NONSOLVABLE FINITE GROUPS ALL OF WHOSE LOCAL SUBGROUPS ARE SOLVABLE[1]

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1. Introduction. The results of this paper grew from an attempt to classify the minimal simple groups. For obvious reasons, this paper is a natural successor to 0.[2] The structure of the proof showed that a larger class of groups could be mastered with some further effort. An easy corollary classifies the minimal simple groups.

In a broad way, this paper may be thought of as a successful translation of the theory of solvable groups to the theory of simple groups. By this is meant that a substantial structure is constructed which makes it possible to exploit properties of solvable groups to obtain delicate information about the structure and embedding of many solvable subgroups of the simple group under consideration. In this way, routine results about solvable groups acquire great power.

In somewhat more detail, the arguments go as follows, apart from numerous special cases which involve groups of small order: Let $\mathfrak{G}$ be a finite group. Let $\mathcal{S}o\ell(\mathfrak{G})$ be the set of all solvable subgroups of $\mathfrak{G}$. Then $\mathcal{S}o\ell(\mathfrak{G})$ is partially ordered by inclusion and we let $\mathcal{M}\mathcal{S}(\mathfrak{G})$ be the set of maximal elements of $\mathcal{S}o\ell(\mathfrak{G})$. Let $\mathcal{M}^*(\mathfrak{G})$ be the set of all elements of $\mathcal{S}o\ell(\mathfrak{G})$ which are contained in precisely one element of

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[2] 0 refers to *Solvability of groups of odd order*, W. Feit and J. Thompson, Pacific J. Math. (3) **13** (1963), and Result X of 0 is here referred to as Result 0.X. Also, as in 0, (B) refers to Theorem B of [26].

$\mathfrak{MS}(\mathfrak{G})$, so that $\mathfrak{M}^*(\mathfrak{G}) \supseteq \mathfrak{MS}(\mathfrak{G})$. The theory of solvable groups makes it possible to prove statements of the sort $\mathfrak{H} \in \mathfrak{M}^*(\mathfrak{G})$, and most of the important technical results of this paper are of this type.

The characterizations of $E_2(3)$ and $S_4(3)$ which emerge are the result of detailed and careful study. These characterizations could be avoided in classifying the minimal simple groups, but the effort this requires is comparable to the characterizations themselves. Furthermore, these characterizations have an independent interest. They are prototypes for the translation referred to above.

A portion of an earlier version of this paper was read by E. C. Dade, whose comments have led to several improvements. Recent results of J. Alperin [**1**], [**2**], G. Glauberman [**16**], [**17**], [**18**], and P. Fong [**14**], have also eased the proofs somewhat. A recent result of C. Sims [**33**] is helpful.

It is somewhat anomalous that the character theory is not used in this paper. The reason for this is that the relevant character theory is in the literature [**8**], [**9**], [**10**], [**11**], [**14**], [**15**], [**16**], [**18**], [**28**], [**35**], [**45**]. This anomaly is in marked contrast with 0, where character theory was needed and was not readily available.

The work is flawed because as yet I have been unable to axiomatize the properties of solvable groups which are "really" needed. To carry out the axiomatization of the various parts of this paper will require several years further study. If this is done, the usual benefits will undoubtedly accrue: stronger theorems, shorter proofs.

This first paper sets the stage. §5 introduces many of the configurations which are relevant to the study of simple groups, and §6 deals with the notion of transitivity.

2. **Notation and definitions.** A *minimal simple group* is a simple group of composite order all of whose proper subgroups are solvable.

Following Alperin [**2**], the subgroup $\mathfrak{H}$ of the group $\mathfrak{G}$ is a *local subgroup* of $\mathfrak{G}$ if and only if, for some prime p, there is a nonidentity p-subgroup $\mathfrak{P}$ of $\mathfrak{G}$ such that $\mathfrak{H} = N(\mathfrak{P})$.

An *N-group* is a group all of whose local subgroups are solvable. Since every nonidentity solvable group contains a nonidentity characteristic p-subgroup for some prime p, it follows that N-groups are precisely those groups such that the normalizer of every nonidentity solvable subgroup is solvable.

An *involution* is a group element of order 2.

A noncyclic group of order 8 with exactly 1 involution is a *quaternion group*. A noncyclic 2-group with exactly 1 involution is a *generalized quaternion group*. A group which is generated by two distinct involutions is a *dihedral group*. A *four-group* is a dihedral group of order 4.

The techniques and results of 0 are used freely here. The terminology and notation of this paper extend that of 0.

Artin's notation [4] for simple groups is used. In addition, $Sz(q)$ is the group of order $q^2(q^2+1)(q-1)$ discovered by Suzuki [37], M_{11} is the Mathieu group of order 7920, and Σ_n, A_n denote the symmetric group and alternating group on n letters. The group of inner automorphisms of the group $\mathfrak{X}$ is $I(\mathfrak{X})$.

The number of conjugacy classes of involutions of $\mathfrak{X}$ is $i(\mathfrak{X})$.

If $\mathfrak{A}$, $\mathfrak{B}$ are permutation groups, $\mathfrak{A} \wr \mathfrak{B}$ is the wreath product of $\mathfrak{A}$ and $\mathfrak{B}$, and if $\mathfrak{A}$, $\mathfrak{B}$ have not been presented as permutation groups, $\mathfrak{A} \wr \mathfrak{B}$ is the wreath product of the regular representations of $\mathfrak{A}$, $\mathfrak{B}$. This is the regularity convention [24] and will be used on occasion.

Let $\mathfrak{S}=\mathfrak{K}/\mathfrak{L}$ be a section of the group $\mathfrak{X}$. There is thus a homomorphism of $N(\mathfrak{K})\cap N(\mathfrak{L})$ into Aut$(\mathfrak{S})$ induced by conjugation. The image of $N(\mathfrak{K})\cap N(\mathfrak{L})$ in Aut$(\mathfrak{S})$ is denoted $A_{\mathfrak{X}}(\mathfrak{S})$. More generally, if $\mathfrak{M}$ is a subgroup of $\mathfrak{S}$, $A_{\mathfrak{M}}(\mathfrak{S})$ denotes the image of $\mathfrak{M}\cap N(\mathfrak{K})\cap N(\mathfrak{L})$ in $A_{\mathfrak{X}}(\mathfrak{S})$. If X is in $N(\mathfrak{K})\cap N(\mathfrak{L})$ and $S=\mathfrak{L}K$ is in $\mathfrak{S}$, then $[X, S]$ denotes $\mathfrak{L}[X, K]$. Similarly, if $\mathfrak{T}=\mathfrak{M}/\mathfrak{N}$ is a section of $\mathfrak{X}$, if $\mathfrak{M}$ normalizes both $\mathfrak{K}$ and $\mathfrak{L}$, and if $[\mathfrak{N}, \mathfrak{K}]\subseteq\mathfrak{L}$, then we will view $\mathfrak{T}$ as a group of operators of $\mathfrak{S}$, and we let $A_{\mathfrak{T}}(\mathfrak{S})=A_{\mathfrak{M}}(\mathfrak{S})$.

If $1=\mathfrak{P}_0\subseteq\mathfrak{N}_0\subseteq\mathfrak{P}_1\subseteq\mathfrak{N}_1\subseteq \cdots \subseteq\mathfrak{P}_r\subseteq\mathfrak{N}_r=\mathfrak{G}$ is the upper π-series for the π-solvable group $\mathfrak{G}$[3] defined via $\mathfrak{N}_n=O_{\pi'}$, ($\mathfrak{G}$ mod $\mathfrak{P}_n$), $\mathfrak{P}_{n+1}=O_\pi(\mathfrak{G}$ mod $\mathfrak{N}_n)$, $n=0, 1, \cdots$, we set $P_\pi^n(\mathfrak{G})=\mathfrak{P}_n/\mathfrak{N}_{n-1}$, $n=1, \cdots, r$, and $Q_\pi^n(\mathfrak{G})=\mathfrak{N}_n/\mathfrak{P}_n$, $n=0, \cdots, r$. Here $r=l_\pi(\mathfrak{G})$ is the π-length of $\mathfrak{G}$. As in 0, the major attention is focussed on $P_p^1(\mathfrak{G})$, $Q_p^1(\mathfrak{G})$ and $Q_p^0(\mathfrak{G})(=O_{p'}(\mathfrak{G}))$.

If $\mathfrak{A}$ is a group of operators of the group $\mathfrak{B}$ and $1=C_{\mathfrak{B}}(\mathfrak{A})$, we say that $\mathfrak{A}$ has no fixed points on $\mathfrak{B}$.

Definition 2.1. The group $\mathfrak{G}$ is *π-reduced* if and only if $O_\pi(\mathfrak{G})=1$. The subgroup $\mathfrak{A}$ of the group $\mathfrak{G}$ is *π-reducible* if and only if $A_{\mathfrak{G}}(\mathfrak{A})$ is π-reduced.

Definition 2.2. $R_\pi(\mathfrak{G})$ is the subgroup of $\mathfrak{G}$ generated by all the normal π-reducible π-subgroups of $\mathfrak{G}$.

Definition 2.3. The subgroup $\mathfrak{A}$ of the group $\mathfrak{G}$ is a π-signalizer of $\mathfrak{G}$ if and only if $|\mathfrak{A}|$ and $|\mathfrak{G}: N(\mathfrak{A})|$ are π'-numbers.

Definition 2.4. A noncyclic p-group $\mathfrak{P}$ is of symplectic type if and only if every characteristic abelian subgroup of $\mathfrak{P}$ is cyclic.

[3] By a π-solvable group, we mean a group each of whose c.f. is either a p-group for some p in π, or a π'-group, that is, we adhere to the terminology of [23], not [26]. A π-separable group is one for which every c.f. is either a π-group or a π'-group.

REMARK. The groups of symplectic type are classified in [24]. This classification is of importance in this paper. If $\mathfrak{P}$ is a p-group of symplectic type, then $\mathfrak{P}$ is the central product of a cyclic group and an extra special group, or $p=2$ and $\mathfrak{P}$ is the central product of an extra special group and a group of maximal class, or $p=2$ and $\mathfrak{P}$ is of maximal class. The explicit nature of the groups of symplectic type will be used frequently.

DEFINITION 2.5. If $\mathfrak{P}$ is a p-group of symplectic type, the *width* of $\mathfrak{P}$ is the largest integer n such that $\mathfrak{P}$ contains an extra special subgroup $\mathfrak{H}$ of order p^{2n+1} such that $\mathfrak{P}=\mathfrak{H}\cdot \boldsymbol{C}(\mathfrak{H})$. If $\mathfrak{P}$ contains no such extra special subgroups, the width of $\mathfrak{P}$ is 0.

DEFINITION 2.6. If $\mathfrak{X}$ is a nilpotent group and $\mathfrak{A}$ is a characteristic abelian subgroup of $\mathfrak{X}$, $\mathcal{B}(\mathfrak{X};\mathfrak{A})$ is the set of all subgroups $\mathfrak{B}$ of $\mathfrak{X}$ such that

(a) $\mathfrak{B}$ char $\mathfrak{X}$.
(b) $\ker(\mathrm{Aut}(\mathfrak{X})\xrightarrow{\mathrm{res}}\mathrm{Aut}(\mathfrak{B}))$ is an abelian $\pi(\mathfrak{X})$-group.
(c) $\mathfrak{A}\subseteq \boldsymbol{Z}(\mathfrak{B})$.
(d) $[\mathfrak{X},\mathfrak{B}]\subseteq \boldsymbol{Z}(\mathfrak{B})$.
(e) $\boldsymbol{D}(\mathfrak{B})\subseteq \boldsymbol{Z}(\mathfrak{B})$.
(f) $\boldsymbol{C}_{\mathfrak{X}}(\mathfrak{B})=\boldsymbol{Z}(\mathfrak{B})$.

We set $\mathcal{B}(\mathfrak{X})=\mathcal{B}(\mathfrak{X};1)$ and observe that $\mathcal{B}(\mathfrak{X})\supseteq\mathcal{B}(\mathfrak{X};\mathfrak{A})$ for every characteristic abelian subgroup $\mathfrak{A}$ of $\mathfrak{X}$.

If $\mathcal{C}$: $\mathfrak{A}=\mathfrak{A}_0\supseteq\mathfrak{A}_1\supseteq\cdots\supseteq\mathfrak{A}_n=1$ is a chain, $\boldsymbol{A}(\mathcal{C})$ denotes the stability group of $\mathcal{C}$, that is, the group of all automorphisms α such that for $i=1,2,\cdots,n$, α fixes each coset of $\mathfrak{A}_i$ in $\mathfrak{A}_{i-1}$. If $\mathfrak{A}$ is a section of $\mathfrak{X}$, set $\boldsymbol{A}_{\mathfrak{X}}(\mathcal{C})=\boldsymbol{A}_{\mathfrak{X}}(\mathfrak{A})\cap\boldsymbol{A}(\mathcal{C})$.

C denotes the field of complex numbers, F_q the field of q elements. If K is a field and $\mathfrak{G}$ is a group, $K\mathfrak{G}$ denotes the group algebra of $\mathfrak{G}$ over K.

Let K be a field of characteristic p. It is well known that the subgroup $\mathfrak{A}$ of the group $\mathfrak{G}$ is represented trivially on every irreducible $K\mathfrak{G}$-module if and only if $\mathfrak{A}$ lies in $\boldsymbol{O}_p(\mathfrak{G})$. Thus, if $\mathfrak{A}$ is a subgroup of $\mathfrak{G}$ which does not lie in $\boldsymbol{O}_p(\mathfrak{G})$, we may define $r_K(\mathfrak{A};\mathfrak{G})$ to be the smallest integer r such that $\mathfrak{G}$ has an r-dimensional irreducible representation over K which does not represent $\mathfrak{A}$ trivially. In particular, $r_K(\mathfrak{A};\mathfrak{G})$ is defined for all fields of characteristic 0, with the convention that $\boldsymbol{O}_0(\mathfrak{G})=1$. We also set $r_q(\mathfrak{A};\mathfrak{G})=r_{F_q}(\mathfrak{A};\mathfrak{G})$.

DEFINITION 2.7. $\mathcal{S}o\ell(\mathfrak{G})$ is the set of solvable subgroups of $\mathfrak{G}$, and $\mathcal{MS}(\mathfrak{G})$ is the set of maximal elements of $\mathcal{S}o\ell(\mathfrak{G})$ under inclusion. $\mathcal{M}^*(\mathfrak{G})$ is the set of all solvable subgroups of $\mathfrak{G}$ which are contained in precisely one element of $\mathcal{MS}(\mathfrak{G})$, and if $\mathfrak{H}\in\mathcal{M}^*(\mathfrak{G})$, $\boldsymbol{M}(\mathfrak{H})$ is the unique element of $\mathcal{MS}(\mathfrak{G})$ which contains $\mathfrak{H}$.

We define for each group $\mathfrak{G}$, the following sets of primes:

$\pi_1(\mathfrak{G}) = \{p \mid$ A S_p-subgroup of $\mathfrak{G}$ is a nonidentity cyclic group$\}$.

$\pi_2(\mathfrak{G}) = \{p \mid$ (i) A S_p-subgroup $\mathfrak{P}$ of $\mathfrak{G}$ is noncyclic. (ii) $\mathcal{S}\mathfrak{cn}_3(\mathfrak{P}) = \emptyset\}$.

$\pi_3(\mathfrak{G}) = \{p \mid$ (i) If $\mathfrak{P}$ is a S_p-subgroup of $\mathfrak{G}$, then $\mathcal{S}\mathfrak{cn}_3(\mathfrak{P}) \neq \emptyset$. (ii) $\mathcal{N}_{\mathfrak{G}}(\mathfrak{P})$ contains a nonidentity subgroup$\}$.

$\pi_4(\mathfrak{G}) = \{p \mid$ (i) If $\mathfrak{P}$ is a S_p-subgroup of $\mathfrak{G}$, then $\mathcal{S}\mathfrak{cn}_3(\mathfrak{P}) \neq \emptyset$. (ii) $\mathcal{N}_{\mathfrak{G}}(\mathfrak{P})$ contains only $1\}$.

As proved in [5], if p is an odd prime in $\pi_2(\mathfrak{G})$, the structure of the S_p-subgroups of $\mathfrak{G}$ is known. Those 2-groups $\mathfrak{T}$ with $\mathcal{S}\mathfrak{cn}_3(\mathfrak{T}) = \emptyset$ are as yet undetermined, an awkward situation.[4]

If p and q are odd primes, we write $p \sim q$ if and only if $\mathcal{S}\mathfrak{o}\ell(\mathfrak{G})$ contains an element which contains elementary subgroups of order p^3 and q^3, otherwise $p \nsim q$. This definition conforms with 0. We wish to extend the relation in a useful fashion. This is difficult. We need the sets $\mathfrak{U}(p)$ explicitly.

Definition 2.8. Let $p \in \pi(\mathfrak{G})$ and $\mathfrak{P}$ be a S_p-subgroup of $\mathfrak{G}$. If every normal abelian subgroup of $\mathfrak{P}$ is cyclic, then $\mathfrak{U}(\mathfrak{P}) = \emptyset$. If $\boldsymbol{Z}(\mathfrak{P})$ is noncyclic, then $\mathfrak{U}(\mathfrak{P}) = \{\mathfrak{A} \mid \mathfrak{A} \subseteq \boldsymbol{Z}(\mathfrak{P}),\ \mathfrak{A}$ is of type $(p, p)\}$. If $\mathfrak{P}$ contains a noncyclic normal abelian subgroup and $\boldsymbol{Z}(\mathfrak{P})$ is cyclic, then $\mathfrak{U}(\mathfrak{P}) = \{\mathfrak{A} \mid \mathfrak{A} \lhd \mathfrak{P},\ \mathfrak{A}$ is of type $(p, p)\}$. $\mathfrak{U}(p) = \bigcup \mathfrak{U}(\mathfrak{P})$, $\mathfrak{P}$ ranging over all the S_p-subgroups of $\mathfrak{G}$. In case we wish to emphasize the dependence on $\mathfrak{G}$, we write $\mathfrak{U}_{\mathfrak{G}}(p)$ for $\mathfrak{U}(p)$.

Definition 2.9. If p is odd, we set $\mathfrak{J}(p) = \{\mathfrak{A} \mid \mathfrak{A}$ is a p-subgroup of $\mathfrak{G}$ and $\mathfrak{A}$ contains a subgroup $\mathfrak{B}$ of type (p, p) such that for each B in $\mathfrak{B}$, $\boldsymbol{C}_{\mathfrak{G}}(B)$ contains an element of $\mathfrak{U}(p)\}$ $\mathfrak{J}(2) = \{\mathfrak{A} \mid \mathfrak{A}$ is a 2-subgroup of $\mathfrak{G}$ and $\mathfrak{A}$ contains a noncyclic abelian subgroup of order $8\}$.

For a prime q, we write $q \sim 2$ and $2 \sim q$ if and only if there is an element of $\mathcal{S}\mathfrak{o}\ell(\mathfrak{G})$ which contains elements of $\mathfrak{J}(q)$ and $\mathfrak{J}(2)$.

Definition 2.10. $\mathcal{E}(p) = \mathcal{E}_{\mathfrak{G}}(p)$ is the set of subgroups $\mathfrak{E}$ of $\mathfrak{G}$ of type (p, p) which centralize every element in $\mathcal{N}_{\mathfrak{G}}(\mathfrak{E}; p')$.

Let $p \in \pi_3(\mathfrak{G}) \cup \pi_4(\mathfrak{G})$ and let $\mathfrak{P}$ be a S_p-subgroup of $\mathfrak{G}$. The sets $\mathfrak{a}_i(\mathfrak{P})$ are relevant. Here, as in 0,

$\mathfrak{a}_1(\mathfrak{P}) = \{\mathfrak{A} \mid$ (i) $\mathfrak{A}$ is a subgroup of $\mathfrak{P}$. (ii) $\mathfrak{A}$ contains some element of $\mathcal{S}\mathfrak{cn}_3(\mathfrak{P})\}$.

$\mathfrak{a}_{i+1}(\mathfrak{P}) = \{\mathfrak{A} \mid$ (i) $\mathfrak{A}$ is a subgroup of $\mathfrak{P}$. (ii) $\mathfrak{A}$ contains a subgroup $\mathfrak{B}$ of type (p, p) such that $\boldsymbol{C}_{\mathfrak{P}}(B) \in \mathfrak{a}_i(\mathfrak{P})$ for all B in $\mathfrak{B}\}$, $i = 1, 2, 3$.

Let $\mathfrak{a}(\mathfrak{P}) = \mathfrak{a}_4(\mathfrak{P})$ and $\mathfrak{a}_i(p) = \bigcup \mathfrak{a}_i(\mathfrak{P})$, $\mathfrak{a}(p) = \bigcup \mathfrak{a}(\mathfrak{P})$, where in both unions, $\mathfrak{P}$ ranges over all the S_p-subgroups of $\mathfrak{G}$.

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[4] The S_2-subgroups of Janko's simple group of order 604,800 are of this type.

If G, $H \in \mathfrak{G}$, we write $G \sim_{\mathfrak{G}} H$ if and only if G and H are $\mathfrak{G}$-conjugate, and similarly for subsets of $\mathfrak{G}$. If there is no danger of confusion, we write $G \sim H$. The negation of $\sim$ is $\nsim$. We are thus using the symbol $\sim$ in two senses, but since a prime is hardly to be confused with an element of a finite group, no confusion is likely. Following Brauer, if $\mathfrak{H}$ is a subgroup of $\mathfrak{G}$ and G, $H \in \mathfrak{H}$ satisfy $G \nsim_{\mathfrak{H}} H$ and $G \sim_{\mathfrak{G}} H$, we say that G and H are *fused* in $\mathfrak{G}$, or that a *fusion* of G and H occurs in $\mathfrak{G}$.

3. Statement of main theorem and corollaries.

MAIN THEOREM. *Each nonsolvable N-group is isomorphic to a group* $\mathfrak{G}$ *such that* $I(\mathfrak{S}) \subseteq \mathfrak{G} \subseteq \mathrm{Aut}(\mathfrak{S})$, *where* $\mathfrak{S}$ *is one of the following N-groups*:

(a) $L_2(q)$, $q>3$.
(b) $Sz(q)$, $q=2^{2n+1}$, $n \geqq 1$.
(c) $L_3(3)$.
(d) M_{11}.
(e) A_7.
(f) $U_3(3)$.

COROLLARY 1. *Every minimal simple group is isomorphic to one of the following minimal simple groups*:

(a) $L_2(2^p)$, *p any prime.*
(b) $L_2(3^p)$, *p any odd prime.*
(c) $L_2(p)$, *p any prime exceeding* 3 *such that* $p^2+1 \equiv 0 \pmod 5$.
(d) $Sz(2^p)$, *p any odd prime.*
(e) $L_3(3)$.

COROLLARY 2. *A finite group is solvable if and only if every pair of its elements generates a solvable group.*

COROLLARY 3. *A finite group is solvable if and only if it does not contain three nonidentity elements A, B, C of pairwise coprime orders such that* $ABC=1$.

COROLLARY 4. *If* $\mathfrak{G}$ *is a nonsolvable group with* $|\pi(\mathfrak{G})|=3$, *then one of the following groups is involved in* $\mathfrak{G}$: $L_2(4)$, $L_2(7)$, $L_2(8)$, $L_2(17)$, $L_3(3)$.

COROLLARY 5. *If every c.f. of the finite group* $\mathfrak{G}$ *is an N-group and n is a divisor of* $|\mathfrak{G}|$ *such that there are exactly n elements in* $\mathfrak{G}$ *of order dividing n, they form a subgroup.*

COROLLARY 6. *If* $\mathfrak{G}$ *is a nonsolvable group, then* $|\pi(\mathfrak{G})| \geqq 3$.

Corollary 6 of Burnside is well known. The interested reader may extract the relevant results from 0 and the present paper to give a new proof of Corollary 6. The other five corollaries are probably new. The possible existence of Corollary 3 was mentioned in [22]. Corollary 5 is a minuscule contribution to an old problem and sheds no light on it. Finally, we state a characterization theorem for $E_2(3)$ and $S_4(3)$.

THEOREM *ES*. *$E_2(3)$ and $S_4(3)$ are the only simple groups $\mathfrak{G}$ such that*

(i) $2, 3 \in \pi_4(\mathfrak{G})$.

(ii) *If $p \in \{2, 3\}$, $\mathfrak{G}_p$ is a S_p-subgroup of $\mathfrak{G}$ and $\mathfrak{A} \in \mathcal{S}\mathfrak{cn}_3(\mathfrak{G}_p)$, then $\mathsf{N}(\mathfrak{A})$ is trivial.*

(iii) *The normalizer of every nonidentity 3-subgroup of $\mathfrak{G}$ is solvable.*

(iv) *The centralizer of every involution of $\mathfrak{G}$ is solvable.*

(v) $2 \sim 3$, *that is, $\mathfrak{G}$ has a solvable subgroup containing*

(a) *a noncyclic abelian subgroup of order* 8,

(b) *an elementary subgroup of type* (3, 3) *each element of which centralizes a subgroup in* $\mathcal{U}(3)$.

4. **Proofs of corollaries.** It is a consequence of results of Dickson [**12**] that the groups listed in (a), (b), (c), (e) of Corollary 1 are minimal simple groups. Suzuki [**37**] has shown that the groups in (d) are minimal simple groups. By Lemma 5.33, $U_3(3) \supset L_2(7)$, so Corollary 1 follows from the Main Theorem.

Corollary 2 is an almost trivial consequence of Corollary 1. Explicit proofs are available for all the groups listed in Corollary 1 [**34**].

In proving Corollary 3, it suffices to show that for each minimal simple group $\mathfrak{G}$, there are elements A, B, C of $\mathfrak{G}$ of pairwise coprime order with $ABC = 1$. As the character tables of all the minimal simple groups have been determined [**12**], [**37**], Corollary 3 may be easily verified. We remark that if $\mathfrak{G} = Sz(q)$, we may choose A, B, C of orders $q-1$, $q-r+1$, $q+r+1$, where $2q = r^2$.

Corollary 4 is a consequence of elementary number theory and Corollary 1.

In proving Corollary 5 for $\mathfrak{G}$, an appeal to a result of Zemlin [**46**] entitles us to assume that $\mathfrak{G}$ is simple. Rust [**31**] has verified Corollary 5 for $L_2(q)$, $L_3(3)$, A_7 and $Sz(q)$. We omit the discussion of M_{11} and $U_3(3)$, which is not difficult.

5. Preliminary lemmas.

5.1. *Inequalities and modules.*

LEMMA 5.1. *Suppose $\mathfrak{P}$ is a p-group and $\mathfrak{A}$ is a subgroup of $\mathfrak{P}$ of order p. Let $\mathfrak{B}$ be an abelian subgroup of $\mathfrak{P}$ containing $\mathfrak{A}$, and set*

$\mathfrak{D}=\boldsymbol{N}(\mathfrak{B})\cap\boldsymbol{N}(\mathfrak{A})$. *Let* Λ *be the set of linear characters of* $\mathfrak{B}$ *which do not have* $\mathfrak{A}$ *in their kernel and let* $\Lambda_1,\cdots,\Lambda_s$ *be the orbits under the action of* $\mathfrak{D}$. *Then*

(a) $r_C(\mathfrak{A};\mathfrak{P})\geqq\min\{|\Lambda_1|,\cdots,|\Lambda_s|\}$.

(b) $r_C(\mathfrak{A};\mathfrak{P})\geqq|\boldsymbol{A}_{\mathfrak{P}}(\mathcal{C})|$ *where* $\mathcal{C}\colon\mathfrak{B}\supseteq\mathfrak{A}\supset 1$.

PROOF. Since $r_C(\mathfrak{A};\mathfrak{P})\geqq r_C(\mathfrak{A};\mathfrak{D})$, we may assume that $\mathfrak{D}=\mathfrak{P}$. Let M be an irreducible $C\mathfrak{P}$-module on which $\mathfrak{A}$ acts nontrivially, and let X be the character afforded by M. By Clifford's theorem, $X_{\mathfrak{B}}=c\sum\lambda_i$, where $\{\lambda_i\}$ is an orbit of linear characters of $\mathfrak{B}$ and c is a positive integer. Since $\mathfrak{A}\lhd\mathfrak{P}$, $\mathfrak{A}$ is not in the kernel of any λ_i, so $\{\lambda_i\}=\Lambda_j$ for some j. Hence dim $M=c|\Lambda_j|\geqq|\Lambda_j|$, proving (a).

Let $\mathfrak{C}$ be the largest subgroup of $\mathfrak{P}$ which stabilizes $\mathcal{C}$. Then $\boldsymbol{A}_{\mathfrak{P}}(\mathcal{C})=\boldsymbol{A}_{\mathfrak{C}}(\mathcal{C})$, and each Λ_i is a union of orbits under $\mathfrak{C}$. To prove (b), it suffices to show that $\boldsymbol{A}_{\mathfrak{C}}(\mathcal{C})$ acts regularly on Λ. Choose $\lambda\in\Lambda$, $\alpha\in\boldsymbol{A}_{\mathfrak{C}}(\mathcal{C})$, $\alpha\neq 1$. Since $\alpha\neq 1$, there is a B in $\mathfrak{B}$ such that $B^\alpha=BA$, where A is a generator for $\mathfrak{A}$. Hence, $\lambda(B^\alpha)\neq\lambda(B)$, so $\lambda\neq\lambda^\alpha$.

LEMMA 5.2. *Suppose* $\mathfrak{H}=\mathfrak{P}\mathfrak{Q}$ *where* $|\mathfrak{P}|=p$ *is an odd prime and* $\mathfrak{Q}$ *is a normal q-subgroup such that* $\mathfrak{Q}'=\boldsymbol{C}_{\mathfrak{Q}}(\mathfrak{P})$, $q\neq p$. *Suppose also that* $\mathrm{cl}(\mathfrak{Q})\leqq 2$ *and* $\mathfrak{Q}'=\boldsymbol{D}(\mathfrak{Q})$. *Suppose V is an $k\mathfrak{H}$-module, where k is a field of characteristic p and* $\boldsymbol{C}_V(\mathfrak{Q})=0$. *Let* $V_0=\boldsymbol{C}_V(\mathfrak{P})$. *Then* dim $V_0\leqq(\dim V)/2$.

PROOF. We proceed by induction on dim V. Suppose V is not irreducible and that W is a proper submodule. Since V_0+W/W is contained in the centralizer of $\mathfrak{P}$ on V/W, we may apply the lemma to W and V/W to complete the proof. Suppose V is irreducible. If $\mathfrak{Q}'=1$, the lemma is trivial, since in this case, V is a free $k\mathfrak{P}$-module, so suppose $\mathfrak{Q}'\neq 1$.

Let $\mathfrak{Q}_0=\boldsymbol{C}_{\mathfrak{Q}}(V)$. Since $p\neq q$, we get $\mathfrak{Q}'\mathfrak{Q}_0/\mathfrak{Q}_0=\boldsymbol{C}_{\mathfrak{Q}/\mathfrak{Q}_0}(\mathfrak{P})$, so that our hypotheses are satisfied with $\mathfrak{H}/\mathfrak{Q}_0$ in the role of $\mathfrak{H}$. We may therefore assume that $\mathfrak{Q}$ acts faithfully on V. Hence, $\boldsymbol{Z}(\mathfrak{H})$ is cyclic, so in particular, $\mathfrak{Q}'$ is cyclic, so $\mathfrak{Q}'$ is of order q. Since $\mathfrak{Q}'=\boldsymbol{C}_{\mathfrak{Q}}(\mathfrak{P})$, we get $\boldsymbol{Z}(\mathfrak{H})=\mathfrak{Q}'$. Suppose $\boldsymbol{Z}(\mathfrak{Q})\supset\mathfrak{Q}'$. Then $\boldsymbol{Z}(\mathfrak{Q})=\mathfrak{Q}'\times\mathfrak{Q}_1$, where $\mathfrak{Q}_1$ admits $\mathfrak{P}$ and $\boldsymbol{C}_{\mathfrak{Q}_1}(\mathfrak{P})=1$. Since $\mathfrak{Q}_1\subseteq\boldsymbol{Z}(\mathfrak{Q})$, we get $\mathfrak{Q}_1\lhd\mathfrak{H}$, so $\boldsymbol{C}_V(\mathfrak{Q}_1)=0$. Replacing $\mathfrak{Q}$ by $\mathfrak{Q}_1$, we are reduced to a previous case. Hence, we may assume that $\boldsymbol{Z}(\mathfrak{Q})=\mathfrak{Q}'$, so $\mathfrak{Q}$ is extra special.

If $\mathfrak{P}$ acts irreducibly on $\mathfrak{Q}/\mathfrak{Q}'=\overline{\mathfrak{Q}}$, then the proof of (B) implies the desired inequality. Suppose $\mathfrak{P}$ acts reducibly on $\mathfrak{Q}/\mathfrak{Q}'$ and that $\mathfrak{Q}_1/\mathfrak{Q}'$ is an irreducible constituent. If $\mathfrak{Q}_1$ is nonabelian, we are in the preceding situation, so suppose that every irreducible constituent of $\mathfrak{P}$ on $\mathfrak{Q}/\mathfrak{Q}'$ corresponds to an abelian subgroup of $\mathfrak{Q}$. We can then choose $\mathfrak{Q}_2$ so that $\mathfrak{Q}_2/\mathfrak{Q}'$ is an irreducible constituent such that $\mathfrak{Q}_1\mathfrak{Q}_2$ is extra special, and we may assume that $\mathfrak{Q}=\mathfrak{Q}_1\mathfrak{Q}_2$. Choose $Q_i\in\mathfrak{Q}_i-\mathfrak{Q}'$, and set $Q=Q_1Q_2$. If $\langle\mathfrak{P}, \mathfrak{P}^Q\rangle=\mathfrak{H}$, we are done, since $C_V(\mathfrak{Q})=0$. Suppose $\langle\mathfrak{P}, \mathfrak{P}^Q\rangle\subset\mathfrak{H}$ for all such Q. Then the mapping $Q_1\mathfrak{Q}'\rightarrow Q_2\mathfrak{Q}'$ can be extended to an isomorphism σ of $\mathfrak{Q}_1/\mathfrak{Q}'=\overline{\mathfrak{Q}}_1$ to $\mathfrak{Q}_2/\mathfrak{Q}'=\overline{\mathfrak{Q}}_2$ as $\mathfrak{P}$-modules. Let $\mathfrak{Q}(\sigma)$ be the inverse image of $\{QQ^\sigma \mid Q\in\overline{\mathfrak{Q}}_1\}$ in $\mathfrak{Q}$. Thus, $\mathfrak{Q}(\sigma)$ admits $\mathfrak{P}$ and $\mathfrak{Q}(\sigma)/\mathfrak{Q}'\cong\mathfrak{Q}_1/\mathfrak{Q}'$ as $\mathfrak{P}$-modules. Since $\mathfrak{Q}(\sigma)$ is abelian, it follows that for all $Q_i\in Q_i$, $[Q_1Q_2, (Q_1Q_2)^P]=1$, where $\mathfrak{P}=\langle P\rangle$. Hence, $[Q_1, Q_2^P]=[Q_1^P, Q_2]$. Replacing Q_2 by Q_2^P, we get $[Q_1, Q_2^{P^2}]=[Q_1^P, Q_2^P]=[Q_1, Q_2]$, so Q_1 centralizes $Q_2^{1-P^2}$. Since p is odd, we get $\mathfrak{Q}'=1$. This contradiction completes the proof.

REMARK. There is a group of order $3^3\cdot 2$ which shows that the oddness assumption in the previous lemma is necessary.

LEMMA 5.3. *Suppose $\mathfrak{P}$ is a p-group whose Frattini subgroup is elementary and central. For each field F of characteristic $\neq p$ and each $F\mathfrak{P}$-module V on which $\mathfrak{P}$ acts faithfully, the following hold*:

(a) *If $p\neq 2$, then* $\dim V\geq m(\mathfrak{P})\,|F(\zeta): F|$.

(b) *If $p=2$, then* $\dim V\geq 2m(\mathfrak{P})/3$.

Here ζ is a primitive pth root of 1 *in an extension field of F.*

PROOF. We assume without loss of generality that $\mathfrak{P}$ acts faithfully on no proper submodule of V. By complete reducibility, $\mathfrak{P}$ has no fixed points on V. Suppose $V=V_1\oplus V_2$, where V_i is a proper submodule, $i=1, 2$. Let $\mathfrak{P}_1=C_{\mathfrak{P}}(V_1)$. Since $\mathfrak{P}_1$ is faithfully represented on V_2 and $\mathfrak{P}/\mathfrak{P}_1$ is faithfully represented on V_1 and since $m(\mathfrak{P}/\mathfrak{P}_1)+m(\mathfrak{P}_1)\geq m(\mathfrak{P})$, the lemma follows by induction on $|\mathfrak{P}|$ in this case. We may therefore assume that V is irreducible, and so $\mathfrak{P}$ is the central product of a cyclic group of order p^{1+e} and an extra special group of order p^{2r+1}, where $e\leq 1$. Hence, $m(\mathfrak{P})=2r+e$ and $\dim V\geq p^r\,|F(\zeta): F|$. The lemma follows.

REMARK. The central product of a cyclic group of order 4 and a quaternion group shows that 2/3 may not be replaced by any larger value in (b).

LEMMA 5.4. *Let $\mathfrak{P}$ be a p-subgroup of the p-solvable group $\mathfrak{S}$ and let $\mathfrak{A}$ be a subgroup of $\mathfrak{P}$ with $\mathfrak{A}\not\subseteq O_p(\mathfrak{S})$. Then for each field F of characteristic p, $r_F(\mathfrak{A}; \mathfrak{S})\geq r_C(\mathfrak{A}; \mathfrak{P})$.*

PROOF. Let V be an irreducible $F\mathfrak{S}$-module of minimal dimension on which $\mathfrak{A}$ acts nontrivially. We must show that $\dim V\geq r_C(\mathfrak{A}; \mathfrak{P})$.

Let $\mathfrak{H}=C_{\mathfrak{S}}(V)$, $\overline{\mathfrak{S}}=\mathfrak{S}/\mathfrak{H}$. Thus, $\overline{\mathfrak{P}}=\mathfrak{P}\mathfrak{H}/\mathfrak{H}\cong\mathfrak{P}/\mathfrak{P}\cap\mathfrak{H}$ is a S_p-subgroup of $\overline{\mathfrak{S}}$ and $\overline{\mathfrak{A}}=\mathfrak{A}\mathfrak{H}/\mathfrak{H}\neq 1$. Since $r_C(\mathfrak{A}; \mathfrak{P})\leq r_C(\overline{\mathfrak{A}}; \overline{\mathfrak{P}})$, we may assume that $\mathfrak{H}=1$. With this normalization, together with char $F=p$, we have $O_p(\mathfrak{S})=1$, so $\mathfrak{P}$ acts faithfully on $O_{p'}(\mathfrak{S})$. We may therefore choose a $\mathfrak{P}$-admissible special q-subgroup $\mathfrak{Q}$ of $O_{p'}(\mathfrak{S})$ on which $\mathfrak{A}$ acts nontrivially, and such that $\mathfrak{Q}/D(\mathfrak{Q})$ is a chief factor of $\mathfrak{P}\mathfrak{Q}$.

Since $\mathfrak{A}\not\subseteq O_p(\mathfrak{Q}\mathfrak{P})$, it follows that when V is viewed as an $F\mathfrak{Q}\mathfrak{P}$-module, there is a c.f. on which $\mathfrak{A}$ acts nontrivially. Hence, dim $V \geqq r_F(\mathfrak{A};\ \mathfrak{P}\mathfrak{Q})$. This inequality entitles us to assume that $\mathfrak{S}=\mathfrak{P}\mathfrak{Q}$.

Define the positive integer s by $|\mathfrak{Q}: D(\mathfrak{Q})| = q^s$, and let K be the algebraic closure of F_q. Then $s \geqq r_K(\mathfrak{A};\ \mathfrak{P})$, since $\mathfrak{A}$ does not centralize $K\otimes_{F_q}\mathfrak{Q}/D(\mathfrak{Q})$. Since $p\neq q$, it is well known that $r_K(\mathfrak{A};\mathfrak{P})=r_C(\mathfrak{A};\mathfrak{P})$. Thus, $s\geqq r_C(\mathfrak{A};\ \mathfrak{P})$. However, if $q=2$, we get the stronger inequality $s\geqq 2r_C(\mathfrak{A};\ \mathfrak{P})$, as $\mathfrak{Q}/D(\mathfrak{Q})$ is not absolutely irreducible for $\mathfrak{P}$. The lemma now follows from Lemma 5.3 with $\mathfrak{Q}$ in the role of $\mathfrak{P}$.

REMARK. Lemma 5.4 is a typical result for p-solvable groups, for the group $\mathfrak{Q}$ in the lemma appears neither in the hypothesis nor conclusion and plays the role of an intermediary, as in (B).

HYPOTHESIS 5.1. (a) p is an odd prime and $\mathfrak{P}$ is a S_p-subgroup of the group $\mathfrak{G}$.

(b) $\mathfrak{A}$ is a normal elementary subgroup of $\mathfrak{P}$ with $m(\mathfrak{A})\geqq 3$.

(c) $\mathfrak{A}\cap Z(\mathfrak{P})=\mathfrak{Z}$ has order p.

(d) $A_{\mathfrak{P}}(\mathfrak{C})=A(\mathfrak{C})$ where $\mathfrak{C}$: $\mathfrak{A}\supset\mathfrak{Z}\supset 1$.

LEMMA 5.5. *Suppose Hypothesis* 5.1 *is satisfied. Choose A in $\mathfrak{A}-\mathfrak{Z}$ and suppose that $\mathfrak{H}$ is a p-solvable subgroup of $\mathfrak{G}$ which contains $C_{\mathfrak{P}}(A)$. Then $\mathfrak{Z}\subseteq O_{p',p}(\mathfrak{H})$.*

PROOF. Let $|\mathfrak{A}| = p^{w+1}$, so that $w\geqq 2$. Let $\mathfrak{P}^*$ be a S_p-subgroup of $\mathfrak{H}$ which contains $C_{\mathfrak{P}}(A)$. Note that since $\mathfrak{A}\lhd\mathfrak{P}$, it follows that $|\mathfrak{P}: C_{\mathfrak{P}}(A)| \leqq p^w$. Since $\mathfrak{P}$ is a S_p-subgroup of $\mathfrak{G}$, we see that

$$(5.1)\qquad |\mathfrak{P}^*: C_{\mathfrak{P}}(A)| \leqq p^w.$$

Let $W=\mathfrak{K}/\mathfrak{L}$ be a chief factor of $\mathfrak{H}$ with $O_{p'}(\mathfrak{H})\subseteq\mathfrak{L}\subset\mathfrak{K}\subseteq O_{p',p}(\mathfrak{H})$. It suffices to show that $\mathfrak{Z}$ centralizes W. Suppose false. Let $\mathfrak{M}=A_{\mathfrak{H}}(W)$, and let $\mathfrak{P}_0$, $\mathfrak{A}_0$, $\mathfrak{Z}_0$ be the images of $C_{\mathfrak{P}}(A)$, $\mathfrak{A}$, $\mathfrak{Z}$ in $\mathfrak{M}$, respectively. Hence, $\mathfrak{Z}_0\cong\mathfrak{Z}$. Since $O_p(\mathfrak{M})=1$, there is a $\mathfrak{P}_0$-invariant special q-subgroup $\mathfrak{Q}$ of $\mathfrak{M}$ on which $\mathfrak{Z}_0$ is faithfully represented, and such that $\mathfrak{P}_0$ is irreducible on $\mathfrak{Q}/D(\mathfrak{Q})$. Since $\mathfrak{Q}$ is faithfully represented on W, there is an irreducible $F_p\mathfrak{P}_0\mathfrak{Q}$-submodule V of W on which $\mathfrak{Q}$ acts nontrivially.

Let $\mathfrak{M}_1=A_{\mathfrak{P}_0\mathfrak{Q}}(V)$ and let $\mathfrak{P}_1$, $\mathfrak{A}_1$, $\mathfrak{Z}_1$, $\mathfrak{Q}_1$ be the images of $\mathfrak{P}_0$, $\mathfrak{A}_0$, $\mathfrak{Z}_0$, $\mathfrak{Q}$ in $\mathfrak{M}_1$ so that $\mathfrak{Z}_1\cong\mathfrak{Z}_0$. Since $\mathfrak{Z}_0\subseteq Z(\mathfrak{P}_0)$, $\mathfrak{Z}_1$ has no fixed points on $\mathfrak{Q}_1/D(\mathfrak{Q}_1)$. Let $V_0=C_V(\mathfrak{Z}_1)$. By Lemma 5.2, we have

$$(5.2)\qquad \dim V_0\leqq(\dim V)/2.$$

Since $\mathfrak{Q}_1/D(\mathfrak{Q}_1)$ is a $F_q C_{\mathfrak{P}}(A)$-module on which $\mathfrak{Z}$ acts nontrivially, $m(\mathfrak{Q}_1) \geqq r_C(\mathfrak{Z};\ C_{\mathfrak{P}}(A))$. If $q=2$, we get the stronger inequality $m(\mathfrak{Q}_1) \geqq 2r_C(\mathfrak{Z};\ C_{\mathfrak{P}}(A))$, since $\mathfrak{Q}_1/D(\mathfrak{Q}_1)$ is not absolutely irreducible in this case. By Lemma 5.1, $r_C(\mathfrak{Z};\ C_{\mathfrak{P}}(A)) \geqq |A_{C_{\mathfrak{P}}(A)}(\mathfrak{C})|$. By Hypothesis 5.1(d), it follows that $|A_{C_{\mathfrak{P}}(A)}(\mathfrak{C})| = p^{w-1}$. Hence,

$$m(\mathfrak{Q}_1) \geqq bp^{w-1}, \tag{5.3}$$

where $b=1$ or 2 according as $q\neq 2$ or $q=2$.

On the other hand, by Lemma 5.3 we have

$$\dim V \geqq am(\mathfrak{Q}_1), \tag{5.4}$$

where $a=1$ or $2/3$ according as $q\neq 2$ or $q=2$.

Now V is a submodule of W, so $V=\mathfrak{K}_0/\mathfrak{L}$ for some subgroup $\mathfrak{K}_0$ of $\mathfrak{K}$. Since $\mathfrak{K}_0 \lhd \lhd \mathfrak{H}$, $\mathfrak{P}^* \cap \mathfrak{K}_0$ is a S_p-subgroup of $\mathfrak{K}_0$, so $\mathfrak{K}_0 = \mathfrak{L}(\mathfrak{P}^*\cap\mathfrak{K}_0)$. Let $\mathfrak{K}_1=\mathfrak{L}(C_{\mathfrak{P}}(A)\cap\mathfrak{K}_0)$, and let $V_1=\mathfrak{K}_1/\mathfrak{L}$. Since $C_{\mathfrak{P}}(A)$ centralizes $\mathfrak{Z}$, we have

$$V_1 \subseteq V_0. \tag{5.5}$$

Since $|\mathfrak{P}^*: C_{\mathfrak{P}}(A)| \leqq p^w$, so also $|\mathfrak{P}^*\cap\mathfrak{K}_0: C_{\mathfrak{P}}(A)\cap\mathfrak{K}_0| \leqq p^w$. Hence,

$$|\mathfrak{K}_0 : \mathfrak{K}_1| \leqq p^w. \tag{5.6}$$

Since $\mathfrak{K}_0/\mathfrak{K}_1 \cong \mathfrak{K}_0/\mathfrak{L}/\mathfrak{K}_1/\mathfrak{L}$, we have

$$\dim(V/V_1) \leqq w. \tag{5.7}$$

Now (5.2), (5.5), and (5.7) yield that $\dim V \leqq w + \dim V_1 \leqq w + \dim V_0 \leqq w + (\dim V)/2$, or $2w \geqq \dim V$. With (5.3) and (5.4), we find that $2w \geqq abp^{w-1}$.

Since p is odd, and $ab \geqq 1$, $w \geqq 2$, we see that $p=3$, $w=2$. Thus, $\mathfrak{M}_1 \subseteq GL(4, 3)$. This forces $q=2$ so that (5.3) yields $m(\mathfrak{Q}_1) \geqq 6$. On the other hand, $GL(4, 3)$ has S_2-subgroups of order 2^9 and of the shape $T \wr Z_2$ where T is a S_2-subgroup of $GL(2, 3)$. Thus, S_2-subgroups of $GL(4, 3)$ have a subgroup of index 2, every subgroup of which is generated by four elements. So $m(\mathfrak{Q}_1) \leqq 5$. This contradiction completes the proof.

LEMMA 5.6. *Suppose the following hold*:

(a) $\mathfrak{S}=\mathfrak{S}_2\mathfrak{S}_3$ *where* $\mathfrak{S}_p$ *is a* S_p-*subgroup of* $\mathfrak{S}$, $p=2, 3$.

(b) $\mathfrak{S}_2$ *is extra special and normal in* $\mathfrak{S}$.

(c) $\mathfrak{S}_3$ *is abelian*, $\mathfrak{S}_3 \neq 1$.

(d) $\mathfrak{S}$ *is faithfully and irreducibly represented as automorphisms of the elementary* 3-*group* $\mathfrak{E}$.

(e) *For each subgroup* $\mathfrak{A}$ *of* $\mathfrak{S}_3$, *define* $3^{a(\mathfrak{A})} = |C_{\mathfrak{E}}(\mathfrak{A}):\ C_{\mathfrak{E}}(\mathfrak{S}_3)|$.

(f) $a(\mathfrak{A}) \leqq m(\mathfrak{S}_3/\mathfrak{A})$ *for all subgroups* $\mathfrak{A}$ *of* $\mathfrak{S}_3$.
Then $\mathfrak{S}_2$ *is a quaternion group.*

PROOF. Let $|\mathfrak{S}_2| = 2^{2n+1}$. Hence, $m(\mathfrak{E}) \geqq 2^n$. Let $3^f = |\boldsymbol{C}_{\mathfrak{E}}(\mathfrak{S}_3)|$. Then $m(\mathfrak{E}) - f = a(\langle 1 \rangle)$, so $m(\mathfrak{E}) - f \leqq m(\mathfrak{S}_3)$, by (f). On the other hand, since $\mathfrak{S}$ is irreducible and faithful on $\mathfrak{E}$, we have $\boldsymbol{O}_3(\mathfrak{S}) = 1$, so $\mathfrak{S}_3$ is faithfully represented on $\mathfrak{S}_2/\mathfrak{S}_2'$. Hence, $m(\mathfrak{S}_3) \leqq n$, so that $m(\mathfrak{E}) - f \leqq n$. If $f > m(\mathfrak{E})/2$, then $\langle \mathfrak{S}_3, \mathfrak{S}_3^S \rangle$ centralizes a nonidentity element of E for each S in $\mathfrak{S}$. Since $\mathfrak{S}_3 \not\triangleright \mathfrak{S}$, we can choose S in $\mathfrak{S}$ so that $\mathfrak{S}_2' \subseteq \langle \mathfrak{S}_3, \mathfrak{S}_3^S \rangle$. This is impossible, since $\mathfrak{S}_2'$ inverts $\mathfrak{E}$. Hence, $f \leqq m(\mathfrak{E})/2$, so $m(\mathfrak{E})/2 \leqq m(\mathfrak{E}) - f \leqq n$, and so $2^{n-1} \leqq n$. This implies that $n \leqq 2$. If $n = 1$, we are done by (c), so suppose $n = 2$. In this case, we get $2 \leqq m(\mathfrak{S}_3)$, so $\mathfrak{S}_3$ is elementary of order 9. Also, $\mathfrak{S}_2$ is the central product of two quaternion groups $\mathfrak{Q}_0$, $\mathfrak{Q}_1$, and if $\mathfrak{B}_i = \boldsymbol{C}_{\mathfrak{S}_3}(\mathfrak{Q}_i)$, then $|\mathfrak{B}_i| = 3$, $i = 0, 1$, and $\mathfrak{S}_3 = \mathfrak{B}_0 \times \mathfrak{B}_1$. Thus $\mathfrak{Q}_0$ is faithfully represented on $\boldsymbol{C}_{\mathfrak{E}}(\mathfrak{B}_0)$, so $|\boldsymbol{C}_{\mathfrak{E}}(\mathfrak{B}_0)| = 9$. Hence, $|\boldsymbol{C}_{\mathfrak{E}}(\mathfrak{S}_3)| = 3$. This means that $a(\langle 1 \rangle) \geqq 3$, against (f). The proof is complete.

LEMMA 5.7. *Suppose* $\mathfrak{S}$ *is a* p*-reduced* p*-solvable group and* k *is a field of characteristic* p. *Suppose* M *is a* $k\mathfrak{S}$*-module on which* $\mathfrak{S}$ *acts faithfully, but that* $\mathfrak{S}$ *acts faithfully on no proper submodule of* M. *Then* M *is completely reducible.*

PROOF. Let N be a maximal submodule of M, and let $\mathfrak{S}_0 = \boldsymbol{C}_{\mathfrak{S}}(N)$. By hypothesis, $\mathfrak{S}_0 \neq 1$. Since $\boldsymbol{O}_p(\mathfrak{S}) = 1$, so also $\boldsymbol{O}_p(\mathfrak{S}_0) = 1$. Let $\mathfrak{H} = \boldsymbol{O}_{p'}(\mathfrak{S}_0)$, $\sigma = \sigma(\mathfrak{H}) = |\mathfrak{H}|^{-1} \sum_{H \in \mathfrak{H}} H$. Since N is a maximal submodule of M, we get $N = M\sigma$. Hence, $M(1-\sigma)$ is an irreducible submodule of M isomorphic to M/N. Thus, every maximal submodule of M is complemented. As M is obviously finitely generated, we may write $M = M_1 \oplus \cdots \oplus M_r \oplus M'$, where $M_1, \cdots, M_r$ are irreducible, and M' is either 0 or has no irreducible summands. If $M' \neq 0$, let N' be a maximal submodule of M'. Then $M_1 \oplus \cdots \oplus M_r \oplus N'$ is a maximal submodule of M, so has a complement M_{r+1}, giving $M = M_1 \oplus \cdots \oplus M_{r+1} \oplus N'$. Hence, $M' \cong M_{r+1} \oplus N'$, against the construction. The proof is complete.

LEMMA 5.8. *Suppose* $\mathfrak{S} = \mathfrak{S}_1 \times \cdots \times \mathfrak{S}_a$, $a \geqq 1$, *where* $\mathfrak{S}_i$ *is a dihedral group of order* $2p_i$, p_i *an odd prime. Suppose also that* M *is an* $F_2\mathfrak{S}$*-module on which* $\mathfrak{S}$ *acts faithfully and* $|M| = 2^m$ *with* $m \leqq 2a$. *Then the following hold:*

(a) $m = 2a$.

(b) M *is completely reducible.*

(c) $M = M_1 \oplus \cdots \oplus M_s$, $\mathfrak{S} = \mathfrak{S}(1) \times \cdots \times \mathfrak{S}(s)$, $\mathfrak{S}(i)$ *centralizes* M_j, $j \neq i$ *and* $S(i)$ *acts faithfully and irreducibly on* M_i, $1 \leqq i \leqq s$.

(d) $|\mathfrak{S}(i)| = 6$ *or* 36 *for each* i.

PROOF. Let N be a submodule of M which is minimal subject to $C_{\mathfrak{S}}(N)=1$. Let $|N|=2^n$. By Lemma 5.7, $N=N_1\oplus\cdots\oplus N_s$, where each N_i is irreducible. For each subset J of $\{1,\cdots,a\}$, let $\mathfrak{S}_J=\langle\mathfrak{S}_j|j\in J\rangle$. Let $\mathfrak{S}^i=C_{\mathfrak{S}}(N_i)$. Since N_i is irreducible, $O_2(\mathfrak{S}/\mathfrak{S}_i)=1$. Hence, $\mathfrak{S}^i=\mathfrak{S}_{J(i)}$ for some subset $J(i)$ of $\{1,\cdots,a\}$, so that $\mathfrak{S}/\mathfrak{S}^i\cong\mathfrak{S}_{J(i)'}$, where $J(i)'=\{1,\cdots,a\}-J(i)$.

Let $\mathfrak{X}$ be the direct product of all the $\mathfrak{S}/\mathfrak{S}^i$. If $X=(X_1,\cdots,X_s)\in\mathfrak{X}$, and $u=u_1+\cdots+u_s$, $u_i\in N_i$, let $uX=u_1X_1+\cdots+u_sX_s$, thereby converting N to a $F_2\mathfrak{X}$-module. Also, $\mathfrak{S}$ is isomorphic to a subgroup of $\mathfrak{X}$, and $\mathfrak{X}=\mathfrak{X}_1\times\cdots\times\mathfrak{X}_b$, where $\mathfrak{X}_i$ is a dihedral group of order $2q_i$, q_i an odd prime, $1\leqq i\leqq b$, $b\geqq a$. Let $\mathfrak{X}^i=(1,\cdots,\mathfrak{S}/\mathfrak{S}^i,\cdots,1)$. By construction, $\mathfrak{X}^i$ centralizes N_j for $i\neq j$ and $\mathfrak{X}^i$ acts faithfully and irreducibly on N_i.

Let $|N_i|=2^{n_i}$, $|\mathfrak{X}^i|_2=2^{a_i}$. Since N_i is a projective $F_2\mathfrak{X}^i$-module, we get $n_i\equiv 0 \pmod{2^{a_i}}$. Also, $a_i\neq 0$, by minimality of N. Hence, $2a\geqq m\geqq n=\sum_{i=1}^s n_i\geqq\sum_{i=1}^s 2^{a_i}\geqq 2\sum_{i=1}^s a_i=2b\geqq 2a$. Thus equality holds throughout. We unravel what this means. First, $N=M$ is completely reducible, and $m=2a$, so that (a) and (b) hold. Next $\mathfrak{S}$ and $\mathfrak{X}$ are isomorphic by an isomorphism which respects the action of $\mathfrak{S}$, $\mathfrak{X}$ on M, that is, $\theta:\mathfrak{X}\to\mathfrak{S}$, and $uX=u(\theta(X))$ for all $X\in\mathfrak{X}$, $u\in M$. Let $\mathfrak{S}(i)=\theta(\mathfrak{X}^i)$, so that $\mathfrak{S}(i)\lhd\mathfrak{S}$, $\mathfrak{S}(i)\cap\mathfrak{S}^i=1$. Hence $\mathfrak{S}=\mathfrak{S}^i\times\mathfrak{S}(i)$, so that $\mathfrak{S}(i)=\mathfrak{S}_{J(i)'}$. Hence, $\mathfrak{S}=\mathfrak{S}_{J(1)'}\times\cdots\times\mathfrak{S}_{J(s)'}$.

Set $M_i=N_i$, so that (c) holds. Since $2^{a_i}=2a_i$, we get $a_i=1$ or 2. If $a_i=1$, then $|\mathfrak{S}(i)|=6$, since $n_i=2$. Suppose $a_i=2$. Then $|\mathfrak{S}(i)|=36$ or 60, since $n_i=4$. If $|\mathfrak{S}(i)|=60$, then $\mathfrak{S}(i)'$ is of order 15 with generator S_i. Since $a_i=2$, some element of $\mathfrak{S}(i)$ inverts S_i. But the characteristic roots of S_i on N_i are λ, λ^2, λ^4, λ^8 for some primitive 15th root of 1 in an extension field of F_2, so S_i is nonreal. Thus, (d) holds.

5.2. *π-reducibility and $R_\pi(\mathfrak{G})$.*

The next lemma explores some easy consequences of Definitions 2.1 and 2.2.

LEMMA 5.9. (i) *$R_\pi(\mathfrak{G})$ is π-reducible for all π, $\mathfrak{G}$.*

(ii) *$R_\pi(\mathfrak{G})$ is abelian for all π, $\mathfrak{G}$.*

(iii) *For each prime p and group $\mathfrak{G}$, $C(R_p(\mathfrak{G}))=C(\Omega_1(R_p(\mathfrak{G})))$.*

(iv) *For each prime p and group $\mathfrak{G}$, $R_p(\mathfrak{G})\subseteq Z(O_p(\mathfrak{G}))$.*

(v) *For each prime p and group $\mathfrak{G}$, define $\mathfrak{Z}_n$, $\mathfrak{C}_n$, $\mathfrak{D}_n$ recursively, as follows: $\mathfrak{Z}_1=Z(O_p(\mathfrak{G}))$, $\mathfrak{C}_1=C(\mathfrak{Z}_1)$, $\mathfrak{D}_1=O_p(\mathfrak{G} \bmod \mathfrak{C}_1)$, $\mathfrak{Z}_{n+1}=\mathfrak{Z}_n\cap C(\mathfrak{D}_n)$, $\mathfrak{C}_{n+1}=C(\mathfrak{Z}_{n+1})$, $\mathfrak{D}_{n+1}=O_p(\mathfrak{G} \bmod \mathfrak{C}_{n+1})$, $n=1,\cdots$. Then $\mathfrak{Z}_1\supseteq\mathfrak{Z}_2\supseteq\cdots$, while $\mathfrak{C}_1\subseteq\mathfrak{D}_1\subseteq\mathfrak{C}_2\subseteq\mathfrak{D}_2\subseteq\cdots$. Also, $\mathfrak{Z}_n=R_p(\mathfrak{G})$ for suitably large n.*

(vi) *If $O_p(\mathfrak{G})\neq 1$, then $R_p(\mathfrak{G})\neq 1$.*

Proof. (i) It suffices to show that if $\mathfrak{A}_1$, $\mathfrak{A}_2$ are normal π-reducible π-subgroups of $\mathfrak{G}$, then $\mathfrak{A}_1\mathfrak{A}_2$ is π-reducible. Let $\mathfrak{C}_i = \boldsymbol{C}(\mathfrak{A}_i)$, $\mathfrak{C} = \boldsymbol{C}(\mathfrak{A}_1\mathfrak{A}_2)$. If $\mathfrak{D}/\mathfrak{C}$ is a normal π-subgroup of $\mathfrak{G}/\mathfrak{C}$, then $\mathfrak{D}\mathfrak{C}_i/\mathfrak{C}_i \cong \mathfrak{D}/\mathfrak{D}\cap\mathfrak{C}_i \cong \mathfrak{D}/\mathfrak{C}/\mathfrak{D}\cap\mathfrak{C}_i/\mathfrak{C}$ is a normal π-subgroup of $\mathfrak{G}/\mathfrak{C}_i$, so $\mathfrak{D}\subseteq\mathfrak{C}_i$, $i=1, 2$. Since $\mathfrak{C} = \mathfrak{C}_1\cap\mathfrak{C}_2$, we have $\mathfrak{D}/\mathfrak{C}=1$, as required.

(ii) By (i), $\mathfrak{R} = \boldsymbol{R}_\pi(\mathfrak{G})$ is π-reducible, so $\mathfrak{G}/\boldsymbol{C}(\mathfrak{R})$ has no nonidentity normal π-subgroups. Since $\mathfrak{R}\boldsymbol{C}(\mathfrak{R})/\boldsymbol{C}(\mathfrak{R}) \cong \mathfrak{R}/\boldsymbol{Z}(\mathfrak{R})$ is a normal π-subgroup of $\mathfrak{G}/\boldsymbol{C}(\mathfrak{R})$, we have $\mathfrak{R} = \boldsymbol{Z}(\mathfrak{R})$, as required.

(iii) Let $\mathfrak{C}_1 = \boldsymbol{C}(\boldsymbol{R}_p(\mathfrak{G}))$, $\mathfrak{C}_2 = \boldsymbol{C}(\Omega_1(\boldsymbol{R}_p(\mathfrak{G})))$. Clearly, $\mathfrak{C}_1\subseteq\mathfrak{C}_2$. By a well-known property of abelian groups, $\mathfrak{C}_2/\mathfrak{C}_1$ is a p-group, so $\mathfrak{C}_2/\mathfrak{C}_1 = 1$, since $\mathfrak{G}/\mathfrak{C}_1$ is p-reduced.

(iv) Clearly, $\boldsymbol{R}_p(\mathfrak{G})\subseteq \boldsymbol{O}_p(\mathfrak{G})$. Also, $\boldsymbol{C}(\boldsymbol{R}_p(\mathfrak{G}))\boldsymbol{O}_p(\mathfrak{G})/\boldsymbol{C}(\boldsymbol{R}_p(\mathfrak{G})) \cong \boldsymbol{O}_p(\mathfrak{G})/\boldsymbol{O}_p(\mathfrak{G})\cap \boldsymbol{C}(\boldsymbol{R}_p(\mathfrak{G}))$ is a normal p-subgroup of $\mathfrak{G}/\boldsymbol{C}(\boldsymbol{R}_p(\mathfrak{G}))$.

(v) It is obvious that $\mathfrak{Z}_1\supseteq\mathfrak{Z}_2\supseteq \cdots$, and that $\mathfrak{C}_n\subseteq\mathfrak{D}_n$. Since $\mathfrak{D}_n$ centralizes $\mathfrak{Z}_{n+1}$, we also have $\mathfrak{D}_n\subseteq\mathfrak{C}_{n+1}$. Suppose $\mathfrak{Z}_n = \mathfrak{Z}_{n+1}$ for some n. This means that $\mathfrak{D}_n$ centralizes $\mathfrak{Z}_n$, that is, $\mathfrak{D}_n\subseteq\mathfrak{C}_n$, so $\mathfrak{D}_n = \mathfrak{C}_n$ and $\mathfrak{Z}_n$ is p-reducible. Furthermore, since $\mathfrak{Z}_n = \mathfrak{Z}_{n+1}$, we have $\mathfrak{C}_n = \mathfrak{C}_{n+1}$ so that $\mathfrak{D}_{n+1} = \mathfrak{D}_n = \mathfrak{C}_n$, which means that $\mathfrak{Z}_{n+1} = \mathfrak{Z}_{n+2}$. Let $\mathfrak{Z} = \mathfrak{Z}_n = \mathfrak{Z}_{n+1}$. Then $\mathfrak{Z}\subseteq \boldsymbol{R}_p(\mathfrak{G})$, since $\mathfrak{Z}$ is a normal p-subgroup of $\mathfrak{G}$. On the other hand, $\mathfrak{Z}_1 \supseteq \boldsymbol{R}_p(\mathfrak{G})$, by (iv) and if $\mathfrak{Z}_r \supseteq \boldsymbol{R}_p(\mathfrak{G})$ for some r, then $\mathfrak{C}_r\subseteq \boldsymbol{C}(\boldsymbol{R}_p(\mathfrak{G})) = \mathfrak{C}$, say. Since $\mathfrak{D}_r/\mathfrak{C}_r$ is a p-group, so is $\mathfrak{D}_r\mathfrak{C}/\mathfrak{C}$ since $\mathfrak{D}_r\mathfrak{C}/\mathfrak{C} \cong \mathfrak{D}_r/\mathfrak{D}_r\cap\mathfrak{C} \cong \mathfrak{D}_r/\mathfrak{C}_r/\mathfrak{D}_r\cap\mathfrak{C}/\mathfrak{C}_r$, so $\mathfrak{D}_r\subseteq\mathfrak{C}$, which implies that $\mathfrak{Z}_{r+1}\supseteq \boldsymbol{R}_p(\mathfrak{G})$. Hence, $\mathfrak{Z}_r\supseteq \boldsymbol{R}_p(\mathfrak{G})$ for all r. Taking $r=n$, we conclude that $\mathfrak{Z}\supseteq \boldsymbol{R}_p(\mathfrak{G})$. As the reverse containment also holds, (v) is proved.

(vi) This is an immediate consequence of (v), since the center of a nonidentity p-group is $\neq 1$.

Lemma 5.10. *If $\mathfrak{G}$ is p-solvable and p'-reduced for some prime p and $\mathfrak{A}$ is a subgroup of $\mathfrak{G}$ which is contained in the center of some S_p-subgroup of $\mathfrak{G}$, then the normal closure $\mathfrak{N}$ of $\mathfrak{A}$ in $\mathfrak{G}$ is p-reducible.*

Proof. In any case, $\mathfrak{N}\subseteq \boldsymbol{Z}(\boldsymbol{O}_p(\mathfrak{G}))$. Let $\mathfrak{P}$ be a S_p-subgroup of $\mathfrak{G}$ with $\mathfrak{A}\subseteq \boldsymbol{Z}(\mathfrak{P})$. Let $\mathfrak{D} = \boldsymbol{O}_p(\mathfrak{G} \bmod \boldsymbol{C}(\mathfrak{N}))$, so that $\mathfrak{D} = (\mathfrak{D}\cap\mathfrak{P})\boldsymbol{C}(\mathfrak{N})$. The given factorization of $\mathfrak{D}$ shows that $\mathfrak{D}$ centralizes $\mathfrak{A}$, so $\mathfrak{D}^G = \mathfrak{D}$ centralizes $\mathfrak{A}^G$ for all G in $\mathfrak{G}$. Hence, $\mathfrak{D}/\boldsymbol{C}(\mathfrak{N}) = 1$, as required.

Lemma 5.11. *Suppose $\mathfrak{S}$ is a p-solvable group and $\mathfrak{A}$ is an elementary p-subgroup of $\mathfrak{S}$. If $\boldsymbol{C}(\mathfrak{A}) = \boldsymbol{C}(\mathfrak{A}_0)$ for every subgroup $\mathfrak{A}_0$ of index p in $\mathfrak{A}$, then $\mathfrak{A}$ centralizes $\boldsymbol{R}_p(\mathfrak{S})$.*

Proof. Let $\mathfrak{C} = \boldsymbol{C}(\boldsymbol{R}_p(\mathfrak{S}))$. Suppose the lemma is false. Let $\mathfrak{D} = \boldsymbol{O}_{p'}(\mathfrak{S} \bmod \mathfrak{C})$ so that $\mathfrak{A}$ does not centralize $\mathfrak{D}/\mathfrak{C}$. Let $\mathfrak{D}_0$ be chosen of least order subject to (a) $\mathfrak{C}\subset\mathfrak{D}_0\subseteq\mathfrak{D}$, (b) $\mathfrak{D}_0$ admits $\mathfrak{A}$, (c) $[\mathfrak{D}_0, \mathfrak{A}]\not\subseteq\mathfrak{C}$. Then $\mathfrak{D}_0/\mathfrak{D}$ is a q-group for some prime $q\neq p$, and

$\mathfrak{A}$ acts irreducibly and nontrivially on $\mathfrak{D}_0/\mathfrak{D}_1$ where $\mathfrak{D}_1 = \boldsymbol{D}(\mathfrak{D}_0 \bmod \mathfrak{C})$. Let $\mathfrak{A}_0 = \boldsymbol{C}_{\mathfrak{A}}(\mathfrak{D}_0/\mathfrak{C}) = \boldsymbol{C}_{\mathfrak{A}}(\mathfrak{D}_0/\mathfrak{D}_1)$, so that $|\mathfrak{A}:\mathfrak{A}_0| = p$. Let $\mathfrak{S}_0 = \mathfrak{A}_0\mathfrak{D}_0/\mathfrak{C} = \mathfrak{A}_0\mathfrak{C}/\mathfrak{C} \times \mathfrak{D}_0/\mathfrak{C}$. Since $\mathfrak{D}_0/\mathfrak{C}$ is represented faithfully on $\boldsymbol{R}_p(\mathfrak{S})$, Lemma 3.7 of [20] implies that $\mathfrak{D}_0/\mathfrak{C}$ is represented faithfully on $\boldsymbol{R}_p(\mathfrak{S}) \cap \boldsymbol{C}(\mathfrak{A}_0) = \mathfrak{B}$, say. By hypothesis, $[\mathfrak{A}, \mathfrak{B}] = 1$. Let $\mathfrak{L} = \mathfrak{C}\mathfrak{A}\mathfrak{D}_0$. Then $\mathfrak{C}\mathfrak{A} \subseteq \boldsymbol{C}_{\mathfrak{L}}(\mathfrak{B}) \lhd \mathfrak{L}$. By the minimality of $\mathfrak{D}_0$, we also get $\mathfrak{D}_0 \subseteq [\mathfrak{D}_0, \mathfrak{A}]\mathfrak{C} \subseteq \boldsymbol{C}(\mathfrak{B})$, so $\mathfrak{L} \subseteq \boldsymbol{C}(\mathfrak{B})$. This contradiction completes the proof.

5.3. *Groups of symplectic type.*

LEMMA 5.12. *If $\mathfrak{T}$ if a 2-group of symplectic type and width w, and if $\mathfrak{T}$ is not extra special, then $\mathfrak{T}$ contains a characteristic subgroup $\mathfrak{T}_0$ such that $\mathfrak{T}_0$ is the central product of a cyclic group of order 4 and an extra special group of width w. If $\mathfrak{A}$ is of odd order >1 and $\mathfrak{A}$ acts faithfully on $\mathfrak{T}$, then $[\mathfrak{T}, \mathfrak{A}]$ is extra special of width $\leqq w$.*

PROOF. Let $\mathfrak{T} = \mathfrak{T}_1\mathfrak{T}_2$, where $[\mathfrak{T}_1, \mathfrak{T}_2] = 1$, $\mathfrak{T}_1$ is extra special of width w and $\mathfrak{T}_2$ is either cyclic or is of maximal class and order >8. First, suppose $\mathfrak{T}_2$ is cyclic. In this case, $\mathfrak{T}_0 = \Omega_2(\mathfrak{T})$ satisfies the demands of the lemma. Suppose $\mathfrak{T}_2$ is of maximal class. Let $\mathfrak{T}_3 = \boldsymbol{D}(\mathfrak{T}) = \boldsymbol{D}(\mathfrak{T}_2)$, $\mathfrak{T}_4 = \boldsymbol{C}(\mathfrak{T}_3)$, $\mathfrak{T}_0 = \Omega_2(\mathfrak{T}_4)$. Since $\mathfrak{T}_4$ is the central product of $\mathfrak{T}_1$ and a cyclic group, again $\mathfrak{T}_0$ satisfies the demands of the lemma.

We next show that $\mathfrak{A}$ centralizes $\mathfrak{T}/\mathfrak{T}_0$. This is clear if $\mathfrak{T}_2$ is cyclic, since $\mathfrak{T}/\mathfrak{T}_0 \cong \mathfrak{T}_2/\mathfrak{T}_2 \cap \mathfrak{T}_0$ is cyclic in this case. We may assume that $\mathfrak{T}_2$ is of maximal class. Let $\mathfrak{T}_3 = \boldsymbol{C}(\boldsymbol{Z}(\mathfrak{T}_0))$. Then $|\mathfrak{T}:\mathfrak{T}_3| = 2$, and $\mathfrak{T}_3/\mathfrak{T}_0$ is cyclic. Thus, $\mathfrak{A}$ centralizes $\mathfrak{T}/\mathfrak{T}_3$ and $\mathfrak{T}_3/\mathfrak{T}_0$, so $\mathfrak{A}$ centralizes $\mathfrak{T}/\mathfrak{T}_0$, as $|\mathfrak{A}|$ is odd.

Let $\mathfrak{U}_0 = [\mathfrak{T}, \mathfrak{A}]$. As we have just shown, $\mathfrak{U}_0 \subseteq \mathfrak{T}_0$, so $\mathfrak{U}_0 = [\mathfrak{T}_0, \mathfrak{A}]$. Since $\boldsymbol{Z}(\mathfrak{T}_0)/\mathfrak{T}_0'$ is of order 2 in the elementary group $\mathfrak{T}_0/\mathfrak{T}_0'$, there is a subgroup $\mathfrak{U}$ of $\mathfrak{T}_0$ which contains $\mathfrak{T}_0'$, admits $\mathfrak{A}$, and such that $\mathfrak{T}_0/\mathfrak{T}_0' = \boldsymbol{Z}(\mathfrak{T}_0)/\mathfrak{T}_0' \times \mathfrak{U}/\mathfrak{T}_0'$. Hence, $[\mathfrak{T}_0, \mathfrak{A}] = [\mathfrak{U}, \mathfrak{A}]$, and $\mathfrak{U}$ is extra special. Since $|\mathfrak{A}| > 1$ and $\mathfrak{A}$ acts faithfully on $\mathfrak{T}$, $\mathfrak{A}$ acts faithfully on $\mathfrak{U}$, and on $\mathfrak{U}_0 = [\mathfrak{U}, \mathfrak{A}]$. It is straightforward to check that $\mathfrak{U}_0$ is extra special.

LEMMA 5.13. *Let $\mathfrak{S}$ be a 2-reduced solvable group, let $\mathfrak{H} = \boldsymbol{O}_2(\mathfrak{S})$ and assume that $\mathfrak{H}$ is of symplectic type and width $n > 1$. Assume also that $\mathfrak{A}$ is an extra special 2-subgroup of S of width $n-1$ and that $\mathfrak{H} \cap \mathfrak{A} = 1$. Then $n = 2$.*

PROOF. Since $\mathfrak{A}$ is faithfully represented on $\boldsymbol{Q}_2^1(\mathfrak{S})$ and since $\mathfrak{A}$ has a unique minimal normal subgroup, there is a q-subgroup $\mathfrak{Q}$ of $\boldsymbol{Q}_2^1(\mathfrak{S})$ on which $\mathfrak{A}$ is faithfully represented. Since the absolutely irreducible faithful representations of $\mathfrak{A}$ are of degree 2^{n-1}, we have $m(\mathfrak{Q}) \geqq 2^{n-1}$.

On the other hand, $\mathfrak{Q} = \mathfrak{H}\tilde{\mathfrak{Q}}/\mathfrak{H}$ for some q-subgroup $\tilde{\mathfrak{Q}}$ of $\mathfrak{S}$ with $\tilde{\mathfrak{Q}} \cong \mathfrak{Q}$. Let $\tilde{\mathfrak{H}} = [\mathfrak{H}, \tilde{\mathfrak{Q}}]$. By Lemma 5.12, $\tilde{\mathfrak{H}}$ is extra special of width $\tilde{n} \leqq n$, so $|\mathfrak{Q}|$ divides $|S_{2n}(2)|_{2'} = (2^2-1)(2^4-1) \cdots (2^{2n}-1)$. Let e be the smallest positive even integer with $2^e \equiv 1 \pmod{q}$, and let $2^e - 1 = q^f q'$, $(q, q') = 1$. Hence, $2^{n-1} \leqq f[2n/e] + [2n/q] + [2n/q^2] + \cdots$. It is clear that $f \leqq e/2$, and so $2^{n-1} < n + 2n/(q-1) \leqq 2n$, which implies $n \leqq 3$. Suppose $n = 3$. The S_3-subgroups of $S_6(2)$ are $Z_3 \wr Z_3$, so that $m(\mathfrak{Q}) \leqq 3$. This violates $2^2 \leqq m(\mathfrak{Q})$. Hence, $n = 2$.

LEMMA 5.14. *Suppose $\mathfrak{S}$ is a $2'$-reduced solvable group and the following hold:*

(a) *$O_2(\mathfrak{S}) = \mathfrak{H}$ is the central product of a cyclic group and an extra special group of width w.*

(b) *If $\mathfrak{S}_2$ is a S_2-subgroup of $\mathfrak{S}$, then $\mathfrak{S}_2/\mathfrak{H}$ is elementary of order 2^w.*

(c) *$O_2^1(\mathfrak{S}) = \mathfrak{B}_1 \times \cdots \times \mathfrak{B}_w$, where*

(i) *$|\mathfrak{B}_i| = p_i$ is a prime, $1 \leqq i \leqq w$,*

(ii) *$\mathfrak{S}_2$ normalizes $\mathfrak{B}_i$, $1 \leqq i \leqq w$.*

Then $p_i = 3$, $1 \leqq i \leqq w$. Furthermore, if $\mathfrak{Q}$ is a $S_{2'}$-subgroup of $O_{2,2'}(\mathfrak{S})$, then $C_{\mathfrak{H}}(\mathfrak{Q}) = Z(\mathfrak{H})$ and $[\mathfrak{H}, \mathfrak{Q}]$ is the central product of extra special groups $\mathfrak{H}_1, \cdots, \mathfrak{H}_s$ such that $\mathfrak{H}_i/\mathfrak{H}'$ is a chief factor of $O_{2,2'}(\mathfrak{S})\mathfrak{S}_2$. If w_i is the width of $\mathfrak{H}_i$, then $w_i \leqq 2$.

PROOF. Let $V = \mathfrak{H}/\mathfrak{H}'$, $V_0 = Z(\mathfrak{H})/\mathfrak{H}'$. Thus, $|V| = 2^{2w+e}$, $|V_0| = 2^e$. Let V_1 be a complement to V_0 in V which admits $\mathfrak{Q}$, and let $W = [V, \mathfrak{Q}] = [V_1, \mathfrak{Q}]$. We assume without loss of generality that $\mathfrak{S} = \mathfrak{Q}\mathfrak{S}_2$. Thus, $\mathfrak{S} = \mathfrak{H}N(\mathfrak{Q})$, so W admits $\mathfrak{S}$, and $C_{\mathfrak{S}}(W) = \mathfrak{H}$. Let $\overline{\mathfrak{S}}_2 = \mathfrak{S}_2/\mathfrak{H}$, an elementary group of order 2^w. Let $\mathfrak{T}_i = C_{\overline{\mathfrak{S}}_2}(\mathfrak{B}_i)$, $1 \leqq i \leqq w$, $\mathfrak{T}^i = \bigcap_{j \neq i} \mathfrak{T}_j$. Thus $\mathfrak{S}/\mathfrak{H} = \mathfrak{B}_1\mathfrak{T}^1 \times \mathfrak{B}_2\mathfrak{T}^2 \times \cdots \times \mathfrak{B}_w\mathfrak{T}^w$, and $\mathfrak{B}_i\mathfrak{T}^i = \mathfrak{D}_i$ is dihedral of order $2p_i$. By Lemma 5.8, we get $m = 2w$, so that $V_1 = W$. Also, $V_1 = X_1 \times \cdots \times X_s$, where X_i is an irreducible $\mathfrak{S}$-group. Let $X_i = \mathfrak{H}_i/\mathfrak{H}'$, $1 \leqq i \leqq s$. Since X_i is irreducible, $\mathfrak{H}_i$ is either abelian or extra special. Since $\mathfrak{S}/\mathfrak{H}$ acts faithfully on no proper submodule of V_1, it follows that neither does $\mathfrak{Q}$. Let $\mathfrak{Q}_i = C_{\mathfrak{Q}}(\mathfrak{X}_i)$, $\mathfrak{Q}^i = \bigcap_{j \neq i} \mathfrak{Q}_j$. Thus $\mathfrak{Q}^i \neq 1$, and $\mathfrak{Q}^i$ acts faithfully on $\mathfrak{H}_i$ and without fixed points on $\mathfrak{X}_i$. Hence, $\mathfrak{H}_i = [\mathfrak{H}, \mathfrak{Q}^i]$ is extra special by Lemma 5.12. If $i \neq j$, then $\mathfrak{Q}^i$ centralizes $\mathfrak{H}_j$, so $[\mathfrak{H}_i, \mathfrak{H}_j, \mathfrak{Q}^i] = [\mathfrak{H}_j, \mathfrak{Q}^i, \mathfrak{H}_i] = 1$, from which we get $[\mathfrak{Q}^i, \mathfrak{H}_i, \mathfrak{H}_j] = 1$. Since $\mathfrak{H}_i = [\mathfrak{Q}^i, \mathfrak{H}_i]$, we get that $\mathfrak{H}_1\mathfrak{H}_2 \cdots \mathfrak{H}_s$ is the central product of $\mathfrak{H}_1, \cdots, \mathfrak{H}_s$. The remaining parts of the lemma follow from Lemma 5.8.

LEMMA 5.15. *Suppose $\mathfrak{S}$ is a $2'$-reduced solvable group and the following hold:*

(a) *$O_2(\mathfrak{S}) = \mathfrak{H}$ is the central product of w dihedral groups of order* 8.

(b) *If* $\mathfrak{S}_2$ *is a* S_2*-subgroup of* $\mathfrak{S}$, *then* $\mathfrak{S}_2/\mathfrak{H}$ *is elementary of order* 2^{w-1}.

(c) $\boldsymbol{O}_{2,2'}(\mathfrak{S})/\mathfrak{H}=\mathfrak{B}_1\times\cdots\times\mathfrak{B}_{w-1}$, *where*

(i) $|\mathfrak{B}_i|=p_i$ *is a prime*, $1\leqq i\leqq w-1$,

(ii) $\mathfrak{S}_2$ *normalizes* $\mathfrak{B}_i$, $1\leqq i\leqq w-1$.

Then one of the following holds:

(α) $p_i=3$ *for all* i.

(β) *There is exactly one value of* i *such that* $p_i\neq 3$, *and for this* i, $p_i=5$. $\mathfrak{H}$ *is the central product of extra special groups* $\mathfrak{H}_1,\cdots,\mathfrak{H}_s$ *such that* $\mathfrak{H}_i\lhd \boldsymbol{O}_{2,2'}(\mathfrak{S})\mathfrak{S}_2$, $\mathfrak{H}_i/\mathfrak{H}'$ *is a chief factor of* $\boldsymbol{O}_{2,2'}(\mathfrak{S})\mathfrak{S}_2$, *and* $A_{\mathfrak{L}}(\mathfrak{H}_1/\mathfrak{H}')$ *is dihedral of order* 10, *where* $L=\boldsymbol{O}_{2,2'}(\mathfrak{S})\mathfrak{S}_2$. *Finally*, $\mathfrak{H}_2\cdots\mathfrak{H}_s$ *is the central product of an odd number of quaternion groups.*

Proof. Let $V=\mathfrak{H}/\mathfrak{H}'$. We may assume that $\mathfrak{S}=\mathfrak{S}_2\mathfrak{Q}$, where $\mathfrak{Q}$ is a $S_{2'}$-subgroup of $\boldsymbol{O}_{2,2'}(\mathfrak{S})$. First, suppose that $\mathfrak{S}/\mathfrak{H}$ is represented faithfully on some proper submodule V_0 of V. Let $W=[V_0,\mathfrak{S}]$. We argue that $|W|\leqq 2^{2(w-1)}$, that is, $|V\colon W|\geqq 4$. Suppose false. Then $|V\colon W|=2$, $W=V_0$. But $V_0=\mathfrak{K}/\mathfrak{H}'$ for some subgroup $\mathfrak{K}$ of $\mathfrak{H}$, and $|\boldsymbol{Z}(\mathfrak{K})|=4$. Thus, $\boldsymbol{Z}(\mathfrak{K})/\mathfrak{H}'=V_1$ is of order 2 and admits $\mathfrak{Q}$. Hence, $[V_0,\mathfrak{Q}]\subset V_0$, against $W=V_0=[V_0,\mathfrak{S}]=[V_0,\mathfrak{Q}]$. Hence $|W|\leqq 2^{2(w-1)}$. Hence, (α) holds by Lemma 5.8.

We may assume that $\mathfrak{S}/\mathfrak{H}$ acts faithfully on no proper submodule of V. In particular, $V=[V,\mathfrak{Q}]$. By Lemma 5.7, $V=V_1\times\cdots\times V_s$, where each V_i is an irreducible $\mathfrak{S}$-group. Let $\mathfrak{Q}_i=\boldsymbol{C}_{\mathfrak{Q}}(V_i)$; $\mathfrak{Q}^i=\bigcap_{j\neq i}\mathfrak{Q}_j$. Thus, $\mathfrak{Q}^i\neq 1$, and $\mathfrak{Q}^i$ acts faithfully on V_i and without fixed points on V_i. Let $V_i=\mathfrak{H}_i/\mathfrak{H}'$. Then for $i\neq j$, $\mathfrak{Q}^i$ centralizes $\mathfrak{H}_j$, so $[\mathfrak{H}_i,\mathfrak{H}_j]=1$, $\mathfrak{H}_i$ is extra special and $\mathfrak{H}$ is the central product of $\mathfrak{H}_1,\cdots,\mathfrak{H}_s$.

Let $\mathfrak{T}=\boldsymbol{N}_{\mathfrak{S}_2}(\mathfrak{Q})$. Then $\mathfrak{T}\cap\mathfrak{H}=\mathfrak{H}'$ and $\mathfrak{T}/\mathfrak{H}'$ is elementary of order 2^{w-1}. Let $\mathfrak{R}=\mathfrak{Q}\mathfrak{T}/\mathfrak{H}'$.

We may assume that $p_i\neq 3$ for some i. Let $\mathfrak{R}_i=\boldsymbol{C}_{\mathfrak{R}}(V_i)$, $1\leqq i\leqq s$, and let $\mathfrak{X}$ be the direct product of the $\mathfrak{R}/\mathfrak{R}_i$. We may convert V to an $\mathfrak{X}$-module in the obvious way. Thus, $\mathfrak{X}=\mathfrak{X}_1\times\cdots\times\mathfrak{X}_b$, where $\mathfrak{X}_i$ is a dihedral group of order $2p_i$, $1\leqq i\leqq b$, and $b\geqq w-1$. If $b>w-1$, then by Lemma 5.8, $p_i=3$ for all i. Hence, $b=w-1$, and so $\mathfrak{X}\cong\mathfrak{R}$. This implies that $\mathfrak{R}=\mathfrak{R}^1\times\cdots\times\mathfrak{R}^s$, where $[\mathfrak{R}^i,V_j]=1$ for $i\neq j$, and $\mathfrak{R}^i\mathfrak{R}_i=\mathfrak{R}^i\times\mathfrak{R}_i=\mathfrak{R}$. Let $|\mathfrak{R}^i|_2=2^{a_i}$, $|V_i|=2^{v_i}$. As we have seen before, $v_i\geqq 2^{a_i}$. Thus, $2w=\sum v_i\geqq\sum 2^{a_i}\geqq 2\sum a_i=2(w-1)$. Suppose $2^{a_i}>2a_i$ for some i. Then $2^{a_i}-2a_i=2$, as the inequalities show, so $a_i=3$, while for $i\neq j$, $2a_j=2^{a_j}$. Since $2^{a_i}>2a_i$, we get $v_j=2^{a_j}$ for all j (including $j=i$). Hence, $v_i=8$. If $j\neq i$, then $v_j=2$ or 4, so that $\mathfrak{R}^j$ is a 2, 3-group. Hence, $\mathfrak{R}^i$ is not a 2, 3-group. Since $v_i=8$ and $\mathfrak{H}_i$ is extra special, $|\mathfrak{R}^i|_{2'}$ divides $(2^2-1)(2^4-1)(2^6-1)(2^8-1)=3^5\cdot 5^2\cdot 7\cdot 17$. If $7\mid |\mathfrak{R}^i|$, then $\mathfrak{R}^i$ does not act irreducibly on V_i. If $17\mid |\mathfrak{R}^i|$, then $O_{2'}(\mathfrak{R}^i)$ acts irreducibly on V_i, so that $O_{2'}(\mathfrak{R}^i)$ is necessarily cyclic. Hence, we get $|O_{2'}(\mathfrak{R}^i)|=3\cdot 5\cdot 17=2^8-1$, so that $O_{2'}(\mathfrak{R}^i)$ permutes transitively V_i-O. This is not the case, since $\mathfrak{H}_i$ is extra special. Hence, $\mathfrak{R}^i$ is a 2, 3, 5-group. If $5^2\mid |\mathfrak{R}^i|$, then $\mathfrak{H}_i$ is the central product of $\mathfrak{H}_{i1}$ and $\mathfrak{H}_{i2}$, where $\mathfrak{H}_{ij}$ is the central product of a quaternion group and a dihedral group such that $\mathfrak{H}_{ij}$ admits $O_5(\mathfrak{R}^i)$. On the one hand, $\mathrm{Aut}(\mathfrak{H}_{ij})$ has no subgroup of order 15, and on the other $O_3(\mathfrak{R}^i)$ normalizes both $\mathfrak{H}_{i1}$ and $\mathfrak{H}_{i2}$. This is impossible, so $|\mathfrak{R}^i|=2^8\cdot 3^2\cdot 5$. Let $O_{2'}(\mathfrak{R}^i)=\mathfrak{A}\times\mathfrak{B}\times\mathfrak{C}$, where $|\mathfrak{A}|=|\mathfrak{B}|=3$, $|\mathfrak{C}|=5$. Since V_i is irreducible, $\mathfrak{C}$ has no fixed points on V_i. We may assume that $C_{V_i}(\mathfrak{A})\neq 1$, $C_{V_i}(\mathfrak{B})\neq 1$. Thus, $O_{2'}(\mathfrak{R}^i)/\mathfrak{A}$ acts faithfully on $C_{V_i}(\mathfrak{A})=\mathfrak{H}_i^*/\mathfrak{H}'$, and $\mathfrak{H}_i^*$ is of width 2. This is impossible, since no extra special 2-group of width 2 has an automorphism of order 15; so this case does not arise.

We conclude that $2^{a_i}=2a_i$ for all i, so $a_i=1$ or 2. By the inequalities, there is i such that $v_i>2^{a_i}$. Since v_i is even, we get $v_i-2^{a_i}=2$, $v_j=2^{a_j}$, for all $j\neq i$.

Since $a_j=1$ or 2 for all j, and since $v_j=2^{a_j}$ for all $j\neq i$, we get that $\mathfrak{R}^i$ is not a 2, 3-group.

Suppose $a_i=2$. Then $v_i=6$. As V_i is irreducible, $5\nmid|\mathfrak{R}^i|$, so we may assume that $|\mathfrak{R}^i|=2^2\cdot 3\cdot 7$. Since V_i is irreducible, $O_p(\mathfrak{R}^i)$ has no fixed points on V_i, $p=3$, 7. We may write $V_i=V_{i1}\times V_{i2}$, where V_{ij} admits $O_7(\mathfrak{R}^i)$, $j=1$, 2, and $|V_{ij}|=8$. Since $O_3(\mathfrak{R}^i)$ has no fixed points on V_i, $O_3(\mathfrak{R}^i)$ does not normalize V_{i1}. Since $O_3(\mathfrak{R}^i)$ centralizes $O_7(\mathfrak{R}^i)$, we get that V_{i1} and V_{i2} are isomorphic $O_7(\mathfrak{R}^i)$-groups. Hence, the elements of $O_7(\mathfrak{R}^i)-\{1\}$ are nonreal in $\mathfrak{R}^i$. This is impossible, so this case does not occur.

Suppose finally that $a_i=1$. Then $v_i=4$. Since $\mathfrak{R}^i$ is not a 2, 3-group, we get $|\mathfrak{R}^i|=10$. Hence, $\mathfrak{H}_i$ is the central product of a quaternion group and a dihedral group, so $\langle\mathfrak{H}_j \mid j\neq i\rangle$ is the central product of an odd number of quaternion groups, by (a). The lemma follows after a suitable relabeling.

5.4. *p-groups, p-solvability and $F(\mathfrak{G})$.*

LEMMA 5.16. *Let $\mathfrak{S}$ be a p-solvable, p'-reduced group. Let $\mathfrak{A}$ be a p-subgroup of $\mathfrak{S}$ and $\mathfrak{Q}$ an element of $\mathsf{M}(\mathfrak{A}; p')$ satisfying $[\mathfrak{A}, \mathfrak{Q}]=\mathfrak{Q}\neq 1$. Let $\mathfrak{H}=O_p(\mathfrak{S})$. Then $[\mathfrak{H}, \mathfrak{A}, \mathfrak{Q}]\neq 1$.*

PROOF. We assume without loss of generality that $\mathfrak{H}$ is elementary. Let $\mathfrak{H}_1=[\mathfrak{H}, \mathfrak{Q}]$, so that $\mathfrak{H}_1=[\mathfrak{H}_1, \mathfrak{Q}]\neq 1$. Suppose that $[\mathfrak{H}_1, \mathfrak{A}, \mathfrak{Q}]=1$. As $\mathfrak{H}_1$ is normalized by $\mathfrak{A}$ and as $\mathfrak{Q}$ has no fixed points on $\mathfrak{H}_1$, we have $[\mathfrak{H}_1, \mathfrak{A}]=1$. Thus, $\mathfrak{A}$ and so $[\mathfrak{A}, \mathfrak{Q}]$ centralize $\mathfrak{H}_1$. But $[\mathfrak{A}, \mathfrak{Q}]=\mathfrak{Q}$. This contradiction completes the proof.

LEMMA 5.17. *Suppose $\mathfrak{H}=\mathfrak{P}\mathfrak{Q}$ where $\mathfrak{Q}$ is a normal 2-subgroup of $\mathfrak{H}$, $\mathfrak{P}$ is a p-group, p an odd prime, and $\mathfrak{Q}=[\mathfrak{P}, \mathfrak{Q}]\neq 1$. Suppose furthermore that $\mathfrak{P}$ centralizes every characteristic subgroup of $\mathfrak{Q}$. Then $\mathfrak{Q}$ is special.*

PROOF. Let $c=\mathrm{cl}(\mathfrak{Q})$. If $c\geqq 3$, then $C_{c-1}(\mathfrak{Q})$ is abelian, so $[C_{c-1}(\mathfrak{Q}), \mathfrak{P}]=1$. This implies that $[\mathfrak{P}, \mathfrak{Q}, C_{c-1}(\mathfrak{Q})]=1$, by the three subgroups lemma. Hence, $C_{c-1}(\mathfrak{Q})\subseteq Z(\mathfrak{Q})$, since $\mathfrak{Q}=[\mathfrak{P}, \mathfrak{Q}]$. This is not the case, so $c\leqq 2$. Hence, $c=2$, since $\mathfrak{Q}$ is obviously nonabelian. Hence, $\mathfrak{Q}'\subseteq Z(\mathfrak{Q})$, so $\mathfrak{Q}'=Z(\mathfrak{Q})$, since $\mathfrak{P}$ has no fixed points on $\mathfrak{Q}/\mathfrak{Q}'$. If $\mathfrak{Q}/\mathfrak{Q}'$ is elementary, we are done. Otherwise, let $\mathfrak{Q}_0=\Omega_2$ ($\mathfrak{Q}$ mod $\mathfrak{Q}'$) so that $\mathfrak{Q}_0/\mathfrak{Q}'$ is of exponent 4. Since $\mathfrak{Q}$ is of class 2, it follows that if X, $Y\in\mathfrak{Q}_0$, then $1=[X^4, Y]=[X^2, Y^2]$ so that $D(\mathfrak{Q}_0)$ is abelian. This is impossible, since $\mathfrak{Q}=[\mathfrak{P}, \mathfrak{Q}]$. The proof is complete.

LEMMA 5.18. *If $\mathfrak{X}$ is nilpotent and $\mathfrak{A}$ is a characteristic abelian subgroup of $\mathfrak{X}$, then $\mathcal{B}(\mathfrak{X}; \mathfrak{A})\neq\varnothing$.* (See Definition 2.6.)

The proof of this lemma is given in Lemma 0.8.2.

LEMMA 5.19. *Suppose $\mathfrak{H}$ is a S_π-subgroup of the solvable group $\mathfrak{S}$ and that 2, $3\in\pi'$. If $\mathfrak{A}\in\mathcal{S}\mathrm{cn}(F(\mathfrak{H}))$, then $C(\mathfrak{A})=\mathfrak{A}\times\mathfrak{D}$ where $\mathfrak{D}$ is a π'-group.*

PROOF. Proceeding by induction on $|\mathfrak{S}|$, we may assume that $O_{\pi'}(\mathfrak{S})=1$. Hence, $F(\mathfrak{S})\subseteq F(\mathfrak{H})$, so $[F(\mathfrak{S}), \mathfrak{A}, \mathfrak{A}]=1$. Let $\mathcal{C}$: $F(\mathfrak{S})=\mathfrak{C}_0\supset\mathfrak{C}_1\supset\cdots\supset\mathfrak{C}_k=1$ be part of a chief series of $\mathfrak{S}$, and let $V_i=\mathfrak{C}_i/\mathfrak{C}_{i+1}$, V_i a p_i-group, $i=0,\cdots,k-1$. Then the $S_{p_i'}$-subgroup of $\mathfrak{A}$ centralizes V_i and the S_{p_i}-subgroup $\mathfrak{A}_i$ of $\mathfrak{A}$ satisfies $[V_i, \mathfrak{A}_i, \mathfrak{A}_i]=1$. Since V_i is p_i-reducible in S, it follows from (B) and $p_i\geqq 5$ that $\mathfrak{A}\subseteq C(V_i)$. As is well known, $\bigcap_{i=0}^{k-1} C(V_i)=F(\mathfrak{S})$, so $\mathfrak{A}\subseteq F(\mathfrak{S})$, $\mathfrak{A}\in\mathcal{S}\mathrm{cn}(F(\mathfrak{S}))$. Suppose $C\in C(\mathfrak{A})$ and $C^{p^n}=1$ for some p in π. We will show that $C\in\mathfrak{A}$. In any case, C stabilizes $F(\mathfrak{S})\supseteq\mathfrak{A}\supseteq 1$, so C centralizes each V_i, so $C\in F(\mathfrak{S})$. Since $\mathfrak{A}\in\mathcal{S}\mathrm{cn}(F(\mathfrak{S}))$, we get $C\in\mathfrak{A}$. Since $\mathfrak{A}$ contains every π-element of $C(\mathfrak{A})$, the lemma follows.

LEMMA 5.20. *Suppose $\mathfrak{M}$ is a normal elementary 2-subgroup of the solvable group $\mathfrak{S}$, and the following hold:*

(a) *$\mathfrak{S}$ contains an elementary subgroup of order p^3 for some odd prime p.*

(b) *$\mathfrak{S}$ contains a cyclic 3-subgroup $\mathfrak{Q}$ with the following properties:*

(i) $\mho^1(\mathfrak{Q})$ *centralizes* $\mathfrak{M}$.

(ii) $[\mathfrak{Q}, \mathfrak{M}]$ *is a four-group.*

Then $\mathfrak{S}$ *contains an elementary subgroup* $\mathfrak{F}$ *of order* p^3 *for some odd prime* p *such that* $\mathbf{C}_{\mathfrak{F}}([\mathfrak{Q}, \mathfrak{M}])$ *is noncyclic.*

PROOF. Let $\mathfrak{H}$ be a $2'$-subgroup of $\mathfrak{S}$ which contains $\mathfrak{Q}$ and is minimal subject to containing an elementary subgroup of order q^3 for some prime q. We suppose without loss of generality that $\mathfrak{H}$ acts irreducibly on $\mathfrak{M}$. Suppose $\mathfrak{H}$ is a 3-group. Then $\mathfrak{H}=\mathfrak{H}_0\mathfrak{Q}$, where $\mathfrak{H}_0 \lhd \mathfrak{H}$, $\mathfrak{H}_0$ is elementary and $|\mathfrak{H}_0| = 3^3$. Thus, $\mathfrak{Q}$ permutes transitively the Wedderburn components of $\mathfrak{H}_0$ on $\mathfrak{M}$. Since $|\mathfrak{H}_0|$ is odd, (b)(ii) implies that there is only 1 Wedderburn component. Hence, $\mathbf{C}_{\mathfrak{H}_0}(\mathfrak{M})$ is noncyclic and we are done. We may therefore suppose that $\mathfrak{S}$ has no elementary subgroup of order 3^3. Thus, $\mathfrak{H}$ is a 3, p-group for some prime p. Since $p \geqq 5$, it follows from Lemma 0.8.5 that $\mathfrak{H}$ is p-closed; $\mathfrak{H}=\mathfrak{H}_p\mathfrak{Q}$ where $\mathfrak{H}_p$ is the S_p-subgroup of $\mathfrak{H}$. Minimality of $\mathfrak{H}$ forces $\mathfrak{H}_p$ to be of exponent p and $|\mathfrak{H}_p'| \leqq p^2$. As above, the irreducible $\mathfrak{H}_p$-subgroups of $\mathfrak{M}$ are pairwise isomorphic. If $|\mathbf{C}_{\mathfrak{H}_p}(\mathfrak{M})| > p$, we are done, so suppose $|\mathbf{C}_{\mathfrak{H}_p}(\mathfrak{M})| \leqq p$. Hence, $\mathfrak{H}_p/\mathbf{C}_{\mathfrak{H}_p}(\mathfrak{M}) = \overline{\mathfrak{H}}_p$ is extra special. Clearly, $\mathfrak{Q}$ does not centralize $\overline{\mathfrak{H}}_p$. Since $[\mathfrak{M}, \mathfrak{Q}]$ is a four-group, we get that $p=7$. Thus, $\overline{\mathfrak{H}}_7$ contains a noncentral subgroup $\mathfrak{R}$ of order 7 such that $|[\mathfrak{R}, \mathfrak{M}]| = 2^3$. This is not the case, since $\mathfrak{M}$ is a free $F_2\mathfrak{R}$-module. The proof is complete.

LEMMA 5.21. *Suppose the following hold:*

(i) $\mathfrak{M}$ *is a solvable group.*

(ii) $\mathfrak{F}$ *is a noncyclic normal elementary 2-subgroup of* $\mathfrak{M}$ *and one of the following holds:*

(a) $\mathfrak{F}$ *is 2-reducible in* $\mathfrak{M}$.

(b) $\mathfrak{F}$ *contains a subgroup* $\mathfrak{F}_0$ *of order* 2 *which is central in* $\mathfrak{M}$ *and* $\mathfrak{F}/\mathfrak{F}_0$ *is 2-reducible in* $\mathfrak{M}$.

(iii) I *is an involution of* $\mathfrak{M}$ *with* $|\mathfrak{F}: \mathbf{C}_{\mathfrak{F}}(I)| = 2$.

(iv) *There is at least one odd prime* p *such that* $\mathfrak{M}$ *contains an elementary subgroup of order* p^3.

Let $\mathfrak{F}^* = [\mathfrak{F}, I]$. *Then there are an odd prime* q *and an elementary subgroup* $\mathfrak{Q}$ *of* $\mathfrak{M}$ *of order* q^3 *such that* $\mathbf{C}_{\mathfrak{Q}}(\mathfrak{F}^*)$ *is noncyclic.*

PROOF. If $\mathfrak{F}$ is 2-reducible in $\mathfrak{M}$, set $\mathfrak{F}_0=1$, otherwise, let $\mathfrak{F}_0$ be the subgroup given in (ii)(b). Let $V=\mathfrak{F}/\mathfrak{F}_0$, $\mathfrak{C}=\mathbf{C}_{\mathfrak{M}}(V)$. If $I\in\mathfrak{C}$, then $[\mathfrak{F}, I]=\mathfrak{F}_0$ and the lemma is clear. We may assume that $I \notin \mathfrak{C}$. Hence, I inverts an element M of $\mathfrak{M}$ of odd prime power order $r^n>1$, such that $M^r\in\mathfrak{C}$, $M\notin\mathfrak{C}$. Since $\mathbf{C}_V(I)$ is of index 2 in V, it follows that $r=3$. Thus, this lemma follows from Lemma 5.20.

LEMMA 5.22. *Suppose p is an odd prime, $\mathfrak{S}$ is a p'-reduced p-solvable group, $\mathfrak{S}_p$ is a S_p-subgroup of $\mathfrak{S}$ and $\mathfrak{A}$ is a normal cyclic subgroup of $\mathfrak{S}$. Then $\mathfrak{A} \subseteq O_p(\mathfrak{S})$.*

PROOF. Suppose false. Since every subgroup of $\mathfrak{A}$ is normal in $\mathfrak{S}_p$, we may assume that $|\mathfrak{A}: \mathfrak{A} \cap O_p(\mathfrak{S})| = p$. Since $[O_p(\mathfrak{S}), \mathfrak{A}, \mathfrak{A}] = 1$, it follows that $p = 3$. Proceeding by induction on $|\mathfrak{S}|$, we may assume that $O_3(\mathfrak{S})$ is elementary and that $\mathfrak{S} = O_{3,3'}(\mathfrak{S})\mathfrak{A}$. Thus, $\mathfrak{S}_p = O_p(\mathfrak{S})\mathfrak{A}$ and $\mathfrak{A}$ is cyclic of order 9. Since $[\mathfrak{A}, O_3(\mathfrak{S})] = \Omega_1(\mathfrak{A})$, it follows that $O_{3,3'}(\mathfrak{S}) = O_3(\mathfrak{S})\mathfrak{Q}$, where $\mathfrak{Q}$ is a quaternion group. Minimality of $\mathfrak{S}$ forces $|O_3(\mathfrak{S})| = 9$. Hence, $\mathfrak{S} = O_3(\mathfrak{S}) \cdot N(\mathfrak{Q})$ and $N(\mathfrak{Q}) \cap O_3(\mathfrak{S}) = 1$. This implies that $\mathfrak{S}_3$ has exponent 3, against the presence of $\mathfrak{A}$. The proof is complete.

LEMMA 5.23. *Suppose $\mathfrak{A}$ is a normal subgroup of the p-group $\mathfrak{P}$ and $\mathfrak{B}$ is an abelian subgroup of $\mathfrak{A}$ which is normal in $\mathfrak{P}$. Then there is an element $\mathfrak{C}$ of $\mathcal{S}cn(\mathfrak{A})$ which contains $\mathfrak{B}$ and is normal in $\mathfrak{P}$.*

PROOF. Let $\mathfrak{C}$ be maximal subject to (a) $\mathfrak{C} \triangleleft \mathfrak{P}$, (b) $\mathfrak{B} \subseteq \mathfrak{C} \subseteq \mathfrak{A}$, (c) $\mathfrak{C}' = 1$. If $\mathfrak{C} \subset C_{\mathfrak{A}}(\mathfrak{C})$, then there is a subgroup $\mathfrak{D}$ of $\mathfrak{A}$ such that $\mathfrak{D} \triangleleft \mathfrak{P}$, $\mathfrak{C} \subset \mathfrak{D} \subseteq C_{\mathfrak{A}}(\mathfrak{C})$, $|\mathfrak{D}: \mathfrak{C}| = p$. Thus, $\mathfrak{D}$ is abelian against the maximality of $\mathfrak{C}$. Hence, $\mathfrak{C} = C_{\mathfrak{A}}(\mathfrak{C}) \in \mathcal{S}cn(\mathfrak{A})$.

LEMMA 5.24. *Suppose p is an odd prime, $\mathfrak{G}$ is a p-solvable group, and $\mathfrak{G}$ has no elementary subgroup of order p^3. Then each chief p-factor of $\mathfrak{G}$ is of order p or p^2.*

PROOF. Let $\mathfrak{G}_p$ be a S_p-subgroup of $\mathfrak{G}$. We assume without loss of generality that $O_{p'}(\mathfrak{G}) = 1$. Let $\mathfrak{H} = O_p(G)$. First, suppose $p \geqq 5$. In this case, the structure of $\mathfrak{G}_p$ is given in 0.3.4 and the lemma follows. So suppose $p = 3$.

Choose $\mathfrak{B} \in \mathcal{B}(\mathfrak{H})$, and let $\mathfrak{A} = \Omega_1(\mathfrak{B})$. Let $\mathfrak{Q}$ be a $S_{3'}$-subgroup of $\mathfrak{G}$. Thus, $\mathfrak{Q}$ is represented faithfully on $\mathfrak{H}$, so by definition of $\mathfrak{B}$, $\mathfrak{Q}$ is represented faithfully on $\mathfrak{B}$. By 0.3.6, $\mathfrak{Q}$ is represented faithfully on $\mathfrak{A}$.

By definition of $\mathfrak{B}$, $\mathrm{cl}(\mathfrak{B}) \leqq 2$, so $\mathfrak{A}$ is of exponent 3. Hence, $m(\mathfrak{A}) \leqq 2$ since $\mathfrak{G}$ has no elementary subgroup of order 27. Hence, $\mathfrak{Q}$ is isomorphic to a subgroup of $GL(2, 3)$, so $\mathfrak{Q}$ is a 2-group. Furthermore, since $\mathfrak{G}_3/\mathfrak{H}$ is represented faithfully on $O_{3,2}(\mathfrak{G})/\mathfrak{H}$ it follows that $|\mathfrak{G}_3: \mathfrak{H}| \leqq 3$.

Let $\mathcal{C}$ be a part of a chief series of $\mathfrak{G}$ from 1 to $\mathfrak{H}$ passing through $\mathfrak{A}$ and through $\mathfrak{H}_0 = \mathfrak{A}C_{\mathfrak{H}}(\mathfrak{A})$. If $|\mathfrak{A}| = 3$, then $\mathfrak{B}$ is cyclic, and $\mathfrak{H}$ is metacyclic, and the lemma is clear. Suppose $|\mathfrak{A}| = 9$. Here we get $|\mathfrak{H}: \mathfrak{H}_0| \leqq 3$. Also, $\mathfrak{H}_0$ has just 4 subgroups of order 3, each of which is central. As is well known [5], $\mathfrak{H}_0$ is metacyclic, and we are done.

Suppose $|\mathfrak{A}| = 27$. Here we get $|\mathfrak{H}:\mathfrak{H}_0| \leqq 3$, and $\mathfrak{H}_0$ is the central product of $\mathfrak{A}$ and a cyclic group. Again the lemma follows since the chief factors of $\mathfrak{G}$ between $\mathfrak{A}$ and $\mathfrak{H}_0$ are of order 3.

LEMMA 5.25. *Suppose $\mathfrak{S}$ is a p-solvable group, p an odd prime, and $\mathfrak{B}$ is a subgroup of $\mathfrak{S}$ of type $\boldsymbol{r}(p, p)$ such that for some B_0 in $\mathfrak{B}$, $\mathfrak{B} = \Omega_1(\mathfrak{P})$, where $\mathfrak{P}$ is a S_p-subgroup of $\boldsymbol{C}(B_0)$. Then every element of $\text{И}(\mathfrak{B}; p')$ is contained in $\boldsymbol{O}_{p'}(\mathfrak{S})$.*

PROOF. We may assume that $\boldsymbol{O}_{p'}(\mathfrak{S}) = 1$ and try to show that 1 is the only element of $\text{И}(\mathfrak{B}; p')$. Suppose $\mathfrak{Y} \in \text{И}(\mathfrak{B}; p')$, $\mathfrak{Y} \neq 1$. We may assume that $\mathfrak{Y}$ is an elementary q-group for some prime $q \neq p$, and that $\mathfrak{B}$ acts irreducibly on $\mathfrak{Y}$. Let $\mathfrak{H} = \boldsymbol{O}_p(\mathfrak{S})$. Thus, $\mathfrak{Y}$ acts faithfully on $\mathfrak{H}$. We may assume that $\mathfrak{S} = \mathfrak{H}\mathfrak{Y}\mathfrak{B}$ and that $\mathfrak{Y}$ centralizes $\boldsymbol{D}(\mathfrak{H})$, and that $\mathfrak{Y}$ has no fixed points on $\mathfrak{H}/\boldsymbol{D}(\mathfrak{H})$. Hence, $\mathfrak{H}$ is of exponent p, by Lemma 0.8.7 and 0.3.6. This implies that $\mathfrak{B}$ is a S_p-subgroup of $\boldsymbol{C}(\mathfrak{B})$, so by Theorem 2 of [**41**], we have $\mathfrak{Y} \subseteq \boldsymbol{O}_{p'}(\mathfrak{S})$. The proof is complete.

5.5. *Groups of low order.*

LEMMA 5.26. *Suppose $\mathfrak{H} = \mathfrak{P}\mathfrak{Q}$, $\mathfrak{P} = \mathfrak{H}'$, $|\mathfrak{P}| = 64$, $|\mathfrak{Q}| = 5$ and $\boldsymbol{Z}(\mathfrak{P})$ is a four-group. Then $\mathfrak{H}$ is isomorphic to the centralizer of an involution of $U_3(4)$.*

PROOF. As is well known [**36**], the centralizers of involutions of $U_3(4)$ satisfy the hypotheses. Thus, it suffices to show that the hypotheses determine $\mathfrak{H}$.

Since $\mathfrak{P} = \mathfrak{H}'$, it follows that $\mathfrak{P}$ is special. Let P be an element of $\mathfrak{P}$ of order 4. Thus, $\mathfrak{P} = \langle P^{\mathfrak{Q}} \rangle$ so we can choose a generator Q of $\mathfrak{Q}$ such that $[P, P^Q] \neq 1$. Let $P_i = P^{Q^i}$, $i = 0, 1, 2, 3$. Thus, $P_3^Q = P_0^{Q^4} = P_0P_1P_2P_3Z$, where $Z \in \boldsymbol{Z}(\mathfrak{P})$. If $X \in \mathfrak{P}$, then

$$P^{X^{-1}Q^iX} = (P[P, X])^{Q^iX} = (P_i[P, X])^X = P_i[P_i, X][P, X].$$

For $i = 0, 1, 2, 3$, set $\tilde{P}_i = P^{X^{-1}Q^iX}$, and

$$\begin{aligned} P^{X^{-1}Q^4X} &= \tilde{P}_3^{X^{-1}QX} = \tilde{P}_0^{X^{-1}Q^4X} = (P_3[P_3, X])^{QX} = (P_0P_1P_2P_3Z[P_3, X])^X \\ &= P_0P_1P_2P_3[P_3, X][P_0P_1P_2P_3, X]Z = \tilde{P}_0\tilde{P}_1\tilde{P}_2\tilde{P}_3[P_3, X]Z. \end{aligned}$$

We can choose X in $\mathfrak{P}$ such that $[P_3, X] = Z$ and replacing Q by $X^{-1}QX$, we may assume at the outset that

$$(*) \qquad P_3^Q = P_0P_1P_2P_3, \qquad P_i = P^{Q^i}, \qquad i = 0, 1, 2, 3.$$

Let $Z_1 = [P, P^Q] = [P_0, P_1] \neq 1$. Since $\mathfrak{Q}$ centralizes $\boldsymbol{Z}(\mathfrak{P})$, we get $Z_1 = [P_1, P_2] = [P_2, P_3] = [P_3, P_0P_1P_2]$, so that $1 = [P_3, P_0P_1]$. If

$[P_3, P_0] \in \langle Z_1 \rangle$, we get that $[P_1, \mathfrak{P}] \subseteq \langle Z_1 \rangle$, against $[P_1, \mathfrak{P}] = \boldsymbol{Z}(\mathfrak{P})$. Hence $[P_3, P_0] = Z_2 \notin \langle Z_1 \rangle$. Hence $[P_1, P_3] = Z_2$ and conjugation by Q^{-1} gives $[P_0, P_2] = Z_2$. Thus, the commutation relations in $\mathfrak{P}$ are

$$(**) \qquad \begin{aligned} [P_0, P_1] &= Z_1, & [P_1, P_2] &= Z_1, \\ [P_0, P_2] &= Z_2, & [P_1, P_3] &= Z_2, \\ [P_0, P_3] &= Z_2, & [P_2, P_3] &= Z_1. \end{aligned}$$

Again, since $\mathfrak{Q}$ centralizes $\boldsymbol{Z}(\mathfrak{P})$, we get

$$P_0^2 = P_1^2 = P_2^2 = P_3^2 = (P_0P_1P_2P_3)^2 = P_0^2P_1^2P_2^2P_3^2 \prod_{0 \leqq i < j \leqq 3} [P_i, P_j] = Z_1Z_2$$

and so

$$(***) \qquad P_i^2 = Z_1Z_2, \qquad 0 \leqq i \leqq 3.$$

Now (*), (**), (***) determine $\mathfrak{H}$.

LEMMA 5.27. *Suppose $\mathfrak{T}$ is a metacyclic 2-group and* $\mathrm{Aut}(\mathfrak{T})$ *is not a 2-group. Then $\mathfrak{T}$ is either a quaternion group or is abelian of type* $(2^n, 2^n)$.

PROOF. We may assume that $\mathfrak{T}$ is nonabelian.

Suppose $|\mathfrak{T}'| > 2$. Then $\mathfrak{T}/\Omega_1(\mathfrak{T}')$ is a nonabelian metacyclic 2-group, so by induction on $|\mathfrak{T}|$, it follows that $\mathfrak{T}/\Omega_1(\mathfrak{T}')$ is a quaternion group. Hence, $\mathfrak{T}$ is a group of maximal class and order 16, so $\mathrm{Aut}(\mathfrak{T})$ is a 2-group. This is not so, by assumption, so $|\mathfrak{T}'| = 2$, and $\mathfrak{T}/\mathfrak{T}'$ is of type $(2^n, 2^n)$. If $n \geqq 2$, then $\boldsymbol{Z}(\mathfrak{T})$ is of type $(2^{n-1}, 2^n)$, so $\boldsymbol{Z}(\mathfrak{T})$ is centralized by every automorphism of $\mathfrak{T}$ of odd order. Hence, $\mathfrak{T}/\mathfrak{A}$ is nonabelian for some subgroup $\mathfrak{A}$ of $\boldsymbol{Z}(\mathfrak{T})$ of order 2, so that $\mathfrak{T}/\mathfrak{A}$ is a quaternion group. This is not the case, since $|\mathfrak{T}| \geqq 2^5$. We conclude that $n = 1$. This implies that $\mathfrak{T}$ is a quaternion group, since a dihedral group of order 8 has no nontrivial automorphisms of odd order.

LEMMA 5.28. *Let $\mathfrak{T}$ be a nonabelian 2-group with $\mathscr{SCN}_3(\mathfrak{T}) = \varnothing$. Suppose $\mathfrak{A}$ is a group of automorphisms of $\mathfrak{T}$ of odd order and $[\mathfrak{A}, \mathfrak{T}] = \mathfrak{T}$. Then $\mathfrak{T}$ is isomorphic to one of the following groups*:

(i) *A quaternion group.*

(ii) *A special group of order* 64 *with exactly* 3 *involutions, each of which is central.*

(iii) *The central product of a quaternion group and a dihedral group of order* 8.

PROOF. First, suppose that $\mathfrak{T}$ is of symplectic type. Since $[\mathfrak{A}, \mathfrak{T}] = \mathfrak{T}$, $\mathfrak{T}$ is extra special, by Lemma 5.12. Since $\mathfrak{Scn}_3(\mathfrak{T}) = \emptyset$, (i) and (iii) are the only possibilities.

Let $\mathfrak{B}$ be a noncyclic characteristic abelian subgroup of $\mathfrak{T}$ of largest order. Notice that $\boldsymbol{Z}(\mathfrak{T}) \subseteq \mathfrak{B}$. Let $\mathfrak{C} = \Omega_2(\mathfrak{B})$, $\mathfrak{D} = \Omega_1(\mathfrak{B})$. Thus, $\mathfrak{D}$ is a four-group and $\boldsymbol{C}_{\mathfrak{T}}(\mathfrak{D})$ is normalized by $\mathfrak{A}$. Since $|\mathfrak{T}: \boldsymbol{C}_{\mathfrak{T}}(\mathfrak{D})| \leqq 2$, it follows that $\mathfrak{D} \subseteq \boldsymbol{Z}(\mathfrak{T})$.

Suppose $\mathfrak{A}$ centralizes $\mathfrak{B}$. In this case, $\mathfrak{A}$ centralizes $\mathfrak{T}/C_{\mathfrak{T}}(\mathfrak{B})$, so that $\mathfrak{B} = Z(\mathfrak{T})$. Let $c = \mathrm{cl}(\mathfrak{T})$. If $c > 2$, then $\boldsymbol{Z}(\mathfrak{T})\,\boldsymbol{C}_{c-1}(\mathfrak{T})$ is a noncyclic characteristic abelian subgroup of $\mathfrak{T}$, against the maximality of $\mathfrak{B}$. Hence, $c = 2$. Since $\mathfrak{T}' \subseteq \mathfrak{B}$, it follows that $\mathfrak{T}$ contains exactly 3 involutions. If $\mathfrak{C}$ is of type (4, 4), then by a result of Alperin [**1**], $\mathfrak{T}$ is metacyclic, so by Lemma 5.27, $\mathfrak{T}$ is abelian, contrary to hypothesis. Hence $\mathfrak{B}$ is of type $(2, 2^n)$, $n \geqq 1$. Since $\mathfrak{T} = [\mathfrak{A}, \mathfrak{T}]$, it follows that $\mathfrak{B} = \mathfrak{T}'$. Suppose $n \geqq 2$. Then $\mathfrak{T}/\mho^1(\mathfrak{T}')$ contains exactly 3 involutions, so by induction on $|\mathfrak{T}|$, $\mathfrak{T}/\mho^1(\mathfrak{T}')$ is special of order 64. But in this case, $\mathfrak{T}'$ is of exponent 2, against $\mathfrak{B} = \mathfrak{T}'$. Hence, $n = 1$. Since $\mathfrak{B} = \mathfrak{T}' = \boldsymbol{Z}(\mathfrak{T})$, it follows that $\boldsymbol{D}(\mathfrak{T}) \subseteq \boldsymbol{Z}(\mathfrak{T})$, so that $\mathfrak{T}$ is special.

Let $|\mathfrak{T}: \mathfrak{T}'| = 2^m$. Since $\mathfrak{T}' = \boldsymbol{D}(\mathfrak{T})$, it follows that every linear character of $\mathfrak{T}$ lies in the rational field. By a result of Schur-Frobenius [**15**], we get $4 = 2^m + \sum \epsilon_\chi \chi(1)$, where χ ranges over all the nonlinear characters of $\mathfrak{T}$. If χ is a nonlinear irreducible character of $\mathfrak{T}$, then $\boldsymbol{Z}(\mathfrak{T}) \cap \ker \chi$ is of order 2, since $\boldsymbol{Z}(\mathfrak{T}) = \mathfrak{T}'$ and $\boldsymbol{Z}(\mathfrak{T})$ is a four-group, while $\boldsymbol{Z}(\mathfrak{T}/\ker \chi)$ is cyclic. Let $\mathfrak{Z}_1, \mathfrak{Z}_2, \mathfrak{Z}_3$ be the subgroups of $\boldsymbol{Z}(\mathfrak{T})$ of order 2, and let $a_i = \sum \epsilon_\chi \chi(1)$, where χ ranges over all the nonlinear irreducible characters of $\mathfrak{T}$ with $\boldsymbol{Z}(\mathfrak{T}) \cap \ker \chi = \mathfrak{Z}_i$, $i = 1, 2, 3$. Thus, $4 = 2^m + a_1 + a_2 + a_3$.

Since $\mathfrak{T}$ admits $\mathfrak{A}$, we get $m \geqq 3$. Hence, we may assume that notation is chosen so that $8 \nmid a_1$. Let $\mathfrak{K} = \mathfrak{T}/\mathfrak{Z}_1$. Then $|\boldsymbol{D}(\mathfrak{K})| = 2$. If $\boldsymbol{Z}(\mathfrak{K})$ is not elementary, then no nonlinear irreducible character of $\mathfrak{K}$ is real, so that $a_1 = 0$. This is not the case, so $\boldsymbol{Z}(\mathfrak{K})$ is elementary. This implies that $\mathfrak{K} = \mathfrak{K}_0 \times \mathfrak{K}_1$, where $\mathfrak{K}_0$ is elementary and $\mathfrak{K}_1$ is extra special of order 2^{2l+1}. Hence $a_1 = \pm |\mathfrak{K}_0| \cdot 2^l$, since $\mathfrak{K}_1$ has just 1 nonlinear irreducible character whose degree is 2^l. Since $8 \nmid a_1$, we get $l \leqq 2$. Suppose $l = 2$. Then $\mathfrak{K}_0 = 1$ and so $|\mathfrak{T}| = 64$. Suppose $l = 1$. In this case $|\mathfrak{K}_0| \leqq 2$, so $|\mathfrak{T}| \leqq 32$. Thus $|\mathfrak{T}| = 32$ or 64.

If $|\mathfrak{T}| = 32$, then since $\mathfrak{T} = [\mathfrak{T}, \mathfrak{A}]$, it follows that $7 \mid |\mathfrak{A}|$. However, $\mathfrak{T}$ is special and no nonabelian special group of order 32 has an automorphism of order 7. Hence $|\mathfrak{T}| = 64$.

Suppose $\mathfrak{A}$ does not centralize $\mathfrak{B}$. Then $\mathfrak{B}$ is of type $(2^n, 2^n)$, $n \geqq 1$. Suppose $n \geqq 2$. Then $\mathfrak{C}$ is of type (4, 4), and by the above mentioned result of Alperin, $\boldsymbol{C}_{\mathfrak{T}}(\mathfrak{C}) = \mathfrak{B}$. Since $\mathfrak{T}$ stabilizes $\mathfrak{C} \supset \mathfrak{D} \supset 1$, it follows that $\mathfrak{T}/\mathfrak{B}$ is elementary of order 4 or 16. It now follows easily that $|\mathfrak{A}|=3$, and that $\mathfrak{A}$ has no fixed points on $\mathfrak{T}$. Hence, $\mathrm{cl}(\mathfrak{T})=2$, and the lemma follows. If $n=1$, the lemma also follows, since in this case $\mathfrak{B}=\boldsymbol{Z}(\mathfrak{T})=\mathfrak{T}'$.

LEMMA 5.29. *Suppose* $\mathfrak{S}=\mathfrak{T}\mathfrak{Q}$, *where* $|\mathfrak{T}|=64$ *and* $|\mathfrak{Q}|=5$. *Let* $\mathfrak{H}=\boldsymbol{O}_2(\mathfrak{S})$ *and assume that* $\mathfrak{H}=\boldsymbol{F}(\mathfrak{S})$ *is the central product of a quaternion group and a dihedral group of order* 8. *Suppose* T *is an involution of* $\mathfrak{T}-\mathfrak{H}$ *and* $\boldsymbol{C}_{\mathfrak{H}}(T)$ *contains a four-group. Then* $\boldsymbol{C}_{\mathfrak{H}}(T)$ *is a four-group.*

PROOF. Let I be an involution of $\boldsymbol{C}_{\mathfrak{H}}(T)-\mathfrak{H}'$; I is available since $|\mathfrak{H}'|=2$. We can assume without loss of generality that T inverts $\mathfrak{Q}$. As $\mathfrak{Q}$ acts irreducibly on $\mathfrak{T}/\mathfrak{T}'$ and as $I\notin\boldsymbol{Z}(\mathfrak{H})$, we may then choose Q such that $[I, I^Q]=Z\neq 1$.

Let $I_1=I$, $I_2=I^Q$, $I_3=I^{Q^2}$, $I_4=I^{Q^3}$. Since $Q^5=1$, it follows that $I_4^Q=I_4I_3I_2I_1Z^a=Z^aI_1I_2I_3I_4$. Since $[I_1, I_2]=Z$, transformation by Q yields $[I_2, I_3]=[I_3, I_4]=Z$. Now $I_2^T=I_1^{QT}=I_1^{Q^{-1}}=I_1I_2I_3I_4Z^a$, $I_3^T=I_1^{Q^2T}=I_1^{Q^3}=I_4$. Thus, I_3I_4 is of order 4 and $\langle I_1, I_3I_4\rangle/\mathfrak{H}'=\boldsymbol{C}_{\mathfrak{H}/\mathfrak{H}'}(T\mathfrak{H}')$. Since $(I_3I_4)^T=I_4I_3\neq I_3I_4$, it follows that $\boldsymbol{C}_{\mathfrak{H}}(T)=\langle I_1, Z\rangle$, a four-group.

LEMMA 5.30. *Suppose* $\mathfrak{S}$ *is a 3-solvable group,* $|\mathfrak{S}:\boldsymbol{O}_{3'}(\mathfrak{S})|=3$, $\mathfrak{S}_3$ *is a* S_3*-subgroup of* $\mathfrak{S}$ *and* $\boldsymbol{O}_{3'}(\mathfrak{S})=[\boldsymbol{O}_{3'}(\mathfrak{S}), \mathfrak{S}_3]$. *Suppose also that* $\mathfrak{S}$ *is faithfully represented as automorphisms of an elementary 3-group* V *and that* $|V:\boldsymbol{C}_V(\mathfrak{S}_3)|=3$. *Then* $\boldsymbol{O}_{3'}(\mathfrak{S})$ *is a quaternion group.*

PROOF. Let $V=V_1\oplus\cdots\oplus V_r$, where each V_i is an indecomposable $\mathfrak{S}$-module. We may assume that $\mathfrak{S}_3$ does not centralize V_1. Since $|V:\boldsymbol{C}_V(\mathfrak{S}_3)|=3$, we get $|[V, \mathfrak{S}_3]|=3$, so that $[V_1, \mathfrak{S}_3]=[V, \mathfrak{S}_3]$. This implies that $\mathfrak{S}_3$ centralizes $V_2\oplus\cdots\oplus V_r$. Since $\boldsymbol{O}_{3'}(\mathfrak{S})=[\boldsymbol{O}_{3'}(\mathfrak{S}), \mathfrak{S}_3]$, it follows that $\boldsymbol{O}_{3'}(\mathfrak{S})$ also centralizes $V_2\oplus\cdots\oplus V_r$. Thus, $\mathfrak{S}$ is faithfully represented on V_1, so we may assume that $V=V_1$ is indecomposable.

By (B), it follows that $|\boldsymbol{O}_{3'}(\mathfrak{S}):\boldsymbol{O}_{3'}(\mathfrak{S})\cap\boldsymbol{C}(\mathfrak{S}_3)|$ is a power of 2. Let $\mathfrak{T}$ be a S_2-subgroup of $\mathfrak{S}$ which is normalized by $\mathfrak{S}_3$. Thus, $[\mathfrak{T}, \mathfrak{S}_3]\neq 1$. Suppose $[\mathfrak{T}, \mathfrak{S}_3]<\boldsymbol{O}_{3'}(\mathfrak{S})$. Then by induction on $|\mathfrak{S}|$, we get that $[\mathfrak{T}, \mathfrak{S}_3]$ is a quaternion group. Hence, $|\boldsymbol{O}_{3'}(\mathfrak{S}):\boldsymbol{O}_{3'}(\mathfrak{S})\cap\boldsymbol{C}(\mathfrak{S}_3)|=4$.

Let $\mathfrak{F}=\bigcap_{S\in\mathfrak{S}}\boldsymbol{O}_{3'}(\mathfrak{S})\cap\boldsymbol{C}(\mathfrak{S}_3)^S$. Thus, $\overline{\mathfrak{S}}=\boldsymbol{O}_{3'}(\mathfrak{S})/\mathfrak{F}$ has a faithful permutation representation on 4 letters, and $\mathfrak{F}\triangleleft\mathfrak{S}$. Since $[\mathfrak{S}_3, \overline{\mathfrak{S}}]=\overline{\mathfrak{S}}$, and since $3\nmid|\overline{\mathfrak{S}}|$ • it follows that $\overline{\mathfrak{S}}$ is a four-group. Now $\mathfrak{F}\subseteq\boldsymbol{C}(\mathfrak{S}_3)$, and $\boldsymbol{C}_{\mathfrak{S}}(\mathfrak{F})\triangleleft\mathfrak{S}$, so that $\boldsymbol{C}_{\mathfrak{S}}(\mathfrak{F})$ contains $[\boldsymbol{O}_{3'}(\mathfrak{S}), \mathfrak{S}_3]=\boldsymbol{O}_{3'}(\mathfrak{S})$. Hence, $\mathfrak{F}\subseteq\boldsymbol{Z}(\mathfrak{S})$. Since V is indecomposable, $\mathfrak{F}$ is cyclic. Also, $\mathfrak{F}\neq 1$, by (B).

Finally, no nonidentity subgroup of $\mathfrak{F}$ has nontrivial fixed points on V. Since $\mathfrak{F}$ normalizes $[V, \mathfrak{S}_3]$, a group of order 3, we get $|\mathfrak{F}| \leqq 2$, so $|\mathfrak{F}| = 2$ and $O_{3'}(\mathfrak{S}) = [\mathfrak{T}, \mathfrak{S}_3]$, against our assumption.

We may now assume that $[\mathfrak{T}, \mathfrak{S}_3] = O_{3'}(\mathfrak{S})$, so that $O_{3'}(\mathfrak{S})$ is a 2-group. By (B), $\mathfrak{S}_3$ centralizes every characteristic abelian subgroup of $O_{3'}(\mathfrak{S})$. Hence, $O_{3'}(\mathfrak{S})$ is special by Lemma 5.17. Since V is indecomposable, $O_{3'}(\mathfrak{S})$ is extra special. It follows from the proof of (B) that $O_{3'}(\mathfrak{S})$ is a quaternion group.

LEMMA 5.31. *Suppose $\mathfrak{H} = \mathfrak{H}_1 \times \mathfrak{H}_2$, where $\mathfrak{H}_i \simeq A_4$, $i = 1, 2$. Let $\mathfrak{T}$ be the S_2-subgroup of $\mathfrak{H}$ and let $\mathfrak{T}_i = \mathfrak{T} \cap \mathfrak{H}_i$, $i = 1, 2$. Let $\mathfrak{Q}$ be a S_3-subgroup of $\mathfrak{H}$ and let $\mathfrak{Q}_i = \mathfrak{Q} \cap \mathfrak{H}_i$, $i = 1, 2$. Let $\mathfrak{Q}_3$, $\mathfrak{Q}_4$ be the remaining subgroups of $\mathfrak{Q}$ of order 3, and let $\mathfrak{B}$ be any subgroup of $\mathfrak{T}$ of order 4. Then one of the following holds:*

(i) $\mathfrak{B} \cap \mathfrak{T}_1 \neq 1$,
(ii) $\mathfrak{B} \cap \mathfrak{T}_2 \neq 1$,
(iii) $\mathfrak{Q}_3 \subseteq N(\mathfrak{B})$,
(iv) $\mathfrak{Q}_4 \subseteq N(\mathfrak{B})$.

PROOF. Suppose neither (i) nor (ii) holds. Then $\mathfrak{B} = \langle II', JJ' \rangle$, where $I, J \in \mathfrak{T}_1^\#$, $I', J' \in \mathfrak{T}_2^\#$, and $I \neq J$, $I' \neq J'$. Since $\mathfrak{Q}_1$ permutes $\mathfrak{T}_1^\#$ transitively by conjugation, we may choose a generator Q_1 of $\mathfrak{Q}_1$ with $I^{Q_1} = J$; since $\mathfrak{Q}_2$ permutes $\mathfrak{T}_2^\#$ transitively, we may choose a generator Q_2 of $\mathfrak{Q}_2$ with $I'^{Q_2} = J'$. Let $Q = Q_1Q_2$, so that $II'^Q = JJ'$. Since $C_{\mathfrak{T}}(Q) = 1$, it follows that $T^{1+Q+Q^2} = 1$ for all T in $\mathfrak{T}$. Hence, Q normalizes $\mathfrak{B}$. Since $\langle Q \rangle = \mathfrak{Q}_3$ or $\mathfrak{Q}_4$, either (iii) or (iv) holds.

LEMMA 5.32. *Suppose $\mathfrak{S} = \mathfrak{A}\mathfrak{T}$ where $\mathfrak{A} \lhd \mathfrak{S}$, $\mathfrak{A}$ is elementary of order 9 and $\mathfrak{A} = C(\mathfrak{A})$. Suppose also that $\mathfrak{T}$ is a dihedral group of order 8. Let M be an irreducible $F_2\mathfrak{S}$-module on which $\mathfrak{S}$ acts faithfully. Then $|M| = 16$ and $\mathfrak{T}$ contains exactly one four-subgroup $\mathfrak{B}$ such that M is a free $F_2\mathfrak{B}$-module.*

PROOF. Let $\mathfrak{T} = \langle T_1, T_2 \rangle$, where T_1, T_2 are involutions, and let $Z = (T_1T_2)^2$. Let $\mathfrak{B}_i = \langle T_i, Z \rangle$, so that $\mathfrak{B}_1$, $\mathfrak{B}_2$ are the only four-subgroups of $\mathfrak{B}$.

Let $\mathfrak{I} = \{T_1, T_1Z, T_2, T_2Z\}$ and for each I in $\mathfrak{I}$, let $\mathfrak{A}(I) = C_{\mathfrak{A}}(I)$. Since no element of $\mathfrak{I}$ inverts $\mathfrak{A}$, it follows that $|\mathfrak{A}(I)| = 3$ for each I in $\mathfrak{I}$. If $\mathfrak{A}(I) = \mathfrak{A}(J)$, then $\langle I, J \rangle$ centralizes $\mathfrak{A}(I)$. Since Z inverts $\mathfrak{A}$, we get $I = J$. Thus, as I ranges over $\mathfrak{I}$, $\mathfrak{A}(I)$ ranges over all subgroups of $\mathfrak{A}$ of order 3.

Since $\mathfrak{A}$ is elementary of order 9, there is A in $\mathfrak{A}^\#$ such that $M_0 = C_M(A) \neq 0$. Changing notation if necessary, we may assume that $\langle A \rangle = \mathfrak{A}(T_1)$. Thus, $\mathfrak{B}_1$ normalizes $\langle A \rangle$ and Z inverts A, so that M_0

admits $\mathfrak{A}\mathfrak{B}_1=\mathfrak{Q}$. Let M_1 be an irreducible $\mathfrak{Q}$-submodule of M_0. Since $\mathfrak{S}$ acts faithfully on the irreducible module M, we get $\mathbf{C}_M(\mathfrak{A})=0$. Since T_1 inverts $\mathfrak{A}/\langle A\rangle$, T_1Z centralizes $\mathfrak{A}/\langle A\rangle$, so $T_1Z\in \mathbf{O}_2(\mathfrak{Q}/\langle A\rangle)$. Thus, T_1Z centralizes M_1, so we may view M_1 as a $F_2\overline{\mathfrak{Q}}$-module, where $\overline{\mathfrak{Q}}=\mathfrak{Q}/\langle A, T_1Z\rangle$. Thus, $|M_1|=4$. Since $M_1+M_1T_2$ admits $\mathfrak{S}$, we get $M_1\oplus M_1T_2=M$, and so $|M|=16$. Since T_1Z centralizes M_1 and since M_1T_2 admits T_1Z, it follows that $|\mathbf{C}_M(T_1Z)|=8$. Hence, M is not a free $F_2\mathfrak{B}_1$-module.

Choose $m\in M_1-\mathbf{C}_{M_1}(Z)$. Then $\langle mV \mid V\in\mathfrak{B}_2\rangle$ contains M_1 and M_1T_2, so coincides with M. Hence, M is a free $F_2\mathfrak{B}_2$-module, and we are done.

Lemma 5.33. $U_3(3)\supset L_2(7)$.

Proof. As is well known [12], $L_2(7)$ has an irreducible complex matrix representation ρ of degree 3 whose character lies in $Q((-7)^{1/2})$. The restriction of ρ to a subgroup $\mathfrak{H}$ of order 21 is absolutely irreducible and so we may assume that ρ_H lies in $Q((-7)^{1/2})$. This already forces ρ to lie in $Q((-7)^{1/2})$, so we may assume at the outset that ρ lies in $Q((-7)^{1/2})$ and has $\mathfrak{p}$-integral entries where $\mathfrak{p}$ is a divisor of 3 in the ring of integers of $Q((-7)^{1/2})$. Reading (mod $\mathfrak{p}$) gives an absolutely irreducible matrix representation σ of $L_2(7)$ in $SL(3, 9)$ with character ϕ, say. Let $\sigma^*(X)={}^t[\sigma(X^{-1})]^-$, where $^-$ denotes the map induced by the generator of Aut F_9. Thus, σ^* and σ have the same character. Since $\mathfrak{H}$ has just 2 irreducible representations of degree 3 in F_9 and since both lie in $U_3(3)$, we may assume at the outset that σ^* and σ agree on $\mathfrak{H}$. This forces $\sigma=\sigma^*$, so $\sigma(L_2(7))\subset U_3(3)$, as required.

5.6. *2-groups, involutions and 2-length.* The next lemma plays a basic role in this work.

Lemma 5.34. *Suppose the subgroup $\mathfrak{T}$ of the solvable group $\mathfrak{S}$ is elementary of order $2^n>1$, and $|\mathbf{F}(\mathfrak{S})|$ is odd. Then $\mathbf{F}(\mathfrak{S})$ contains a subgroup $\mathfrak{A}$ with the properties*

(a) $\mathfrak{A}=\mathfrak{A}_1\times\mathfrak{A}_2\times\cdots\times\mathfrak{A}_n$, *where $\mathfrak{A}_i$ is of prime order,*

(b) *$\mathfrak{A}_i$ is $\mathfrak{T}$-invariant*, $1\leq i\leq n$,

(c) $\mathbf{C}_{\mathfrak{T}}(\mathfrak{A})=1$,

(d) *if $\mathfrak{T}_i=\bigcap_{j\neq i}\mathbf{C}_{\mathfrak{T}}(\mathfrak{A}_j)$, and $\mathfrak{D}_i=\mathfrak{A}_i\mathfrak{T}_i$, then $\mathfrak{D}_i$ is a dihedral group and $\mathfrak{T}\mathfrak{A}=\mathfrak{D}_1\times\cdots\times\mathfrak{D}_n$.*

Proof. By hypothesis and 0.3.3, $\mathbf{C}_{\mathfrak{T}}(\mathbf{F}(\mathfrak{S}))=1$. Let $\mathfrak{A}$ be a $\mathfrak{T}$-invariant subgroup of $\mathbf{F}(\mathfrak{S})$ minimal subject to (c). Since $\mathfrak{A}$ is nilpotent, we may choose a $\mathfrak{T}$-invariant subgroup $\tilde{\mathfrak{A}}_1$ of prime index in $\mathfrak{A}$. Choose I in $\mathbf{C}_{\mathfrak{T}}(\tilde{\mathfrak{A}}_1)^\#$. Then $\mathfrak{A}_1=[\mathfrak{A}, I]$ is a $\mathfrak{T}$-invariant normal subgroup of $\mathfrak{A}$, and $[\mathfrak{A}, I, I] = [\mathfrak{A}, I]$. The second equality implies that $\mathfrak{A} = \mathfrak{A}_1 \times \tilde{\mathfrak{A}}_1$. Let $\mathfrak{T}_1 = C_{\mathfrak{T}}(\mathfrak{A}_1)$. Since $\mathrm{Aut}(\mathfrak{A}_1)$ is cyclic, $\mathfrak{T} = \mathfrak{T}_1 \times \langle I \rangle$, so if we set $\mathfrak{D}_1 = \langle \mathfrak{A}_1, I \rangle$, $\tilde{\mathfrak{D}}_1 = \langle \mathfrak{T}_1, \tilde{\mathfrak{A}}_1 \rangle$, we get $\mathfrak{T}\mathfrak{A} = \mathfrak{D}_1 \times \tilde{\mathfrak{D}}_1$. Hence, we are done by induction.

LEMMA 5.35. *Let $\mathfrak{M}$ be a subgroup of the group $\mathfrak{S}$ such that*
(a) $|\mathfrak{M}|$ *is even.*
(b) $\mathfrak{M}$ *contains the centralizer of each of its involutions.*
(c) $\bigcap_{S \in \mathfrak{S}} \mathfrak{M}^S$ *is of odd order.*
Then $i(\mathfrak{S}) = 1$.

Let $\mathfrak{T}$ be a S_2-subgroup of $\mathfrak{M}$ and let I be an involution in $\boldsymbol{Z}(\mathfrak{T})$. If in addition to (a), (b), (c) *we also have*
(d) $\boldsymbol{N}(\mathfrak{T}) \subseteq \mathfrak{M}$,
then
(i) $i(\mathfrak{M}) = 1$,
(ii) $\mathfrak{M}$ *contains a subgroup $\mathfrak{M}_0$ of odd order such that* $\mathfrak{M} = \mathfrak{M}_0 \boldsymbol{C}_{\mathfrak{M}}(I)$.

PROOF. Let J be a fixed involution of $\mathfrak{M}$, and let K be any involution of $\mathfrak{S}$ which is not conjugate to J. Then JK is of even order, so J and K commute with an involution L. Applying (b) successively to J and L, we have $K \in \mathfrak{M}$. Thus, $\mathfrak{M}$ contains the normal closure of K in $\mathfrak{S}$, contrary to (c).

Suppose now that (a)–(d) all hold and J is an involution of $\mathfrak{M}$. Then $J = S^{-1}IS$ for some S in $\mathfrak{S}$, so $\mathfrak{T}^S \subseteq \mathfrak{M}$, $\mathfrak{T}^S = \mathfrak{T}^M$, M in $\mathfrak{M}$, so $SM^{-1} \in \boldsymbol{N}(\mathfrak{T}) \subseteq \mathfrak{M}$, so $S \in \mathfrak{M}$, that is, $i(\mathfrak{M}) = 1$.

Let $\mathfrak{C} = \boldsymbol{C}_{\mathfrak{M}}(I)S$ be a coset of $\boldsymbol{C}_{\mathfrak{M}}(I)$ with $S \notin \mathfrak{M}$.[5] Suppose J, K are distinct involutions in $\mathfrak{C}$. Let $M = JK$. Thus, $M^2 \neq 1$ and M centralizes I. Also, $J \in \boldsymbol{C}^*(M)$, so that I and J are not conjugate in $\boldsymbol{C}^*(M)$. Hence, IJ is of even order, so that I and J commute with a common involution I^*. Applying (b) successively to I and I^*, we get $J \in \mathfrak{M}$, against $S \notin \mathfrak{M}$. Hence, each coset of $\boldsymbol{C}_{\mathfrak{M}}(I)$ in $\mathfrak{S} - \mathfrak{M}$ contains at most one involution. Notice that (a) and (c) imply that $\mathfrak{M} \subset \mathfrak{S}$. Since $\mathfrak{M}$ contains exactly $m = |\mathfrak{M} : \boldsymbol{C}_{\mathfrak{M}}(I)|$ involutions and $\mathfrak{S}$ contains exactly $|\mathfrak{S} : \boldsymbol{C}_{\mathfrak{M}}(I)|$ involutions, it follows that each coset $\mathfrak{M}S$ of $\mathfrak{M}$ in $\mathfrak{S} - \mathfrak{M}$ contains exactly m involutions $I_1, \cdots, I_m$. If $m = 1$, then $\mathfrak{M} = \boldsymbol{C}_{\mathfrak{M}}(I)$ and we may take $\mathfrak{M}_0 = 1$. Suppose $m > 1$. Let $\mathfrak{M}_0 = \mathfrak{M} \cap \langle I_1, \cdots, I_m \rangle$ so that $\mathfrak{M}_0$ is normalized by I_1, and $1, I_1 I_2, \cdots, I_1 I_m$ is a set of representatives for the cosets of $\boldsymbol{C}_{\mathfrak{M}}(I)$ in $\mathfrak{M}$. Hence, $\mathfrak{M} = \mathfrak{M}_0 \boldsymbol{C}_{\mathfrak{M}}(I)$. Clearly, $|\mathfrak{M}_0|$ is odd, since I_1 commutes with no involution of $\mathfrak{M}$. The proof is complete.

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[5] The argument here appeared in Feit [11].

LEMMA 5.36. *Let $\mathfrak{S}$ be a 2′-reduced solvable group and let J be an involution of $\mathfrak{S}-O_2(\mathfrak{S})$. Then there is an element Q of $\mathfrak{S}$ of odd prime order which is inverted by J.*

Lemma 5.36 is a special case of Corollary 1 of [**39**], and will be used very often in this work.

LEMMA 5.37. *If $\mathfrak{X}$ is a 2-group and $\mathfrak{X}'$ is elementary and central then $\Omega_2(\mathfrak{X})$ is of exponent 4 and* $\ker(\mathrm{Aut}(\mathfrak{X})\xrightarrow{\mathrm{res}}\mathrm{Aut}(\Omega_2(\mathfrak{X})))$ *is a 2-group.*

PROOF. The first assertion follows from the fact that $[X^2, Y] = [X, Y]^2=1$ for all X, Y in $\mathfrak{X}$, and the second from Lemma 5.17.

LEMMA 5.38. (a) *Suppose $\mathfrak{G}$ is a finite group of even order with no subgroup of index 2.*

(i) *Let $\mathfrak{G}_2$ be a S_2-subgroup of $\mathfrak{G}$ and let $\mathfrak{M}$ be a maximal subgroup of $\mathfrak{G}_2$. Then for each involution I of $\mathfrak{G}$, there is an element G in $\mathfrak{G}$ such that $I^G\in\mathfrak{M}$.*

(ii) *For each $\mathfrak{U}$ in $\mathcal{U}(2)$ and each involution I of $\mathfrak{G}$, some conjugate of I centralizes $\mathfrak{U}$ (see Definition 2.8).*

(b) *Suppose $\mathfrak{T}$ is an elementary 2-group and $\mathfrak{A}$ is a group of odd order which normalizes $\mathfrak{T}$ and has no fixed points on $\mathfrak{T}$. Suppose $I\in\mathfrak{T}$ and $\mathfrak{T}_0$ is of index 2 in $\mathfrak{T}$. Then there is an element A in A with $I^A\in\mathfrak{T}_0$.*

PROOF. (a)(i). Let t be the transfer of $\mathfrak{G}$ into $\mathfrak{G}_2/\mathfrak{M}$. Thus, $t(I) =\mathfrak{M}/\mathfrak{M}$. Since $|\mathfrak{G}: \mathfrak{G}_2|$ is odd, the number of cosets $\mathfrak{C}=\mathfrak{G}_2G$ of $\mathfrak{G}_2$ in $\mathfrak{G}$ which satisfy $\mathfrak{C}I=\mathfrak{C}$ is also odd. Since $t(I)=\mathfrak{M}\prod_G GIG^{-1}$ where G ranges over a set of representatives of the cosets fixed by I, we can find G in $\mathfrak{G}$ with $GIG^{-1}\in\mathfrak{M}$, as required.

Both (a)(ii) and (b) are consequences of (a)(i).

LEMMA 5.39. *Suppose p is an odd prime, $\mathfrak{P}$ is a p-group and $\mathfrak{T}$ is an elementary subgroup of* $\mathrm{Aut}(\mathfrak{P})$ *of order* 8. *Let $\hat{\mathfrak{T}}=\{T \mid T\in\mathfrak{T}^\#,\ C_{\mathfrak{P}}(T)$ is noncyclic$\}$. Suppose that $\mathfrak{T}$ contains a proper subgroup $\mathfrak{T}_0$ which is disjoint from $\hat{\mathfrak{T}}$ and such that $TT'\in\mathfrak{T}_0$ for all T, $T'\in\hat{\mathfrak{T}}$. Then*

(a) *$\mathfrak{P}$ is abelian.*

(b) $m(\mathfrak{P})=3$.

(c) $|\hat{\mathfrak{T}}|=3$ *and* $\langle\hat{\mathfrak{T}}\rangle=\mathfrak{T}$.

(d) $\mathfrak{T}_0=\{TT' \mid T, T'\in\hat{\mathfrak{T}}\}$.

PROOF. Clearly, $m(\mathfrak{P})\geqq 3$, since $GL(2, p)$ contains no elementary subgroup of order 8. Our hypotheses guarantee that if T, $T'\in\hat{\mathfrak{T}}$, then $TT'\notin\hat{\mathfrak{T}}$.

We first treat the case $\mathfrak{P}'=1$. Let $\mathfrak{T}^\#=\{T_1, \cdots, T_7\}$, let $\mathfrak{P}_0 =\Omega_1(\mathfrak{P})$, let $m(P)=m$, and define a_i via $p^{a_i}=|C_{\mathfrak{P}_0}(T_i)|$. If $C_{\mathfrak{P}}(\mathfrak{T})\neq 1$, then $\hat{\mathfrak{T}} \cup \{1\}$ contains a four-subgroup of $\mathfrak{T}$ and so $\mathfrak{T}_0$ does not exist. Hence, $C_{\mathfrak{P}}(\mathfrak{T})=1$. By Satz 2.3 of [44] applied to $\mathfrak{T}$ acting on $\mathfrak{P}_0$, we have $1=p^{-6m+2\Sigma a_i}$, so that $3m=\sum a_i$. Clearly, $a_i \leqq 3$ for all i since $\hat{\mathfrak{T}} \cup \{1\}$ contains no four-subgroup. Assume by way of contradiction that $m \geqq 4$. Suppose $a_1=3$. Then each involution of $\mathfrak{T}/\langle T_1\rangle$ centralizes a subgroup of $C_{\mathfrak{P}_0}(T_1)$ of order p. Let $\mathfrak{P}_0=C_{\mathfrak{P}_0}(T_1)\times\mathfrak{P}^*$, where $\mathfrak{P}^*$ admits $\mathfrak{T}$. Let $\tilde{\mathfrak{P}}$ be a $\mathfrak{T}$-subgroup of $\mathfrak{P}^*$ of order p and let $\tilde{\mathfrak{T}}=C_{\mathfrak{T}}(\tilde{\mathfrak{P}})$. Then $\tilde{\mathfrak{T}} \subseteq \hat{\mathfrak{T}} \cup \{1\}$ and $\tilde{\mathfrak{T}}$ is a four-subgroup of $\mathfrak{T}$. This is impossible, so that $a_i \leqq 2$, $i=1, 2, \cdots, 7$. Since $\hat{\mathfrak{T}}$ is contained in a coset of $\mathfrak{T}_0$ it follows that $|\hat{\mathfrak{T}}| \leqq |\mathfrak{T}_0| \leqq 4$. Thus, $\sum a_i \leqq 4\cdot 2+3<3m$, against the preceding equality. Hence $m=3$ so that (b) holds.

Let $\mathfrak{P}=\mathfrak{P}_1\times\mathfrak{P}_2\times\mathfrak{P}_3$, where $\mathfrak{P}_i$ is an indecomposable $\mathfrak{T}$-group. Let λ_i be the character of $\mathfrak{T}$ on $\mathfrak{P}_i$, and let $\mathfrak{T}_i=C_{\mathfrak{T}}(\mathfrak{P}_i)$, and set $\mathfrak{T}_{ij}=\mathfrak{T}_i\cap\mathfrak{T}_j$. If $i\neq j$, then $\mathfrak{T}_{ij}$ is of order 2; $\mathfrak{T}_{ij}=\langle T_{ij}\rangle$. Also, T_{12}, T_{23}, T_{31} all lie in $\hat{\mathfrak{T}}$ and are distinct since $\mathfrak{T}\subseteq \mathrm{Aut}(\mathfrak{P})$. If σ is any permutation of $\{1, 2, 3\}$, then $T_{\sigma(1)\sigma(2)}$ inverts $\mathfrak{P}_{\sigma(3)}$ and centralizes $\mathfrak{P}_{\sigma(1)}\times\mathfrak{P}_{\sigma(2)}$. Also, $T_{\sigma(1)\sigma(2)}T_{\sigma(2)\sigma(3)}$ centralizes $\mathfrak{P}_{\sigma(2)}$ and inverts both $\mathfrak{P}_{\sigma(1)}$ and $\mathfrak{P}_{\sigma(3)}$, while $T_{12}T_{23}T_{31}$ inverts $\mathfrak{P}$. Thus, (c) and (d) also hold.

We may now assume that $\mathfrak{P}'\neq 1$. We apply the portion of the proof already completed to $\mathfrak{P}/\mathfrak{P}'$. Let $\lambda_1, \lambda_2, \lambda_3$ be the characters of $\mathfrak{T}$ on $\mathfrak{P}/\mathfrak{P}'$ defined previously. We may assume that $\mathfrak{P}'$ is of order p. Then $\Omega_1(\mathfrak{P})$ is of exponent p. If $|\Omega_1(\mathfrak{P})|=p^4$, then we may assume by induction on $|\mathfrak{P}|$ that $\mathfrak{P}=\Omega_1(\mathfrak{P})$, since by 0.3.6, $\mathfrak{T}$ is faithfully represented on $\Omega_1(\mathfrak{P})$. In this case, $\mathfrak{P}$ is the direct product of a group of order p and a nonabelian group of order p^3 and it is straightforward to verify that $\mathfrak{T}_0$ does not exist. Hence, $\Omega_1(\mathfrak{P})$ is of order p^3 and is elementary. Hence, $\mathfrak{P}'\subseteq \mho^1(P)$ so that the character of $\mathfrak{T}$ on $\mathfrak{P}'$ is one of $\lambda_1, \lambda_2, \lambda_3$, say λ_1, so that $\lambda_1=\lambda_2\lambda_3$. This implies that T_{23} centralizes $\mathfrak{P}$, against our hypothesis. The proof is complete.

LEMMA 5.40. *Assume the following*:
(i) $\mathfrak{S}$ *is a solvable group.*
(ii) $i(\mathfrak{S})=1$.
(iii) *A S_2-subgroup $\mathfrak{T}$ of $\mathfrak{S}$ contains more than one involution. Then $\mathfrak{S}$ has 2-length* 1.

PROOF. We assume without loss of generality that $\boldsymbol{O}_{2'}(\mathfrak{S})=1$. Let $\mathfrak{T}_0=\boldsymbol{O}_2(\mathfrak{S})$. We must show that $\mathfrak{T}_0=\mathfrak{T}$.

Let $\mathfrak{M}=\Omega_1(\boldsymbol{Z}(\mathfrak{T}_0))$. Since $i(\mathfrak{S})=1$, $\mathfrak{M}$ contains all the involutions of $\mathfrak{S}$. Let $|\mathfrak{M}|=q=2^m$, and set $\mathfrak{C}=\boldsymbol{C}(\mathfrak{M})$.

Let $\mathfrak{Q}$ be a $S_{2'}$-subgroup of $\mathfrak{S}$ and let $\mathfrak{Q}_0=\mathfrak{Q}\cap\boldsymbol{O}_{2,2'}(\mathfrak{S})$. We assume by way of contradiction that $\mathfrak{T}_0\subset\mathfrak{T}$. Then $\boldsymbol{N}(\mathfrak{Q}_0)$ contains an involution since $\mathfrak{S}=\mathfrak{T}_0\boldsymbol{N}(\mathfrak{Q}_0)$. Since all involutions of $\mathfrak{S}$ are in

$\mathfrak{M}$, $\mathbf{C}_{\mathfrak{M}}(\mathfrak{Q}_0) \neq 1$. Since $\mathbf{C}_{\mathfrak{M}}(\mathfrak{Q}_0) \lhd \mathfrak{S}$, it follows that $\mathfrak{Q}_0$ centralizes $\mathfrak{M}$. Hence $\mathfrak{Q}_0$ centralizes every abelian subgroup of $\mathfrak{T}_0$ which $\mathfrak{Q}_0$ normalizes. By Lemma 5.17, $\mathfrak{T}_1 = [\mathfrak{Q}_0, \mathfrak{T}_0]$ is special. Since $\mathfrak{T}_1 \lhd \mathfrak{S}$, it follows that $\boldsymbol{Z}(\mathfrak{T}_1) = \mathfrak{M}$.

We argue that $|\mathfrak{T}_1| = q^n$ for some $n \geqq 2$. Namely, if $W \in \mathfrak{M}^\#$ and there are exactly r solutions to the equation $X^2 = W$ with X in $\mathfrak{T}_1$, then r does not depend on W, so $|\mathfrak{T}_1| = r(q-1)+q$. Hence, $q-1$ divides $|\mathfrak{T}_1| - 1$, which implies the assertion.

Let A be the $q-1$ by $q-1$ matrix whose rows are indexed by the elements of $\mathfrak{M}^\#$ and whose columns are indexed by the hyperplanes of $\mathfrak{M}$, and where the (i, j) entry is 1 if i is contained in j and is 0 otherwise. Then ${}^tAA = xI + yN$, where N is the matrix with 1 in each entry, $x+y = q/2-1$, $y = q/4-1$. Since N has rank 1 and $q-1$ is the only nonzero characteristic root of N, it follows that ${}^tAA$ is nonsingular. Hence, A is nonsingular. We view $\mathfrak{S}$ as a permutation group on the elements of $\mathfrak{M}^\#$, and as a permutation group on the hyperplanes of $\mathfrak{M}$. By a result of Brauer (0.3.12), both representations of $\mathfrak{S}$ are transitive.

We next argue that $n=3$. Namely, $q = \sum \epsilon_\chi \chi(1)$, where χ ranges over the irreducible characters of $\mathfrak{T}_1$ and $\epsilon_\chi = 0$, 1 or -1 [**15**]. We have $\epsilon_\chi = 1$ for all the q^{n-1} linear characters of $\mathfrak{T}_1$. Let $\mathfrak{U}$ be a hyperplane of $\mathfrak{M}$ and $\overline{\mathfrak{T}}_1 = \mathfrak{T}_1/\mathfrak{U}$. Since $\mathfrak{Q}_0$ centralizes $\mathfrak{M}$, $\mathfrak{Q}_0$ acts on $\overline{\mathfrak{T}}_1$. Since $\mathfrak{T}_1 = [\mathfrak{T}_1, \mathfrak{Q}_0]$, so also $\overline{\mathfrak{T}}_1 = [\overline{\mathfrak{T}}_1, \mathfrak{Q}_0]$. Now $|\overline{\mathfrak{T}}_1'| = 2$, and $\overline{\mathfrak{T}}_1' = \mathbf{C}_{\overline{\mathfrak{T}}_1}(\mathfrak{Q}_0)$. Since $\mathfrak{Q}_0$ normalizes $\boldsymbol{Z}(\overline{\mathfrak{T}}_1)$, it follows that $\boldsymbol{Z}(\overline{\mathfrak{T}}_1)$ is elementary of order $2q_0$, say, this condition being the definition of q_0. Thus, $\overline{\mathfrak{T}}_1$ is the direct product of an elementary group of order q_0 and an extra special group of order $2a^2$, this condition being the definition of a. It follows that $\epsilon = \epsilon_\chi = \epsilon_{\chi'} \neq 0$ for all the nonlinear irreducible characters χ, χ' of $\overline{\mathfrak{T}}_1$, of which there are q_0. Since $\mathfrak{S}$ permutes transitively the hyperplanes of $\mathfrak{M}$, we get $q = q^{n-1} + \epsilon q_0(q-1)a$. Here we also have $q_0 a^2 = q^{n-1} = |\overline{\mathfrak{T}}_1 : \overline{\mathfrak{T}}_1'|$. This already shows that $\epsilon = -1$, and so $q = q_0a^2 - q_0(q-1)a = q_0a(a-q+1)$. Since $a-q+1$ is odd, and q is a power of 2, we get $q=a$, $q_0 = 1$, $n=3$.

Suppose $\mathfrak{T}_1 \subset \mathfrak{T}_0$. Let $\mathfrak{T}_2/\mathfrak{T}_1$ be a chief factor of $\mathfrak{S}$ with $\mathfrak{T}_2 \subseteq \mathfrak{T}_0$. Then $\mathfrak{Q}_0$ centralizes $\mathfrak{T}_2/\mathfrak{T}_1$ so $\mathfrak{T}_2 = \mathfrak{T}_1 \mathbf{C}_{\mathfrak{T}_2}(\mathfrak{Q}_0)$. Let $\mathfrak{T}_3/\mathfrak{M} = \mathbf{C}_{\mathfrak{T}_1/\mathfrak{M}}(\mathbf{C}_{\mathfrak{T}_2}(\mathfrak{Q}_0))$. Then $\mathbf{C}_{\mathfrak{T}_2}(\mathfrak{Q}_0)$ centralizes $\mathfrak{T}_3$, by the three subgroups lemma and the equality $\mathfrak{T}_3 = [\mathfrak{T}_3, \mathfrak{Q}_0]$. Also, $\mathfrak{T}_3 \lhd \mathfrak{S}$. Choose $T \in \mathbf{C}_{\mathfrak{T}_1}(\mathfrak{Q}_0) - \mathfrak{T}_1$. Then $T^2 \in \mathfrak{M}^\#$, so $T^2 = T_3^2$ for some T_3 in $\mathfrak{T}_3$. Hence, TT_3 is an involution of $\mathfrak{T}_0 - \mathfrak{M}$. This is impossible, so $\mathfrak{T}_1 = \mathfrak{T}_0$.

Let $|\mathfrak{T}_0 : \mathfrak{T}_0'| = 2^l$ so that $l = 2m$. Let $M = \mathfrak{T}_0/\mathfrak{T}_0'$ so that M is a $F_2\mathfrak{Q}$-module. Let N be an irreducible submodule of M. Suppose $N \subset M$. Let $|N| = q^*$. If $q^* < q$, then $\mathfrak{Q}$ has a subgroup $\mathfrak{Q}^*$ of index

$<q-1$ such that $C_N(\mathfrak{Q}^*)\neq 0$. Hence, $C_{\mathfrak{T}_0}(\mathfrak{Q}^*)\neq 1$, so that $C_M(Q^*)\neq 1$. This implies that $\mathfrak{Q}$ is not transitive on $\mathfrak{M}^\#$. We conclude that $q^*\geqq q$. Since this inequality holds for every irreducible submodule N of M, we get $M=N\oplus N'$, where $|N|=|N'|=q$.

Let k be the algebraic closure of F_2, and let $M^*=k\otimes M$. Since $|\mathfrak{Q}|$ is odd, it follows that the irreducible submodules of M^* have odd dimension. By the previous paragraph, we get that every irreducible submodule of M^* has a dimension which is a proper divisor of $2m$, so has dimension $\leqq m$.

Since $|\mathfrak{Q}|$ is odd, a result of Ito [28] implies that for each $p>m$, $\mathfrak{Q}$ has a normal abelian S_p-subgroup. Since $C_{\mathfrak{Q}}(\mathfrak{Q}_0)=Z(\mathfrak{Q}_0)$, it follows that $\mathfrak{Q}_0$ contains all the normal abelian S-subgroups of $\mathfrak{Q}$. Hence, since $\mathfrak{Q}_0\subseteq\mathfrak{C}$, it follows that all the prime divisors of $|\mathfrak{S}:\mathfrak{C}|$ are $\leqq m$. By an elementary number theoretic result [4], we get $m=6$. Hence $|\mathfrak{S}:\mathfrak{C}|$ is a $\{2, 3, 5\}$-number, against $2^6-1=3^2\cdot 7$. The proof is complete.

LEMMA 5.41. *Suppose the following hold*:

(a) $\mathfrak{S}$ *is a solvable group.*

(b) $O_{2'}(\mathfrak{S})=1$.

(c) $\mathfrak{S}$ *contains a noncyclic abelian subgroup of order* 8.

(d) *If* $\mathfrak{K}$ *is any proper subgroup of* $\mathfrak{S}$ *of index a power of* 2, *then* $\mathfrak{K}$ *contains no noncyclic abelian subgroup of order* 8.

Let $\mathfrak{T}$ *be a* S_2*-subgroup of* $\mathfrak{S}$. *Then* $\mathfrak{T}\lhd\mathfrak{S}$ *and one of the following holds*:

(i) $\mathfrak{T}'=1$.

(ii) $\mathfrak{T}$ *is extra special of width at least* 2.

(iii) $\mathfrak{T}$ *contains a quaternion subgroup* $\mathfrak{T}_0$ *of index* 2 *and* $\mathfrak{T}=\mathfrak{T}_0C_{\mathfrak{T}}(\mathfrak{T}_0)$.

(iv) $\mathfrak{T}$ *is special and* $Z(\mathfrak{T})$ *is a four-group.*

PROOF. If $\mathfrak{T}'=1$, the lemma follows from the containment $C(O_2(\mathfrak{S}))\subseteq O_2(\mathfrak{S})$.

Let $\mathfrak{T}_0=O_2(\mathfrak{S})$. First, suppose that $\mathfrak{T}_0$ contains a noncyclic abelian subgroup of order 8. By (d), $\mathfrak{T}_0=\mathfrak{T}$.

Let $c=\mathrm{cl}(\mathfrak{T})$, and assume that $c\geqq 3$. Then $C_{c-1}(\mathfrak{T})$ is abelian so is of order 4 or is cyclic. But then (d) is violated in $C_{\mathfrak{S}}(C_{c-1}(\mathfrak{T}))$. Thus $c=2$.

Case 1. $|Z(\mathfrak{T})|=2$. Since $|Z(\mathfrak{T})|=2$, $D(\mathfrak{T})\subseteq Z(\mathfrak{T})$, so $D(\mathfrak{T})=Z(\mathfrak{T})$. Thus, $\mathfrak{T}$ is extra special. Clearly, the width of $\mathfrak{T}$ is at least 2, so (ii) holds.

Case 2. $Z(\mathfrak{T})$ is cyclic of order at least 4.

Clearly, $\mathfrak{T}/\boldsymbol{Z}(\mathfrak{T})$ is a chief factor of $\mathfrak{S}$, so $|\mathfrak{T}'|=2$. Hence $\Omega_2(\mathfrak{T})$ is of exponent 4, so $\mathfrak{T}=\Omega_2(\mathfrak{T})$. Thus, $\boldsymbol{Z}(\mathfrak{T})/\mathfrak{T}'$ is a direct factor of $\mathfrak{T}/\mathfrak{T}'$, so $\mathfrak{T}/\mathfrak{T}'=\boldsymbol{Z}(\mathfrak{T})/\mathfrak{T}'\times\mathfrak{T}_1/\mathfrak{T}'$, with $\mathfrak{T}_1\lhd\mathfrak{S}$. Hence, $\mathfrak{T}_1$ contains no noncyclic abelian subgroup of order 8 and $\mathfrak{T}_1/\mathfrak{T}'$ is a chief factor of $\mathfrak{S}$. It follows that $\mathfrak{T}_1$ is a quaternion group, so (iii) holds.

Case 3. $\boldsymbol{Z}(\mathfrak{T})$ is noncyclic.

Since $\boldsymbol{Z}(\mathfrak{T})$ is noncyclic, $\boldsymbol{Z}(\mathfrak{T})$ is a four-group. Hence, $\mathfrak{T}/\boldsymbol{Z}(\mathfrak{T})$ is a chief factor of $\mathfrak{S}$. If $\boldsymbol{Z}(\mathfrak{T})=\mathfrak{T}'$, then $\mathfrak{T}$ is special and (iv) holds. We may assume that $|\mathfrak{T}'|=2$. Hence, $\boldsymbol{Z}(\mathfrak{T})=\boldsymbol{Z}(\mathfrak{S})$. Also, $\boldsymbol{Z}(\mathfrak{T})/\mathfrak{T}'$ is a direct factor of $\mathfrak{T}/\mathfrak{T}'$, so $\mathfrak{T}/\mathfrak{T}'=\boldsymbol{Z}(\mathfrak{T})/\mathfrak{T}'\times\mathfrak{T}_1/\mathfrak{T}'$, with $\mathfrak{T}_1\lhd\mathfrak{S}$. Hence, $\mathfrak{T}_1$ is a quaternion group, so that (iii) holds.

Finally, suppose that $\boldsymbol{O}_2(\mathfrak{S})$ contains no noncyclic abelian subgroup of order 8. If $\boldsymbol{O}_2(\mathfrak{S})$ contains no four-subgroup, then $\boldsymbol{O}_2(\mathfrak{S})$ is necessarily a quaternion group, so $\mathfrak{S}$ contains no noncyclic abelian subgroup of order 8. Let $\mathfrak{N}$ be a four-subgroup of $\boldsymbol{O}_2(\mathfrak{S})$. Clearly, $\mathfrak{N}$ is self-centralizing in $\boldsymbol{O}_2(\mathfrak{S})$, so $\boldsymbol{O}_2(\mathfrak{S})$ is either of maximal class or $\boldsymbol{O}_2(\mathfrak{S})=\mathfrak{N}$. Since the only 2-group of maximal class which admits a nonidentity automorphism of odd order is the quaternion group, we get $\boldsymbol{O}_2(\mathfrak{S})=\mathfrak{N}$. But then $\mathfrak{S}$ contains no noncyclic abelian subgroup of order 8. The proof is complete.

Lemma 5.42. *Suppose the following hold*:

(a) *$\mathfrak{S}$ is a solvable group which is faithfully represented as automorphisms of the elementary 3-group $\mathfrak{E}$.*

(b) $\boldsymbol{O}_3(\mathfrak{S})=1$.

(c) *$\mathfrak{Z}$ is a subgroup of $\mathfrak{S}$ of order 3 such that $\mathfrak{E}=\mathfrak{E}_0\times\mathfrak{E}_1$ where $\mathfrak{Z}$ centralizes $\mathfrak{E}_1$, $|\mathfrak{E}_0|=27$, and $\mathfrak{E}_0$ is an indecomposable $\mathfrak{Z}$-module.*

(d) *$\mathfrak{Z}$ normalizes but does not centralize the four-subgroup $\mathfrak{Q}$ of $\mathfrak{S}$.*

(e) *$\mathfrak{Z}$ is the center of a S_3-subgroup of $\mathfrak{S}$.*

Then $[\boldsymbol{O}_{3'}(\mathfrak{S}), \mathfrak{Z}]$ is either a four-group or is the central product of two quaternion groups. Also, $\mathfrak{E}=\mathfrak{E}_0^*\times\mathfrak{E}_1^*$ where $\mathfrak{E}_1^*$ is centralized by $[\boldsymbol{O}_{3'}(\mathfrak{S}), \mathfrak{Z}]$ and $\mathfrak{E}_0^*$ is an irreducible faithful $F_3[\boldsymbol{O}_{3'}(\mathfrak{S}), \mathfrak{Z}]$-module.

Proof. By (e), $\mathfrak{Z}\subseteq\boldsymbol{O}_{3',3}(\mathfrak{S})$, so $\mathfrak{Q}\subseteq\boldsymbol{O}_{3'}(\mathfrak{S})$. We assume therefore without loss of generality that $\boldsymbol{O}_{3'}(\mathfrak{S})=[\boldsymbol{O}_{3'}(\mathfrak{S}), \mathfrak{Z}]$, $\mathfrak{S}=\boldsymbol{O}_{3'}(\mathfrak{S})\mathfrak{Z}$.

Let $\mathfrak{F}$ be the normal closure of $\mathfrak{Q}$ in $\mathfrak{S}$ and set $\mathfrak{K}=\boldsymbol{O}_{3'}(\mathfrak{S})$. For each prime q, let $\mathfrak{K}_q$ be a S_q-subgroup of $\mathfrak{K}$ normalized by $\mathfrak{Z}$. If $q\neq 2$ and $q\neq 13$, it follows immediately from (B) that $\mathfrak{Z}$ centralizes $\mathfrak{K}_q$. Since $\mathfrak{K}=[\mathfrak{K}, \mathfrak{Z}]$ and since $\mathfrak{K}$ is solvable, it follows that $\mathfrak{K}$ is a 2, 13-group. Since $[\mathfrak{K}_{13}, \mathfrak{Z}]$ is of order at most 13, it follows that $\mathfrak{K}_2\lhd\mathfrak{K}$. Thus, $\mathfrak{F}$ is a 2-group.

Suppose $[\mathfrak{K}_{13}, \mathfrak{Z}]\neq 1$. Then $[\mathfrak{K}_{13}, \mathfrak{Z}]=\tilde{\mathfrak{K}}$ is of order 13 and $[\tilde{\mathfrak{K}}, \mathfrak{E}]$

is of order 27. This implies that $\tilde{\mathfrak{K}}$ centralizes $\mathfrak{K}_2$. Hence, $\mathfrak{Q}$ centralizes $\mathfrak{E}$, against (a). Hence, $\mathfrak{Z}$ centralizes $\mathfrak{K}_{13}$, so $\mathfrak{K}$ is a 2-group.

Case 1. $\mathfrak{K}$ contains a noncyclic characteristic elementary abelian subgroup $\mathfrak{A}$.

Let $\mathfrak{A}_0 = \mathfrak{A} \cap \boldsymbol{Z}(\mathfrak{K})$. We may assume that $\mathfrak{A}_0 = \Omega_1(\boldsymbol{Z}(\mathfrak{K}))$.

First, suppose that $[\mathfrak{A}_0, \mathfrak{Z}] \neq 1$. Then $\mathfrak{A}_1 = [\mathfrak{A}_0, \mathfrak{Z}]$ is a four-group and $[\mathfrak{E}, \mathfrak{A}_1]$ is of order 27 and is normalized by $\mathfrak{K}\mathfrak{Z}$, while $\boldsymbol{C}_{\mathfrak{E}}(\mathfrak{A}_1)$ is centralized by $\mathfrak{K}\mathfrak{Z}$. Hence, $\mathfrak{K}'$ centralizes $\mathfrak{E}$, so $\mathfrak{K}' = 1$. Hence, $\mathfrak{K}$ is a four-group and the lemma holds.

Next, suppose that $\mathfrak{Z}$ centralizes $\mathfrak{A}_0$. Let $\mathfrak{E} = M_1 \times \cdots \times M_s$, where M_i is the join of all the irreducible $\mathfrak{K}$-submodules of $\mathfrak{E}$ which are isomorphic to one of its irreducible constituents. Then $\mathfrak{Z}$ permutes the M_i. If $\mathfrak{Z}$ permutes M_1, M_2, M_3 transitively, then $|M_1| = 3$, and $\mathfrak{Z}$ centralizes $M_4 \times \cdots \times M_s$. In this case, $\mathfrak{K}'$ centralizes $\mathfrak{E}$, so $\mathfrak{K}' = 1$, against $[\mathfrak{A}_0, \mathfrak{Z}] = 1$. Hence, $\mathfrak{Z}$ normalizes each M_i. Hence, we may assume that $\mathfrak{Z}$ centralizes $M_2 \times \cdots \times M_s$ and so $\mathfrak{S}$ has a faithful irreducible representation. Hence, $\mathfrak{A}_0$ is cyclic, so $\boldsymbol{Z}(\mathfrak{K})$ is cyclic.

Since $\mathfrak{Z}$ has no fixed points on $\mathfrak{K}/\mathfrak{K}'$, it follows that $\mathfrak{Z}$ has no fixed points on $\mathfrak{A} \cap \boldsymbol{Z}_2(\mathfrak{K})/\mathfrak{A}_0$. Hence, $\mathfrak{A} \cap \boldsymbol{Z}_2(\mathfrak{K})$ is elementary of order 8. Let $\mathfrak{C} = \boldsymbol{C}_{\mathfrak{K}}(\mathfrak{A} \cap \boldsymbol{Z}_2(\mathfrak{K}))$. Then $\mathfrak{C} \lhd \mathfrak{S}$, and $\boldsymbol{D}(\mathfrak{C})$ centralizes $[\mathfrak{A} \cap \mathfrak{Z}_2(\mathfrak{K}), \mathfrak{Z}, \mathfrak{E}]$. Hence, $\boldsymbol{D}(\mathfrak{C}) = 1$, since $\boldsymbol{D}(\mathfrak{C}) \lhd \mathfrak{S}$. Since $\mathfrak{K}/\mathfrak{C}$ is a four-group, and since $\boldsymbol{Z}(\mathfrak{K})$ is cyclic, it follows that $\mathfrak{C} = \mathfrak{A} \cap \boldsymbol{Z}_2(\mathfrak{K})$. Thus, $|\mathfrak{K}| = 2^5$ and $\mathfrak{K}' = \boldsymbol{Z}(\mathfrak{K})$ is of order 2. In this case, $\mathfrak{K}$ is the central product of two quaternion groups and we are done.

Case 2. Every characteristic abelian subgroup of $\mathfrak{K}$ is cyclic.

Since $\mathfrak{K} = [\mathfrak{K}, \mathfrak{Z}]$, Lemma 5.12 implies that $\mathfrak{K}$ is extra special. The width of $\mathfrak{K}$ is at least 2 since $\mathfrak{Q}$ exists. It follows that the width of $\mathfrak{K}$ is two. Since $\langle \mathfrak{Q}, \mathfrak{K}' \rangle$ is elementary of order 8, $\mathfrak{K}$ is the central product of 2 quaternion groups. The proof is complete.

Lemma 5.43. *Suppose $\mathfrak{G}$ is a group with no subgroup of index* 2, *$\mathfrak{T}$ is a S_2-subgroup of $\mathfrak{G}$ and $\mathfrak{N} = \boldsymbol{N}(\mathfrak{T})$.*

(a) $\langle \boldsymbol{Z}(\mathfrak{T}) \cap \mathfrak{T}', \ \Omega_1(\boldsymbol{Z}(\mathfrak{T})) \rangle = (\boldsymbol{Z}(\mathfrak{T}) \cap \mathfrak{T}') \times \mathfrak{F}$, *where* $\mathfrak{F} \lhd \mathfrak{N}$ *and* $\mathfrak{F} \cap \boldsymbol{Z}(\mathfrak{N}) = 1$.

(b) *Suppose $\mathfrak{T}$ is isomorphic to a subgroup of $GL(3, q)$, q is an odd prime power and $\mathfrak{T}$ contains an elementary subgroup of order* 8. *Then* $\mathfrak{T}' = 1$.

Proof. (a) If $\Omega_1(\boldsymbol{Z}(\mathfrak{T})) \subseteq \mathfrak{T}'$, the lemma is trivial, so suppose $\Omega_1(\boldsymbol{Z}(\mathfrak{T})) \not\subseteq \mathfrak{T}'$. By complete reducibility of $\Omega_1(\boldsymbol{Z}(\mathfrak{T}))$, it follows that $\Omega_1(\boldsymbol{Z}(\mathfrak{T})) = (\Omega_1(\boldsymbol{Z}(\mathfrak{T})) \cap \mathfrak{T}') \times \mathfrak{F}$, where $\mathfrak{F} \lhd \mathfrak{N}$. Thus, it suffices to show that $\mathfrak{F} \cap \boldsymbol{Z}(\mathfrak{N}) = 1$. Suppose false, and I is an involution of $\mathfrak{F} \cap \boldsymbol{Z}(\mathfrak{N})$. Let $\mathfrak{T}_0$ be a subgroup of $\mathfrak{T}$ which contains $\mathfrak{T}'$ and is maximal subject to $I \notin \mathfrak{T}_0$. Thus, $\mathfrak{T}/\mathfrak{T}_0$ is cyclic and every involution of $\mathfrak{T} - \mathfrak{T}_0$ lies in

$\mathfrak{T}_0 I$. Let $I=I_1, \cdots, I_m$ be all the involutions of $\mathfrak{T}-\mathfrak{T}_0$ which are $\mathfrak{G}$-conjugate to I. Suppose $I_i \in \boldsymbol{Z}(\mathfrak{T})$. Then I and I_i are conjugate in $\mathfrak{N}$, so $I=I_i$. Hence, m is odd. It follows that if t is the transfer of $\mathfrak{G}$ into $\mathfrak{T}/\mathfrak{T}_0$, then $t(I)=\mathfrak{T}_0 I$, against $2 \nmid |\mathfrak{G}: \mathfrak{G}'|$.

(b) Let V be the underlying space on which $GL(3, q)$ acts, and assume without loss of generality that $\mathfrak{T} \subseteq GL(3, q)$. If $m(\boldsymbol{Z}(\mathfrak{T})) \geqq 3$, then V is the direct sum of 1-dimensional $\mathfrak{T}$-subspaces and so $\mathfrak{T}'=1$. We may assume that $m(\boldsymbol{Z}(\mathfrak{T})) \leqq 2$. Let $\mathfrak{F}$ be an elementary subgroup of $\mathfrak{T}$ of order 8. Hence, $\mathfrak{F}$ contains an element which inverts V. Thus, $\Omega_1(\boldsymbol{Z}(\mathfrak{T})) \not\subseteq \mathfrak{T}'$. Thus, by (a), $\boldsymbol{Z}(\mathfrak{T})$ contains a four-group $\mathfrak{N}$ with $\mathfrak{N} \cap \mathfrak{T}'=1$. Since $\mathfrak{T}' \neq 1$, $m(\boldsymbol{Z}(\mathfrak{T})) \geqq 3$. The proof of (b) is complete.

REMARK. The group $U_3(3)$ shows that with $\mathfrak{G}=U_3(3)$, $\mathfrak{N}$, $\mathfrak{T}$ as in Lemma 5.43, we cannot conclude that $\mathfrak{N}$ has no fixed points on $\boldsymbol{Z}(\mathfrak{T})/\boldsymbol{Z}(\mathfrak{T}) \cap \mathfrak{T}'$, since $\boldsymbol{Z}(\mathfrak{T})$ is cyclic of order 4 and $\boldsymbol{Z}(\mathfrak{T}) \cap \mathfrak{T}'$ is of order 2.

LEMMA 5.44. *Suppose* $\mathfrak{D}=gp\langle D, T \mid D^n=T^2=1,\ TDT=D^{-1}\rangle$ *is a dihedral group of order* $2n$ *with* n *odd*, $n>1$, *and that* $\mathfrak{D}$ *is represented as automorphisms of an abelian* 2*-group* $\mathfrak{A}$. *Suppose also that* $\mathfrak{A} \cap \boldsymbol{C}(D) =1$. *Let* $\mathfrak{A}_0=\mathfrak{A} \cap \boldsymbol{C}(T)$. *Then* $\mathfrak{A}=\mathfrak{A}_0 \times \mathfrak{A}_0^D$.

PROOF. Set $\mathfrak{B}=\langle \mathfrak{A}_0, \mathfrak{A}_0^D\rangle$. If $A \in \mathfrak{A}_0 \cap \mathfrak{A}_0^D$, then both T and $D^{-1}TD$ centralize A, so that $TD^{-1}TD=D^2$ centralizes A. As D^2 is also a generator for $\langle D\rangle$, we get $A=1$. Thus, $\mathfrak{B}=\mathfrak{A}_0 \times \mathfrak{A}_0^D$. Let $\mathfrak{B}_1=\mathfrak{A}_0^{D^{-1}} \times \mathfrak{A}_0^D$. For each A in $\mathfrak{A}_0$, we get $A^{D^{-1}}A^D \in \mathfrak{A}_0$; so $\mathfrak{A}_0 \subseteq \mathfrak{B}_1$. Hence, $\mathfrak{A}_0^D \subseteq \mathfrak{A}_0 \times \mathfrak{A}_0^{D^2}$, so that $\mathfrak{B}$ admits $\mathfrak{D}$. For each A in $\mathfrak{A}$, we have $A^T =A^{1+T}A^{-1}$, and $A^{1+T} \in \mathfrak{A}_0$. Thus, T inverts $\mathfrak{A}/\mathfrak{B}$. This implies that $\mathfrak{D}$ centralizes $\mathfrak{A}/\mathfrak{B}$, and so $\mathfrak{A}=\mathfrak{B}$, as required.

LEMMA 5.45. *Suppose* V *is an elementary abelian* 2*-group and* $\mathfrak{S}$ *is a solvable subgroup of* $\operatorname{Aut}(V)$ *with* $O_2(\mathfrak{S})=1$. *Let* P *be an element of* $\mathfrak{S}$ *of order* 4, *and assume that the minimal polynomial of* P *on* V *divides* $(x-1)^3$. *Let* $V=V_1 \oplus V_2 \oplus \cdots \oplus V_t$, *where each* V_i *is an indecomposable* $\langle P\rangle$*-module, and let* $|V_i|=2^{d_i}$, $1 \leqq i \leqq t$, *notation being chosen so that* $d_1 \geqq d_2 \geqq \cdots \geqq d_t$. *Then* $d_2=3$.

PROOF. Suppose false. Since $P^2 \neq 1$, it follows that $d_1 \geqq 3$. Thus, our hypothesis guarantees that $d_1=3$, $d_i \leqq 2$, $2 \leqq i \leqq t$.

We proceed by induction on $|\mathfrak{S}|$. We get that $\mathfrak{S}=\mathfrak{Q}\langle P\rangle$, where $\mathfrak{Q}$ is a special q-group and where P^2 inverts $\mathfrak{Q}/\mathfrak{Q}'$ and P^2 centralizes $\mathfrak{Q}'$. Since P is an exceptional element in the sense of Hall-Higman [26], it follows that $\mathfrak{Q}$ is nonabelian, and that $q=3$. Since we are proceeding by way of contradiction, we may also assume that V is an irreducible S-module.

Since P^2 is an involution, we have $\mathfrak{Q}=\mathfrak{J}\mathfrak{Q}'$, where $\mathfrak{J}$ is the set of elements of $\mathfrak{Q}$ which are inverted by P^2. Since $\mathfrak{Q}$ is nonabelian, we may choose Q_1, Q_2 in $\mathfrak{J}$ such that Q_1 and Q_2 do not commute.

Since $d_1=3$, $d_i \leqq 2$ for $i=2, \cdots, t$, it follows that $C_V(P^2)$ is a hyperplane of V. Since $\langle P^2, Q_1, Q_2\rangle = \langle P^2, Q_1^{-1}P^2Q_1, Q_2^{-1}P^2Q_2\rangle$, it follows that $\langle Q_1, Q_2\rangle$ centralizes a subgroup W of index 8 in V. Since the S_3-subgroups of $\mathrm{Aut}(V/W)$ are abelian, it follows that $[Q_1, Q_2]$ centralizes V/W and W, so centralizes V. The proof is complete.

LEMMA 5.46. *Suppose $\mathfrak{S}$ is a solvable $\{2, p, q\}$-group where p, q are distinct odd primes. Let $\{\mathfrak{S}_2, \mathfrak{S}_p, \mathfrak{S}_q\}$ be a Sylow system for $\mathfrak{S}$ and assume the following*:

(a) *$\mathfrak{S}_p$ is a minimal normal subgroup of $\mathfrak{S}$ of order p or p^2.*

(b) *$|\mathfrak{S}_q| = q$ and $\mathfrak{S}_p = [\mathfrak{S}_p, \mathfrak{S}_q]$.*

(c) *$O_2(\mathfrak{S})=1$.*

Let E be an irreducible $F_2\mathfrak{S}$-module on which $\mathfrak{S}_p$ acts faithfully. Let $E_0 = C_E(\mathfrak{S}_2)$. Then one of the following holds:

(i) $|E_0| \geqq 8$.

(ii) *$|E_0| = 4$ and $\mathfrak{S}_q$ does not normalize E_0.*

PROOF. Let $E=E_1 \oplus \cdots \oplus E_s$, where the E_i are the Wedderburn components of E as $\mathfrak{S}_p$-module.

First, suppose $|\mathfrak{S}_p| = p$. Then $\mathfrak{S}_2\mathfrak{S}_q$ is cyclic and $\mathfrak{S}$ is a Frobenius group. Since $q \geqq 3$, we get that (i) holds.

We may now assume that $|\mathfrak{S}_p| = p^2$. Here $s > 1$ and $\mathfrak{S}_2\mathfrak{S}_q$ permutes $\{E_1, \cdots, E_s\}$ transitively.

Case 1. s is a power of 2.

Let $\mathfrak{L}_i$ be the stabilizer of E_i in $\mathfrak{S}$, so that $q \mid |\mathfrak{L}_i|$, $|\mathfrak{S}:\mathfrak{L}_i| = s$. We choose notation so that $\mathfrak{S}_q \subset L_1$. Let $\tilde{\mathfrak{S}}_2 = \mathfrak{S}_2 \cap \mathfrak{L}_1$, so that $|\mathfrak{S}_2 : \tilde{\mathfrak{S}}_2| = s$. Let $\mathfrak{T}$ be a transversal to $\tilde{\mathfrak{S}}_2$ in $\mathfrak{S}_2$.

If $|C_{E_1}(\tilde{\mathfrak{S}}_2)| \geqq 8$, let e_1, e_2, e_3 be linearly independent elements of $C_{E_1}(\tilde{\mathfrak{S}}_2)$. Let $e_i^* = \sum_{t \in \mathfrak{T}} e_i t$. Then e_1^*, e_2^*, e_3^* are linearly independent elements of E_0, so (i) holds. We may assume that $|C_{E_1}(\tilde{\mathfrak{S}}_2)| \leqq 4$. By induction, we get $|C_{E_1}(\tilde{\mathfrak{S}}_2)| = 4$ and $\mathfrak{S}_q$ does not normalize $C_{E_1}(\tilde{\mathfrak{S}}_2)$. Hence, $\tilde{\mathfrak{S}}_2 \ntriangleleft \tilde{\mathfrak{S}}_2\mathfrak{S}_q$, and in particular, $\tilde{\mathfrak{S}}_2\mathfrak{S}_q$ is nonabelian. On the other hand, $\mathfrak{L}_1$ acts reducibly on $\mathfrak{S}_p$, since $C_{\mathfrak{S}_p}(E_1)$ is of order p and is normal in $\mathfrak{L}_1$. Hence, $\mathfrak{L}_1'$ centralizes $\mathfrak{S}_p$. This is not the case since $\mathfrak{S}_p = F(\mathfrak{S})$.

Case 2. s is not a power of 2.

Since $\mathfrak{S}_p \subseteq \mathfrak{L}_1$, it follows that $|\mathfrak{S}:\mathfrak{L}_1| = 2^a q$ for some $a \geqq 0$. Let $\mathfrak{O}_1, \cdots, \mathfrak{O}_r$ be the orbits of $\{E_1, \cdots, E_s\}$ under $\mathfrak{S}_2$. Let $E^i = \oplus E_j$, where E_j ranges over $\mathfrak{O}_i$. Thus, $E = E^1 \oplus \cdots \oplus E^r$ as $\mathfrak{S}_2$-module. If $r \geqq 3$, then (i) holds. Hence, we may assume that $r=2$. If $|C_{E^i}(\mathfrak{S}_2)|$

$\geqq 4$ for some i, then again (i) holds, so we may assume that $|C_{E^i}(\mathfrak{S}_2)| = 2$, $i=1, 2$, so that $E_0 = C_{E^1}(\mathfrak{S}_2) \oplus C_{E^2}(\mathfrak{S}_2)$ is of order 4.

Let $\mathfrak{K}_0 = \bigcap_{i=1}^{s} \mathfrak{K}_i$, so that $\mathfrak{S}/\mathfrak{K}_0$ acts faithfully as a permutation group on $\{E_1, \cdots, E_s\}$. Since $\mathfrak{K}_i$ is a q'-group, it follows that $\mathfrak{S}_q\mathfrak{K}_0/\mathfrak{K}_0$ is represented regularly. Since $r=2$, $\mathfrak{S}_2$ has 2 orbits, of size 2^a and $2^a(q-1)$. Hence, $\mathfrak{S}/\mathfrak{K}_0$ is represented as a primitive group on $\{E_1, \cdots, E_s\}$ since every set of primitivity has a multiple of q elements. Hence, $a=0$, since $\mathfrak{S}$ is solvable. So $\mathfrak{S}/\mathfrak{K}_0$ is a Frobenius group of order $q(q-1)$. Since $|\mathfrak{S}_p| = p^2$, we get $q=3$.

Since $s=q$, $\mathfrak{S}_2$ normalizes $C_{\mathfrak{S}_p}(E_1)$. Hence, $\mathfrak{S}_2$ acts reducibly on $\mathfrak{S}_p$, so $\mathfrak{S}_2$ is abelian. Hence $\mathfrak{S}_3 \lhd \mathfrak{S}_2\mathfrak{S}_3$, since $\mathfrak{S}_2\mathfrak{S}_3$ is isomorphic to a subgroup of $GL(2, p)$. Since $\mathfrak{S}/\mathfrak{K}_0 \cong \sum_3$, $\mathfrak{S}_2\mathfrak{S}_3$ is not 2-closed. Let $\hat{\mathfrak{S}}_2 = O_2(\mathfrak{S}_2\mathfrak{S}_3)$, so that $\mathfrak{S}_2\mathfrak{S}_3/\hat{\mathfrak{S}}_2 \simeq \sum_3$. By Lemma 5.7 of [20], $\mathfrak{S}_3$ is represented faithfully on $C_E(\hat{\mathfrak{S}}_2)$. Let $\hat{E} = [C_E(\hat{\mathfrak{S}}_2), \mathfrak{S}_3] \neq 0$, and let F be a minimal $\mathfrak{S}_2\mathfrak{S}_3$-submodule of $\hat{E}$. Thus, $|F| = 4$, $|C_F(\mathfrak{S}_2)| = 2$. Hence, (ii) holds.

Remark. (i) need not hold. There is a group of order $2^3 \cdot 3 \cdot 5^2$ for which a module E exists violating (i). We may take $|E| = 2^{12}$.

Lemma 5.47. *Suppose $\mathfrak{S} = \mathfrak{S}_1 \times \cdots \times \mathfrak{S}_a$, $a \geqq 1$, and $\mathfrak{S}_i$ is a dihedral group of order $2p_i$, where p_i is an odd prime, $1 \leqq i \leqq a$. Suppose also that M is a $k\mathfrak{S}$-module, $\mathfrak{A}$ is an S_2-subgroup of $\mathfrak{S}$, k is of characteristic 2, and $C_M(\mathfrak{S}_i') = 0$ for $1 \leqq i \leqq a$. Then M is a free $k\mathfrak{A}$-module. If $m(1-A)(1-B) = 0$ for all $m \in M$ and all A, $B \in \mathfrak{A}$, then $a=1$.*

Proof. We assume without loss of generality that k is algebraically closed. Let $\sigma = |\mathfrak{S}'|^{-1} \sum_{S \in \mathfrak{S}'} S$. Then $M = M\sigma + M(1-\sigma)$, so that $M = M(1-\sigma)$, since $\mathfrak{S}'$ centralizes $M\sigma$, while $C_M(\mathfrak{S}') = 0$. We may further assume that M is irreducible. Let $M = M_1 \oplus \cdots \oplus M_s$, where M_i is an irreducible $k\mathfrak{S}'$-module. Let χ_i be the character of $\mathfrak{S}'$ afforded by M_i. Let $\mathfrak{U}_i = \ker \chi_i$, and let $\mathfrak{A}_i$ be the stabilizer of χ_i in $\mathfrak{A}$. Then $\mathfrak{A}_i$ normalizes $\mathfrak{U}_i$ and $[\mathfrak{S}', \mathfrak{A}_i] \subseteq \mathfrak{U}_i$. Suppose $\mathfrak{A}_i \neq 1$. Then $[\mathfrak{S}', \mathfrak{A}_i] \neq 1$, so that $\mathfrak{U}_i$ contains some $\mathfrak{S}_j'$. This is not the case, since $\mathfrak{U}_i$ centralizes M_i, while $C_M(\mathfrak{S}_j') = 0$. Hence, $\mathfrak{A}_i = 1$. Thus, $\mathfrak{A}$ permutes transitively and regularly the M_i, so M is a free $k\mathfrak{A}$-module. If $m(1-A)(1-B) = 0$ for all A, $B \in \mathfrak{A}$, then $a=1$ by the freeness of M.

The next two lemmas deal with the following delicate situation: $\mathfrak{S}$ is a group, $1 \subseteq \mathfrak{S}_0 \subseteq \mathfrak{S}_1 \subset \mathfrak{S}$ is a chain of normal subgroups of $\mathfrak{S}$ such that

(a) $\mathfrak{S}_0$ is an elementary 2-group.

(b) $\mathfrak{S}_1/\mathfrak{S}_0$ is a 3-group.

(c) $|\mathfrak{S}:\mathfrak{S}_1| = 2$ and X is an involution of $\mathfrak{S} - \mathfrak{S}_1$.

Furthermore, M is an $F_2\mathfrak{S}$-module and N is a submodule of M such that

(α) $\mathfrak{S}_0$ stabilizes $M \supset N \supset 0$.
(β) $\mathfrak{S}$ centralizes N.
(γ) $|M: \mathbf{C}_M(X)| = 2^f$.
(δ) M_0 is a subgroup of M of index 2 which contains $\mathbf{C}_M(X)$.
(ϵ) $|N| \leqq 4$.

Let $\hat{M}$ be the set of all m in M such that some element of $\mathfrak{S}$ of order 3 centralizes m.

LEMMA 5.48. *Suppose $\mathfrak{S}_1/\mathfrak{S}_0$ is elementary of order 27. Then the following hold:*
(a) *If $f \leqq 2$, then $M(1-X) \subseteq \hat{M}$.*
(b) *If $f = 3$, then $M_0(1-X) \cap \hat{M} \neq 0$.*

PROOF. Let $\mathfrak{P}$ be an S_3-subgroup of $\mathfrak{S}$. If $\mathfrak{P}$ does not act faithfully on M, both (a) and (b) are clear, since $\hat{M} = M$. So we may assume that $\mathfrak{P}$ acts faithfully on M.

Case 1. $[\mathfrak{S}_1, X] \subseteq \mathfrak{S}_0$.

Here $\langle \mathfrak{S}_0, X\rangle$ is a normal S_2-subgroup of $\mathfrak{S}$, so that $N + M(1-X)$ admits $\mathfrak{P}$. Since the order of $N + M(1-X)$ is at most 32, some element of $\mathfrak{P}^\#$ centralizes $N + M(1-X)$, so $N + M(1-X) \subseteq \hat{M}$, and we are done.

Case 2. $[\mathfrak{S}_1, X] \not\subseteq \mathfrak{S}_0$.

By Lemma 5.36, there is an element R of $\mathfrak{S}$ of order 3 which is inverted by X. Let $\hat{\mathfrak{S}} = \{S \mid S \in \mathfrak{S}, S \text{ is of order 3}, S \text{ is inverted by } X\}$. Suppose there is S in $\hat{\mathfrak{S}}$ such that $M(1-S)$ has order >4. Let $\tilde{M} = M(1-S)$. Thus, $\tilde{M}$ is a free $F_2\langle X\rangle$-module, so $|\tilde{M}| = 2^{2f_0}$, with $f_0 \leqq 3$. Thus, $f_0 = 2$ or 3. Since $\mathfrak{P}$ centralizes N, we get that $\tilde{M} \cap N = 0$. Let $\mathfrak{C} = \mathbf{C}_{\mathfrak{S}}(S)$. Then $\mathfrak{C} \cap \mathfrak{S}_0$ normalizes $\tilde{M}$, so centralizes $\tilde{M}$, as $\mathfrak{S}_0$ stabilizes $M \supset N \supset 0$. Thus, $A_{\mathfrak{C}}(\tilde{M})$ is $\mathfrak{Z}$-closed. If $|\tilde{M}| = 2^4$, then some element of $\mathfrak{S}$ of order 3 centralizes $\tilde{M}$. If $f = 2$, then $\tilde{M} \supseteq M(1-X)$, and we are done. If $f = 3$, then $\tilde{M} \cap M_0(1-X) \neq 0$, and we are done. We may assume that $|\tilde{M}| = 2^6$. In this case, $f = 3$, and we may assume that $\mathfrak{P} \subseteq \mathfrak{C}$ and that $\mathfrak{P}$ acts faithfully on $\tilde{M}$. Furthermore, $\tilde{M}$ contains $M(1-X)$. Hence, $\tilde{M} = \tilde{M}_1 \oplus \tilde{M}_2 \oplus \tilde{M}_3$, where $\tilde{M}_i$ is an irreducible $F_2\mathfrak{P}$-module. Since X permutes $\tilde{M}_1$, $\tilde{M}_2$, $\tilde{M}_3$, we may assume that X normalizes $\tilde{M}_1$. Since $\tilde{M}$ is a free $F_2\langle X\rangle$-module, and $\tilde{M}_1$ is a summand, $\tilde{M}_1$ is also free. Let $U = M_0(1-X)$, so that $|U| = 4$. Since X centralizes U, we get $U \subseteq \tilde{M}_1(1-X) \oplus \tilde{M}_2 \oplus \tilde{M}_3$. Hence, $U \cap \tilde{M}_2 \oplus \tilde{M}_3 \neq 0$. Since $\tilde{M}_2 \oplus \tilde{M}_3 \subseteq \hat{M}$, we are done.

We may now assume that for each S in $\hat{\mathfrak{S}}$, $M(1-X)$ is of order 4. Suppose X inverts a subgroup $\mathfrak{R}_0$ of order 9. Then $[M, \mathfrak{R}_0]$ is of order 16 (as usual, we use commutation notation as well as additive notation), and $[M, \mathfrak{R}_0]$ admits a S_3-subgroup of $\mathfrak{S}$. Since $[M, \mathfrak{R}_0] \subseteq \hat{M}$, we again are done. We may now assume that X inverts no subgroup of $\mathfrak{S}$ of order 9. In this case, S_3-subgroups of $[\mathfrak{S}_1, X]$ are of order 3, by Lemma 5.36. Let $\mathfrak{T}=\langle\mathfrak{S}_0, X\rangle$, $\mathfrak{S}^*=N_{\mathfrak{S}}(T)$, so that $|\mathfrak{S}:\mathfrak{S}^*|=3$. Also, $\mathfrak{S}^*$ normalizes $M(1-X)+N$. Since $\mathfrak{S}^*$ centralizes N, there is a subgroup $\mathfrak{S}_1^*$ of $\mathfrak{S}^*$ of order 3 such that $|C(\mathfrak{S}_1^*)\cap(M(1-X)+N):N|$ $\geqq 4$. Hence, $\mathfrak{S}_1^*$ centralizes $M(1-X)+N$, a group of order $\leqq 32$. The proof is complete.

LEMMA 5.49. *If* $|\mathfrak{S}_1:\mathfrak{S}_0|>27$, *then the following hold*:
(a) *If* $f\leqq 2$, *then* $M(1-X)\subset\hat{\mathfrak{M}}$.
(b) *If* $f=3$, *then* $M_0(1-X)\cap\hat{\mathfrak{M}}\neq 0$.

PROOF. By Lemma 5.48, we may assume that $\mathfrak{S}$ has no elementary subgroup $\mathfrak{P}$ of order 27 such that $\mathfrak{S}_0\mathfrak{P}$ admits X. Since $\mathfrak{S}/\mathfrak{S}_0$ is supersolvable, we may assume that $|\mathfrak{S}_1:\mathfrak{S}_0|=3^4$. Let $\mathfrak{P}$ be a S_3-subgroup of $\mathfrak{S}$, so that $\mathfrak{S}_1=\mathfrak{S}_0\mathfrak{P}$. Again, since $\mathfrak{S}/\mathfrak{S}_0$ is supersolvable, we may assume that $\mathcal{SCN}_3(\mathfrak{P})=\varnothing$.

First, suppose that X centralizes a subgroup $\mathfrak{H}/\mathfrak{S}_0$ of $\mathfrak{S}_1/\mathfrak{S}_0$ of order 9. Then $\mathfrak{H}$ normalizes $N+M(1-X)$, so a S_3-subgroup of $\mathfrak{H}$ does not act faithfully on $N+M(1-X)$, since $f\leqq 3$ and $\mathfrak{H}$ centralizes N. We may assume that $|C_{\mathfrak{S}_1/\mathfrak{S}_0}(X)|\leqq 3$.

We assume without loss of generality that $\mathfrak{P}$ acts faithfully on M. If X inverts $\mathfrak{S}_1/\mathfrak{S}_0$, then since $|\mathfrak{P}|=3^4$, we get $f\geqq 4$. Hence, we may assume that $|C_{\mathfrak{S}_1/\mathfrak{S}_0}(X)|=3$.

We next argue that X inverts a subgroup $\mathfrak{L}/\mathfrak{S}_0$ of order 9. Namely, $\mathfrak{S}/\mathfrak{S}_0$ is supersolvable, so $\mathfrak{S}_1/\mathfrak{S}_0$ contains an abelian subgroup of order 27 which admits X. The existence of $\mathfrak{L}$ follows.

Since X inverts $\mathfrak{L}/\mathfrak{S}_0$, there is a subgroup $\mathfrak{Q}$ of $\mathfrak{L}$ of order 9 which is inverted by X. Hence, $C_{\mathfrak{S}}(\mathfrak{Q})$ contains a subgroup of order 27, and $N_{\mathfrak{S}}(\mathfrak{Q})$ contains X. Also, $[M,\mathfrak{Q}]$ admits $N(\mathfrak{Q})$. If $|[M,\mathfrak{Q}]|\leqq 2^4$, then $|[M,\mathfrak{Q}]|=2^4$ and so $[M,\mathfrak{Q}]\subseteq\hat{\mathfrak{M}}$, from which the lemma follows. We may therefore assume that $|[M,\mathfrak{Q}]|=2^6$, $f=3$. By our construction of $\mathfrak{L}/\mathfrak{S}_0$, we get that $27\,|\,|C_{\mathfrak{S}}(\mathfrak{Q})|$. If $\mathfrak{Q}$ is cyclic, then since $GL(6, 2)$ has no abelian subgroup of order 27 and exponent 9, we get that $[M,\mathfrak{Q}]\subseteq\hat{\mathfrak{M}}$. So suppose $\mathfrak{Q}$ is elementary of order 9. Let $\mathfrak{L}^*/\mathfrak{S}_0$ be an abelian subgroup of $\mathfrak{S}_1/\mathfrak{S}_0$ of order 27 which admits X with $\mathfrak{L}\subset\mathfrak{L}^*$. Since $\mathcal{SCN}_3(\mathfrak{P})=\varnothing$, $\mathfrak{L}^*/\mathfrak{S}_0$ is of type (3, 9). Since $GL(6, 2)$ has no abelian subgroup of order 27 and exponent 9, we get $[M,\mathfrak{Q}]$ $\subseteq\hat{\mathfrak{M}}$, and we are done.

LEMMA 5.50. *Suppose* $\mathfrak{S}=\mathfrak{T}\mathfrak{Q}$ *is a solvable group, where* $\mathfrak{T}$ *is a* S_2-*subgroup of* $\mathfrak{S}$ *and* $\mathfrak{Q}=\mathfrak{E}\mathfrak{F}$ *is a Frobenius group with the following properties*:

(a) *The Frobenius kernel* $\mathfrak{F}$ *of* $\mathfrak{Q}$ *is of odd prime order* q *and is permutable with* $\mathfrak{T}$.

(b) *The complement* $\mathfrak{E}$ *of* $\mathfrak{F}$ *in* $\mathfrak{Q}$ *is of odd prime order* p *and is permutable with* $\mathfrak{T}$.

Then one of the following holds:

(i) $\mathfrak{T} \lhd \mathfrak{S}$.

(ii) $\mathfrak{F} \lhd \mathfrak{S}$.

(iii) $C_{\mathfrak{T}}(\mathfrak{E})$ *contains a four-group.*

(iv) $Z(\mathfrak{S})$ *contains a unique involution.*

Proof. Suppose (iii) does not hold, and $|Z(\mathfrak{S})|$ is even. Let $\mathfrak{Z}$ be the S_2-subgroup of $Z(\mathfrak{S})$, so that $\mathfrak{Z} \neq 1$. Since (iii) fails, $\mathfrak{Z}$ is cyclic, so (iv) holds. Thus, proceeding by way of contradiction we may assume that (i), (ii), (iii) fail and that $|Z(\mathfrak{S})|$ is odd.

Let $\mathfrak{H} = O_2(\mathfrak{S})$. Since $\mathfrak{Q}$ is a Frobenius group with kernel $\mathfrak{F}$, it follows that $\mathfrak{H} = O_{q'}(\mathfrak{S})$. Since $\mathfrak{F}$ is a S_q-subgroup of $\mathfrak{S}$, $\mathfrak{H}\mathfrak{F} \lhd \mathfrak{S}$. Since (ii) fails, $\mathfrak{F} \not\lhd \mathfrak{H}\mathfrak{F}$. Hence, $\mathfrak{H} = F(\mathfrak{S})$.

Since (i) does not hold, $\mathfrak{H} \subset \mathfrak{T}$. Since $\mathfrak{S}'$ centralizes the chief factor $\mathfrak{H}\mathfrak{F}/\mathfrak{H}$, while $\mathfrak{E}$ does not, we have $\mathfrak{S}' \cap \mathfrak{E} = 1$. Hence $\mathfrak{T}\mathfrak{F} \lhd \mathfrak{S}$.

Let $|\mathfrak{S} : \mathfrak{H}| = 2^a pq$, so that $\mathfrak{S}/\mathfrak{H}$ is a Frobenius group with kernel $\mathfrak{H}\mathfrak{F}/\mathfrak{H}$. Since (i) fails, $a \geqq 1$. Since (iii) fails, $C_{\mathfrak{H}}(\mathfrak{E})$ is either cyclic or generalized quaternion, so every section of $C_{\mathfrak{H}}(\mathfrak{E})$ is generated by 2 elements.

Let $\mathfrak{H}_0 = [\mathfrak{H}, \mathfrak{F}]$. Thus, $\mathfrak{H}$ and $N(\mathfrak{F})$ normalize $\mathfrak{H}_0$. Since $\mathfrak{S} = \mathfrak{H} \cdot N(\mathfrak{F})$, $\mathfrak{H}_0 \lhd \mathfrak{S}$. Let $\mathfrak{H}_0/\mathfrak{H}_1 = V$ be a chief factor of $\mathfrak{S}$. Thus, V is a faithful $F_2\mathfrak{S}/\mathfrak{H}$-module, so if $\mathfrak{K}$ is a complement to $\mathfrak{H}\mathfrak{F}/\mathfrak{H}$ in $\mathfrak{S}/\mathfrak{H}$ which contains $\mathfrak{E}\mathfrak{H}/\mathfrak{H}$, then V is a free $F_2\mathfrak{K}$-module. In particular, $|V| = 2^{c \cdot 2^a p}$ for some positive integer c, and $|C_V(\mathfrak{E})| = 2^{c \cdot 2^a}$. By a previous remark, we get $c = a = 1$.

Now $C_{\mathfrak{H}_1}(\mathfrak{E}) \lhd C_{\mathfrak{H}_0}(\mathfrak{E})$, and $C_{\mathfrak{H}_0}(\mathfrak{E})/C_{\mathfrak{H}_1}(\mathfrak{E})$ is of order 4, being incident to $C_V(\mathfrak{E})$. Since (iii) fails, it follows that $C_{\mathfrak{H}_1}(\mathfrak{E})$ is cyclic.

Suppose $\mathfrak{H}_3 \subseteq \mathfrak{H}_2 \subseteq \mathfrak{H}_1$, and $\mathfrak{H}_2/\mathfrak{H}_3 = W$ admits $N(\mathfrak{F})$ as an irreducible group of operators. If $[\mathfrak{H}_2, \mathfrak{F}] \not\subseteq \mathfrak{H}_3$, then $C_W(\mathfrak{E})$ contains a four-group, against the cyclicity of $C_{\mathfrak{H}_1}(\mathfrak{E})$. Hence $[\mathfrak{H}_2, \mathfrak{F}] \subseteq \mathfrak{H}_3$, and so $[\mathfrak{H}_1, \mathfrak{F}] = 1$.

Since $\mathfrak{H}_0 = [\mathfrak{H}_0, \mathfrak{F}]$ and $\mathfrak{H}_0/\mathfrak{H}_1$ is a chief factor, we get $\mathfrak{H}_1 = D(\mathfrak{H}_0)$. By Lemma 0.8.7, $D(\mathfrak{H}_0) \subseteq Z(\mathfrak{H}_0)$. By Lemma 5.17, $\mathfrak{H}_0$ is special. (Note that the hypotheses of Lemma 5.17 hold since (iii) fails and $a = 1$.)

Since $\mathfrak{H}_0/\mathfrak{H}_1$ is a chief factor of $\mathfrak{S}$, we get $[\mathfrak{H}, \mathfrak{H}_0] \subseteq \mathfrak{H}_1$. By the 3 subgroups lemma, $\mathfrak{H}$ centralizes $\mathfrak{H}_0' = D(\mathfrak{H}_0)$.

Since $C_V(\mathfrak{E})$ is a four-group, $C_{\mathfrak{H}_0}(\mathfrak{E})$ is a quaternion group. Thus, $D(\mathfrak{H}_0) \cap C(\mathfrak{E})$ is of order 2, and is centralized by $\mathfrak{H}$, $\mathfrak{F}$ and $N(\mathfrak{E})$. Since $\mathfrak{S} = \mathfrak{H}\mathfrak{F}N(\mathfrak{E})$, (iv) holds.

5.7. *Factorizations.*

LEMMA 5.51. *Let $\mathfrak{G}_p$ be a S_p-subgroup of the group $\mathfrak{G}$ and let p, q be distinct primes. Suppose the following hold:*

(a) *$A_{\mathfrak{G}}(\mathfrak{H})$ is p-solvable for every p-subgroup $\mathfrak{H}$ of $\mathfrak{G}$.*

(b) *$q \mid |A_{\mathfrak{G}}(\mathfrak{H})|$ for some p-subgroup $\mathfrak{H}$ of $\mathfrak{G}$.*

(c) *$\{p, q\} \neq \{2, 3\}$.*

Then $\mathfrak{G}_p$ contains a normal subgroup $\mathfrak{H}$ such that $q \mid |A_{\mathfrak{G}}(\mathfrak{H})|$.

PROOF. Let $\mathfrak{K} = \{\mathfrak{H} \mid \mathfrak{H}$ is a p-subgroup of $\mathfrak{G}$ and $q \mid |A_{\mathfrak{G}}(\mathfrak{H})|\}$. Choose $\mathfrak{H}$ in $\mathfrak{K}$ so that $|N(\mathfrak{H}) \cap \mathfrak{G}_p|$ is maximal, and with this restriction maximize $|\mathfrak{H}|$. Then $\mathfrak{H}_0 = N(\mathfrak{H}) \cap \mathfrak{G}_p$ is a S_p-subgroup of $N(\mathfrak{H})$. Among all subgroups of $N(\mathfrak{H})$ which cover $N(\mathfrak{H})/C(\mathfrak{H})$, let $\mathfrak{R}$ be minimal. Then $\mathfrak{R} \cap C(\mathfrak{H})$ is nilpotent, so $\mathfrak{R}$ is p-solvable. Let $\mathfrak{L} = \mathfrak{R}\mathfrak{H}$. Then $\mathfrak{L}$ contains $\mathfrak{H}_0^N$ for some N in $N(\mathfrak{H})$, so we may assume that $\mathfrak{L} \supset \mathfrak{H}_0$. Let $\mathfrak{Q}$ be a S_q-subgroup of $\mathfrak{L}$ permutable with $\mathfrak{H}_0$. Let $\mathfrak{S} = \mathfrak{H}_0\mathfrak{Q}$, and set $\mathfrak{Q}_0 = O_q(\mathfrak{S})$. Then $\bar{\mathfrak{S}} = \mathfrak{Q}\mathfrak{H}_0/\mathfrak{Q}_0$ satisfies the hypotheses of Theorem 1 of [39], so either $C_{\bar{\mathfrak{S}}}(Z(\mathfrak{H}_0))$ or $N_{\bar{\mathfrak{S}}}(J(\mathfrak{G}_0))$ has no normal p-complement. Maximality of $\mathfrak{H}_0$ forces $\mathfrak{H}_0 = \mathfrak{G}_p$, as desired.

LEMMA 5.52. *Suppose the following hold:*

(a) *$\mathfrak{S} = \mathfrak{S}_2\mathfrak{S}_3$, where $\mathfrak{S}_p$ is a S_p-subgroup of $\mathfrak{S}$, $p = 2, 3$.*

(b) *$O_2(\mathfrak{S}) = 1$.*

(c) *$\mathfrak{S}_2$ is extra special of width at least 2.*

(d) *$O_3(\mathfrak{S})\mathfrak{S}_2 \lhd \mathfrak{S}$.*

(e) *$\mathfrak{S}$ contains a minimal normal subgroup $\mathfrak{E}$ such that $C_{\mathfrak{S}_2}(\mathfrak{E}) = 1$.*

Then $J(\mathfrak{S}_3) \lhd \mathfrak{S}$.

PROOF. The lemma will follow from the containment $J(\mathfrak{S}_3) \subseteq O_3(\mathfrak{S})$, which we will establish.

By (e), $C(\mathfrak{E})$ is a 3-group, so $O_3(\mathfrak{S}) = C(\mathfrak{E})$. For any subset $\mathfrak{L}$ of $\mathfrak{S}$, let $\bar{\mathfrak{L}}$ be the image of $\mathfrak{L}$ in $A_{\mathfrak{S}}(\mathfrak{E})$.

Let $d = \max m(\mathfrak{A})$, $\mathfrak{A}$ ranging over all the abelian subgroups of $\mathfrak{S}_3$, and choose an abelian subgroup $\mathfrak{A}$ of S_3 with $m(\mathfrak{A}) = d$. We must show that $\mathfrak{A} \subseteq O_3(\mathfrak{S})$. Suppose false.

Let $\mathfrak{R} = \bar{\mathfrak{A}}\bar{\mathfrak{S}}_2$, and in Lemma 5.6, let $\mathfrak{R}$ play the role of $\mathfrak{S}$, $\bar{\mathfrak{A}}$ the role of $\mathfrak{S}_3$, $\bar{\mathfrak{S}}_2$ the role of $\mathfrak{S}_2$. Thus, (a)–(d) of Lemma 5.6 hold; (e) is simply definition, so it suffices to verify (f) to complete the proof of this lemma.

Let $\mathfrak{A}_0$ be any subgroup of $\mathfrak{A}$. Then $\langle \mathfrak{A}_0, C_{\mathfrak{E}}(\mathfrak{A}_0)\rangle = \mathfrak{F}$ is abelian, so $m(\mathfrak{F}) = m(\mathfrak{A}_0) + m(C_{\mathfrak{E}}(\mathfrak{A}_0)) - m(\mathfrak{A}_0 \cap C_{\mathfrak{E}}(\mathfrak{A}_0)) \leqq d$, or $m(C_{\mathfrak{E}}(\mathfrak{A}_0)) - m(\mathfrak{A}_0 \cap C_{\mathfrak{E}}(\mathfrak{A}_0)) \leqq d - m(\mathfrak{A}_0) \leqq m(\mathfrak{A}/\mathfrak{A}_0)$. Since $\mathfrak{A}_0 \cap C_{\mathfrak{E}}(\mathfrak{A}_0) \subseteq C_{\mathfrak{E}}(\mathfrak{A})$, we have $m(C_{\mathfrak{E}}(\mathfrak{A}_0)) - m(C_{\mathfrak{E}}(\mathfrak{A})) \leqq m(C_{\mathfrak{E}}(\mathfrak{A}_0)) - m(\mathfrak{A}_0 \cap C_{\mathfrak{E}}(\mathfrak{A}_0)) \leqq m(\mathfrak{A}/\mathfrak{A}_0)$. Thus, (f) holds.

LEMMA 5.53. *Suppose* $\mathfrak{S}_2$ *is a* S_2*-subgroup of the solvable group* $\mathfrak{S}$. *Assume also that* $\mathfrak{S}$ *is a* $\{3, 5\}'$*-group and that* $O_{2'}(\mathfrak{S})=1$. *Let* $\mathfrak{N}_1 = C(Z(\mathfrak{S}_2))$, $\mathfrak{N}_2=N(J_0(\mathfrak{S}_0))$, $\mathfrak{N}_3=N(Z(J_1(\mathfrak{S}_2)))$. *Here we have set* $d=\max m(\mathfrak{A})$, $\mathfrak{A}$ *ranging over all the abelian subgroups of* $\mathfrak{S}_2$, *and* $J_i(\mathfrak{S}_2)=\{\mathfrak{B} \mid \mathfrak{B}\subseteq\mathfrak{S}_2,\ \mathfrak{B}'=1,\ m(\mathfrak{B})\geqq d-i\}$. *Then for each permutation* σ *of* $\{1, 2, 3\}$, $\mathfrak{S}=\mathfrak{N}_{\sigma(1)}\cdot\mathfrak{N}_{\sigma(2)}$.

PROOF. By Lemma 0.7.7, we may assume that $\mathfrak{S}=\mathfrak{S}_2\mathfrak{Q}$, where $\mathfrak{Q}$ is a q-group for some odd prime q, and that $l_2(\mathfrak{S})\leqq 2$. Here we are also using Lemma 0.7.3 to conclude that $O_{2'}(\tilde{\mathfrak{S}})=1$ whenever $\mathfrak{S}_2\subseteq\tilde{\mathfrak{S}}\subseteq\mathfrak{S}$.

By Theorem 1 of [43], we have $\mathfrak{S}=\mathfrak{N}_1\mathfrak{N}_2=\mathfrak{N}_2\mathfrak{N}_1$. In particular, $\mathfrak{N}_3\subseteq\mathfrak{N}_1\mathfrak{N}_2$, $\mathfrak{N}_3\subseteq\mathfrak{N}_2\mathfrak{N}_1$. By Lemma 0.8.6, it suffices to show that $\mathfrak{N}_1\subseteq\mathfrak{N}_2\mathfrak{N}_3\cap\mathfrak{N}_3\mathfrak{N}_2$, and that $\mathfrak{N}_2\subseteq\mathfrak{N}_1\mathfrak{N}_3\cap\mathfrak{N}_3\mathfrak{N}_1$. Proceeding by induction, we may assume that $\mathfrak{S}=\mathfrak{N}_i$ for some $i\in\{1, 2\}$.

Let $\mathfrak{Q}_j=\mathfrak{Q}\cap\mathfrak{N}_j$. Thus, $\mathfrak{N}_j=\mathfrak{S}_2\mathfrak{Q}_j$, $j=1, 2, 3$. Set $\mathfrak{H}=O_2(\mathfrak{S})$, and let $\mathfrak{Q}^*$ be a subgroup of $\mathfrak{Q}$ such that $\mathfrak{H}\mathfrak{Q}/\mathfrak{H}\mathfrak{Q}^*$ is a chief factor of $\mathfrak{S}$. By induction, we get $\mathfrak{Q}^*\subseteq\mathfrak{Q}_{\sigma(1)}\mathfrak{Q}_{\sigma(2)}$ for all permutations σ of $\{1, 2, 3\}$. If $\mathfrak{Q}_j\not\subseteq\mathfrak{Q}^*$, then since $\mathfrak{S}_2\mathfrak{Q}_j$ is a group and $\mathfrak{H}\mathfrak{Q}/\mathfrak{H}\mathfrak{Q}^*$ is a chief factor we get $\mathfrak{Q}=\mathfrak{Q}_j\mathfrak{Q}^*=\mathfrak{Q}^*\mathfrak{Q}_j$.

Suppose $i=1$. In this case, we must show that $\mathfrak{Q}=\mathfrak{Q}_2\mathfrak{Q}_3$. Suppose false. By the previous paragraph we get $\mathfrak{Q}^*=\mathfrak{Q}_2\mathfrak{Q}_3$. Our induction hypothesis implies that $\mathfrak{Q}^*=D(\mathfrak{Q})$. In particular, $J_0(\mathfrak{S}_2)\not\subseteq\mathfrak{H}$. Let $\mathfrak{A}$ be an abelian subgroup of $\mathfrak{S}_2$ with $m(\mathfrak{A})=d$, $\mathfrak{A}\not\subseteq\mathfrak{H}$. We assume without loss of generality that $\mathfrak{A}=C_{\mathfrak{S}_2}(\mathfrak{A})$. Thus, $Z(J_1(\mathfrak{S}_2))\subseteq\mathfrak{A}$. Let $\tilde{\mathfrak{Q}}$ be a subgroup of $\mathfrak{Q}$ such that (a) $\mathfrak{A}$ normalizes $\mathfrak{H}\tilde{\mathfrak{Q}}$, (b) $[\mathfrak{A}, \tilde{\mathfrak{Q}}]\not\subseteq\mathfrak{H}\mathfrak{Q}^*$, (c) $\tilde{\mathfrak{Q}}$ is minimal subject to (a) and (b). The minimality of $\tilde{\mathfrak{Q}}$ implies that $\mathfrak{A}$ acts nontrivially and irreducibly on $\mathfrak{H}\tilde{\mathfrak{Q}}/\mathfrak{H}D(\tilde{\mathfrak{Q}})$. Let $\mathfrak{A}_0=C_{\mathfrak{A}}(\mathfrak{H}\tilde{\mathfrak{Q}}/\mathfrak{H})$, so that $\mathfrak{A}/\mathfrak{A}_0$ is cyclic. Let $\mathfrak{S}^*=\mathfrak{H}\tilde{\mathfrak{Q}}\mathfrak{A}$. Since $\mathfrak{H}\subseteq\mathfrak{S}^*$, it follows that $O_{2'}(\mathfrak{S}^*)=1$. Let $\mathfrak{H}^*=O_2(\mathfrak{S}^*)$. Thus, $\mathfrak{A}_0\subseteq\mathfrak{H}^*$ and $\mathfrak{A}\cap\mathfrak{H}^*=\mathfrak{A}_0$. Let $\mathfrak{K}$ be the subgroup of $\mathfrak{H}^*$ generated by all the abelian subgroups $\mathfrak{B}$ of $\mathfrak{H}^*$ with $m(\mathfrak{B})\geqq d-1$. Hence, $\mathfrak{A}_0\subseteq\mathfrak{K}\subseteq\mathfrak{H}^*\subseteq\mathfrak{S}_2$. Let $\mathfrak{C}=C_{\mathfrak{S}^*}(\mathfrak{K})\lhd\mathfrak{S}^*$. Since $O_{2'}(\mathfrak{S}^*)=1$, so also $O_{2'}(\mathfrak{C})=1$. Let $\mathfrak{C}_0=O_2(\mathfrak{C})$. For each C in $\mathfrak{C}_0$, we get that $\langle\mathfrak{A}_0, C\rangle$ is abelian, since $\mathfrak{A}_0\subseteq\mathfrak{K}$, $C\in C(\mathfrak{K})$. Since $m(\langle\mathfrak{A}_0, C\rangle)\geqq d-1$, it follows that $Z(J_1(\mathfrak{S}_2))\subseteq Z(\mathfrak{C}_0)$.

Since $\tilde{\mathfrak{Q}}\not\subseteq\mathfrak{Q}^*$, by (b), it follows that $\mathfrak{S}$ does not normalize $Z(J_1(\mathfrak{S}_2))$. Hence, $\tilde{\mathfrak{Q}}$ does not centralize $Z(\mathfrak{C}_0)$, so $\tilde{\mathfrak{Q}}$ does not centralize $\mathfrak{M}=\Omega_1(Z(\mathfrak{C}_0))$. On the other hand, if $\mathfrak{M}_0=\mathfrak{M}\cap\mathfrak{A}_0$, and $|\mathfrak{M}:\mathfrak{M}_0|=2^w$, then $\langle\mathfrak{A}_0, \mathfrak{M}\rangle$ is an abelian group with $m(\langle\mathfrak{A}_0, \mathfrak{M}\rangle)=m(\mathfrak{A}_0)+w\geqq d-1+w$. Hence, $w\leqq 1$. Thus, $\mathfrak{M}_0$ is of index at most 2 in $\mathfrak{M}$. Since $\mathfrak{M}_0\subseteq\mathfrak{A}_0\subseteq\mathfrak{A}$, $\mathfrak{A}$ centralizes a hyperplane of $\mathfrak{M}$. Since $q\geqq 7$, and since $\mathfrak{H}\tilde{\mathfrak{Q}}=\mathfrak{H}[\tilde{\mathfrak{Q}}, \mathfrak{A}]$, it follows that $\tilde{\mathfrak{Q}}$ centralizes $\mathfrak{M}$. This contradiction shows that $i\neq 1$.

Suppose $i=2$. Here we must show that $\mathfrak{S}=\mathfrak{N}_1\mathfrak{N}_3$. Suppose false. Then $\mathfrak{Q}\neq\mathfrak{Q}_1\mathfrak{Q}_3$, and $\mathfrak{Q}^*\subseteq\mathfrak{Q}_1\mathfrak{Q}_3$. Since $\mathfrak{H}\mathfrak{Q}/\mathfrak{H}\mathfrak{Q}^*$ is a chief factor of S, we get $\mathfrak{Q}^*=\mathfrak{Q}_1\mathfrak{Q}_3$. Our induction hypothesis forces $\mathfrak{Q}^*=D(\mathfrak{Q})$. In particular, $J_1(\mathfrak{S}_2)\not\subseteq\mathfrak{H}$. Let $\mathfrak{A}$ be an abelian subgroup of $\mathfrak{S}_2$ with $m(A)\geqq d-1$ and with $\mathfrak{A}\not\subseteq\mathfrak{H}$. Let $\tilde{\mathfrak{Q}}$ be a subgroup of $\mathfrak{Q}$ such that (a) $\mathfrak{A}$ normalizes $\mathfrak{H}\tilde{\mathfrak{Q}}$, (b) $[\mathfrak{A},\tilde{\mathfrak{Q}}]\not\subseteq\mathfrak{H}\mathfrak{Q}^*$, (c) $\tilde{\mathfrak{Q}}$ is minimal subject to (a) and (b). The minimality of $\tilde{\mathfrak{Q}}$ implies that $\mathfrak{A}$ acts nontrivially and irreducibly on $\mathfrak{H}\tilde{\mathfrak{Q}}/\mathfrak{H}D(\tilde{\mathfrak{Q}})$. Let $\mathfrak{A}_0=C_{\mathfrak{A}}(\mathfrak{H}\tilde{\mathfrak{Q}}/\mathfrak{H})$, so that $\mathfrak{A}/\mathfrak{A}_0$ is cyclic, $m(\mathfrak{A}_0)\geqq d-2$. Let $\mathfrak{S}^*=\mathfrak{H}\tilde{\mathfrak{Q}}\mathfrak{A}$, $\mathfrak{H}^*=O_2(\mathfrak{S}^*)$, $\mathfrak{Z}=Z(\mathfrak{H}^*)\supseteq Z(\mathfrak{S}_2)$. Since $\tilde{\mathfrak{Q}}\not\subseteq\mathfrak{Q}^*$, it follows that $\tilde{\mathfrak{Q}}$ does not centralize $Z(\mathfrak{S}_2)$, so does not centralize $\mathfrak{Z}$, so does not centralize $\mathfrak{M}=\Omega_1(\mathfrak{Z})$. Let $\mathfrak{M}_0=\mathfrak{M}\cap\mathfrak{A}_0$, $|\mathfrak{M}:\mathfrak{M}_0|=2^w$. Then $\langle\mathfrak{A}_0,\mathfrak{M}\rangle$ is an abelian group with $m(\langle\mathfrak{A}_0,\mathfrak{M}\rangle)=m(\mathfrak{A}_0)+w\geqq d-2+w$. Hence, $w\leqq 2$. Thus, $\mathfrak{A}$ centralizes a subgroup of $\mathfrak{M}$ of index 4. Since $q\geqq 7$, it follows that $\tilde{\mathfrak{Q}}$ centralizes $\mathfrak{M}$. The proof is complete.

We can salvage something for the small primes.

LEMMA 5.54. *Suppose* $\mathfrak{S}=\mathfrak{T}\mathfrak{P}$ *is a solvable group,* $O_{2'}(\mathfrak{S})=1$, $\mathfrak{T}$ *is a* S_2*-subgroup of* $\mathfrak{S}$ *and* $\mathfrak{P}$ *is a cyclic p-group of order* $p^a>5$, *p odd. Let* $\mathfrak{N}_1=C(Z(\mathfrak{T}))$, $\mathfrak{N}_2=N(J(\mathfrak{T}))$, $\mathfrak{N}_3=N(Z(J_1(\mathfrak{T})))$, *and let* $\mathfrak{Q}=\mho^1(\mathfrak{P})$. *Then for each permutation* σ *of* $\{1, 2, 3\}$, $\mathfrak{Q}=\mathfrak{Q}\cap\mathfrak{N}_{\sigma(1)}\cdot\mathfrak{Q}\cap\mathfrak{N}_{\sigma(2)}$.

PROOF. Let $\mathfrak{H}=O_2(\mathfrak{S})$, so that $\mathfrak{T}/\mathfrak{H}$ is cyclic of order dividing $p-1$. If $J_1(\mathfrak{T})\subseteq\mathfrak{H}$, then $\mathfrak{S}=\mathfrak{N}_2=\mathfrak{N}_3$ and we are done. We may assume that $J_1(\mathfrak{T})\not\subseteq\mathfrak{H}$. Let $d=\max m(\mathfrak{A})$, where $\mathfrak{A}$ ranges over all the abelian subgroups of $\mathfrak{T}$ and let $\mathfrak{B}$ be an abelian subgroup of $\mathfrak{T}$ with $m(\mathfrak{B})\geqq d-1$, $\mathfrak{B}\not\subseteq\mathfrak{H}$. We will show that $\mathfrak{Q}$ centralizes $\mathfrak{N}=\Omega_1(Z(\mathfrak{H}))$. Let $\mathfrak{N}_0=\mathfrak{N}\cap\mathfrak{B}$. Since $m(\mathfrak{B}\cap\mathfrak{H})\geqq d-2$, it follows that $|\mathfrak{N}:\mathfrak{N}_0|\leqq 4$. Thus, the involution of $\mathfrak{T}/\mathfrak{H}$ centralizes $\mathfrak{N}_0$. Since $[\mathfrak{N},\mathfrak{P}]$ is a free $F_2\mathfrak{T}/\mathfrak{H}$-module, it follows that $|[\mathfrak{N},\mathfrak{P}]|\leqq 2^4$, so $\mathfrak{Q}$ centralizes $[\mathfrak{N},\mathfrak{P}]$, and so centralizes $\mathfrak{N}$. Hence, $\mathfrak{Q}$ centralizes $Z(\mathfrak{H})$, and so centralizes its subgroup $Z(\mathfrak{T})$.

To complete the proof, it suffices to show that $\mathfrak{Q}=\mathfrak{Q}\cap\mathfrak{N}_2\cdot\mathfrak{Q}\cap\mathfrak{N}_3$. Suppose false. Then $J(\mathfrak{T})\not\subseteq\mathfrak{H}$, and we may assume that $\mathfrak{B}$ is an abelian subgroup of $\mathfrak{T}$ with $\mathfrak{B}\not\subseteq\mathfrak{H}$, $m(\mathfrak{B})=d$. Let $\mathfrak{B}_0=\mathfrak{B}\cap\mathfrak{H}$ so that $m(\mathfrak{B}_0)\geqq d-1$, $\mathfrak{B}_0\subseteq J_1(\mathfrak{T})$. Let $\mathfrak{H}_0=\langle\mathfrak{C}\mid\mathfrak{C}'=1,\ \mathfrak{C}\subseteq\mathfrak{H},\ m(\mathfrak{C})\geqq d-1\rangle$, so that $\mathfrak{H}_0\subseteq J_1(\mathfrak{T})$, $\mathfrak{H}_0\lhd\mathfrak{S}$. Let $\mathfrak{H}_1=\mathfrak{C}_{\mathfrak{S}}(\mathfrak{H}_0)$, so that $Z(J_1(\mathfrak{T}))\subseteq\mathfrak{H}_1$. Since $O_{2'}(\mathfrak{T})=1$, so also $O_{2'}(\mathfrak{H}_1)=1$. Let $\mathfrak{H}_2=O_2(\mathfrak{H}_1)$. Thus, for each H in $\mathfrak{H}_2$, $m(\langle H,\mathfrak{B}_0\rangle)\geqq d-1$, so that $Z(J_1(\mathfrak{T}))$ centralizes $\langle H,\mathfrak{B}_0\rangle$. Hence, $Z(J_1(\mathfrak{T}))\subseteq Z(\mathfrak{H}_2)\lhd\mathfrak{S}$. Let $\mathfrak{M}=\Omega_1(Z(\mathfrak{H}_2))$. Since $\langle\mathfrak{M},\mathfrak{B}_0\rangle$ is abelian, it follows that $|\mathfrak{M}:\mathfrak{M}\cap\mathfrak{B}_0|\leqq 2$, so the involution of $\mathfrak{B}/\mathfrak{B}_0$ centralizes a hyperplane of $\mathfrak{M}$. This implies that if $\mathfrak{M}=\mathfrak{M}_0\neq\mathfrak{M}_1\supset\cdots\supset\mathfrak{M}_s=1$ is part of a chief series for $\mathfrak{S}$, then $\mathfrak{Q}$ centralizes each $\mathfrak{M}_i/\mathfrak{M}_{i+1}$, so $\mathfrak{Q}$ centralizes $\mathfrak{M}$, so centralizes $\boldsymbol{Z}(\mathfrak{H}_2)$, so centralizes $\boldsymbol{Z}(J_1(\mathfrak{T}))$. The proof is complete.

5.8. *Miscellaneous.* We need a generalization of Lemma 0.8.11.

LEMMA 5.55. *Suppose $\mathfrak{A}$ is a π-group, $\mathfrak{B}$ is a π'-group and $\mathfrak{A}$ normalizes $\mathfrak{B}$. Then* $[\mathfrak{B}, \mathfrak{A}, \mathfrak{A}]=[\mathfrak{B}, \mathfrak{A}]$.

PROOF.[6] Choose A in $\mathfrak{A}$, B in $\mathfrak{B}$, and set $C=[B, A]$, so that $B^A=BC$. For each $n=1, 2, \cdots$, define X_n by $B^{A^n}=BC^nX_n$, so that $X_1=1$. Suppose $X_n\in[\mathfrak{B}, \mathfrak{A}, \mathfrak{A}]$ for some n. Then $B^{A^{n+1}}=BC\cdot C^n\cdot[C^n, A]X_n^A$, so that $X_{n+1}\in[\mathfrak{B}, \mathfrak{A}, \mathfrak{A}]$. Taking $n=|\mathfrak{A}|$ shows that $C^n\in[\mathfrak{B}, \mathfrak{A}, \mathfrak{A}]$. Since $\langle C\rangle=\langle C^n\rangle$, we get $C\in[\mathfrak{B}, \mathfrak{A}, \mathfrak{A}]$, and so $[\mathfrak{B}, \mathfrak{A}]\subseteq[\mathfrak{B}, \mathfrak{A}, \mathfrak{A}]\subseteq[\mathfrak{B}, \mathfrak{A}]$, as required.

LEMMA 5.56. *Let $\mathfrak{G}_p$ be a S_p-subgroup of the finite group $\mathfrak{G}$. Let $\mathfrak{H}=\mathfrak{G}_p\cap O_{p',p}(\mathfrak{G})$, $\mathfrak{N}=N(\mathfrak{H})$, $\mathfrak{D}=O_{p'}(\mathfrak{G})$.*

(a) *If Y is a p-element of $\mathfrak{N}$, $D\in\mathfrak{D}$, and $Y^D\in\mathfrak{N}$, then $D=D_1D_2$ where $D_1\in C_{\mathfrak{D}}(Y)$ and $D_2\in C_{\mathfrak{D}}(\mathfrak{H})$.*

(b) *Elements of $\mathfrak{G}_p$ are $\mathfrak{G}$-conjugate only if they are $\mathfrak{N}$-conjugate.*

PROOF. (a) For each $i=0, 1, \cdots$, let $D_i=[Y^i, D]=Y^{-i}\cdot D^{-1}Y^iD$. Since Y and Y^D are in $\mathfrak{N}$, so also $D_i\in\mathfrak{N}$, for all i. Since $D_i\in\mathfrak{D}$, we get $D_i\in\mathfrak{D}\cap\mathfrak{N}=C_{\mathfrak{D}}(\mathfrak{H})$. Since Y normalizes $\mathfrak{H}$, we get $D_i^{Y^j}\in C_{\mathfrak{D}}(\mathfrak{H})$ for all i, j. Let $\mathfrak{D}_0=\langle D_i^{Y^j}\mid i, j=0, 1, \cdots\rangle$. Then, $\mathfrak{D}_0\subseteq C_{\mathfrak{D}}(\mathfrak{H})$ and $\mathfrak{D}_0$ admits Y. Since $Y^{-1}\mathfrak{D}_0D^{-1}Y=\mathfrak{D}_0Y^{-1}D^{-1}Y=\mathfrak{D}_0D^{-1}$, it follows that Y centralizes some element D_1^{-1} of $\mathfrak{D}_0D^{-1}$. Thus, $D_1^{-1}=D_2D^{-1}$ with $D_2\in\mathfrak{D}_0$. Hence, $D=D_1D_2$, as required.

(b) Suppose X, $Y\in\mathfrak{G}_p$, $G\in\mathfrak{G}$ and $X=Y^G$. Notice that $\mathfrak{G}_p\subseteq\mathfrak{N}$. By the Frattini argument, we have $G=DN$, $D\in\mathfrak{D}$, $N\in\mathfrak{N}$. Hence, $X^{N^{-1}}=Y^D$. Thus, Y is a p-element of $\mathfrak{N}$ and $Y^D\in\mathfrak{N}$. Hence, $D=D_1D_2$, $D_1\in C(Y)$, $D_2\in C_{\mathfrak{D}}(\mathfrak{H})$, by (a). Hence, $X^{N^{-1}}=Y^{D_1D_2}=Y^{D_2}$, so that $X=Y^{D_2N}$ with $D_2N\in\mathfrak{N}$.

The following lemma involves easy consequences of 0.

LEMMA 5.57. *Suppose $\mathfrak{G}$ is a π-separable group. Then*

(a) *$\mathfrak{G}$ satisfies D_π.*

(b) *Let $\mathfrak{G}_\sigma$ be a S_σ-subgroup of $\mathfrak{G}$, $\sigma\in\{\pi, \pi'\}$. Then $N(\mathfrak{G}_\pi)\cap N(\mathfrak{G}_{\pi'})$ covers every central factor of $\mathfrak{G}$.*

PROOF. Suppose $\mathfrak{H}$, $\mathfrak{K}$ are maximal π-subgroups of $\mathfrak{G}$. If $\mathfrak{M}$ is a minimal normal subgroup of $\mathfrak{G}$, we may assume that $\mathfrak{H}\mathfrak{M}/\mathfrak{M}$ and $\mathfrak{K}\mathfrak{M}/\mathfrak{M}$ are contained in S_π-subgroups of $\mathfrak{G}/\mathfrak{M}$ and that $\mathfrak{G}/\mathfrak{M}$ satisfies D_π. If $\mathfrak{M}$ is a π-group, we get that $\mathfrak{H}$ and $\mathfrak{K}$ are conjugate. Suppose $\mathfrak{M}$ is not a π-group, so is a π'-group. By the Schur-Zassenhaus the-

[6] The proof has been supplied by N. Blackburn.

orem, $\mathfrak{G}$ satisfies E_π. Let $\mathfrak{L}$ be a S_π-subgroup of $\mathfrak{G}$. We assume without loss of generality that $\langle\mathfrak{H}, \mathfrak{K}\rangle\subseteq\mathfrak{L}\mathfrak{M}$, so we may assume that $\mathfrak{L}\mathfrak{M}=\mathfrak{G}$. Since one of $\mathfrak{L}$, $\mathfrak{M}$ is solvable, $\mathfrak{G}$ satisfies C_π. Let $\mathfrak{L}_0=\mathfrak{H}\mathfrak{M}\cap\mathfrak{L}$, so that $\mathfrak{H}$ and $\mathfrak{L}_0$ are S_π-subgroups of $\mathfrak{H}\mathfrak{M}$. Since $\mathfrak{H}\mathfrak{M}$ satisfies C_π, we get that $\mathfrak{H}=\mathfrak{L}_0^M$ for some M in $\mathfrak{M}$, so $\mathfrak{H}\subseteq\mathfrak{L}^M$. By maximality of $\mathfrak{H}$, we have $\mathfrak{H}=\mathfrak{L}^M$. By symmetry, $\mathfrak{K}=\mathfrak{L}^{M'}$ for some M' in $\mathfrak{M}$. This completes a proof of (a), and of course, we also get that $\mathfrak{G}$ satisfies $D_{\pi'}$.

Let $\mathfrak{H}/\mathfrak{K}$ be a central factor of $\mathfrak{G}$. Let $|\mathfrak{H}:\mathfrak{K}|=p$. By symmetry, we may assume that $p\in\pi$. Let $\mathfrak{R}=\mathfrak{H}\mathfrak{G}_{\pi'}$, $\mathfrak{S}=\mathfrak{K}\mathfrak{G}_{\pi'}$. Since $\mathfrak{H}\cap\mathfrak{G}_{\pi'}=\mathfrak{K}\cap\mathfrak{G}_{\pi'}$, we have $|\mathfrak{R}:\mathfrak{S}|=p$, $\mathfrak{S}\triangleleft\mathfrak{R}$. Since $\mathfrak{S}$ satisfies $D_{\pi'}$, it follows that $\mathfrak{R}=\mathfrak{S}\mathfrak{N}$, where $\mathfrak{N}=\boldsymbol{N}_{\mathfrak{R}}(\mathfrak{G}_{\pi'})$, so that $\mathfrak{R}=\mathfrak{K}\mathfrak{N}$, as $\mathfrak{N}\supseteq\mathfrak{G}_{\pi'}$. Since $\mathfrak{G}=\mathfrak{G}_{\pi'}\mathfrak{G}_\pi$, it follows that $\mathfrak{N}=\mathfrak{G}_{\pi'}\mathfrak{L}$, where $\mathfrak{L}=\mathfrak{G}_\pi\cap\mathfrak{N}$. Hence $\mathfrak{R}=\mathfrak{K}\mathfrak{N}=\mathfrak{K}\mathfrak{L}\mathfrak{G}_{\pi'}=\mathfrak{H}\mathfrak{G}_{\pi'}$. Hence, $\mathfrak{L}\not\subseteq\mathfrak{K}\mathfrak{G}_{\pi'}$. Choose L in $\mathfrak{L}-\mathfrak{K}\mathfrak{G}_{\pi'}$. Thus, $L=L_1L_2$, where $L_1\in\mathfrak{H}$, $L_2\in\mathfrak{G}_{\pi'}$, $L_1\notin\mathfrak{K}$. Since L and L_2 normalize $\mathfrak{G}_{\pi'}$, we get $L_1\in\boldsymbol{N}(\mathfrak{G}_{\pi'})$. Since $L_1\in\mathfrak{H}-\mathfrak{K}$, the order of L_1 is a multiple of p. Let $L_1=K_1K_2=K_2K_1$, where K_1 is a p-element, K_2 is a p'-element. Then K_1 is a power of L_1, so $K_1\in\boldsymbol{N}(\mathfrak{G}_{\pi'})$. Since $K_2\in\mathfrak{K}$, we get $K_1\in\mathfrak{H}-\mathfrak{K}$. Now $\langle K_1\rangle$ and $\mathfrak{G}_\pi\cap(\langle K_1\rangle\mathfrak{G}_{\pi'})$ are S_p-subgroups of $\langle K_1\rangle\mathfrak{G}_{\pi'}$, so we may choose Y in $\mathfrak{G}_{\pi'}$, such that $Y^{-1}K_1Y\in\mathfrak{G}_\pi$. Since $\mathfrak{H}$ and $\mathfrak{K}$ are normal in $\mathfrak{G}$, we have $X=Y^{-1}K_1Y\in\mathfrak{H}-\mathfrak{K}$. Since $\langle K_1\rangle G_{\pi'}$ normalizes $\mathfrak{G}_{\pi'}$, we get that $X\in\boldsymbol{N}(\mathfrak{G}_{\pi'})\cap\mathfrak{G}_\pi\subseteq\boldsymbol{N}(\mathfrak{G}_{\pi'})\cap\boldsymbol{N}(\mathfrak{G}_\pi)$, so (b) holds.

LEMMA 5.58. *Suppose p, q are distinct primes and $\mathfrak{S}=\mathfrak{P}\mathfrak{Q}$, where $\mathfrak{P}$ is a normal S_p-subgroup of $\mathfrak{S}$ and $\mathfrak{Q}$ is elementary of order q^2. Let $\{\mathfrak{X}_1,\cdots,\mathfrak{X}_r\}$ be the set of all subgroups of $\mathfrak{Q}$ of order q which have nontrivial fixed points on $\mathfrak{P}$. If $r=2$, then $\mathfrak{P}=\boldsymbol{C}_{\mathfrak{P}}(\mathfrak{X}_1)\times\boldsymbol{C}_{\mathfrak{P}}(\mathfrak{X}_2)$.*

PROOF. We proceed by induction on $|\mathfrak{P}|$. Let $\mathfrak{N}$ be a minimal normal subgroup of $\mathfrak{S}$. (Notice that $\boldsymbol{C}_{\mathfrak{P}}(\mathfrak{Q})=1$, since $r=2<q+1$, and $q+1$ is the number of subgroups of $\mathfrak{Q}$ of order q.) Hence, $\mathfrak{N}\subseteq\boldsymbol{Z}(\mathfrak{P})$, and $\mathfrak{N}$ is an irreducible $\mathfrak{Q}$-group. We may assume notation is chosen so that $\mathfrak{N}\subseteq\boldsymbol{C}(\mathfrak{X}_1)$. Since $r=2$, it follows that $\mathfrak{Q}$ acts faithfully on $\mathfrak{P}$. Hence, $\mathfrak{N}\subset\mathfrak{P}$, and $\mathfrak{X}_1$ acts faithfully on $\mathfrak{P}/\mathfrak{N}$. If $\mathfrak{X}_2$ centralizes $\mathfrak{P}\mathfrak{N}$, then $\boldsymbol{C}_{\mathfrak{P}}(\mathfrak{X}_2)$ is a complement to $\mathfrak{N}$ in $\mathfrak{P}$, so that $\mathfrak{P}=\mathfrak{N}\times\boldsymbol{C}_{\mathfrak{P}}(\mathfrak{X}_2)$. Since $\boldsymbol{C}_{\mathfrak{P}}(\mathfrak{Q})=1$, we get $\mathfrak{N}=\boldsymbol{C}_{\mathfrak{P}}(\mathfrak{X}_1)$, and we are done. We may assume that $\mathfrak{X}_2$ acts faithfully on $\mathfrak{P}/\mathfrak{N}$.

If $X\in\mathfrak{Q}^\#$ and $\boldsymbol{C}_{\mathfrak{P}/\mathfrak{N}}(X)\neq1$, then $\boldsymbol{C}_{\mathfrak{P}}(X)\neq1$, so $X\in\mathfrak{X}_1\cup\mathfrak{X}_2$. Thus, by our induction hypothesis, $\mathfrak{P}/\mathfrak{N}$ is the direct product of $\boldsymbol{C}_{\mathfrak{P}/\mathfrak{N}}(\mathfrak{X}_1)$ and $\boldsymbol{C}_{\mathfrak{P}/\mathfrak{N}}(\mathfrak{X}_2)$. In particular, $\boldsymbol{C}_{\mathfrak{P}}(\mathfrak{X}_1)\triangleleft\mathfrak{P}$. Since $\mathfrak{P}=\langle\boldsymbol{C}_{\mathfrak{P}}(\mathfrak{X}_1), \boldsymbol{C}_{\mathfrak{P}}(\mathfrak{X}_2)\rangle=\boldsymbol{C}_{\mathfrak{P}}(\mathfrak{X}_1)\boldsymbol{C}_{\mathfrak{P}}(\mathfrak{X}_2)$, it suffices to show that $[\boldsymbol{C}_{\mathfrak{P}}(\mathfrak{X}_2), \boldsymbol{C}_{\mathfrak{P}}(\mathfrak{X}_1)]=1$. Notice that $\boldsymbol{C}_{\mathfrak{P}}(\mathfrak{X}_2)=[\boldsymbol{C}_{\mathfrak{P}}(\mathfrak{X}_2), \mathfrak{X}_1]$. By our induction hypothesis, $[\boldsymbol{C}_{\mathfrak{P}}(\mathfrak{X}_2), \boldsymbol{C}_{\mathfrak{P}}(\mathfrak{X}_1)]\subseteq\mathfrak{N}$, so $[\boldsymbol{C}_{\mathfrak{P}}(\mathfrak{X}_2), \boldsymbol{C}_{\mathfrak{P}}(\mathfrak{X}_1), \mathfrak{X}_1]=1$. Since $[\boldsymbol{C}_{\mathfrak{P}}(\mathfrak{X}_1), \mathfrak{X}_1]$

$=1$, we also get $[C_{\mathfrak{P}}(\mathfrak{X}_1), \mathfrak{X}_1, C_{\mathfrak{P}}(\mathfrak{X}_2)]=1$. By the three subgroups lemma, we get $[\mathfrak{X}_1, C_{\mathfrak{P}}(\mathfrak{X}_2), C_{\mathfrak{P}}(\mathfrak{X}_1)]=1$, and since $[\mathfrak{X}_1, C_{\mathfrak{P}}(\mathfrak{X}_2)] = C_{\mathfrak{P}}(\mathfrak{X}_2)$, the proof is complete.

LEMMA 5.59. *Suppose $\mathfrak{Q}$ is a p'-group contained in the p-solvable group $\mathfrak{G}$ and $p \mid |A_{\mathfrak{G}}(\mathfrak{Q})|$. Then $\mathfrak{Q}$ does not normalize any S_p-subgroup of $\mathfrak{G}$.*

PROOF. Let $\mathfrak{P}$ be a S_p-subgroup of $N(\mathfrak{Q})$. By hypothesis, $[\mathfrak{P}, \mathfrak{Q}]\neq 1$. Suppose by way of contradiction that $\mathfrak{G}_p$ is a S_p-subgroup of $\mathfrak{G}$ which $\mathfrak{Q}$ normalizes. We assume without loss of generality that $\mathfrak{Q}=[\mathfrak{P}, \mathfrak{Q}]$.

Let $\mathfrak{M}$ be a minimal normal subgroup of $\mathfrak{G}$. Thus, $\mathfrak{G}_p\mathfrak{M}/\mathfrak{M}$ is a S_p-subgroup of $\mathfrak{G}/\mathfrak{M}$ normalized by $\mathfrak{Q}\mathfrak{M}/\mathfrak{M}$. By induction on $|\mathfrak{G}|$, we conclude that $A_{\mathfrak{G}/\mathfrak{M}}(\mathfrak{Q}\mathfrak{M}/\mathfrak{M})$ is a p'-group. Hence, $[\mathfrak{Q}, \mathfrak{P}]\subseteq\mathfrak{M}$. Since $\mathfrak{Q}=[\mathfrak{Q}, \mathfrak{P}]$, we get $\mathfrak{Q}\subseteq\mathfrak{M}$. Since $1\neq\mathfrak{Q}$, and $\mathfrak{M}$ is a minimal normal subgroup of $\mathfrak{G}$, we conclude that $\mathfrak{M}$ is a p'-group, by the p-solvability of $\mathfrak{G}$. Hence, $[\mathfrak{Q}, \mathfrak{G}_p]\subseteq\mathfrak{G}_p\cap\mathfrak{M}=1$. Hence, $|C(\mathfrak{Q})|_p = |\mathfrak{G}|_p$, so that $|N(\mathfrak{Q}): C(\mathfrak{Q})|_p=1$, against $p \mid |\mathfrak{A}_{\mathfrak{G}}(\mathfrak{Q})|$. The proof is complete.

6. A transitivity theorem.

LEMMA 6.1. *Suppose $\mathfrak{G}$ is a finite group, $\mathfrak{B}\in\mathfrak{U}_{\mathfrak{G}}(p)$, p is a prime, and $B\in\mathfrak{B}^{\#}$. Then the following hold:*

(a) *If $C(B)$ is p-solvable, then $\mathfrak{B}\subseteq O_{p',p}(C(B))$.*

(b) *If $C(B)$ is p-solvable and $O_{p'}(C(B))=1$ for all B in $\mathfrak{B}^{\#}$, then $\mathfrak{B}$ centralizes every element of $\mathsf{M}_{\mathfrak{G}}(\mathfrak{B}; p')$.*

PROOF. We restrict our attention to the proof of (a), since (b) is an immediate consequence of (a). Let $\mathfrak{C}=C(B)$ and let $\mathfrak{G}_p$ be a S_p-subgroup of $N(\mathfrak{B})$. Since $\mathfrak{B}\in\mathfrak{U}(p)$, $\mathfrak{G}_p$ is a S_p-subgroup of $\mathfrak{G}$. Let $\mathfrak{P}=C_{\mathfrak{G}_p}(B)$, and let $\mathfrak{P}^*$ be a S_p-subgroup of $\mathfrak{C}$ which contains $\mathfrak{P}$. Suppose $\mathfrak{B}\triangleleft\mathfrak{P}^*$. Then $\mathfrak{B}/\langle B\rangle\subseteq Z(\mathfrak{P}^*/\langle B\rangle)$ and since $O_{p',p}(\mathfrak{C}) = O_{p',p}(\mathfrak{C} \bmod \langle B\rangle)$, the lemma follows. We may therefore assume that $\mathfrak{B}\ntriangleleft\mathfrak{P}^*$. In particular, $\mathfrak{P}\subset\mathfrak{P}^*$, so $\mathfrak{B}\nsubseteq Z(\mathfrak{G}_p)$. By definition of $\mathfrak{U}(p)$, it follows that $Z(\mathfrak{G}_p)$ is cyclic.

Now $\mathfrak{B}\subseteq\Omega_1(Z(\mathfrak{P}))$, so if $p=2$, the cyclicity of $Z(\mathfrak{G}_p)$ forces $\mathfrak{B} =\Omega_1(Z(\mathfrak{P}))$, so that $\mathfrak{B}$ char $\mathfrak{P}\triangleleft\mathfrak{P}^*$, against the previous argument. Hence, $p\neq 2$. If $p\geqq 5$, then since $[\mathfrak{P}^*, \mathfrak{B}, \mathfrak{B}]\subseteq[\mathfrak{P}, \mathfrak{B}]=1$, the lemma follows from (b). Hence, $p=3$. Since $\mathfrak{B}\ntriangleleft\mathfrak{P}^*$, it follows that $\mathfrak{F} =\Omega_1(Z(\mathfrak{P}))$ is elementary of order 27, and of course $\mathfrak{B}\subset\mathfrak{F}$.

Since $\mathfrak{P}\subseteq\mathfrak{G}_p$, it follows that $\mathfrak{B}=\langle B\rangle\times\langle Z\rangle$, where $\langle Z\rangle=\Omega_1(Z(\mathfrak{G}_p))$. Thus, $Z\notin O_{3',3}(\mathfrak{C})$. Let $\mathfrak{Q}$ be a S_2-subgroup of $\mathfrak{C}$ permutable with $\mathfrak{P}^*$. By (B), we get that $Z\notin O_{3',3}(\mathfrak{P}^*\mathfrak{Q})$. In particular, Z does not centralize $\mathfrak{P}^*\cap O_{3',3}(\mathfrak{P}^*\mathfrak{Q})=\mathfrak{P}_0$. Hence, $\mathfrak{P}^*=\mathfrak{P}_0\mathfrak{P}$ since $|\mathfrak{P}^*:\mathfrak{P}|=3$. This means that $ZO_{3',3}(\mathfrak{P}^*\mathfrak{P})\in \boldsymbol{Z}(O_{3',3}(\mathfrak{P}^*\mathfrak{P})\mathfrak{P}^*/O_{3',3}(\mathfrak{P}^*\mathfrak{Q}))$. Hence $[\boldsymbol{D}_3^1(\mathfrak{P}^*\mathfrak{Q}),\langle Z\rangle]=\mathfrak{Q}_0$ is normalized by $\mathfrak{P}^*$. Since $\mathfrak{P}_0\cong \boldsymbol{P}_3'(\mathfrak{P}^*\mathfrak{Q})$, and since $\boldsymbol{C}_{\mathfrak{P}_0}(Z)=\mathfrak{P}_0\cap\mathfrak{P}$ is of index 3 in $\mathfrak{P}_0$, it follows from Lemma 5.30 that $\mathfrak{Q}_0$ is a quaternion group. Let $\tilde{\mathfrak{P}}=\boldsymbol{C}_{\mathfrak{P}^*}(\mathfrak{Q}_0)$. Thus, $\mathfrak{P}^*=\tilde{\mathfrak{P}}\langle Z\rangle$ and $\tilde{\mathfrak{P}}\cap\langle Z\rangle=1$. In particular, Z is not in the Frattini subgroup of $\mathfrak{P}^*$. All the more so, $Z\notin\boldsymbol{D}(\mathfrak{P})$. On the other hand, $\boldsymbol{D}(\mathfrak{P})\lhd\mathfrak{G}_p$ and Z lies in every nonidentity normal subgroup of $\mathfrak{G}_p$ since $\boldsymbol{Z}(\mathfrak{G}_p)$ is cyclic. Hence, $\boldsymbol{D}(\mathfrak{P})=1$, so $\mathfrak{P}=\mathfrak{F}$ is of order 27, while $\mathfrak{G}_p\simeq Z_3\wr Z_3$.

We are now in a position to play off $\mathfrak{P}_0$ against $\mathfrak{F}$. Namely, $\mathfrak{P}_0$ is a nonabelian group of order 27 and exponent 3, and since $\mathfrak{Q}_0$ is a quaternion group, it follows from Lemma 5.57 that $\boldsymbol{N}(\mathfrak{P}^*)$ contains a 2-element T which neither inverts nor centralizes $\mathfrak{F}$. But $\mathfrak{F}$ char $\mathfrak{P}^*$, so T normalizes $\mathfrak{F}$. Let $\mathfrak{A}=\boldsymbol{A}_{\mathfrak{G}}(\mathfrak{F})$ and let $\mathfrak{A}_0$ be the subgroup of $\mathfrak{A}$ whose elements have determinant 1 on $\mathfrak{F}$. Then $|\mathfrak{A}:\mathfrak{A}_0|=1$ or 2 and $\mathfrak{A}=\mathfrak{A}_0\boldsymbol{N}_{\mathfrak{A}}(\langle Z\rangle)=\mathfrak{A}_0\boldsymbol{N}_{\mathfrak{A}}(\mathfrak{B})$.

A S_3-subgroup $\mathfrak{A}_3$ of $\mathfrak{A}$ is of order 3 and acts indecomposably on $\mathfrak{F}$. On the other hand, we argue that $13\mid|\mathfrak{A}|$. Suppose false. Then $\mathfrak{A}$ is a 2,3-group, hence is solvable. (Clearly, we do not need Burnside here.) Since $\mathfrak{G}_p$ and $\mathfrak{P}^*$ both normalize $\mathfrak{F}$, $\mathfrak{A}$ is not 3-closed. Since $\mathfrak{A}_3$ is indecomposable on $\mathfrak{F}$, it follows that $\boldsymbol{O}_{3'}(\mathfrak{A}_0)$ is a four-group which is a chief factor of $\mathfrak{A}$. Let $\boldsymbol{O}_{3'}(\mathfrak{A}_0)\cong\mathfrak{L}/\boldsymbol{C}(\mathfrak{F})$ under the natural projection. Then clearly, $\boldsymbol{N}(\mathfrak{F})=\mathfrak{L}\boldsymbol{N}(\mathfrak{G}_p)\subseteq\mathfrak{L}\mathfrak{N}(\langle Z\rangle)$. Hence, we can choose L in $\mathfrak{L}$ so that $B^L=Z^{\pm1}$, since $\langle B\rangle=\Omega_1(\mathfrak{Z}(\mathfrak{P}^*))$, $\langle Z\rangle=\Omega_1(\boldsymbol{Z}(\mathfrak{G}_p))$. Since $L^2\in\boldsymbol{C}(\mathfrak{F})$, L normalizes $\langle B, B^L\rangle=\mathfrak{B}$. This implies that $\mathfrak{B}\lhd\boldsymbol{N}(\mathfrak{F})$, since $\boldsymbol{O}_{3'}(\mathfrak{A}_0)$ is a chief factor of $\mathfrak{A}$. But this means that $\boldsymbol{O}_{3'}(\mathfrak{A}_0)$ normalizes $\mathfrak{B}$, which is not the case, since Aut($\mathfrak{B}$) contains no subgroup isomorphic to A_4. Hence, $13\mid|\mathfrak{A}|$. Thus, $|\mathfrak{A}|=13\cdot3\cdot2^a$ with $a\leq5$. By Sylow's theorem, $\mathfrak{A}$ is 13-closed. Since the 13-elements of $GL(3,3)$ are nonreal, it follows that $a=0$ or 1. In any case, every involution of $\mathfrak{A}$ inverts $\mathfrak{F}$. This violates the existence of T and completes the proof of the lemma.

HYPOTHESIS 6.1. (a) $\mathfrak{A}$ is a π-subgroup of $\mathfrak{G}$ and $q\in\pi'$.

(b) If $\mathfrak{B}$ is any minimal normal subgroup of $\mathfrak{A}$, then $\boldsymbol{C}(\mathfrak{B})$ is π-solvable.

(c) If $\mathfrak{Q}$ is any nonidentity element of $\mathsf{N}(\mathfrak{A};q)$, then $\boldsymbol{N}(\mathfrak{Q})$ is π-solvable.

(d) If $\mathfrak{Q}\in\mathsf{N}(\mathfrak{A};q)$ and $\mathfrak{S}$ is a π-solvable subgroup of $\mathfrak{G}$ which contains $\mathfrak{Q}\mathfrak{A}$, then $\mathfrak{Q}\subseteq\boldsymbol{O}_{\pi'}(\mathfrak{S})$.

LEMMA 6.2. *Under Hypothesis* 6.1, *let* $\mathfrak{Q}$, $\mathfrak{Q}_1$ *be maximal elements of* $\mathsf{N}(\mathfrak{A};q)$. *If* $\mathfrak{Q}$ *and* $\mathfrak{Q}_1$ *are not conjugate by any element of* $\boldsymbol{C}(\mathfrak{A})$, *then for each minimal normal subgroup* $\mathfrak{B}$ *of* $\mathfrak{A}$, *either* $C_{\mathfrak{Q}}(\mathfrak{B})=1$ *or* $C_{\mathfrak{Q}_1}(\mathfrak{B})=1$.

PROOF. Suppose false. Let $\mathfrak{C}=C_{\mathfrak{Q}}(\mathfrak{B})$, $\mathfrak{C}_1=C_{\mathfrak{Q}_1}(\mathfrak{B})$. By Hypothesis 6.1, $\langle\mathfrak{C},\mathfrak{C}_1\rangle\subseteq O_{\pi'}(C(\mathfrak{B}))$. Since $O_{\pi'}(C(\mathfrak{B}))$ char $C(\mathfrak{B})\lhd N(\mathfrak{B})$, we may therefore choose C in $C(A)$ so that $\langle\mathfrak{C}_1^C,\mathfrak{C}\rangle$ is a q-group. We assume without loss of generality that among all triples $(\mathfrak{Q},\mathfrak{Q}_1,\mathfrak{A})$ which violate the lemma, $\mathfrak{Q}\cap\mathfrak{Q}_1$ is maximal. By the preceding argument $\mathfrak{Q}\cap\mathfrak{Q}_1=\mathfrak{Q}_0\neq 1$. By Hypothesis 6.1, $\langle N_{\mathfrak{Q}}(\mathfrak{Q}_0), N_{\mathfrak{Q}_1}(\mathfrak{Q}_0)\rangle\subseteq O_{\pi'}(N(\mathfrak{Q}_0))$, and we may therefore choose D in $C(\mathfrak{A})$ so that $\langle N_{\mathfrak{Q}}(\mathfrak{Q}_0), N_{\mathfrak{Q}_1}(\mathfrak{Q}_0)^D\rangle$ is a q-group, violating the maximality of $\mathfrak{Q}\cap\mathfrak{Q}_1$ and completing the proof.

HYPOTHESIS 6.2. (a) $\mathfrak{A}$ is a nilpotent π-subgroup of $\mathfrak{G}$, $\pi=\pi(\mathfrak{A})$, $q\in\pi'$.

(b) $C(\mathfrak{A})=Z(\mathfrak{A})\times\mathfrak{D}$, where $\mathfrak{D}$ is a π'-group.

(c) For each p in π and r in $\pi-\{p\}$, the S_p-subgroup $\mathfrak{A}_p$ of $\mathfrak{A}$ centralizes every element of $\text{И}(\mathfrak{A}_p; r)$.

(d) $\mathfrak{F}$ is a set of normal subgroups of $\mathfrak{A}$ with the following properties:

(i) If $\mathfrak{F}\in\mathcal{F}$, then every element of $\text{И}_{N(\mathfrak{F})}(\mathfrak{A}; q)$ is in $O_{\pi'}(N(\mathfrak{F}))$.

(ii) For each p in π, $\mathfrak{F}$ contains a nonidentity p-subgroup.

(e) One of the following holds:

(i) $|\pi|\geqq 2$.

(ii) There is a noncyclic abelian subgroup $\mathfrak{U}$ of $\mathfrak{A}$ such that every nonidentity subgroup of $\mathfrak{U}$ is in $\mathfrak{F}$.

LEMMA 6.3. *Assume that Hypothesis* 6.2 *is satisfied and that* $(\pi, q, \mathfrak{A})$ *satisfies conditions* (a), (b), (c) *of Hypothesis* 6.1. *Then* $(\pi, q, \mathfrak{A})$ *satisfies Hypothesis* 6.1.

PROOF. For each $\mathfrak{Q}$ in $\text{И}(\mathfrak{A}; q)$, let $\mathcal{S}=\mathcal{S}(\mathfrak{Q})$ be the set of π-solvable subgroups $\mathfrak{S}$ of $\mathfrak{G}$ which contain $\mathfrak{Q}\mathfrak{A}$ and satisfy $\mathfrak{Q}\nsubseteq O_{\pi'}(\mathfrak{S})$. We must show that $\mathcal{S}$ is empty for all $\mathfrak{Q}$. Suppose false and $\mathfrak{Q}$ is of minimal order subject to $\mathcal{S}(\mathfrak{Q})\neq\emptyset$. Choose $\mathfrak{S}$ in $\mathcal{S}(\mathfrak{Q})$.

Let $\mathfrak{Q}_0=\mathfrak{Q}\cap O_{\pi'}(\mathfrak{S})$. By minimality of $\mathfrak{Q}$, $\mathfrak{Q}/\mathfrak{Q}_0$ is a chief $\mathfrak{Q}\mathfrak{A}$-factor. Let $\mathfrak{A}_0=C_{\mathfrak{A}}(\mathfrak{Q}/\mathfrak{Q}_0)$. Since $C_{\mathfrak{Q}}(\mathfrak{A}_0)$ covers $\mathfrak{Q}/\mathfrak{Q}_0$, minimality of $\mathfrak{Q}$ implies that $\mathfrak{A}_0$ centralizes $\mathfrak{Q}$. Since $\mathfrak{Q}$ does not centralize $O_{\pi',\pi}(\mathfrak{S})/O_{\pi'}(\mathfrak{S})=\mathfrak{K}$ and since $\mathfrak{K}$ is solvable, $\mathfrak{Q}$ does not centralize $F(\mathfrak{K})$. Thus, there is a prime p such that $\mathfrak{Q}\mathfrak{A}$ normalizes the p-subgroup $\mathfrak{B}$ of $\mathfrak{K}$ and $[\mathfrak{Q},\mathfrak{B}]\neq 1$. We assume that $\mathfrak{B}$ is minimal with this property. Since $(|\mathfrak{A}|, |O_{\pi'}(\mathfrak{S})|)=1$, there is a p-subgroup $\mathfrak{P}$ of $O_{\pi',\pi}(\mathfrak{S})$ which is incident with $\mathfrak{B}$ and is normalized by $\mathfrak{A}$. By construction, $p\in\pi$, and by Hypothesis 6.2(c), the $S_{p'}$-subgroup of $\mathfrak{A}$ centralizes $\mathfrak{P}$, so centralizes $\mathfrak{B}$, so centralizes $\mathfrak{Q}$. Minimality of $\mathfrak{B}$

and the three subgroups lemma implies that $\mathfrak{A}_0$ centralizes $\mathfrak{B}$, so centralizes $\mathfrak{P}$. Suppose $\mathfrak{A}_0$ contains an element $\mathfrak{F}$ of $\mathfrak{F}$. Then $\langle \mathfrak{Q}, \mathfrak{P}, \mathfrak{A}\rangle \subseteq \boldsymbol{N}(\mathfrak{F})$, so $\mathfrak{Q} \subseteq \boldsymbol{O}_{\pi'}(\boldsymbol{N}(\mathfrak{F}))$, by Hypothesis 6.2(d). Hence, $[\mathfrak{Q}, \mathfrak{P}] \subseteq \boldsymbol{O}_{\pi'}(\boldsymbol{N}(\mathfrak{F})) \cap \boldsymbol{O}_{\pi',\pi}(\mathfrak{S})$, a π'-group. But this means that $\mathfrak{Q}$ centralizes $\mathfrak{B}$, against our construction. Hence, $\mathfrak{A}_0$ contains no element of $\mathfrak{F}$. In particular, $\mathfrak{A}$ is a p-group.

Since $\mathfrak{A}$ is a p-group, Hypothesis 6.2(e)(ii) holds. Since $\mathfrak{U}$ is a noncyclic abelian group, $\mathfrak{U} \cap \mathfrak{A}_0 = \mathfrak{U}_0 \neq 1$. But then $\mathfrak{U}_0 \in \mathfrak{F}$, against the preceding argument. The proof is complete.

THEOREM 6.1 (TRANSITIVITY THEOREM). *Suppose $\mathfrak{G}$ is an N-group, $\mathfrak{G}_p$ is a S_p-subgroup of $\mathfrak{G}$, p and q are distinct primes, and $\mathfrak{A}$ is a normal subgroup of $\mathfrak{G}_p$ with the following properties:*

(i) $\boldsymbol{C}_{\mathfrak{G}_p}(\mathfrak{A}) = \boldsymbol{Z}(\mathfrak{A})$.

(ii) *$\boldsymbol{Z}(\mathfrak{A})$ contains an element $\mathfrak{B}$ of $\mathfrak{U}(\mathfrak{G}_p)$.*

Then

(a) *for each pair of maximal elements $\mathfrak{Q}$, $\mathfrak{Q}_1$ of $\mathsf{M}(\mathfrak{A}; q)$ which are not conjugate under $\boldsymbol{C}(\mathfrak{A})$ and each Z in $\boldsymbol{Z}(\mathfrak{A})^\#$, either $\boldsymbol{C}_{\mathfrak{Q}}(Z) = 1$ or $\boldsymbol{C}_{\mathfrak{Q}_1}(Z) = 1$.*

(b) *$\boldsymbol{C}(\mathfrak{A})$ permutes transitively by conjugation the maximal elements of $\mathsf{M}(\mathfrak{A}; q)$ provided $m(\boldsymbol{Z}(\mathfrak{A})) \geqq 3$.*

PROOF. Clearly, (b) is a consequence of (a), so we restrict attention to (a). It suffices to show that $(p, q, \mathfrak{A})$ satisfies Hypothesis 6.1. Since $\mathfrak{G}$ is an N-group, conditions (a), (b), (c) are satisfied. Thus, it suffices to show that with $\mathfrak{F}$ the set of nonidentity subgroups of $\mathfrak{B}$ Hypothesis 6.2 is satisfied. It is clear that all parts of Hypothesis 6.2 are satisfied except possibly (d)(i).

Choose $\mathfrak{B}_0 \in \mathfrak{F}$ and let $\mathfrak{Q}$ be any element of $\mathsf{M}(\mathfrak{A}; q)$ centralized by $\mathfrak{B}_0$. Let $\mathfrak{H} = \boldsymbol{N}(\mathfrak{B}_0)$. By Lemma 6.1, $\mathfrak{B} \subseteq \boldsymbol{O}_{p',p}(\mathfrak{H})$. Hence $[\mathfrak{Q}, \mathfrak{B}] \subseteq \boldsymbol{O}_{p'}(\mathfrak{H})$. Thus, it suffices to show that $\boldsymbol{C}_{\mathfrak{Q}}(\mathfrak{B}) \subseteq \boldsymbol{O}_{p'}(\mathfrak{H})$. By Lemma 0.7.8, we may assume that $\mathfrak{B}_0 = \mathfrak{B}$. In this case, $\mathfrak{G}_p \subseteq \mathfrak{H}$. Let $\mathfrak{P}_0 = \boldsymbol{O}_{p',p}(\mathfrak{H}) \cap \mathfrak{G}_p$. Hence, $[\boldsymbol{O}_{p',p}(\mathfrak{H}), \mathfrak{A}] \subseteq [\mathfrak{P}_0, \mathfrak{A}]\boldsymbol{O}_{p'}(\mathfrak{H}) \subseteq \mathfrak{A}\boldsymbol{O}_{p'}(\mathfrak{H})$. Hence $[\boldsymbol{P}_p^1(\mathfrak{H}), \mathfrak{A}, \mathfrak{Q}] = 1$. By Lemma 5.16, we get $[\mathfrak{Q}, \mathfrak{A}] \subseteq \boldsymbol{O}_{p'}(\mathfrak{H})$. Thus, we may assume that $\mathfrak{A}$ centralizes $\mathfrak{Q}$. But in this case, we get $\mathfrak{Q} \subseteq \boldsymbol{O}_{p'}(\mathfrak{H})$ by Lemma 3.7 of [**20**]. The proof is complete.

COROLLARY 6.1. *Let $\mathfrak{P}$ be a S_p-subgroup of $\mathfrak{G}$, $p \in \pi_4(\mathfrak{G})$, $\mathfrak{A} \in \mathscr{S}\mathrm{cn}_3(\mathfrak{P})$. If $(p, q, \mathfrak{A})$ satisfies Hypothesis 6.1, then $\mathsf{M}(\mathfrak{A}; q)$ is trivial.*

PROOF. Let $\mathfrak{Q}$ be a maximal element of $\mathsf{M}(\mathfrak{A}; q)$, and let $N \in \boldsymbol{N}(\mathfrak{A})$. Then $\mathfrak{Q}^N$ is a maximal element of $\mathsf{M}(\mathfrak{A}; q)$. As $m(\mathfrak{A}) \geqq 3$, there is an element A of $\mathfrak{A}$ of order p such that $\boldsymbol{C}(A) \cap \mathfrak{Q}^N \neq 1$, $\boldsymbol{C}(A) \cap \mathfrak{Q} \neq 1$. Thus, Lemma 6.1 implies that $\mathfrak{Q} = \mathfrak{Q}^{NC}$ for some C in $\boldsymbol{C}(\mathfrak{A})$. Hence,

$N(\mathfrak{A}) = (N(\mathfrak{A}) \cap N(\mathfrak{Q})) \cdot C(\mathfrak{A})$, so that $N(\mathfrak{Q})$ contains a S_p-subgroup of $\mathfrak{G}$. Since $p \in \pi_4(\mathfrak{G})$, we have $\mathfrak{Q} = 1$, as required.

HYPOTHESIS 6.3. (a) $(p, q, \mathfrak{A})$ satisfies Hypothesis 6.1, p is an odd prime.

(b) $\mathfrak{A} \in \mathcal{S}cn_3(\mathfrak{P})$ and $\mathfrak{P}$ is a S_p-subgroup of $\mathfrak{G}$.

(c) $\mathsf{M}(\mathfrak{A}; q)$ is not trivial, and $\mathfrak{Q}$ is a maximal element of $\mathsf{M}(\mathfrak{A}; q)$ which is normalized by $\mathfrak{P}$.

(d) $\mathfrak{B} = V(\mathrm{ccl}_{\mathfrak{G}}(\mathfrak{A}); \mathfrak{P})$.

(e) The normalizer of every nonidentity p-subgroup of $\mathfrak{G}$ is p-solvable.

(f) If $\mathfrak{H}$ is any p-solvable subgroup of $\mathfrak{G}$ and $\mathfrak{H}_p$ is a S_p-subgroup of $\mathfrak{H}$, then every element of $\mathcal{S}cn(\mathfrak{H}_p)$ is contained in $O_{p',p}(\mathfrak{H})$.

COROLLARY 6.2. *Suppose Hypothesis* 6.3 *is satisfied. Then Lemmas* 0.17.n *hold,* $1 \leqq n \leqq 4$. *If the word "proper" in Lemmas* 0.17.5 *and* 0.17.6 *is replaced by "p-solvable," these lemmas hold, too.*

PROOF. Hypothesis 6.3 guarantees that the relevant subgroups are p-solvable. Thus, (f) is sufficient to carry out the proofs of the designated lemmas.

HYPOTHESIS 6.4. (a) $\mathfrak{H}_p$ is a S_p-subgroup of $\mathfrak{G}$, $\mathfrak{A} \in \mathcal{S}cn(\mathfrak{H}_p)$, $q \in \pi(\mathfrak{G}), q \neq p$.

(b) $\mathfrak{U}$ is a normal elementary subgroup of $\mathfrak{H}_p$, $\mathfrak{U} \subseteq \mathfrak{A}$.

(c) $\mathfrak{U}$ centralizes every element of $\mathsf{M}(\mathfrak{A}; q)$.

(d) If $\mathfrak{Q}$ is a nonidentity element of $\mathsf{M}(\mathfrak{A}; q)$, then $N(\mathfrak{Q})$ is p-solvable.

(e) If $1 \subset \mathfrak{A}_0 \subseteq \mathfrak{A}$, then $C(\mathfrak{A}_0)$ is p-solvable.

LEMMA 6.4. *Suppose Hypothesis* 6.4 *is satisfied and* $\mathfrak{B}$ *is a normal elementary subgroup of* $\mathfrak{H}_p$ *of order* p^2 *with* $\mathfrak{B} \subseteq \mathfrak{A}$. *Suppose also that* $\{p, q\} \neq \{2, 3\}$. *Then* $\mathfrak{U} \cap \mathfrak{B}$ *centralizes every element of* $\mathsf{M}(\mathfrak{B}; q)$

PROOF. If false, we may choose V in $\mathfrak{B}^\#$ so that $\mathfrak{U} \cap \mathfrak{B}$ does not centralize every element of $\mathsf{M}_{\mathfrak{C}}(\mathfrak{B}; q)$, where $\mathfrak{C} = C(V)$. Let $\mathfrak{Q}$ be an element of $\mathsf{M}_{\mathfrak{C}}(\mathfrak{B}; q)$ which is not centralized by $\mathfrak{U} \cap \mathfrak{B}$. We may assume that $\mathfrak{Q} = [\mathfrak{Q}, \mathfrak{B}]$. If $\mathfrak{Q} \subseteq O_{p'}(\mathfrak{C})$, then $\mathfrak{U} \cap \mathfrak{B}$ centralizes no S_q-subgroup of $O_{p'}(\mathfrak{C})$, while some S_q-subgroup of $O_{p'}(\mathfrak{C})$ is in $\mathsf{M}(\mathfrak{A}; q)$. This is impossible so $\mathfrak{Q} \subseteq O_{p'}(\mathfrak{C})$. Hence, $\mathfrak{B} \cap O_{p',p}(\mathfrak{C}) = \langle V \rangle$. Since $\mathfrak{B}$ centralizes a subgroup of $P_p^1(\mathfrak{C})$ of index p, it follows from (B) that $\{p, q\} = \{2, 3\}$. The proof is complete.

HYPOTHESIS 6.5. (a) $\mathfrak{H}$ is a $S_{p,q}$-subgroup of $\mathfrak{G}$, $\pi = \{p, q\} = \pi(\mathfrak{H}) \neq \{2, 3\}$.

(b) $\mathfrak{K} = O_p(\mathfrak{H}) \neq 1$, $\mathfrak{L} = O_q(\mathfrak{H}) \neq 1$.

(c) $\mathfrak{K}$ contains an elementary subgroup $\mathfrak{U}$ of order p^2 which is normal in some S_p-subgroup of $\mathfrak{H}$.

(d) The normalizer of every nonidentity π-subgroup of $\mathfrak{G}$ is π-solvable.

Lemma 6.5. *Under Hypothesis* 6.5, $\mathfrak{K}$ *centralizes every element of* $\text{И}(\mathfrak{K}; q)$.

Proof. Enlarge $\langle \mathfrak{U}, \Omega_1(\boldsymbol{Z}(\mathfrak{K}))\rangle = \mathfrak{U}^*$ to an element $\mathfrak{A}$ of $\mathcal{SCN}(\mathfrak{H}_p)$, $\mathfrak{H}_p$ being a S_p-subgroup of $\boldsymbol{N}_{\mathfrak{H}}(\mathfrak{U})$.

We will show that Hypothesis 6.2 is satisfied with our present $\mathfrak{A}$ in the role of $\mathfrak{A}$, with p in the role of π, and with q in the role of q, where we let $\mathfrak{F}$ be the set of nonidentity subgroups of $\mathfrak{U}$.

Since $\mathfrak{A} \in \mathcal{SCN}(\mathfrak{H}_p)$ and since $\mathfrak{H}_p$ is a S_p-subgroup of $\mathfrak{G}$, Hypothesis 6.2(b) holds. Hypothesis 6.2(c) holds vacuously. Thus, it suffices to verify Hypothesis 6.2(d).

By Lemma 0.7.4, $\mathfrak{L}$ is a maximal element of $\text{И}_{\mathfrak{H}}(\mathfrak{A}, q)$. Since $\mathfrak{H} \subseteq \boldsymbol{N}(\mathfrak{L})$ and since $\boldsymbol{N}(\mathfrak{L})$ is p-solvable, $\mathfrak{L}$ is a maximal element of $\text{И}_{\mathfrak{G}}(\mathfrak{A}; q)$.

Let $\mathfrak{U}_0$ be a subgroup of $\mathfrak{U}$ of order p, and let $\mathfrak{N} = \boldsymbol{N}(\mathfrak{U}_0)$. Let $\mathfrak{Q}$ be an element of $\text{И}_{\mathfrak{N}}(\mathfrak{A}; q)$. We must show that $\mathfrak{Q} \subseteq \boldsymbol{O}_{p'}(\mathfrak{N})$. Suppose false and $\mathfrak{Q}$ is minimal with this property. Let $\mathfrak{Q}_0 = \mathfrak{Q} \cap \boldsymbol{O}_{p'}(\mathfrak{N})$. Then $\mathfrak{Q}/\mathfrak{Q}_0$ is a chief $\mathfrak{Q}\mathfrak{A}$-factor. Let $\mathfrak{A}_0 = \boldsymbol{C}_{\mathfrak{A}}(\mathfrak{Q}/\mathfrak{Q}_0)$. Minimality of $\mathfrak{Q}$ forces $\mathfrak{A}_0$ to centralize $\mathfrak{Q}$. Suppose $\mathfrak{H}_p \cap \mathfrak{N}$ is a S_p-subgroup of $\mathfrak{N}$. Then another application of Lemma 0.7.4 yields $\mathfrak{Q} \subseteq \boldsymbol{O}_{p'}(\mathfrak{N})$. Thus, $\mathfrak{H}_p \cap \mathfrak{N}$ is of index p in a S_p-subgroup $\mathfrak{N}_p$ of $\mathfrak{N}$. By Lemma 0.7.8, $\mathfrak{Q} \not\subseteq \boldsymbol{O}_{p'}(\boldsymbol{C}(\mathfrak{A}_0))$. Suppose $\mathfrak{A}_0$ contains an element Z of $\boldsymbol{Z}(\mathfrak{H}_p)^{\#}$. Then $\boldsymbol{C}(\mathfrak{A}_0) \subseteq \boldsymbol{C}(Z)$, so $\mathfrak{Q} \not\subseteq \boldsymbol{O}_{p'}(\boldsymbol{C}(Z))$. However, $\mathfrak{H}_p$ is a S_p-subgroup of $\boldsymbol{C}(Z)$, so Lemma 0.7.4 is violated. Hence, no such Z is available. As $\mathfrak{A}/\mathfrak{A}_0$ is cyclic, it follows that $\boldsymbol{Z}(\mathfrak{H}_p)$ is cyclic and $\Omega_1(\boldsymbol{Z}(\mathfrak{H}_p)) \not\subseteq \mathfrak{A}_0$. Since $\boldsymbol{Z}(\mathfrak{H}_p)$ is cyclic, $\mathfrak{U} \in \mathcal{U}(\mathfrak{H}_p)$. By Lemma 6.1, we get $\mathfrak{U} \subseteq \boldsymbol{O}_{p',p}(\mathfrak{N})$. Hence, $[\mathfrak{Q}, \mathfrak{U}] \subseteq \boldsymbol{O}_{p'}(\mathfrak{N})$. But by minimality of $\mathfrak{Q}$, we also have $\mathfrak{Q} = [\mathfrak{Q}, \mathfrak{U}]$.

We have shown that Hypothesis 6.2 is satisfied. Hence, Hypothesis 6.1 is satisfied. Let $\mathfrak{Q}^*$ be any maximal element of $\text{И}(\mathfrak{A}; q)$. We can then choose U in $\mathfrak{U}^{\#}$ such that $\boldsymbol{C}_{\mathfrak{Q}^*}(U) \neq 1$. Since $\mathfrak{U}$ centralizes $\mathfrak{L}$, it follows that $\mathfrak{Q}^* = \mathfrak{L}^C$ for some C in $\boldsymbol{C}(\mathfrak{A})$, by Lemma 6.2. Hence, $\mathfrak{A} \cap \mathfrak{K}$ centralizes every element of $\text{И}(\mathfrak{A}; q)$.

It remains to show that $\mathfrak{K}$ centralizes every element of $\text{И}(\mathfrak{K}; q)$. Suppose $\mathfrak{Q} \in \text{И}(\mathfrak{K}; q)$. By Lemma 6.4, $\mathfrak{U}$ centralizes $\mathfrak{Q}$. We will show that $\Omega_1(\boldsymbol{Z}(\mathfrak{K}))$ centralizes $\mathfrak{Q}$. Since $\mathfrak{Q} \subseteq \boldsymbol{N}(\mathfrak{U})$, Lemma 5.16 implies that $[\mathfrak{Q}, \Omega_1(\boldsymbol{Z}(\mathfrak{K}))] \subseteq \boldsymbol{O}_{p'}(\boldsymbol{N}(\mathfrak{U}))$. Since $\mathfrak{A}$ normalizes some S_q-subgroup of $\boldsymbol{O}_{p'}(\boldsymbol{N}(\mathfrak{U}))$, and since $\Omega_1(\boldsymbol{Z}(\mathfrak{K}))$ centralizes every element of

$\text{И}(\mathfrak{A}; q)$, it follows that $\Omega_1(\boldsymbol{Z}(\mathfrak{R}))$ centralizes every q-subgroup of $\boldsymbol{O}_{p'}(\boldsymbol{N}(\mathfrak{U}))$ which it normalizes. Hence, $[\mathfrak{Q}, \Omega_1(\boldsymbol{Z}(\mathfrak{R}))]$ is centralized by $\Omega_1(\boldsymbol{Z}(\mathfrak{R}))$, so $\mathfrak{Q}$ is centralized by $\Omega_1(\boldsymbol{Z}(\mathfrak{R}))$. Since $\mathfrak{Q} \subseteq \boldsymbol{N}(\Omega_1(\boldsymbol{Z}(\mathfrak{R})))$, and since $\mathfrak{H}$ is a $S_{p,q}$-subgroup of $\boldsymbol{N}(\Omega_1(\boldsymbol{Z}(\mathfrak{R})))$, $\mathfrak{R}$ centralizes $\mathfrak{Q}$ by Lemma 0.7.5. The proof is complete.

The following lemma is a careful rephrasing of the argument in Lemma 0.20.3.

LEMMA 6.6. *Suppose $\mathfrak{G}$ is a finite group, p, q are distinct primes and $\boldsymbol{N}(\mathfrak{A})$ is p, q-solvable for every nonidentity p, q-subgroup $\mathfrak{A}$ of $\mathfrak{G}$. Let $\mathfrak{H}$ be a maximal p, q-subgroup of $\mathfrak{G}$ with Sylow system $\{\mathfrak{H}_p, \mathfrak{H}_q\}$. Let $\mathfrak{F}_p = \boldsymbol{O}_p(\mathfrak{H})$, $\mathfrak{F}_q = \boldsymbol{O}_q(\mathfrak{H})$. Suppose that $\mathfrak{F}_p \neq 1$ and $\mathfrak{F}_q \neq 1$. Let $\mathfrak{P}$ be a S_p-subgroup of $\mathfrak{G}$ which contains $\mathfrak{H}_p$. Then*

(i) *$\mathfrak{H}_p$ is a S_p-subgroup of every p, q-subgroup of $\mathfrak{G}$ which contains $\mathfrak{H}_p\mathfrak{F}_q$.*

(ii) *One of the following holds:*

(a) *$\mathfrak{P}$ is faithfully represented on some element of $\text{И}(\mathfrak{P}; q)$.*

(b) *$\mathfrak{H}_p$ contains no element of $\mathfrak{U}(\mathfrak{P})$ and $|\mathfrak{F}_p| = p$.*

(c) *$\mathfrak{H}_p = \mathfrak{P}$.*

(d) *$p \in \pi_2(\mathfrak{G})$.*

(iii) *If $p = 2$, then one of the following holds:*

(a) *$2 \notin \pi_4(\mathfrak{G})$.*

(b) *$|\mathfrak{G} : \mathfrak{G}'|$ is even.*

PROOF. (i) Since $\mathfrak{H}$ is a maximal p, q-subgroup of $\mathfrak{G}$, $\mathfrak{H}$ is a $S_{p,q}$-subgroup of the normalizer of every nonidentity normal subgroup of $\mathfrak{H}$. Let $\mathfrak{R}$ be a p, q-subgroup of $\mathfrak{G}$ which contains $\mathfrak{H}_p\mathfrak{F}_q$ with Sylow system $\{\mathfrak{R}_p, \mathfrak{R}_q\}$ where $\mathfrak{H}_p \subseteq \mathfrak{R}_p$, $\mathfrak{F}_q \subseteq \mathfrak{R}_q$. We must show that $\mathfrak{H}_p = \mathfrak{R}_p$. Let $\mathfrak{A}_p = \boldsymbol{O}_p(\mathfrak{R})$, $\mathfrak{A}_q = \boldsymbol{O}_q(\mathfrak{R})$.

We first show that $\mathfrak{A}_p \subseteq \mathfrak{H}_p$. Namely, $\boldsymbol{C}_{\mathfrak{A}_p}(\mathfrak{F}_p)$ is a p-subgroup of $\boldsymbol{N}(\mathfrak{F}_p)$ which admits $\mathfrak{F}_q$. As $\mathfrak{F}_q$ centralizes all the p-subgroups of $\text{И}_{\boldsymbol{N}(\mathfrak{F}_p)}(\mathfrak{F}_q)$, it follows that $\mathfrak{F}_q$ centralizes $\boldsymbol{C}_{\mathfrak{A}_p}(\mathfrak{F}_p)$. Hence $\mathfrak{F}_q$ centralizes $\mathfrak{A}_p$, by Lemma 3.7 of [20]. Thus, $\mathfrak{A}_p \subseteq \mathfrak{H}_p$, since $\mathfrak{A}_p\mathfrak{H}_p$ is a p-subgroup of $\boldsymbol{N}(\mathfrak{F}_q)$.

Since $\mathfrak{F}_p$ centralizes $\boldsymbol{C}_{\mathfrak{A}_q}(\mathfrak{F}_q)$, it follows that $\mathfrak{F}_q\mathfrak{A}_q \in \text{И}_{\boldsymbol{N}(\mathfrak{F}_p)}(\mathfrak{H}_p; q)$. Since $\mathfrak{H}$ is a maximal p, q-subgroup of $\mathfrak{G}$, it follows that $\mathfrak{F}_q\mathfrak{A}_q = \mathfrak{F}_q$, that is, $\mathfrak{A}_q \subseteq \mathfrak{F}_q$.

Since $\mathfrak{A}_q \subseteq \mathfrak{F}_q$ and $\mathfrak{F}_q$ centralizes $\mathfrak{A}_p$, it follows that $\boldsymbol{Z}(\mathfrak{F}_q) \subseteq \boldsymbol{Z}(\mathfrak{A}_q)$. Let $\mathfrak{C} = \boldsymbol{C}_{\mathfrak{R}}(\boldsymbol{Z}(\mathfrak{A}_q)) \lhd \mathfrak{R}$. Since $\boldsymbol{Z}(\mathfrak{F}_q) \subseteq \boldsymbol{Z}(\mathfrak{A}_q)$, it follows that $\mathfrak{C} \subseteq \boldsymbol{C}(\boldsymbol{Z}(\mathfrak{F}_q))$. Thus, $\mathfrak{C}$ is a p, q-subgroup of $\boldsymbol{N}(\boldsymbol{Z}(\mathfrak{F}_q))$ which contains $\mathfrak{F}_p$. By Lemma 0.7.5, $\mathfrak{F}_p \subseteq \boldsymbol{O}_p(\mathfrak{C})$. Since $\boldsymbol{O}_p(\mathfrak{C})$ char $\mathfrak{C} \lhd \mathfrak{R}$, it follows that $\mathfrak{F}_p \subseteq \mathfrak{A}_p$. Now let $\mathfrak{D} = \boldsymbol{C}_{\mathfrak{R}}(\mathfrak{A}_p) \lhd \mathfrak{R}$. Then $\mathfrak{D} \subseteq \boldsymbol{N}(\mathfrak{F}_p)$, so $\mathfrak{F}_q \subseteq \boldsymbol{O}_q(\mathfrak{D})$. Hence, $\mathfrak{F}_q \subseteq \mathfrak{A}_q$. Since we already have the reverse containment it follows that $\mathfrak{A}_q = \mathfrak{F}_q \lhd \mathfrak{R}$, so $\mathfrak{H}_p = \mathfrak{R}_p$ as required.

(ii) We assume by way of contradiction that (a)–(d) all fail. Obviously, $\mathfrak{P}$ is noncyclic, since (c) fails. Since (d) fails, we have $p \in \pi_3(\mathfrak{G}) \cup \pi_4(\mathfrak{G})$. Let $\mathfrak{A} \in \mathcal{S}\mathcal{C}\mathfrak{n}_3(\mathfrak{P})$ and let $\mathfrak{A}_0 = \mathfrak{A} \cap \mathfrak{H}_p$. If $\mathfrak{A}_0 = \mathfrak{A}$, then by the transitivity theorem, $\mathfrak{H}_p = \mathfrak{P}$. Hence, $\mathfrak{A}_0 \subset \mathfrak{A}$. If $\mathfrak{A}_0 \cap \mathfrak{F}_p \neq 1$, then $\mathfrak{A}_0 \cap \mathfrak{F}_p \cap \boldsymbol{Z}(\mathfrak{H}_p) \neq 1$, so (i) is violated in $\boldsymbol{C}(\mathfrak{A}_0 \cap \mathfrak{F}_p \cap \boldsymbol{Z}(\mathfrak{H}_p))$. Hence, $\mathfrak{A}_0 \cap \mathfrak{F}_p = 1$.

Suppose $\mathfrak{B}$ is an element of $\mathcal{U}(\mathfrak{P})$ with $\mathfrak{B} \subseteq \mathfrak{H}_p$. We assume without loss of generality that $\mathfrak{B} \subseteq \mathfrak{A}_0$. Then $\mathfrak{B} \cap \mathfrak{F}_p = 1$, so $\mathfrak{B} \cap \boldsymbol{Z}(\mathfrak{P})$ does not centralize $\mathfrak{F}_q$. Choose Z in $\mathfrak{B} \cap \boldsymbol{Z}(\mathfrak{P})^\#$. As $\mathfrak{B} \cap \mathfrak{F}_p = 1$, we can choose B in $\mathfrak{B}^\#$ such that Z does not centralize $\mathfrak{F}_q \cap \boldsymbol{C}(B)$. Let $\mathfrak{P}^*$ be a S_p-subgroup of $\boldsymbol{C}(B)$ which contains $\boldsymbol{C}_{\mathfrak{P}}(B)$. Thus, $\mathfrak{A} \subseteq \mathfrak{P}^*$. Let $\mathfrak{Q}^*$ be a S_q-subgroup of $\boldsymbol{C}(B)$ which is permutable with $\mathfrak{P}^*$. By Lemma 6.1(a), $\mathfrak{B} \subseteq \boldsymbol{O}_{p',p}(\boldsymbol{C}(B))$, and so Z does not centralize $\boldsymbol{O}_q(\mathfrak{P}^*\mathfrak{Q}^*)$. Let $\mathfrak{Q}(Z)$ be any maximal element of $\mathsf{M}(\mathfrak{A}; q)$ which contains $\boldsymbol{O}_q(\mathfrak{P}^*\mathfrak{Q}^*)$. Then Z does not centralize $\mathfrak{Q}(Z)$.

On the other hand, if Z_0 is any element of $\boldsymbol{Z}(\mathfrak{P})$ of order p and $Z_0 \notin \langle Z \rangle$, then $\langle Z, Z_0 \rangle \in \mathcal{U}(\mathfrak{P})$ and $\langle Z, Z_0 \rangle \subseteq \mathfrak{H}_p$. Hence, for each Z in $\Omega_1(\boldsymbol{Z}(\mathfrak{P}))^\#$, there is a maximal element $\mathfrak{Q}(Z)$ of $\mathsf{M}(\mathfrak{A}; q)$ such that $[\mathfrak{Q}(Z), Z] \neq 1$. By Lemma 6.2, all these $\mathfrak{Q}(Z)$ are conjugate under $\boldsymbol{C}(\mathfrak{A})$. Hence, $\boldsymbol{Z}(\mathfrak{P})$ is faithfully represented on each $\mathfrak{Q}(Z)$. Let $\tilde{\mathfrak{Q}}$ be a fixed maximal element of $\mathsf{M}(\mathfrak{A}; q)$. Then $\boldsymbol{N}(\mathfrak{A}) = (\boldsymbol{N}(\mathfrak{A}) \cap \boldsymbol{N}(\tilde{\mathfrak{Q}})) \cdot \boldsymbol{C}(\mathfrak{A})$, so that there is D in $\boldsymbol{C}(A)$ such that $\mathfrak{P}^D$ normalizes $\tilde{\mathfrak{Q}}$. Since $\boldsymbol{Z}(\mathfrak{P}^D) = \boldsymbol{Z}(\mathfrak{P})^D = \boldsymbol{Z}(\mathfrak{P})$, it follows that $\mathfrak{P}$ is faithfully represented on $\tilde{\mathfrak{Q}}^{D^{-1}}$, so (a) holds. This is not the case, so $\mathfrak{H}_p$ contains no element of $\mathcal{U}(\mathfrak{P})$. Since (b) fails, $|\mathfrak{F}_p| > p$.

Let $\mathfrak{B} \in \mathcal{U}(P)$, $\mathfrak{B}_0 = \mathfrak{B} \cap \mathfrak{A}_0$, so that $\mathfrak{B}_0 = \Omega_1(\boldsymbol{Z}(\mathfrak{P}))$ is of order p and $\mathfrak{B}$ normalizes $\mathfrak{H}_p$. If $\mathfrak{F}_p \cap \boldsymbol{C}(\mathfrak{B}) \neq 1$, then $\mathfrak{F}_p \cap \boldsymbol{C}(\mathfrak{B}) \cap \boldsymbol{Z}(\mathfrak{H}_p) \neq 1$, so (i) is violated. Hence, $\mathfrak{F}_p \cap \boldsymbol{C}(\mathfrak{B}) = 1$. This is not the case, since $|\mathfrak{P}: \boldsymbol{C}_{\mathfrak{P}}(\mathfrak{B})| = p$ and $|\mathfrak{F}_p| > p$. This completes the proof of (ii).

(iii) Suppose (a) and (b) fail. Since $2 \in \pi_4(\mathfrak{G})$, (ii)(a) fails, as do (c) and (d). Thus, (ii)(b) holds. Since $|\mathfrak{G}: \mathfrak{G}'|$ is odd, it follows from Lemma 5.38(a)(ii) that $\boldsymbol{N}(\mathfrak{F}_2)$ contains an element $\mathfrak{U}$ of $\mathcal{U}(2)$. Since $2 \in \pi_4(\mathfrak{G})$, Lemmas 6.1 and 6.2 imply that $\mathfrak{U}$ centralizes every element of $\mathsf{M}(\mathfrak{U}; q)$. This is impossible since $\mathfrak{H}_2/\mathfrak{F}_2$ is faithfully represented on $\mathfrak{F}_q$. The proof is complete.

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