Nonsingularity on Scaled Factor Circulant Matrices

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Abstract

In this paper, $M_n(F)$ will denote the set of all $n \times n$ matrices over any field. Matrixs A, $R \in M_n(F)$ are called scaled factor circulant matrix and basic scaled factor circulant matrix respectively, where $AR = RA$. We give th ree discriminations by using only the elements in the first row of the scaled factor circulant matrix and the constants $d_1, d_2, , d_n$ in the diagonal matrix D on non singularity.

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1   Introduction

Because of structure and many good properties of Circulant matrices, they have in recent years become one of the most important and active research field of applied mathematic and computation mathematic increasingly. They have been applied in many areas such as signal processing, digital image processing, coding Theory, and have a wide range of interesting applications[4,5] as an important class of special matrices. Scaled factor circulant matrix, a kind of Circulant matrices have have a widely range of application in signal processing, code theoryimage processing, and so on.

J.L.Stuart and J.R.Weaver gave the definition of the Scaled factor circulant matrix and diiscussed the properties of them in [8]. Jiangzhao Lin gave three

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discriminations by using only the elements in the first row of the scaled factor circulant matrix and the constants d_1, d_2, $\cdots$, d_n in the diagonal matrix D on non singularity in [1]. Motivated by [1], we give the discriminations by using only the element a_{max}, a_{min} and the constants d_1, d_2, $\cdots$, d_n in the diagonal matrix D on non singularity continuously, where a_{max}, a_{min} are the maximum and minimun of the elements in the first row of the scaled factor circulant matrix respectively.

Throughout of the paper, let R be the basis scaled circulant factor matrix over F:

$$\begin{pmatrix} 0 & d_1 & 0 & \cdots & 0 \\ 0 & 0 & d_2 & \cdots & 0 \\ 0 & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & d_{n-1} \\ d_n & 0 & 0 & \cdots & 0 \end{pmatrix} \tag{1.1}$$

where$\prod_{j=1}^{n} dj0$.It is easily verified that the polynomial $g(x) = x^n - d_1 d_2 \cdots d_n$ is both the minimal polynomial and the characteristic polynomial of the matrix R.Moreever, R is nonderogatory.

Let also

$$f(x) = \sum_{i=0}^{n-1} a_i k_i^{-1} x^i \tag{1.2}$$

, where $k_i = d_1 d_2 \cdots d_n$, $i = 1, 2, \cdots, n-1$, $k_0 = 1, \omega_j (0 \leq j \leq n-1)$ are the mutually different roots of equation $x^n - d_1 d_2 \cdots d_n = 0$. The polynomial (1.2) will be called the representor of the scaled factor circulant matrix A.

Definition 1.1. *An $n \times n$ matrix A over F is called a scaled factor circulant matrix if A commutes with R ,that is,*

$$AR = RA, \tag{1.2}$$

where R is given in(1.1).

2   Preliminary Notes

In this section , we present some lemmas that are important to our main results.

Lemma 2.1[1]. *$A \in RCM_n$ if and only if A=f(R) .*

Lemma 2.2[2]. *Let $A = scacirc_R(a_0, a_1, \cdots, a_{n-1})$ be a scaled factor circulant matrix over the field F. Then*

$$\sigma(A) = \{\lambda_j | \lambda_j = f(d\omega^j) = a_0 + \sum_{i=1}^{n-1} a_i (\prod_{t=1}^{i} d_t)^{-1} (d\omega^j)^i | \ 0 \leq j \leq n-1\} \tag{2.1}$$

is the spectrum of A and

$$A = f(R) = a_0 I + \sum_{i=1}^{n-1} a_i (\prod_{t=1}^{i} d_t)^{-1} \mathcal{R}^i, \tag{2.2}$$

Lemma 2.3[1]. *Suppose* $A, B \in RCM_n$, *then* $AB = BA \in RCM_n$.

Lemma 2.4[1]. *Suppose* $A, \in RCM_n$, *if* A *is nonsingular then* $A^{-1} \in RCM_n$.

3 Main Results

Our main results are following theorems.

Theorem 3.1. *Suppose* $A = scacirc_R(a_0, a_1, \cdots, a_{n-1}) \in RCM_n$, *if for* $\forall M \in \{1, 2, \cdots, n\}$,

$$\sum_{i=0,i\neq M}^{n-1} \frac{\mid d_1 d_2 \cdots d_M \mid}{\mid d_1 d_2 \cdots d_i \mid} \sqrt[n]{\mid d_1 d_2 \cdots d_n \mid^{i-M}} < 1, \tag{3.1}$$

then A is nonsingular.

Proof. If A is singular, then by [1, Theorem 1], we have

$$\mid a_M \mid \leq \sum_{i=0,i\neq M}^{n-1} \mid a_i \mid \frac{\mid d_1 d_2 \cdots d_M \mid}{\mid d_1 d_2 \cdots d_i \mid} \sqrt[n]{\mid d_1 d_2 \cdots d_i \mid^{i-M}}$$

for $M = 1, 2, \cdots, n$. So

$$\begin{aligned} \mid a_M \mid &\leq \sum_{i=0,i\neq M}^{n-1} \mid a_i \mid \frac{\mid d_1 d_2 \cdots d_M \mid}{\mid d_1 d_2 \cdots d_i \mid} \sqrt[n]{\mid d_1 d_2 \cdots d_i \mid^{i-M}} \\ &\leq \sum_{i=0,i\neq M}^{n-1} a \frac{\mid d_1 d_2 \cdots d_M \mid}{\mid d_1 d_2 \cdots d_i \mid} \sqrt[n]{\mid d_1 d_2 \cdots d_i \mid^{i-M}} \end{aligned}$$

hence

$$\begin{aligned} 1 &\leq \sum_{i=0,i\neq M}^{n-1} \frac{a}{\mid a_M \mid} \frac{\mid d_1 d_2 \cdots d_M \mid}{\mid d_1 d_2 \cdots d_i \mid} \sqrt[n]{\mid d_1 d_2 \cdots d_i \mid^{i-M}} \\ &\leq \sum_{i=0,i\neq M}^{n-1} \frac{\mid d_1 d_2 \cdots v \mid}{\mid d_1 d_2 \cdots d_i \mid} \sqrt[n]{\mid d_1 d_2 \cdots d_i \mid^{i-M}} \end{aligned}$$

which contradicts to inequality (3.1), therefore A is nonsingular.

Corollary 3.2. *Suppose* $A = scacirc_R(a_0, a_1, \cdots, a_{n-1}) \in RCM_n$, *if for* $\forall M \in \{1, 2, \cdots, n\}$,

$$\sum_{i=0}^{n-1} \frac{\mid d_1 d_2 \cdots d_M \mid}{\mid d_1 d_2 \cdots d_i \mid} \sqrt[n]{\mid d_1 d_2 \cdots d_n \mid^{i-M}} < 0, \tag{3.2}$$

then A is nonsingular.

Corollary 3.3. *Suppose* $A = scacirc_R(a_0, a_1, \cdots, a_{n-1}) \in RCM_n$, *if*

$$\sqrt{n[|\ 1 - \frac{\sqrt[n]{|\ d_1 d_2 \cdots d_n\ |^M}}{|\ d_1 d_2 \cdots d_M\ |}\ |^2 + \sum_{i=0, i \neq M}^{n-1} |\ \frac{\sqrt[n]{|\ d_1 d_2 \cdots d_n\ |^i}}{d_1 d_2 \cdots d_i}\ |^2]} < 1, \qquad (3.3)$$

then A is nonsingular.

Proof. The proof is similar to Lemma 1. It is easy to see

$$\sqrt{\frac{|1 - \frac{\sqrt[n]{|d_1 d_2 \cdots d_n|^M}}{|d_1 d_2 \cdots d_M|}|^2 + \sum_{i=0, i \neq M}^{n-1} |\frac{\sqrt[n]{|d_1 d_2 \cdots d_n|^i}}{d_1 d_2 \cdots d_i}|^2}{n}}$$

$$\geq \quad \frac{|1 - \frac{\sqrt[n]{|d_1 d_2 \cdots d_n|^M}}{|d_1 d_2 \cdots d_M|}| + \sum_{i=0, i \neq M}^{n-1} |\frac{\sqrt[n]{|d_1 d_2 \cdots d_n|^i}}{d_1 d_2 \cdots d_i}|}{n}$$

so

$$\sqrt{n[|\ 1 - \frac{\sqrt[n]{|d_1 d_2 \cdots d_n|^M}}{|d_1 d_2 \cdots d_M|}\ |^2 + \sum_{i=0, i \neq M}^{n-1} |\ \frac{\sqrt[n]{|d_1 d_2 \cdots d_n|^i}}{d_1 d_2 \cdots d_i}\ |^2]}$$

$$\geq \quad |\ 1 - \frac{\sqrt[n]{|d_1 d_2 \cdots d_n|^M}}{|d_1 d_2 \cdots d_M|}\ | + \sum_{i=0, i \neq M}^{n-1} |\ \frac{\sqrt[n]{|d_1 d_2 \cdots d_n|^i}}{d_1 d_2 \cdots d_i}\ |$$

$$\geq \quad 1 - |\ \frac{\sqrt[n]{|d_1 d_2 \cdots d_n|^M}}{|d_1 d_2 \cdots d_M|}\ | + \sum_{i=0, i \neq M}^{n-1} |\ \frac{\sqrt[n]{|d_1 d_2 \cdots d_n|^i}}{d_1 d_2 \cdots d_i}\ |$$

by above inequality and (3.3), we have

$$1 - |\ \frac{\sqrt[n]{|\ d_1 d_2 \cdots d_n\ |^M}}{|\ d_1 d_2 \cdots d_M\ |}\ | + \sum_{i=0, i \neq M}^{n-1} |\ \frac{\sqrt[n]{|\ d_1 d_2 \cdots d_n\ |^i}}{d_1 d_2 \cdots d_i}\ | < 1$$

hence

$$|\ \frac{\sqrt[n]{|\ d_1 d_2 \cdots d_n\ |^M}}{|\ d_1 d_2 \cdots d_M\ |}\ | > \sum_{i=0, i \neq M}^{n-1} |\ \frac{\sqrt[n]{|\ d_1 d_2 \cdots d_n\ |^i}}{d_1 d_2 \cdots d_i}\ |$$

,

$$\sum_{i=0, i \neq M}^{n-1} \frac{|\ d_1 d_2 \cdots d_M\ |}{|\ d_1 d_2 \cdots d_i\ |} \sqrt[n]{|\ d_1 d_2 \cdots d_n\ |^{i-M}} < 1$$

therefore *A* is nonsingular.

Theorem 3.4. *Suppose* $A = scacirc_R(a_0, a_1, \cdots, a_{n-1}) \in RCM_n$, *if for* $\forall M \in \{1, 2, \cdots, n\}$,

$$\frac{|\ d_1 d_2 \cdots d_M\ |}{\sqrt[n]{|\ d_1 d_2 \cdots d_n\ |^M}} < \frac{na}{n-1}, \qquad (3.2)$$

and

$$\frac{\mid d_1 d_2 \cdots d_i \mid}{\sqrt[n]{\mid d_1 d_2 \cdots d_n \mid^i}} < na, \ i = 1, 2, \cdots, n-1, \ i \neq M \tag{3.3}$$

then A is nonsingular.

Proof. If A is singular, then by [1, Theorem 2], we have

$$\mid 1 - \frac{\sqrt[n]{\mid d_1 d_2 \cdots d_n \mid^M}}{d_1 d_2 \cdots d_M} a_M \mid \leq \frac{1}{n}, \tag{3.4}$$

or

$$\frac{\sqrt[n]{\mid d_1 d_2 \cdots d_n \mid^i}}{\mid d_1 d_2 \cdots d_i \mid} \mid a_i \mid \leq \frac{1}{n}, \tag{3.5}$$

When inequality (3.4) is hold, then

$$\frac{\sqrt[n]{\mid d_1 d_2 \cdots d_n \mid^M}}{d_1 d_2 \cdots d_M} a_M \leq 1 - \frac{1}{n}, \tag{3.6}$$

or

$$\frac{\sqrt[n]{\mid d_1 d_2 \cdots d_n \mid^M}}{d_1 d_2 \cdots d_M} a_M \geq 1 + \frac{1}{n} \geq -(1 - \frac{1}{n}), \tag{3.7}$$

so

$$1 - \frac{1}{n} \leq \mid \frac{\sqrt[n]{\mid d_1 d_2 \cdots d_n \mid^M}}{d_1 d_2 \cdots d_M} a_M \mid \leq \frac{\sqrt[n]{\mid d_1 d_2 \cdots d_n \mid^M}}{\mid d_1 d_2 \cdots d_M \mid} \mid a \mid,$$

hence

$$\frac{\mid d_1 d_2 \cdots d_M \mid}{\sqrt[n]{\mid d_1 d_2 \cdots d_n \mid^M}} \geq \frac{na}{n-1}$$

this contradicts to inequality (3.2).

On the other hand, when inequality (3.5) is hold, it is easy to see

$$\frac{1}{n} \leq \frac{\sqrt[n]{\mid d_1 d_2 \cdots d_n \mid^i}}{\mid d_1 d_2 \cdots d_i \mid} \mid a_i \mid \leq \frac{\sqrt[n]{\mid d_1 d_2 \cdots d_n \mid^i}}{\mid d_1 d_2 \cdots d_i \mid} \mid a \mid$$

so

$$\frac{\mid d_1 d_2 \cdots d_i \mid}{\sqrt[n]{\mid d_1 d_2 \cdots d_n \mid^i}} \geq na, \ i = 1, 2, \cdots, n-1, \ i \neq M$$

this contradicts to inequality (3.3), therefore A is nonsingular.

Theorem 3.5. *Suppose* $A = scacirc_R(a_0, a_1, \cdots, a_{n-1}) \in RCM_n$, *if for* $\forall M \in \{1, 2, \cdots, n\}$,

$$\frac{\mid d_1 d_2 \cdots d_M \mid}{\sqrt[n]{\mid d_1 d_2 \cdots d_n \mid^M}} < min\{\frac{1}{(n-1)na}, \frac{na}{n-1}\} \tag{3.8}$$

then A is nonsingular.

Proof. If A is singular, then by Theorem 3.1, we have

$$1 \leq \sum_{i=0,i\neq M}^{n-1} \frac{\mid d_1 d_2 \cdots d_M \mid}{\mid d_1 d_2 \cdots d_i \mid} \sqrt[n]{\mid d_1 d_2 \cdots d_i \mid^{i-M}} \tag{3.9}$$

by Theorem 3.4, we have

$$\frac{\mid d_1 d_2 \cdots d_M \mid}{\sqrt[n]{\mid d_1 d_2 \cdots d_n \mid^M}} \geq \frac{na}{n-1} \tag{3.10}$$

or

$$\frac{\mid d_1 d_2 \cdots d_i \mid}{\sqrt[n]{\mid d_1 d_2 \cdots d_n \mid^i}} \geq na, \ i = 1, 2, \cdots, n-1, \ i \neq M \tag{3.11}$$

When (3.9) and (3.11) is hold, then

$$\begin{aligned} 1 \quad &\leq \sum_{i=0,i\neq M}^{n-1} \frac{|d_1 d_2 \cdots d_M|}{|d_1 d_2 \cdots d_i|} \sqrt[n]{\mid d_1 d_2 \cdots d_i \mid^{i-M}} \\ &\leq \frac{|d_1 d_2 \cdots d_M|}{\sqrt[n]{|d_1 d_2 \cdots d_n|^M}} n(n-1)a \end{aligned}$$

this contradicts to inequality (3.8), therefore A is nonsingular.

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