conditions (2.2) and (2.8) reduce to $c_1 + c_2 \neq 0$. Thus, we obtain the following corollary:

Corollary 2.1. *Let $T_1, T_2 \in \mathbb{C}^{n\times n}$ be two k-potent matrices for some natural $k > 1$ such that $T_1^{k-1}T_2 = T_2^{k-1}T_1$ or $T_2T_1^{k-1} = T_1T_2^{k-1}$. If a linear combination $d_1T_1 + d_2T_2$ is nonsingular for some $d_1, d_2 \in \mathbb{C}^*$ satisfying $d_1 + d_2 \neq 0$, then $c_1T_1 + c_2T_2$ is nonsingular for all $c_1, c_2 \in \mathbb{C}^*$ satisfying $c_1 + c_2 \neq 0$.*

When $P_1, P_2 \in \mathbb{C}^{n\times n}$ are idempotent, a necessary and sufficient condition for $P_1 - P_2$ to be nonsingular is that $P_1 + P_2$ and $I_n - P_1P_2$ are nonsingular (see [7]). In [8] Baksalary et al. strengthened this result by taking any linear combination of the form $c_1P_1 + c_2P_2$ instead of $P_1 + P_2$, where $c_1 + c_2 \neq 0$.

Observe that if $T_1, T_2 \in \mathbb{C}^{n\times n}$ are two k-potent matrices for some natural $k > 1$ that satisfy condition (2.1), then $\mathcal{N}(c_1T_1 + c_2T_2) = \mathcal{N}(T_1) \cap \mathcal{N}(T_2)$ for every $c_1, c_2 \in \mathbb{C}^*$ satisfying (2.2). Since it is evident that $\mathcal{N}(T_1) \cap \mathcal{N}(T_2) \subseteq \mathcal{N}(T_1 - T_2)$, we get that the nonsingularity of $T_1 - T_2$ implies the nonsingularity of $c_1T_2 + c_2T_2$. A similar result holds when T_1, T_2 satisfy condition (2.7) and scalars c_1, c_2 satisfy (2.8).

In the following theorem, a similar result than [8, Th. 2] is established for k-potent matrices.

Theorem 2.5. *Let $T_1, T_2 \in \mathbb{C}^{n\times n}$ be two k-potent matrices such that $I_n - T_1^{k-1}T_2^{k-1}$ is nonsingular. If there exist $c_1, c_2 \in \mathbb{C}^*$ such that $c_1T_1 + c_2T_2$ is nonsingular, then $T_1 - T_2$ is also nonsingular.*

Proof. Let $x \in \mathcal{N}(T_1 - T_2)$. Premultiplying $T_1x = T_2x$ by T_1^{k-1} and by T_2^{k-1} we get $T_1x = T_1^{k-1}T_2x$ and $T_2^{k-1}T_1x = T_2x$. Now we have,

$$
\begin{aligned}
(I_n - T_1^{k-1}T_2^{k-1})(c_1T_1 + c_2T_2)x &= c_1T_1x + c_2T_2x - T_1^{k-1}T_2^{k-1}(c_1T_1 + c_2T_2)x \\
&= c_1T_1x + c_2T_2x - c_1T_1^{k-1}T_2^{k-1}T_1x - c_2T_1^{k-1}T_2x \\
&= c_1T_1x + c_2T_2x - c_1T_1^{k-1}T_2x - c_2T_1x \\
&= c_1T_1x + c_2T_2x - c_1T_1x - c_2T_1x \\
&= c_2(T_2x - T_1x) = 0. \qquad (2.9)
\end{aligned}
$$

Under the assumption that $c_1T_1 + c_2T_2$ and $I_n - T_1^{k-1}T_2^{k-1}$ are nonsingular, (2.9) yields $x = 0$. This means that $\mathcal{N}(T_1 - T_2) = \{0\}$. This completes the proof. □

For arbitrary two k-potent matrices $T_1, T_2 \in \mathbb{C}^{n\times n}$, it is possible that $T_1 - T_2$ is nonsingular, and however, $I_n - T_1^{k-1}T_2^{k-1}$ is singular, even if T_1 and T_2 commute. The following example (in $\mathbb{C}^{1\times 1}$) shows this: Let $T_1 = 1$ and $T_2 = -1$. Obviously, T_1 and T_2 are 3-potent (in fact, they satisfy $T_1^2 = T_2^2 = 1$) and they commute. We have that $T_1 - T_2 = 2$ is nonsingular, and however, $1 - T_1^2T_2^2 = 0$ is singular.

Now, we use different approach to prove the next theorem.

Theorem 2.6. *Let $T_1, T_2 \in \mathbb{C}^{n\times n}$ two commuting k-potent matrices for some natural $k > 1$. If there exists $a \in \mathbb{C}^*$ such that $T_1 + aT_2$ is nonsingular, then $c_1T_1 + c_2T_2$ and $c_1I_n + c_2T_1T_2$ are nonsingular for all $c_1, c_2 \in \mathbb{C}^*$ with $-c_1/c_2 \notin \sqrt[k-1]{1}$.*

Proof. As T_1, T_2 are diagonalizable and commuting, they are simultaneously diagonalable, i.e., there is a nonsingular matrix S such that $S^{-1}T_1S$ and $S^{-1}T_2S$ are diagonal matrices. Let us define $(\lambda_i)_{i=1}^n$ and $(\mu_i)_{i=1}^n$ in such a way

$$T_1 = S\,\mathrm{diag}(\lambda_1, \ldots, \lambda_n)S^{-1} \qquad \text{and} \qquad T_2 = S\,\mathrm{diag}(\mu_1, \ldots, \mu_n)S^{-1}.$$

Since T_1 and T_2 are k-potent, we get $\lambda_i, \mu_i \in \{0\} \cup \sqrt[k-1]{1}$ for all $i = 1, \ldots, n$. The nonsingularity of $T_1 + aT_2 = S\,\mathrm{diag}(\lambda_1 + a\mu_1, \ldots, \lambda_n + a\mu_n)S^{-1}$ entails

$$\lambda_i + a\mu_i \neq 0 \qquad \forall\, i = 1, \ldots, n. \tag{2.10}$$

Pick any $c_1, c_2 \in \mathbb{C}^*$ such that $-c_1/c_2 \notin \sqrt[k-1]{1}$. Since

$$\det(c_1T_1 + c_2T_2) = \prod_{i=1}^n (c_1\lambda_i + c_2\mu_i) \qquad \text{and} \qquad \det(c_1I_n + c_2T_1T_2) = \prod_{i=1}^n (c_1 + c_2\lambda_i\mu_i),$$

in order to prove the theorem, it is enough to prove $c_1\lambda_i + c_2\mu_i \neq 0$ and $c_1 + c_2\lambda_i\mu_i \neq 0$ for all $i = 1, \ldots, n$.

Assume that there exists $j \in \{1, \ldots, n\}$ with $c_1\lambda_j + c_2\mu_j = 0$. If $\lambda_j = 0$, then $\mu_j = 0$, which can not occur in view of (2.10). In a similar way, we can prove $\mu_j \neq 0$. Thus $-c_1/c_2 = \mu_j/\lambda_j \in \sqrt[k-1]{1}$ (because the set of $(k-1)$ roots of the unity is a multiplicative group), which is unfeasible with the choice of c_1, c_2. Analogously, we can prove that $c_1 + c_2\lambda_i\mu_i \neq 0$ for all $i = 1, \ldots, n$. The theorem is therefore proved. □

If $P_1, P_2 \in \mathbb{C}^{n\times n}$ are two idempotent matrices, a necessary and sufficient condition for $P_1 + P_2$ to be nonsingular is that $\mathcal{R}(P_1) \cap \mathcal{R}[P_2(I - P_1)] = \{0\}$ and $\mathcal{N}(P_1) \cap \mathcal{N}(P_2) = \{0\}$ (see [3, Cor. 4] and [7, Th. 3.2]). In a more general case (see [8, Th. 3]), a necessary and sufficient condition for $c_1P_1 + c_2P_2$ is nonsingular is that $\mathcal{N}(P_1) \cap \mathcal{N}(P_2) = \{0\}$ and $\mathcal{R}[P_1(I_n - P_2)] \cap \mathcal{R}[P_2(I_n - P_1)] = \{0\}$, where $c_1, c_2 \in \mathbb{C}^*$ satisfy $c_1 + c_2 \neq 0$. For k-potent matrices a similar result is given in the following theorem.

Theorem 2.7. *Let $T_1, T_2 \in \mathbb{C}^{n\times n}$ be two k-potent matrices for some integer positive $k > 1$ and let $c_1, c_2 \in \mathbb{C}^*$ with $c_1 + c_2 \neq 0$. Then*

(i) *If $c_1T_1 + c_2T_2$ is nonsingular, then $\mathcal{N}(T_1) \cap \mathcal{N}(T_2) = \{0\}$.*

(ii) *If $\mathcal{R}[T_1(I - T_1^{k-2}T_2)] \cap \mathcal{R}[T_2(I - T_2^{k-2}T_1)] = \{0\}$ and $\mathcal{N}(T_1) \cap \mathcal{N}(T_2) = \{0\}$, then $c_1T_1 + c_2T_2$ is nonsingular.*

Proof. (i) It is evident that $\mathcal{N}(T_1) \cap \mathcal{N}(T_2) \subseteq \mathcal{N}(c_1T_1 + c_2T_2)$. If $c_1T_1 + c_2T_2$ is nonsingular, then $\mathcal{N}(c_1T_1 + c_2T_2) = \{0\}$ and the conclusion follows.

(ii) We will prove that $\mathcal{N}(c_1T_1 + c_2T_2) = \{0\}$. If $x \in \mathcal{N}(c_1T_1 + c_2T_2)$, then we have

$$c_1T_1x + c_2T_2x = 0. \tag{2.11}$$

Premultiplying (2.11) by T_1^{k-1} and by T_2^{k-1}, we obtain, respectively,

$$c_1T_1x = -c_2T_1^{k-1}T_2x \qquad \text{and} \qquad c_1T_2^{k-1}T_1x = -c_2T_2x. \tag{2.12}$$

From (2.11) and the second relation of (2.12) we obtain $c_1T_1x = -c_2T_2x = c_1T_2^{k-1}T_1x$. Hence, $T_1x = T_2^{k-1}T_1x$ holds. The first relation of (2.12) and (2.11) lead to $-c_2T_2x = c_1T_1x = -c_2T_1^{k-1}T_2x$, and therefore, $T_2x = T_1^{k-1}T_2x$. Now, we have from the first equality of (2.12)

$$(c_1 + c_2)T_1x = c_1T_1x + c_2T_1x = -c_2T_1^{k-1}T_2x + c_2T_1x = c_2T_1(I_n - T_1^{k-2}T_2)x. \tag{2.13}$$

We obtain from (2.11) and using $T_1x = T_2^{k-1}T_1x$

$$(c_1 + c_2)T_1x = c_1T_1x + c_2T_1x = -c_2T_2x + c_2T_2^{k-1}T_1x = -c_2T_2(I_n - T_2^{k-2}T_1)x. \tag{2.14}$$

Expressions (2.13) and (2.14) imply $T_1x \in \mathcal{R}[T_1(I - T_1^{k-2}T_2)] \cap \mathcal{R}[T_2(I - T_2^{k-2}T_1)]$. The first condition of item (ii) implies $T_1x = 0$. In the same way we can get $T_2x = 0$. Hence $x \in \mathcal{N}(T_1) \cap \mathcal{N}(T_2)$, and the second condition leads to $x = 0$, thus establishing the nonsingularity of $c_1T_1 + c_2T_2$. The theorem is proved. $\square$

The following theorem shows that the nonsingularity of the linear combination $T = c_1T_1 + c_2T_2$, when T_1, T_2 are k-potent matrices, is also related to the nonsingularity of the following two linear combinations: $c_1T_1T_2^{k-1} + c_2T_2T_1^{k-1}$ and $c_1T_2^{k-1}T_1 + c_2T_1^{k-1}T_2$.

Theorem 2.8. *Let $T_1, T_2 \in \mathbb{C}^{n\times n}$ be two k-potent matrices for some integer positive $k > 1$ and $c_1, c_2 \in \mathbb{C}^*$, then the following statements are equivalent:*

(i) $c_1T_1T_2^{k-1} + c_2T_2T_1^{k-1}$ *is nonsingular.*

(ii) $c_1T_2^{k-1}T_1 + c_2T_1^{k-1}T_2$ *is nonsingular.*

(iii) $c_1T_1 + c_2T_2$ *and* $I_n - T_1^{k-1} - T_2^{k-1}$ *are nonsingular.*

Proof. The result is an immediate consequence of the equalities

$$(c_1T_1 + c_2T_2)(I_n - T_1^{k-1} - T_2^{k-1}) = -(c_1T_1T_2^{k-1} + c_2T_2T_1^{k-1})$$

and

$$(I_n - T_1^{k-1} - T_2^{k-1})(c_1T_1 + c_2T_2) = -(c_1T_2^{k-1}T_1 + c_2T_1^{k-1}T_2).$$

$\square$

3 Group invertibility of a linear combination of two k-potent matrices

This section is devoted to study the group invertibility of a linear combination of two k-potent matrices. Next we review some well known facts about the group inverse of a matrix (for a deeper insight, the interested reader can consult [14, section 4.4]). Let $A \in \mathbb{C}^{n\times n}$, if there exists $X \in \mathbb{C}^{n\times n}$ such that

$$AXA = A, \qquad XAX = X, \qquad AX = XA, \tag{3.15}$$

then it is said that A is group invertible. It can be proved that for a given square matrix A, there is at most one matrix X satisfying (3.15) and is denoted by $A^{\#}$. The group inverse does not exist for all square matrices: a square matrix A has a group inverse if and only if $\mathrm{rk}(A) = \mathrm{rk}(A^2)$. It is straightforward to prove that A is group invertible if and only if A^* is group invertible, and in this case, one has $(A^{\#})^* = (A^*)^{\#}$. Also, it should be evident that if $A \in \mathbb{C}^{n\times n}$ and $S \in \mathbb{C}^{n\times n}$ is nonsingular, then A is group invertible if and only if SAS^{-1} is group invertible, and in this situation, one has $(SAS^{-1})^{\#} = SA^{\#}S^{-1}$.

Recently, there has been interest in giving formulae for the Drazin inverse of a sum of two matrices (or two operators in a Hilbert space) under some conditions, see [15, 16, 17, 18] and references therein. The group inverse is a particular case of the Drazin inverse. Here, we will discuss if a linear combination of two k-potent matrices is group invertible, and when this answer is affirmative, we will give an expression for the group inverse of this linear combination.

Next representation will be useful for further results. Let $T_1, T_2 \in \mathbb{C}^{n\times n}$ be two k-potent matrices for some natural $k > 1$. Since T_1 is k-potent, this matrix is diagonalizable and its spectrum is contained in $\{0\} \cup \sqrt[k-1]{1}$. Therefore, there exists a nonsingular $S \in \mathbb{C}^{n\times n}$ such that

$$T_1 = S \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} S^{-1}, \qquad A = \mathrm{diag}(\lambda_1, \ldots, \lambda_r), \qquad \lambda_i^{k-1} = 1 \text{ for } i = 1, \ldots, r. \tag{3.16}$$

Note that A is nonsingular and $A^{k-1} = I_r$. Now, we can write T_2 as follows:

$$T_2 = S \begin{bmatrix} B & C \\ D & E \end{bmatrix} S^{-1}, \qquad B \in \mathbb{C}^{r\times r},\ E \in \mathbb{C}^{(n-r)\times(n-r)}. \tag{3.17}$$

Theorem 3.1. *Let $T_1, T_2 \in \mathbb{C}^{n\times n}$ be two k-potent matrices for some natural $k > 1$ such that $T_1^{k-1}T_2 = T_2^{k-1}T_1$ and let $c_1, c_2 \in \mathbb{C}^*$. If T_1 or T_2 are nonsingular, then*

(i) *$c_1T_1 + c_2T_2$ is nonsingular if and only if $c_1 + c_2 \neq 0$. In this case,*

(a) *If T_1 is nonsingular, then*

$$(c_1 + c_2)(c_1T_1 + c_2T_2)^{-1} = T_1^{-1} + c_2c_1^{-1}T_1^{-1}(I_n - T_2^{k-1}).$$

(b) *If T_2 is nonsingular, then*

$$(c_1+c_2)(c_1T_1+c_2T_2)^{-1} = T_2^{-1} + c_1c_2^{-1}T_2^{-1}(I_n - T_1^{k-1}).$$

(ii) *$c_1T_1+c_2T_2$ is group invertible for all $c_1, c_2 \in \mathbb{C}^*$.*

(a) *If T_1 is nonsingular, then $(T_1-T_2)^\# = T_1^{-2}(T_1-T_2)$.*

(b) *If T_2 is nonsingular, then $(T_2-T_1)^\# = T_2^{-2}(T_2-T_1)$.*

Proof. By symmetry, we can suppose that T_2 is nonsingular (the expressions of items (i.a) and (ii.a) have the same proof as the expressions of items (i.b) and (ii.b), and we will not give them). The nonsingularity of T_2 entails $T_2^{k-1} = I_n$, hence $T_1^{k-1}T_2 = T_2^{k-1}T_1$ reduces to $T_1^{k-1}T_2 = T_1$. Let us write T_1 and T_2 as in (3.16) and (3.17). Thus, $T_1^{k-1}T_2 = T_1$ yields $B = A$ and $C = 0$. Therefore,

$$T_2 = S\begin{bmatrix} A & 0 \\ D & E \end{bmatrix}S^{-1} \quad \text{and} \quad c_1T_1+c_2T_2 = S\begin{bmatrix} (c_1+c_2)A & 0 \\ c_2D & c_2E \end{bmatrix}S^{-1} \quad \forall\, c_1,c_2 \in \mathbb{C}^*. \tag{3.18}$$

A simple induction argument shows that there exists a sequence $(D_m)_{m=1}^\infty \subset \mathbb{C}^{(n-r)\times r}$ such that

$$T_2^m = S\begin{bmatrix} A^m & 0 \\ D_m & E^m \end{bmatrix}S^{-1} \qquad \forall\, m \in \mathbb{N}.$$

Since $T_2^{k-1} = I_n$ we get $E^{k-1} = I_{n-r}$. In particular we obtain that E is nonsingular.

(i) Since A and E are nonsingular and $c_2 \neq 0$, the representation of $c_1T_1+c_2T_2$ in (3.18) permits affirm that $c_1T_1+c_2T_2$ is nonsingular if and only if $c_1+c_2 \neq 0$.

If $c_1, c_2 \in \mathbb{C}^*$ satisfy $c_1+c_2 \neq 0$, then from (3.18) it is straightforward to see

$$T_2^{-1} = S\begin{bmatrix} A^{-1} & 0 \\ -E^{-1}DA^{-1} & E^{-1} \end{bmatrix}S^{-1}$$

and

$$(c_1T_1+c_2T_2)^{-1} = S\begin{bmatrix} (c_1+c_2)^{-1}A^{-1} & 0 \\ -(c_1+c_2)^{-1}E^{-1}DA^{-1} & c_2^{-1}E^{-1} \end{bmatrix}S^{-1}. \tag{3.19}$$

Hence

$$\begin{aligned}(c_1+c_2)(c_1T_1+c_2T_2)^{-1} &= S\begin{bmatrix} A^{-1} & 0 \\ -E^{-1}DA^{-1} & \frac{c_1+c_2}{c_2}E^{-1} \end{bmatrix}S^{-1} \\ &= S\left\{\begin{bmatrix} A^{-1} & 0 \\ -E^{-1}DA^{-1} & E^{-1} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & \left(\frac{c_1+c_2}{c_2}-1\right)E^{-1} \end{bmatrix}\right\}S^{-1} \\ &= T_2^{-1} + \frac{c_1}{c_2}S\begin{bmatrix} 0 & 0 \\ 0 & E^{-1} \end{bmatrix}S^{-1}. \end{aligned} \tag{3.20}$$

On the other hand, by (3.16) and (3.19) we have

$$T_2^{-1}(I_n - T_1^{k-1}) = S\begin{bmatrix} A^{-1} & 0 \\ -E^{-1}DA^{-1} & E^{-1} \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & I_{n-r} \end{bmatrix}S^{-1} = S\begin{bmatrix} 0 & 0 \\ 0 & E^{-1} \end{bmatrix}S^{-1}. \quad (3.21)$$

Hence (3.20) and (3.21) yield the formula of item (i.b).

(ii) If $c_1, c_2 \in \mathbb{C}^*$ satisfy $c_1 + c_2 \neq 0$, by the previous item we have that $c_1T_1 + c_2T_2$ is nonsingular, in particular $c_1T_1 + c_2T_2$ has group inverse.

Now, we want to prove that $c_1T_1 + c_2T_2$ has group inverse when $c_1, c_2 \in \mathbb{C}^*$ satisfy $c_1 + c_2 = 0$. Since for any square matrix X and any scalar $c \in \mathbb{C}^*$ we have that X is group invertible if and only if cX is group invertible, we can suppose that $c_1 = -1$ and $c_2 = 1$, or in other words, we must prove that $T_2 - T_1$ is group invertible. In order to do this, let us define

$$M = \begin{bmatrix} 0 & 0 \\ D & E \end{bmatrix} \qquad \text{and} \qquad N = \begin{bmatrix} 0 & 0 \\ E^{-2}D & E^{-1} \end{bmatrix}.$$

It is easy to check $N = M^{\#}$ by the definition of the group inverse. Since from (3.18) we have $T_2 - T_1 = SMS^{-1}$, we get that $T_2 - T_1$ is group invertible and $(T_2 - T_1)^{\#} = (SMS^{-1})^{\#} = SM^{\#}S^{-1} = SNS^{-1}$.

From the first equality of (3.19) we get that

$$T_2^{-2} = S\begin{bmatrix} A^{-2} & 0 \\ X & E^{-2} \end{bmatrix}S^{-1}$$

for some matrix $X \in \mathbb{C}^{(n-r)\times r}$, thus

$$T_2^{-2}(T_2 - T_1) = S\begin{bmatrix} A^{-2} & 0 \\ X & E^{-2} \end{bmatrix}\begin{bmatrix} 0 & 0 \\ D & E \end{bmatrix}S^{-1} = S\begin{bmatrix} 0 & 0 \\ E^{-2}D & E^{-1} \end{bmatrix}S^{-1} = SNS^{-1}.$$

The formula of item (ii.b) is proved, and thus, the proof of this theorem is finished. □

Theorem 3.2. *Let $T_1, T_2 \in \mathbb{C}^{n\times n}$ be two k-potent matrices for some natural $k > 1$ such that $T_2T_1^{k-1} = T_1T_2^{k-1}$ and let $c_1, c_2 \in \mathbb{C}^*$. If T_1 or T_2 are nonsingular, then*

(i) *$c_1T_1 + c_2T_2$ is nonsingular if and only if $c_1 + c_2 \neq 0$. In this case,*

(a) *If T_1 is nonsingular, then*

$$(c_1 + c_2)(c_1T_1 + c_2T_2)^{-1} = 2(T_1 + T_2)^{-1} + \frac{c_2 - c_1}{c_1}(I_n - T_2^{k-1})T_1^{-1}.$$

(b) *If T_2 is nonsingular, then*

$$(c_1 + c_2)(c_1T_1 + c_2T_2)^{-1} = 2(T_1 + T_2)^{-1} + \frac{c_1 - c_2}{c_2}(I_n - T_1^{k-1})T_2^{-1}.$$

(ii) *$c_1T_1 + c_2T_2$ is group invertible for all $c_1, c_2 \in \mathbb{C}^*$.*

(a) *If* T_1 *is nonsingular, then* $(T_1 - T_2)^{\#} = (T_1 - T_2)T_1^{-2}$.

(b) *If* T_2 *is nonsingular, then* $(T_2 - T_1)^{\#} = (T_2 - T_1)T_2^{-2}$.

Proof. Recall the following two elementary facts:

(1) A matrix X is nonsingular if and only X^* is nonsingular and in this case, $(X^{-1})^* = (X^*)^{-1}$.

(2) A matrix X has group inverse if and only X^* has group inverse and in this case $(X^{\#})^* = (X^*)^{\#}$.

If we use Theorem 3.1 for matrices T_1^*, T_2^* and scalars $\overline{c_1}, \overline{c_2}$, this theorem should be easy to prove. □

Another result concerning the group invertibility of a linear combination of two k-potent matrices is the following.

Theorem 3.3. *Let* $T_1, T_2 \in \mathbb{C}^{n\times n}$ *be two* k*-potent matrices for some natural* $k > 1$ *such that* $T_1T_2 = 0$ *and let* $c_1, c_2 \in \mathbb{C}^*$*. Then*

(i) $\mathcal{N}(c_1T_1+c_2T_2) = \mathcal{N}(T_1+T_2)$ *and* $\mathcal{R}(c_1T_1+c_2T_2) = \mathcal{R}(T_1+T_2)$*. In particular,* $c_1T_1+c_2T_2$ *if nonsingular if and only if* $T_1 + T_2$ *is nonsingular, and in this case, we have*

$$(c_1T_1 + c_2T_2)^{-1} = c_1^{-1}(T_1 + T_2)^{-1} + (c_2^{-1} - c_1^{-1})T_2^{k-2}(I_n - T_1^{k-1}). \tag{3.22}$$

(ii) $c_1T_1 + c_2T_2$ *is group invertible and*

$$(c_1T_1 + c_2T_2)^{\#} = c_1^{-1}(I_n - T_2^{k-1})T_1^{k-2} + c_2^{-1}T_2^{k-2}(I_n - T_1^{k-1}). \tag{3.23}$$

Proof. Let us write T_1 and T_2 as in (3.16) and (3.17). Using $T_1T_2 = 0$ and the nonsingularity of A lead to $B = 0$ and $C = 0$. Hence

$$T_2 = S \begin{bmatrix} 0 & 0 \\ D & E \end{bmatrix} S^{-1}.$$

Therefore,

$$c_1T_1 + c_2T_2 = S \begin{bmatrix} c_1A & 0 \\ c_2D & c_2E \end{bmatrix} S^{-1}, \qquad T_1 + T_2 = S \begin{bmatrix} A & 0 \\ D & E \end{bmatrix} S^{-1}. \tag{3.24}$$

A standard induction argument shows that

$$T_2^m = S \begin{bmatrix} 0 & 0 \\ E^{m-1}D & E^m \end{bmatrix} S^{-1} \qquad \forall\, m \in \mathbb{N}. \tag{3.25}$$

Since $T_2^k = T_2$ we get

$$E^{k-1}D = D \qquad \text{and} \qquad E^k = E. \tag{3.26}$$

(i) From $T_1T_2 = 0$ we get $T_1^{k-1}T_2 = 0$ and $T_1T_2^{k-1} = 0$. Now, $\mathcal{N}(c_1T_1+c_2T_2) = \mathcal{N}(T_1+T_2)$ and $\mathcal{R}(c_1T_1 + c_2T_2) = \mathcal{R}(T_1 + T_2)$ follow from Theorems 2.1 and 2.3.

Assume that $T_1 + T_2$ is nonsingular. From the second equality of (3.24) we get that E is nonsingular (recall that A was already nonsingular). It is easy to see

$$(c_1T_1 + c_2T_2)^{-1} = S \begin{bmatrix} c_1^{-1}A^{-1} & 0 \\ -c_1^{-1}E^{-1}DA^{-1} & c_2^{-1}E^{-1} \end{bmatrix} S^{-1}$$

and

$$(T_1 + T_2)^{-1} = S \begin{bmatrix} A^{-1} & 0 \\ -E^{-1}DA^{-1} & E^{-1} \end{bmatrix} S^{-1}.$$

Hence

$$(c_1T_1 + c_2T_2)^{-1} - c_1^{-1}(T_1 + T_2)^{-1} = (c_2^{-1} - c_1^{-1})S \begin{bmatrix} 0 & 0 \\ 0 & E^{-1} \end{bmatrix} S^{-1}.$$

In order to prove formula (3.22), it is enough to prove $T_2^{k-2}(I_n - T_1^{k-1}) = S(0 \oplus E^{-1})S^{-1}$. We will distinguish to cases: $k = 2$ and $k > 2$.

If $k = 2$, then $T_2^{k-2}(I_n - T_1^{k-1}) = I_n - T_1 = S(0 \oplus I_{n-r})S^{-1}$ because from (3.16) we get $T_1 = S(A \oplus 0)S^{-1} = S(I_r \oplus 0)S^{-1}$. Moreover, having in mind the nonsingularity of E and the second relation of (3.26) we get $E = I_{n-r}$. Therefore, $T_2^{k-2}(I_n - T_1^{k-1}) = S(0 \oplus E^{-1})S^{-1}$ holds.

If $k > 2$, we can apply (3.25) to compute T_2^{k-2}. From the decomposition (3.16) we get $T_1^{k-1} = S(A^{k-1} \oplus 0)S^{-1} = S(I_r \oplus 0)S^{-1}$, and therefore

$$T_2^{k-2}(I_n - T_1^{k-1}) = S \begin{bmatrix} 0 & 0 \\ E^{k-3}D & E^{k-2} \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & I_{n-r} \end{bmatrix} S^{-1} = S \begin{bmatrix} 0 & 0 \\ 0 & E^{k-2} \end{bmatrix} S^{-1}$$

Now, the expression $T_2^{k-2}(I_n - T_1^{k-1}) = S(0 \oplus E^{-1})S^{-1}$ follows from $E^k = E$ and the nonsingularity of E.

(ii) In order to prove that $c_1T_1 + c_2T_2$ is group invertible, let us define

$$M = \begin{bmatrix} c_1A & 0 \\ c_2D & c_2E \end{bmatrix} \qquad \text{and} \qquad N = \begin{bmatrix} c_1^{-1}A^{-1} & 0 \\ -c_1^{-1}E^{k-2}DA^{-1} & c_2^{-1}E^{k-2} \end{bmatrix}.$$

Next, we will prove $MN = NM$. From the first relation of (3.26) we get $DA^{-1}-E^{k-1}DA^{-1} = 0$, and therefore we have

$$MN = \begin{bmatrix} I_r & 0 \\ 0 & E^{k-1} \end{bmatrix} = NM.$$

Now, the equality $MNM = M$ follows from (3.26) and the equality $NMN = N$ follows from $E^{k-2}E^{k-1} = E^{k-2}$. Thus, we have proved that M is group invertible and $M^\# = N$. Since $c_1T_1 + c_2T_2 = SMS^{-1}$ we get the group invertibility of $c_1T_1 + c_2T_2$ and $(c_1T_1 + c_2T_2)^\# = SNS^{-1}$. Observe

$$SNS^{-1} = c_1^{-1}S \begin{bmatrix} A^{-1} & 0 \\ -E^{k-2}DA^{-1} & 0 \end{bmatrix} S^{-1} + c_2^{-1}S \begin{bmatrix} 0 & 0 \\ 0 & E^{k-2} \end{bmatrix} S^{-1}. \tag{3.27}$$

From representations (3.16) and (3.25) we get

$$\begin{aligned}(I_n - T_2^{k-1})T_1^{k-2} &= S\left\{\left[\begin{array}{cc} I_r & 0 \\ 0 & I_{n-r}\end{array}\right] - \left[\begin{array}{cc} 0 & 0 \\ E^{k-2}D & E^{k-1}\end{array}\right]\right\}\left[\begin{array}{cc} A^{-1} & 0 \\ 0 & 0\end{array}\right]S^{-1} \\ &= S\left[\begin{array}{cc} I_r & 0 \\ -E^{k-2}D & I_{n-r}-E^{k-1}\end{array}\right]\left[\begin{array}{cc} A^{-1} & 0 \\ 0 & 0\end{array}\right]S^{-1} \\ &= S\left[\begin{array}{cc} A^{-1} & 0 \\ -E^{k-2}DA^{-1} & 0\end{array}\right]S^{-1}. \qquad (3.28)\end{aligned}$$

Now, we are going to prove

$$T_2^{k-2}(I_n - T_1^{k-1}) = S\left[\begin{array}{cc} 0 & 0 \\ 0 & E^{k-2}\end{array}\right]S^{-1}. \qquad (3.29)$$

We distinguish two cases: $k = 2$ and $k > 2$. When $k = 2$ we have from (3.16) the equality $T_1 = S(I_r \oplus 0)S^{-1}$, hence

$$T_2^{k-2}(I_n - T_1^{k-1}) = I_n - T_1 = S(0 \oplus I_{n-r})S^{-1} = S(0 \oplus E^{k-2})S^{-1}.$$

If $k > 2$, then we can apply (3.25). Now, using (3.16) we get

$$T_2^{k-2}(I_n - T_1^{k-1}) = S\left[\begin{array}{cc} 0 & 0 \\ E^{k-3}D & E^{k-2}\end{array}\right]\left[\begin{array}{cc} 0 & 0 \\ 0 & I_{n-r}\end{array}\right]S^{-1} = S\left[\begin{array}{cc} 0 & 0 \\ 0 & E^{k-2}\end{array}\right]S^{-1}.$$

Now, it is enough to consider (3.27), (3.28), and (3.29) to get (3.23). The theorem is thus proved. □

References

[1] J. Benítez, N. Thome. $\{k\}$-Group Periodic Matrices. SIAM. J. Matrix Anal. & Appl. 28 (2006), 9-25.

[2] C.Y. Deng, H.K. Du, Q.H. Li. Generalized n-idempotents and Hyper-generalized n-idempotents. Northeast. Math. J. 22(4)(2006), 387-394.

[3] J. Groß, G. Trenkler. Nonsingularity of the difference of two oblique projectors. SIAM J.Matrix Anal. & Appl. 21 (1999), 390-395.

[4] D. Buckholtz, Inverting the difference of Hilbert space projections. Amer. Math. Monthly. 104 (1997), 60-61.

[5] D. Buckholtz, Hilbert space idempotents and involutions. Proc. Amer. Math. Soc. 128 (2000), 1415-1418.

[6] J. J. Koliha, V. Rakočević. On the norm of idempotents in C*-algebras. Rocky Mountain J. Math. 34 (2004), 685-697.

[7] J. J. Koliha, V. Rakočević, I. Straškraba. The difference and sum of projectors. Linear Algebra Appl. 388 (2004), 279-288.

[8] J.K. Baksalary, O.M. Baksalary. Nonsingularity of linear combinations of idempotent matrcies. Linear Algebra Appl. 388 (2004), 25-29.

[9] J.J. Koliha, V. Rakočević. The nullity and rank of linear combinations of idempotent matrices. Linear Algebra Appl. 418 (2006), 11-14.

[10] H.K. Du, X.Y. Yao, C.Y. Deng. Invertibility of linear combinations of two idempotents. Proc. Amer. Math. Soc. 134 (2005), 1451-1457.

[11] A.M. Mathai, S.B. Provost. Quadratic Forms in Random Variables: Theory and Applications. Dekker, 1992.

[12] C.R. Rao, S.K. Mitra. Generalized Inverse of Matrices and Its Applications. Wiley, 1971.

[13] R. Piziak, P.L. Odell, R. Hahn. Constructing projections on sums and intersections. Comput. Math. Appl. 37 (1999), 67-74.

[14] A. Ben-Israel, T. Greville. Generalized inverses. Theory and applications Springer, 2003.

[15] R.E. Hartwig, G.R.Wang, Y.Wei. Some additive results on Drazin inverse. Linear Algebra Appl. 322 (2001), 207-217.

[16] D.S. Djordjević, Y. Wei. Additive results for the generalized Drazin inverse. J. Austral. Math. Soc. 73 (2002), 115-125.

[17] D.S. Cvetković-Ilić, D.S. Djordjević, Y. Wei. Additive results for the generalized Drazin inverse in a Banach algebra. Linear Algebra Appl. 418 (2006), 53-61.

[18] C.Y. Deng. The Drazin inverses of sum and difference of idempotents. Linear Algebra Appl. 430 (2009), 1282-1291.