

## NONSOLVABLE FINITE GROUPS ALL OF WHOSE LOCAL SUBGROUPS ARE SOLVABLE, II

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**In this second paper, the bulk of the work is devoted to characterizing  $E_2(3)$  and  $S_4(3)$ . These two groups are "almost"  $N$ -groups and it is relevant to treat them separately. The actual characterizations (Theorems 8.1 and 9.1) are very technical but the hypotheses deal with the structure and embedding in a simple group of certain  $\{2, 3\}$ -subgroups.**

This paper is a continuation of an earlier paper.<sup>1</sup> The bibliographical references are to I.

### 7. Groups in which 1 is the only $p$ -signalizer.

**DEFINITION 7.1.**  $\mathcal{U}^*(p) = \{\mathfrak{B} \mid \text{(i) } \mathfrak{B} \text{ is a subgroup of } \mathfrak{G} \text{ of type } (p, p). \text{ (ii) } N(\mathfrak{B}) \text{ contains a } S_p\text{-subgroup of } \mathfrak{G}.\}$

**HYPOTHESIS 7.1.** (i)  $p$  is a prime and if  $\mathfrak{B} \in \mathcal{U}^*(p)$ , then no  $S_p$ -subgroup of  $C(\mathfrak{B})$  normalizes any nonidentity  $p'$ -subgroup of  $\mathfrak{G}$ .

(ii) The centralizer of every nonidentity  $p$ -subgroup of  $\mathfrak{G}$  is  $p$ -solvable.

Lemmas 7.1, 7.2, 7.3 are proved under Hypothesis 7.1.

**LEMMA 7.1.** (i)  $\mathcal{U}(p) \subseteq \mathcal{E}(p)$ . (See Definitions 2.8 and 2.10 of I).  
 (ii) If  $p \geq 5$ , then  $\mathcal{U}^*(p) \subseteq \mathcal{E}(p)$ .  
 (iii) If  $p = 3$  and if no element of  $\mathcal{U}(3)$  centralizes a quaternion subgroup of  $\mathfrak{G}$ , then  $\mathcal{U}^*(3) \subseteq \mathcal{E}(3)$ .

*Proof.* If  $p$  is odd, choose  $\mathfrak{B} \in \mathcal{U}^*(p)$ , while if  $p = 2$ , choose  $\mathfrak{B} \in \mathcal{U}(2)$ . We must show that either  $\mathfrak{B}$  centralizes every element of  $\mathcal{N}(\mathfrak{B}; p')$  or  $p = 3$ ,  $\mathfrak{B} \in \mathcal{U}^*(3) - \mathcal{U}(3)$  and some element of  $\mathcal{U}(3)$  centralizes a quaternion subgroup of  $\mathfrak{G}$ .

Let  $\mathfrak{P}$  be a  $S_p$ -subgroup of  $N(\mathfrak{B})$ , so that  $\mathfrak{P}$  is a  $S_p$ -subgroup of  $\mathfrak{G}$ . Proceeding by way of contradiction, let  $\mathfrak{Q}$  be an element of  $\mathcal{N}(\mathfrak{B}; p')$  minimal subject to  $[\mathfrak{Q}, \mathfrak{B}] \neq 1$ . Then  $\mathfrak{Q}$  is a  $q$ -group for some prime  $q \neq p$ ,  $\mathfrak{Q} = [\mathfrak{Q}, \mathfrak{B}]$ , and  $\mathfrak{B}_0 = C_{\mathfrak{P}}(\mathfrak{Q})$  has order  $p$ . Let  $\mathfrak{C} = C(\mathfrak{B}_0)$ ,  $\mathfrak{C}_1 = C_{\mathfrak{P}}(\mathfrak{B}_0)$ , and let  $\mathfrak{P}^*$  be a  $S_p$ -subgroup of  $\mathfrak{C}$  containing  $\mathfrak{C}_1$ . Hypothesis 7.1 implies that  $O_p(\mathfrak{C}) = 1$ . Let  $\mathfrak{P}_0 = O_p(\mathfrak{C})$ . If  $[\mathfrak{P}_0, \mathfrak{B}] \subseteq \mathfrak{B}$ , then

<sup>1</sup> *Non-solvable finite groups all of whose local subgroups are solvable*, I, Bull. Amer. Math. Soc. **74** (1968), 383-437, which will be referred to as I.

Lemma 5.16 is violated. Hence, we have  $|\mathfrak{P}^* : \mathbb{C}_1| = |\mathfrak{P}_0 : \mathfrak{P}_0 \cap \mathbb{C}_1| = p$  and  $[\mathfrak{P}_0, \mathfrak{B}] \not\subseteq \mathfrak{B}$ .

Suppose  $\mathfrak{B} \subseteq \mathfrak{P}_0$ . Then  $\Omega = [\Omega, \mathfrak{B}] \subseteq \mathfrak{P}_0$ , so  $\Omega = 1$ . This is not the case, so  $\mathfrak{B} \not\subseteq \mathfrak{P}_0$ . By Lemma 6.1, it follows that  $\mathfrak{B} \notin \mathcal{Z}(p)$ . Hence, by construction,  $p$  is odd. Since  $\mathfrak{B} \not\subseteq \mathfrak{A}_0$ ,  $(\mathfrak{B})$  implies that  $p \leq 3$ . Thus,  $p = 3$  and  $\mathfrak{B} \in \mathcal{Z}^*(3) - \mathcal{Z}(3)$ . By definition of  $\mathcal{Z}(3)$  and  $\mathcal{Z}^*(3)$ , it follows that  $Z(\mathfrak{B})$  is non cyclic and  $\mathfrak{B}$  is not contained in the center of any  $S_3$ -subgroup of  $\mathfrak{G}$ .

Since  $[\mathfrak{P}_0, \mathfrak{B}, \mathfrak{B}] = 1$  and since  $\mathfrak{B} \not\subseteq \mathfrak{P}_0$ , it follows that  $\mathfrak{B} \not\subseteq O_3(\mathbb{C}_{2,3})$ , where  $\mathbb{C}_{2,3}$  is a  $S_{2,3}$ -subgroup of  $\mathbb{C}$  containing  $\mathfrak{P}^*$ .

Since  $\mathfrak{B}_0 \not\subseteq Z(\mathfrak{B})$ , we have  $\mathfrak{B} = \mathfrak{B}_0 \times \mathfrak{B}_1$ , where  $\mathfrak{B}_1 \subseteq Z(\mathfrak{B})$ . Since  $\mathfrak{B}_0 \subseteq Z(\mathbb{C})$ , we have  $\mathfrak{B}_0 \subseteq O_3(\mathbb{C}_{2,3})$ , and so  $\mathfrak{B}_1 \not\subseteq O_3(\mathbb{C}_{2,3})$ .

Let  $\mathfrak{H}$  be a subgroup of  $\mathbb{C}_{2,3}$  such that

- (a)  $\mathfrak{P}^* \subseteq \mathfrak{H}$ .
- (b)  $\mathfrak{B}_1 \not\subseteq O_3(\mathfrak{H})$ .
- (c)  $\mathfrak{H}$  is minimal subject to (a) and (b).

Let  $\mathfrak{H}_1 = O_3(\mathfrak{H})$ . Since the fixed point subspace of  $\mathfrak{B}_1$  on  $\mathfrak{H}_1/D(\mathfrak{H}_1)$  is of codimension 1, Lemma 5.30 implies that  $\mathfrak{H} = \mathfrak{P}^* \Omega^*$ , where  $\Omega^*$  is a quaternion group and  $|\mathfrak{P}^* : \mathfrak{H}_1| = 3$ , so that  $\mathfrak{H}_1 \mathfrak{B}_1 = \mathfrak{P}^*$ . Since  $\mathfrak{B}_1$  centralizes  $D(\mathfrak{H}_1)$ , so does  $\Omega^*$ . Let  $C_{\mathfrak{H}_1}(\Omega^*) = \mathfrak{P}_1^*$ . Thus,  $\mathfrak{P}_1^*$  is a normal subgroup of  $\mathfrak{H}$  and  $|\mathfrak{H}_1 : \mathfrak{P}_1^*| = 9$ .

Let  $\mathfrak{P}_2^* = [\mathfrak{H}_1, \Omega^*]$ . Then  $\mathfrak{P}_2^*$  is generated by 2 elements and  $\mathfrak{P}_2^* \cap \mathfrak{P}_1^*$  is of index 9 in  $\mathfrak{P}_2^*$ . Hence,  $\mathfrak{P}_2^*$  is either elementary of order 9 or is a nonabelian group of exponent 3 and order 27. Furthermore,  $\mathfrak{H}_1 = \mathfrak{P}_1^* \mathfrak{P}_2^*$ ,  $\mathfrak{P}_1^* \cap \mathfrak{P}_2^* = D(\mathfrak{P}_2^*)$ , and  $[\mathfrak{P}_1^*, \mathfrak{P}_2^*] = 1$ .

Since  $\mathfrak{B}_1 \not\subseteq \mathfrak{H}_1$ , it follows that  $\mathbb{C}_1 = \mathfrak{B}_1 \times (\mathbb{C}_1 \cap \mathfrak{H}_1)$ . Hence,  $D(\mathbb{C}_1) = D(\mathbb{C}_1 \cap \mathfrak{H}_1) \subseteq \mathfrak{P}_1^*$ . We will show that  $D(\mathbb{C}_1) = 1$ . Suppose false. Let  $\mathbb{C}^* = C(D(\mathbb{C}_1))$ , so that  $\mathbb{C}^*$  is 3-solvable. Since  $\mathbb{C}^* \triangleleft N(D(\mathbb{C}_1))$ , it follows that  $\mathbb{C}^* \mathfrak{B}$  is 3-solvable. Since  $\mathfrak{B}_1 \subseteq Z(\mathfrak{B})$ , we have  $\mathfrak{B}_1 \subseteq O_3(\mathbb{C}^* \mathfrak{B})$ . Since  $\Omega^*$  centralizes  $D(\mathfrak{H}_1)$ , it follows that  $\langle \mathfrak{B}_1, \Omega^* \rangle \subseteq \mathbb{C}^*$ . Thus,  $\langle \mathfrak{B}_1, \Omega^* \rangle$  is 3-closed. This is impossible, since  $\langle \mathfrak{B}_1, \Omega^* \rangle$  covers  $\mathfrak{H}/\mathfrak{H}_1$ .

If  $\mathfrak{P}_2^*$  is nonabelian, then  $\Omega^*$  centralizes  $Z(\mathfrak{P}_2^*)$ . Since  $\Omega^*$  is a quaternion group, we are done in this case.

We may now assume that  $\mathfrak{P}_2^*$  is abelian, so elementary of order 9. Thus,  $\mathfrak{H}_1 = \mathfrak{P}_1^* \times \mathfrak{P}_2^*$ ,  $\mathfrak{P}_1^*$  and  $\mathbb{C}_1$  are elementary and  $Z(\mathfrak{P}^*) = \mathfrak{P}_1^* \times \mathfrak{B}$ , where  $\mathfrak{B} = \mathfrak{P}_2^* \cap Z(\mathfrak{P}^*)$ . Notice that  $\mathfrak{B}_0 \subseteq \mathfrak{P}_1^*$ . If  $\mathfrak{P}_1^* \supset \mathfrak{B}_0$ , then since every subgroup of  $\mathfrak{P}_1^*$  of type (3,3) is in  $\mathcal{Z}(3)$  and since the quaternion group  $\Omega^*$  centralizes  $\mathfrak{P}_1^*$ , we are done. We may therefore assume that  $\mathfrak{P}_1^* = \mathfrak{B}_0$ . Hence,  $Z(\mathfrak{B})$  has order 9,  $|\mathfrak{P}^*| = 3^4$ ,  $|\mathbb{C}_1| = 3^3$ . Also,  $Z(\mathfrak{P}^*) = \mathfrak{B}_0 \times \mathfrak{B}$ . Let  $B$  be a generator for  $\mathfrak{B}_0$  and let  $I$  be the involution of  $\Omega^*$ . Then  $I$  inverts  $\mathfrak{B}$  and centralizes  $B$ .

Let  $\mathfrak{N} = \langle \mathfrak{B}, \mathfrak{P}^* \rangle \subseteq N(\mathbb{C}_1)$ . Since  $SL(3, 3)$  is a minimal simple group, it follows that  $N(\mathbb{C}_1)$  is solvable. As  $O_3(\mathfrak{N}) = 1$ , we have  $\mathbb{C}_1 = C(\mathbb{C}_1)$ . Since  $[\mathbb{C}_1, \mathfrak{P}^*, \mathfrak{P}^*] = 1$ , it follows that  $\mathfrak{N}$  contains a normal subgroup

$\mathfrak{B}^*$  of order 3 and that  $S_2$ -subgroups of  $\mathfrak{N}$  are quaternion. Now  $\mathfrak{B}^* \not\subseteq \mathfrak{B}$ , since  $\mathfrak{B}$  does not normalize  $\mathfrak{B}_0$  and  $\mathfrak{B}^*$  centralizes no subgroup of  $\mathfrak{B}$  other than 1 and  $\mathfrak{B}_0$ . Suppose  $\mathfrak{B}^* = \mathfrak{B}$ . Since  $\mathfrak{B} = \mathfrak{B}^{*'}$ , it follows that  $S_3$ -subgroups of  $\mathfrak{N}/\mathfrak{B}^*$  are abelian. Thus  $\mathfrak{N}$  is 3-closed. But this is impossible since  $\mathfrak{B} \neq \mathfrak{B}^*$ . Hence,  $\mathfrak{B}^*$  is a subgroup of  $Z(\mathfrak{B}^*)$  of order 3 which is different from  $\mathfrak{B}$  and from  $\mathfrak{B}_0$ . Since  $Z(\mathfrak{B}^*) = \mathfrak{B}_0 \times \mathfrak{B}$ , there is a generator for  $\mathfrak{B}^*$  of the shape  $BV$ , where  $V$  is a generator for  $\mathfrak{B}$ .

Let  $J$  be any involution of  $\mathfrak{N}$ . Then  $JVJ = V^{-1}$  and  $JBVJ = BV$ . Now  $I$  and  $J$  both normalize  $\mathfrak{B}^*$  and  $\mathfrak{B}^{*'} = \mathfrak{B}$ . Hence,  $\langle I, J \rangle$  maps onto an abelian subgroup of  $A_{\mathfrak{G}}(Z(\mathfrak{B}^*))$ , which implies that  $J$  normalizes  $Z(\mathfrak{B}^*) \cap C(I) = \mathfrak{B}_0$ . Hence,  $JBJ = B^f$  for some integer  $f$ , and the previous equations yield  $V^2 = 1$ , which is not the case. The proof is complete.

**HYPOTHESIS 7.2.** (i) If  $\mathfrak{A} \in \mathcal{L}_{\mathfrak{G}}(p)$ , then  $\mathfrak{N}(\mathfrak{A})$  contains only 1.

(ii) If  $\mathfrak{B}$  is of order  $p$  and is in the center of some  $S_p$ -subgroup of  $\mathfrak{G}$ , then  $O_p(\mathfrak{M})$  is of symplectic type and width  $w$ , where  $\mathfrak{M} = N(\mathfrak{B})$ .

**LEMMA 7.2.** *Suppose Hypothesis 7.2 is satisfied and that if  $p = 2$ , then  $w \geq 3$ , while if  $p = 3$ , then  $w \geq 2$ . Let  $\mathfrak{B}$  be a subgroup of  $O_p(\mathfrak{M})$  of type  $(p, p)$  which contains  $\mathfrak{B}$ . Then  $\mathfrak{B} \in \mathcal{E}(p)$ .*

*Proof.* Let  $\mathcal{V}$  be the set of subgroups of  $O_p(\mathfrak{M})$  which violate the lemma. Let  $\mathcal{V}_0$  be the subset of those  $\mathfrak{B}$  in  $\mathcal{V}$  which centralize at least one element  $\mathfrak{B}$  of  $\mathcal{U}(p)$  which  $\mathfrak{B} \subset \mathfrak{B} \subset O_p(\mathfrak{M})$ . If  $\mathcal{V}_0 \neq \emptyset$ , choose  $\mathfrak{B} \in \mathcal{V}_0$ , while if  $\mathcal{V}_0 = \emptyset$ , choose  $\mathfrak{B}$  in  $\mathcal{V}$ .

Let  $\mathfrak{G} = O_p(\mathfrak{M})$ ,  $\mathfrak{G}_0 = C_{\mathfrak{G}}(\mathfrak{B})$ . We first argue that  $\mathfrak{N}(\mathfrak{G}_0; p')$  is trivial. Namely,  $\mathfrak{M}$  is  $p$ -solvable with  $O_{p'}(\mathfrak{M}) = 1$ , so  $C_{\mathfrak{M}}(\mathfrak{G}) = \mathfrak{Z}(\mathfrak{G})$ . This implies that  $C_{\mathfrak{M}}(\mathfrak{G}_0)$  is a  $p$ -group. Hence,  $\mathfrak{N}_{\mathfrak{M}}(\mathfrak{G}_0; p')$  is trivial. Suppose  $\mathfrak{R} \in \mathfrak{N}(\mathfrak{G}_0; p')$ . It suffices to show that  $\mathfrak{R} \subseteq \mathfrak{M}$ . If  $\mathfrak{G}_0$  contains an element  $\mathfrak{B}$  of  $\mathcal{U}(p)$  with  $\mathfrak{B} \subset \mathfrak{B}$ , then by Lemma 7.1 we get that  $\mathfrak{B}$  centralizes  $\mathfrak{R}$ . Hence,  $\mathfrak{R} \subseteq C(\mathfrak{B}) \subseteq \mathfrak{M}$ . If no such elements of  $\mathcal{U}(p)$  are available, then by construction,  $\mathcal{V}_0 = \emptyset$ . But  $\mathfrak{G} \triangleleft \mathfrak{M}$ , so if  $\mathfrak{B}$  is a  $S_p$ -subgroup of  $\mathfrak{M}$ , then  $\mathfrak{G}$  contains an element  $\mathfrak{B}$  of  $\mathcal{U}(\mathfrak{B})$ . Let  $\mathfrak{G}_1 = C_{\mathfrak{G}}(\mathfrak{B})$  so that  $|\mathfrak{G} : \mathfrak{G}_1| = p$ . If  $\mathfrak{G}_0 \cap \mathfrak{G}_1$  contains more than one subgroup of order  $p$ , then there is a subgroup  $\mathfrak{B}^*$  of  $\mathfrak{G}_0 \cap \mathfrak{G}_1$  of type  $(p, p)$  which contains  $\mathfrak{B}$ . Since  $\mathcal{V}_0 = \emptyset$ ,  $\mathfrak{B}^* \in \mathcal{E}(p)$ , so  $\mathfrak{R} \subseteq C(\mathfrak{B}^*) \subseteq C(\mathfrak{B}) \subseteq \mathfrak{M}$ . Suppose  $\mathfrak{G}_0 \cap \mathfrak{G}_1$  contains only one subgroup of order  $p$ . Then by hypothesis, we have  $p \geq 5$ , and so  $\mathfrak{G}$  is of width 1 and is a  $S_p$ -subgroup of  $\mathfrak{G}$ . Hypothesis 7.1 guarantees in this case that  $\mathfrak{N}(\mathfrak{G}_0; p')$  is trivial, so  $\mathfrak{R} = 1$ . We have thus shown that  $\mathfrak{N}(\mathfrak{G}_0; p')$  is trivial.

Choose  $\mathfrak{Q}$  in  $\mathfrak{N}(\mathfrak{B}; p')$  minimal subject to  $[\mathfrak{B}, \mathfrak{Q}] \neq 1$ . Then  $\mathfrak{Q} = [\mathfrak{B}, \mathfrak{Q}]$  and  $\mathfrak{B}_0 = C_{\mathfrak{B}}(\mathfrak{Q})$  is of order  $p$ . Clearly,  $\mathfrak{B}_0 \neq \mathfrak{B}$ . Let  $\mathfrak{M}_1 = N(\mathfrak{B}_0)$

so that  $\mathfrak{M}_1$  is  $p$ -solvable. By the preceding argument,  $O_p(\mathfrak{M}_1) = 1$ . Hence,  $\mathfrak{Z} \not\subseteq O_p(\mathfrak{M}_1)$ , so that  $\mathfrak{G}_0$  contains an extra special subgroup  $\mathfrak{G}^*$  of width  $w - 1$  with  $\mathfrak{G}^* \cap O_p(\mathfrak{M}_1) = 1$ .

Let  $\mathfrak{X} = R_p(\mathfrak{M}_1)$  (see Definition 2.2),  $\mathfrak{Y} = C_{\mathfrak{M}_1}(\mathfrak{X})$ . Suppose  $\mathfrak{Z} \subseteq \mathfrak{Y}$ . Since  $\Omega = [\Omega, \mathfrak{Z}]$  and  $\mathfrak{Y} \triangleleft \mathfrak{M}_1$ , we have  $\Omega \subseteq \mathfrak{Y}$ . Let  $\mathfrak{X}_p$  be a  $S_p$ -subgroup of  $\mathfrak{M}_1$  which contains  $\mathfrak{G}^*$ . Then  $\mathfrak{G}^*$  centralizes  $Z(\mathfrak{X}_p)$ , so  $\mathfrak{Y}\mathfrak{G}^*$  centralizes  $Z(\mathfrak{X}_p)$ . Let  $\mathfrak{F}$  be a  $S_p$ -subgroup of  $\mathfrak{G}$  containing  $\mathfrak{X}_p$ . Then  $Z(\mathfrak{F}) \subseteq Z(\mathfrak{X}_p)$ , so  $\mathfrak{G}^*$  is contained in a conjugate  $\tilde{\mathfrak{M}}$  of  $\mathfrak{M}$ ,  $\tilde{\mathfrak{M}} = N(\Omega_1(Z(\mathfrak{F})))$ . Furthermore, since  $\mathfrak{Z}\Omega \subseteq \mathfrak{Y} \subseteq \tilde{\mathfrak{M}}$ ,  $\mathfrak{G}^*$  is faithfully represented on  $\Omega_1^*(\tilde{\mathfrak{M}})$ . Let  $\tilde{\mathfrak{G}} = O_p(\tilde{\mathfrak{M}})$ , and let  $\tilde{\mathfrak{G}} = \tilde{\mathfrak{G}}_0 \supset \tilde{\mathfrak{G}}_1 \supset \dots \supset \tilde{\mathfrak{G}}_k = 1$  be part of a chief series for  $\tilde{\mathfrak{M}}$ . Then since  $\mathfrak{Z} \not\subseteq O_p(\tilde{\mathfrak{M}})$ , it follows that  $\mathfrak{Z}$  does not centralize  $\tilde{\mathfrak{G}}_n/\tilde{\mathfrak{G}}_{n+1}$  for at least one value of  $n$ ,  $0 \leq n < k$ . Hence,  $|\tilde{\mathfrak{G}}_n : \tilde{\mathfrak{G}}_{n+1}| = p^{a_n}$  where  $a_n \geq r_p(\mathfrak{Z}; \tilde{\mathfrak{M}})$ . Then by Lemma 5.4,  $r_p(\mathfrak{Z}; \tilde{\mathfrak{M}}) \geq r_c(\mathfrak{Z}; \mathfrak{F})$ . Clearly,

$$r_c(\mathfrak{Z}; \mathfrak{F}) \geq r_c(\mathfrak{Z}; \mathfrak{G}^*) = p^{w-1}.$$

On the other hand,  $\tilde{\mathfrak{G}}_n$  is a subgroup of  $\tilde{\mathfrak{G}}$ , so  $2w \geq a_n$ . Hence  $2w \geq p^{w-1}$ . If  $p \geq 5$ , then  $w = 1$  is forced, so every  $p$ -solvable subgroup of  $\mathfrak{G}$  has  $p$ -length at most 1. This is absurd, so  $p \leq 3$ . If  $p = 3$ , then  $w = 2$ , since  $w \geq 2$  by hypothesis. It is clear that this is impossible since  $\mathfrak{G}^*$  is faithfully represented on  $\Omega_3^*(\tilde{\mathfrak{M}})$ . If  $p = 2$ , then  $w = 3$  or  $w = 4$ , since by hypothesis  $w \geq 3$ . This is also impossible by Lemma 5.13. We have shown that  $\mathfrak{Z} \not\subseteq \mathfrak{Y}$ .

Since  $\mathfrak{X}$  is a  $p$ -group,  $\mathfrak{X} \subseteq \mathfrak{M}^G$  for some  $G$  in  $\mathfrak{G}$ . Then  $\mathfrak{X} \cap \mathfrak{G}^G$  is an abelian subgroup of  $\mathfrak{G}^G$ , so  $m(\mathfrak{X} \cap \mathfrak{G}^G) \leq w + 1 + e$ , where  $e = 0$  if  $p$  is odd and  $e = 1$  if  $p = 2$ .

If  $p$  is odd, then  $\mathfrak{X}/\mathfrak{X} \cap \mathfrak{G}^G$  is faithfully represented on the Frattini quotient group of  $\Omega_1(\mathfrak{G}^G)$ , and this latter group is generated by  $2w$  elements. If  $p = 2$ , write  $O_{2,2}(\mathfrak{M}^G) = \mathfrak{G}^G \cdot \mathfrak{R}$  where  $|\mathfrak{R}|$  is odd. Then  $[\mathfrak{R}, \mathfrak{G}^G]$  is generated by  $2w$  elements and  $\mathfrak{X}/\mathfrak{X} \cap \mathfrak{G}^G$  is faithfully represented on the Frattini quotient group of  $[\mathfrak{R}, \mathfrak{G}^G]$ . Thus, by a result of Schur [32], we have  $m(\mathfrak{X}/\mathfrak{X} \cap \mathfrak{M}^G) \leq w^2$ , that is,

$$(7.1) \quad m(\mathfrak{X}) \leq w^2 + w + 1 + e.$$

If  $w = 1$ , then  $p \geq 5$  implies that  $O_p(\mathfrak{M})$  is a  $S_p$ -subgroup of  $\mathfrak{G}$ . So every  $p$ -solvable subgroup of  $\mathfrak{G}$  has  $p$ -length at most 1, a contradiction. Hence,  $w \geq 2$ .

There is an elementary subgroup  $\mathfrak{X}_1$  of  $\mathfrak{X}$  such that  $A_{\mathfrak{G}}(\mathfrak{X}_1)$  contains a subgroup  $\tilde{\mathfrak{G}}^*\tilde{\Omega}$ , where  $\tilde{\Omega} \triangleleft \tilde{\mathfrak{G}}^*\tilde{\Omega}$  is special and  $\tilde{\mathfrak{G}}^* \cong \mathfrak{G}^*$  operates faithfully and irreducibly on  $\tilde{\Omega}/D(\tilde{\Omega})$ . Also,  $\tilde{\mathfrak{G}}^*\tilde{\Omega}$  acts irreducibly on  $\mathfrak{X}_1$ .

Assume that  $p$  is odd.

Since  $\tilde{\mathfrak{G}}^*$  is extra special of width  $w - 1$ , it follows that  $m(\tilde{\Omega}) \geq p^{w-1}a$ , where  $a = |F_q(\zeta) : F_q|$ . Here,  $\tilde{\Omega}$  is a  $q$ -group and  $\zeta$  is a primitive

$p^{\text{th}}$  root of 1 in an extension field of the prime field  $F_q$ . By Lemma 5.3,  $m(\mathfrak{X}) \geq m(\tilde{\mathfrak{Q}})b$ , where  $b = 2/3$  if  $q = 2$  and  $b = |F_p(\tau):F_p|$  if  $q$  is odd. Here  $\tau$  is a primitive  $q^{\text{th}}$  root of 1 in an extension field of the prime field  $F_p$ . Together with (7.1), we get  $abp^{w-1} \leq w^2 + w + 1$ . Clearly,  $ab > 1$ . Suppose  $w \geq 4$ . Then  $3^{w-1} \leq p^{w-1} < w^2 + w + 1$ , a contradiction. Suppose  $w = 3$ . If  $p \geq 5$ , then  $5^2 = 5^{w-1} \leq p^{w-1} < 3^2 + 3 + 1$ , a contradiction. Thus,  $p = 3$  and  $q = 2$ . Since  $p = 3, w = 3$ , it follows that  $\mathfrak{M}^G/\mathfrak{G}^G$  is isomorphic to a 3-solvable subgroup of the 6 by 6 symplectic group over  $F_3$ . It follows readily that  $\mathfrak{M}^G/\mathfrak{G}^G$  has no elementary subgroup of order  $3^4$ . Thus, in this case,  $m(\mathfrak{X}) \leq 4 + m(\mathfrak{X} \cap \mathfrak{G}^G) \leq 8$ , against  $(4/3) \cdot 3^3 = 12 = abp^{w-1} \leq m(\mathfrak{X})$ . Hence  $w = 2$ . We now get  $abp^{w-1} < 7$ , so  $p \leq 5$ . Suppose  $p = 5$  and  $q$  is odd. Then  $ab \geq 2$ , so that  $10 < 7$ . Suppose  $p = 5$  and  $q = 2$ . Then  $a = 4$ , so  $ab = 8/3$ . We get  $(8/3) \cdot 5 < 7$ . Hence,  $p = 3$ . Suppose  $q$  is odd. Since  $a$  is characterized as the smallest positive integer  $n$  with  $3^n \equiv 1 \pmod{q}$ , it follows that  $a \geq 3$ , so  $ab \geq 3$ . This gives  $9 \leq abp^{w-1} < 7$ . Hence,  $q = 2$ . Since  $p = 3, w = 2$ , it follows that  $\mathfrak{M}^G/\mathfrak{G}^G$  has no elementary subgroup of order 27. Hence,  $m(\mathfrak{X}) \leq 3 + 2 = 5$ . In particular,  $m(\mathfrak{X}_1) \leq 5$ . Suppose first that  $\tilde{\mathfrak{Q}}$  is abelian. Since  $Z(\tilde{\mathfrak{G}}^*) = \tilde{\mathfrak{Z}}$  acts without fixed points on  $\tilde{\mathfrak{Q}}$ , it follows that  $C_{\tilde{\mathfrak{G}}^*}(\lambda) \cap \tilde{\mathfrak{Z}} = 1$  for every non trivial character  $\lambda$  of  $\tilde{\mathfrak{Q}}$ . So  $|\tilde{\mathfrak{G}}^*:C_{\tilde{\mathfrak{G}}^*}(\lambda)| \geq 9$  for all  $\lambda \neq 1$ . Hence,  $m(\mathfrak{X}_1) \geq 9$ , a contradiction. Suppose  $\tilde{\mathfrak{Q}}$  is nonabelian. Let  $\mathfrak{X}_2$  be a subgroup of  $\mathfrak{X}_1$  on which  $\tilde{\mathfrak{Q}}$  acts irreducibly. Thus  $m(\mathfrak{X}_2) \geq 2$ , since  $\tilde{\mathfrak{Q}}$  does not centralize  $\mathfrak{X}_2$ . Since  $m(\mathfrak{X}_1) \leq 5$ , and  $p = 3$ , it follows that  $\mathfrak{X}_2 = \mathfrak{X}_1$  is an irreducible  $\tilde{\mathfrak{Q}}$ -group. Thus,  $\tilde{\mathfrak{Q}}$  is extra special. But  $m(\tilde{\mathfrak{Q}}) = 6$ , since  $q = 2$  and  $|\tilde{\mathfrak{G}}^*| = 27$ . This yields  $m(\mathfrak{X}_1) \geq 2^3$ . All possibilities have led to contradictions. So  $p = 2$ .

Since  $\tilde{\mathfrak{G}}^*$  is extra special of width  $w - 1$ , we get that  $m(\tilde{\mathfrak{Q}}) \geq 2^{w-1}$ . Now Lemma 5.3(a) applied with  $\tilde{\mathfrak{Q}}$  in the role of  $\mathfrak{P}$ ,  $\mathfrak{X}_1$  in the role of  $V$ , yields  $m(\mathfrak{X}_1) \geq 2^w$ . On the other hand,  $\mathfrak{B}_0$  is a normal subgroup of  $\mathfrak{M}_1$  of order 2, so  $m(\mathfrak{X}) \geq 1 + 2^w$ .

Let  $\mathfrak{C}$  be an elementary subgroup of  $\mathfrak{X}$  with  $m(\mathfrak{C}) = 2^w + 1$ , let  $\mathfrak{C}_0 = \mathfrak{C} \cap \mathfrak{G}^G$ , and let  $\mathfrak{C}_1$  be a complement to  $\mathfrak{C}_0$  in  $\mathfrak{C}$ . Since  $m(\mathfrak{E}_0) \leq w + 2$ , we get  $m(\mathfrak{C}_1) = a \geq 2^w - 1 - w$ . Since  $\mathfrak{C}_1$  acts faithfully on  $O_{2,2}(\mathfrak{M}^G)/\mathfrak{G}^G$ , Lemma 5.34 implies that  $O_{2,2}(\mathfrak{M}^G)/\mathfrak{G}^G$  has a subgroup  $\hat{\mathfrak{D}}/\mathfrak{G}^G$  which admits  $\mathfrak{C}_1$  and such that  $\mathfrak{C}_1\hat{\mathfrak{D}}/\mathfrak{G}^G$  is the direct product of a dihedral groups of order twice an odd prime. Let  $\mathfrak{R}$  be a  $S_2$ -subgroup of  $\hat{\mathfrak{D}}$ . By Lemma 5.12,  $[\mathfrak{G}^G, \mathfrak{R}] = \mathfrak{R}$  is extra special of width  $w_1 \leq w$ . Since  $\mathfrak{G}^G$  is the central product of  $\mathfrak{R}$  and  $C_{\mathfrak{G}^G}(\mathfrak{R})$ , and since  $\mathfrak{C}_1\hat{\mathfrak{D}} = \mathfrak{R} \cdot N_{\mathfrak{C}_1\hat{\mathfrak{D}}}(\mathfrak{R})$ , it follows that if  $M = \mathfrak{R}/D(\mathfrak{R})$ , then  $A_{\mathfrak{C}_1\hat{\mathfrak{D}}}(M)$  has a subgroup which is the direct product of  $a$  dihedral groups of order twice an odd prime. Let  $m(M) = m$ . Then  $m = 2w_1 \leq 2w$ . Since  $w \geq 3$  by hypothesis, we get  $w < 2^w - 1 - w \leq a$ , and so  $2w < 2a$ , whence  $m < 2a$ .

This violates Lemma 5.8, and completes the proof.

**HYPOTHESIS 7.3.** (i)  $p$  is odd.

(ii)  $\mathfrak{B}$  is a  $S_p$ -subgroup of  $\mathfrak{G}$ ,  $\mathfrak{A}$  is a normal elementary subgroup of  $\mathfrak{B}$  with  $m(\mathfrak{A}) \geq 3$ ,  $Z(\mathfrak{B})$  is cyclic, and  $A_{\mathfrak{G}}(\mathcal{E}) = A(\mathcal{E})$ , where  $\mathcal{E}: \mathfrak{A} \supset \mathfrak{B} \cap Z(\mathfrak{B}) \supset 1$ . Also,  $\mathfrak{A} \triangleleft N(Z(\mathfrak{B}) \cap \mathfrak{A})$ .

**LEMMA 7.3.** *Suppose Hypothesis 7.3 is satisfied. Let  $\mathfrak{Z} = Z(\mathfrak{B}) \cap \mathfrak{A}$ . Then each subgroup of  $\mathfrak{A}$  of type  $(p, p)$  which contains  $\mathfrak{Z}$  is in  $\mathcal{E}(p)$ .*

*Proof.* The lemma is an immediate consequence of Lemma 5.5, together with Hypothesis 7.1 (i).

**HYPOTHESIS 7.4.** (i)  $\mathfrak{G}$  is simple.

(ii)  $\{2, 3\} \subseteq \pi_4(\mathfrak{G})$ .

(iii) The centralizer of every involution of  $\mathfrak{G}$  is solvable.

(iv) The normalizer of every nonidentity 3-subgroup of  $\mathfrak{G}$  is solvable.

(v) If  $\mathfrak{A} \in \mathcal{S}_{\text{sol}}(2) \cup \mathcal{S}_{\text{sol}}(3)$ , then  $\mathfrak{N}(\mathfrak{A})$  contains only 1.

*All remaining lemmas in this section are proved under Hypothesis 7.4.*

**DEFINITION 7.2.**

- $$\mathcal{N} = \{(\mathfrak{A}, \mathfrak{B}) \mid \begin{array}{l} 1. \ \mathfrak{A} \text{ is a 2-subgroup of } \mathfrak{G}. \\ 2. \ \mathfrak{B} \text{ is a 3-subgroup of } \mathfrak{G}. \\ 3. \ \langle \mathfrak{A}, \mathfrak{B} \rangle \text{ is not solvable.} \end{array} \}$$

We remark that in the following lemmas, Lemma 7.1 may be invoked, since Hypothesis 7.4 implies that Hypothesis 7.1 is satisfied for  $p = 2$  and for  $p = 3$ .

**LEMMA 7.4.** *If  $\mathfrak{A}$  is a four-subgroup of  $\mathfrak{G}$  which centralizes every element of  $\mathfrak{N}(\mathfrak{A}; 3)$  and  $\mathfrak{B}$  is a subgroup of  $\mathfrak{G}$  of type  $(3, 3)$  which centralizes every element of  $\mathfrak{N}(\mathfrak{B}; 2)$ , then  $(\mathfrak{A}, \mathfrak{B}) \in \mathcal{N}$ .*

*Proof.* Notice that if  $G, H \in \mathfrak{G}$ , then the pair  $(\mathfrak{A}^G, \mathfrak{B}^H)$  satisfies the hypothesis of the lemma.

Suppose the lemma is false and  $\mathfrak{A}, \mathfrak{B}$  are chosen so that  $\langle \mathfrak{A}, \mathfrak{B} \rangle$  is minimal. It follows as in Lemma 0.10.2 that  $\langle \mathfrak{A}, \mathfrak{B} \rangle = \mathfrak{A} \times \mathfrak{B}$ . We may then choose  $A$  in  $\mathfrak{A}^{\#}$  such that  $E(A)$  contains an element  $\mathfrak{A}_1$  of  $\mathcal{Z}(2)$ . Hence,  $\langle \mathfrak{A}_1, \mathfrak{B} \rangle$  is solvable. Thus, we may assume that  $\mathfrak{A} \in \mathcal{Z}(2)$ . Let  $\mathfrak{N} = N(\mathfrak{A})$ . Since  $2 \in \pi_4$ , we have  $O_2(\mathfrak{N}) = 1$ . This is absurd since

$\mathfrak{B}$  centralizes  $O_2(\mathfrak{N})$  and  $\mathfrak{N}$  is solvable. The proof is complete.

We set

$$\begin{aligned} \mathfrak{G}_1 &= \{G \mid G \in \mathfrak{G}, C(G) \text{ is solvable.}\} \\ \mathfrak{G}_p &= \{G \mid G \in \mathfrak{G}, C(G) \text{ contains an elementary subgroup } \mathfrak{E} \text{ of} \\ &\quad \text{order } p^2 \text{ which centralizes every element of } \mathfrak{N}(\mathfrak{E}; q)\}, \\ &\quad p = 2, 3, q = 2, 3, p \neq q. \end{aligned}$$

We conclude from Lemma 7.4 that

$$(7.2) \quad \mathfrak{G}_1 \cap \mathfrak{G}_2 \cap \mathfrak{G}_3 = \emptyset.$$

There are some subtle consequence of (7.2).

**DEFINITION 7.3.**

- $\mathcal{D} = \{\mathfrak{B} \mid$ 
  1.  $\mathfrak{B}$  is a noncyclic elementary 3-subgroup of  $\mathfrak{G}$ .
  2. Every element of  $\mathfrak{B}$  centralizes an element of  $\mathcal{Z}(3)$ .
  3.  $\mathfrak{B}$  centralizes every abelian subgroup in  $\mathfrak{N}(\mathfrak{B}; 2)\}$ .

**LEMMA 7.5.** *Suppose  $\mathfrak{A} \in \mathcal{Z}(2)$ ,  $\mathfrak{B} \in \mathcal{D}$  and  $\mathfrak{X}$  is a 2, 3-subgroup of  $\mathfrak{G}$  which contains  $\langle \mathfrak{A}, \mathfrak{B} \rangle$ . Let  $\mathfrak{X}_2$  be a  $S_2$ -subgroup of  $\mathfrak{X}$ . Then  $\mathcal{MS}(\mathfrak{G})$  (see Definition 2.7) contains an element  $\mathfrak{M}$  such that*

- (a)  $O_2(\mathfrak{M}) = 1$ .
- (b)  $O_2(\mathfrak{M})$  is the central product of  $[O_2(\mathfrak{M}), \mathfrak{B}]$ , which is extra special of width  $w = 2, 3$  or 4, and of  $C_{O_2(\mathfrak{M})}(\mathfrak{B})$ , which is either cyclic or of maximal class  $\geq 3$ .
- (c)  $[O_2(\mathfrak{M}), \mathfrak{B}]$  is the central product of  $w$   $\mathfrak{B}$ -invariant quaternion groups  $\mathfrak{Q}_1, \dots, \mathfrak{Q}_w$  whose centralizers in  $\mathfrak{B}$  are  $w$  distinct subgroups of order 3. In particular, no element of  $\mathfrak{B}^\#$  centralizes any four-subgroup of  $[O_2(\mathfrak{M}), \mathfrak{B}]$ . If  $w > 2$ , then  $C_{O_2(\mathfrak{M})}(\mathfrak{B}) = [O_2(\mathfrak{M}), \mathfrak{B}]'$  is the center of  $O_2(\mathfrak{M})$ .
- (d)  $\mathfrak{X}_2 \subset \mathfrak{M}$ .
- (e)  $\mathfrak{B} \subset \mathfrak{M}$ , and if  $\mathfrak{Q}$  is a quaternion subgroup of  $\mathfrak{X}_2$  which is normalized by  $\mathfrak{B}$  but is not centralized by  $\mathfrak{B}$ , then  $\mathfrak{Q} \subset O_2(\mathfrak{M})$ .
- (f) If  $J$  is an involution of  $\mathfrak{M} \cap C(\mathfrak{B})$ , then  $J \in O_2(\mathfrak{M})$ . If  $\mathfrak{M}$  contains a  $S_2$ -subgroup of  $C(J)$  (e.g., if  $C(J) = \mathfrak{X}$ ), then  $C(J) \subseteq \mathfrak{M}$ .
- (g)  $\mathfrak{M}$  contains a  $S_2$ -subgroup of  $\mathfrak{G}$ .

*Proof.* Let  $\mathcal{S}$  be the set of 2, 3-subgroups of  $\mathfrak{G}$  which contain  $\langle \mathfrak{B}, \mathfrak{X}_2 \rangle$ . Choose  $\mathfrak{S}$  in  $\mathcal{S}$  so that  $|\mathfrak{S}|_2$  is maximal. Let  $\mathfrak{S}_p$  be a  $S_p$ -subgroup of  $\mathfrak{S}$ ,  $p = 2, 3$ , chosen so that  $\mathfrak{X}_2 \subseteq \mathfrak{S}_2, \mathfrak{B} \subseteq \mathfrak{S}_3$ . Let  $\mathfrak{A}_1, \dots, \mathfrak{A}_m$  be all the elements of  $\mathcal{Z}(2)$  in  $\mathfrak{S}_2$ . By Lemma 7.1, each  $\mathfrak{A}_i$  centralizes  $O_3(\mathfrak{S})$ , so by Lemma 7.4,  $|O_3(\mathfrak{S})| \leq 3$ . In particular,  $\mathfrak{B}$  is not contained in  $O_2(\mathfrak{S})$  and  $\mathfrak{B}$  centralizes  $O_3(\mathfrak{S})$ . Since  $\mathfrak{B} \not\subseteq F(\mathfrak{S}), \mathfrak{B}$

does not centralize  $O_2(\mathfrak{C})$ . Hence,  $O_2(\mathfrak{C})$  is nonabelian since  $\mathfrak{B} \in \mathcal{D}$ . Let  $\mathfrak{R} = O_2(\mathfrak{C})' \cap \mathfrak{Z}(O_2(\mathfrak{C}))$ , so that  $\mathfrak{R} \neq 1$ . Let  $\mathfrak{N} = N(\mathfrak{R})$ ,  $\mathfrak{C} = C(\mathfrak{R})$ , and observe that  $\mathfrak{B} \subseteq \mathfrak{C}$ . Since the centralizer of every involution is solvable,  $\mathfrak{C}$  is solvable. Let  $\mathfrak{C}^*$  be a  $S_2$ -subgroup of  $\mathfrak{N}$  which contains  $\mathfrak{C}_2$ . Then  $\mathfrak{C}\mathfrak{C}_2^*$  is solvable. By  $D_{2,3}$  in  $\mathfrak{C}\mathfrak{C}_2^*$  and maximality of  $|\mathfrak{C}|_2$ , it follows that  $\mathfrak{C}_2^* = \mathfrak{C}_2$ . Hence, if  $\mathfrak{C}_2^{**}$  is a  $S_2$ -subgroup of  $\mathfrak{G}$  containing  $\mathfrak{C}_2$ , then  $\mathfrak{C}_2$  contains every element of  $\mathcal{Z}(\mathfrak{C}_2^{**})$ . We may therefore assume that  $\mathfrak{A}_1 \in \mathcal{Z}(\mathfrak{C}_2^{**})$ .

Since  $\mathfrak{A}_1$  centralizes  $O_3(\mathfrak{C})$ , it follows that  $\mathfrak{A}_1 \cap \mathbf{Z}(\mathfrak{C}_2^{**}) \subseteq \mathbf{Z}(O_2(\mathfrak{C}))$ . Since  $\mathfrak{B}$  centralizes  $\mathbf{Z}(O_2(\mathfrak{C}))$ , maximality of  $|\mathfrak{C}|_2$  guarantees that  $\mathfrak{C}_2 = \mathfrak{C}_2^{**}$  is a  $S_2$ -subgroup of  $\mathfrak{G}$ .

Let  $\mathfrak{C} \subseteq \mathfrak{M} \in \mathcal{MS}(\mathfrak{G})$ . Thus, (g) holds, as does (d). Since  $\mathfrak{M}$  contains a  $S_2$ -subgroup of  $\mathfrak{G}$ , and since 1 is the only 2-signalizer of  $\mathfrak{G}$ , it follows that  $O_2(\mathfrak{M}) = 1$ , and (a) holds. Let  $\mathfrak{H} = O_2(\mathfrak{M})$ . Suppose  $\mathfrak{H}$  contains a noncyclic characteristic abelian subgroup  $\mathfrak{H}_0$ . Then  $\mathfrak{B}$  centralizes  $\mathfrak{H}_0\mathbf{Z}(\mathfrak{H})$  and  $\mathfrak{H}_0\mathbf{Z}(\mathfrak{H})$  contains an element of  $\mathcal{Z}(\mathfrak{C}_2)$ . This violates Lemma 7.4.

Clearly,  $\mathfrak{H}$  is noncyclic, since  $\mathfrak{H} = F(\mathfrak{M})$  and  $\mathfrak{M}$  is solvable. Thus,  $\mathfrak{H}$  is of symplectic type. The width  $w$  of  $\mathfrak{H}$  is at least 2, since  $\mathfrak{B}$  is faithfully represented on  $\mathfrak{H}$ .

Suppose  $w \geq 3$  and  $B \in \mathfrak{B}^*$  centralizes a four-subgroup  $\mathfrak{B}$  of  $\mathfrak{H}$  with  $\Omega_1(\mathbf{Z}(\mathfrak{H})) \subset \mathfrak{B}$ . By Lemma 7.2,  $\mathfrak{B}$  centralizes every element of  $\mathcal{N}(\mathfrak{B}; 2')$ . Since  $C(B)$  contains an element of  $\mathcal{Z}(3)$ , (7.2) is violated. Thus, if  $w \geq 3$ , then no element of  $\mathfrak{B}^*$  centralizes any four-subgroup of  $\mathfrak{H}$ . This immediately implies that  $\mathfrak{H}$  is extra special and  $w \leq 4$ . Now (b) and (c) follow from Lemma 5.12.

We next prove the first assertion of (f). Let  $J$  be an involution of  $\mathfrak{M} \cap C(\mathfrak{B})$ . If  $w = 2$ , then  $\mathfrak{B}$  is a  $S_2$ -subgroup of  $\mathfrak{M}$  and (f) is clear. Suppose  $w \geq 3$ . In this case,  $\mathfrak{H}$  is extra special, so  $\mathfrak{H} \cap C(\mathfrak{B}) = \mathfrak{H}'$  is of order 2. Let  $\mathfrak{B}_0$  be any subgroup of  $\mathfrak{B}$  of order 3. Since  $J$  centralizes  $\mathfrak{B}_0$ , it follows that  $J$  normalizes  $C_{\mathfrak{H}}(\mathfrak{B}_0)$ . We will show that  $J$  centralizes  $C_{\mathfrak{H}}(\mathfrak{B}_0)$ . This is clear if  $C_{\mathfrak{H}}(\mathfrak{B}_0) = \mathfrak{H}'$ , so suppose  $C_{\mathfrak{H}}(\mathfrak{B}_0) \supset \mathfrak{H}'$ . Since  $\mathfrak{H}$  is extra special, so is  $C_{\mathfrak{H}}(\mathfrak{B}_0)$ , so  $C_{\mathfrak{H}}(\mathfrak{B}_0)$  is a quaternion group on which  $\mathfrak{B}/\mathfrak{B}_0$  is faithfully represented. Since a quaternion group has no automorphism of order 6,  $J$  necessarily centralizes  $C_{\mathfrak{H}}(\mathfrak{B}_0)$ . Hence,  $J$  centralizes  $\langle C_{\mathfrak{H}}(\mathfrak{B}_0) \mid \mathfrak{B}_0 \subset \mathfrak{B}, |\mathfrak{B}_0| = 3 \rangle = \mathfrak{H}$ , so  $J \in \mathfrak{H}$ . This proves the first assertion of (f).

Now for the second assertion of (f). If  $w > 2$ , then  $\langle J \rangle = \mathfrak{H}'$ , by what we have just shown, together with (c). So suppose  $w = 2$  and  $\langle J \rangle \not\triangleleft \mathfrak{M}$ . Let  $\mathfrak{H}_0 = [\mathfrak{H}, \mathfrak{B}]$ ,  $\mathfrak{H}_1 = C_{\mathfrak{H}}(\mathfrak{B})$ . Thus,  $J \in \mathfrak{H}_1$ ,  $J \neq Z$ , where  $Z$  is central involution of  $\mathfrak{H}_1$ . Since  $w = 2$ ,  $\mathfrak{B}$  is a  $S_2$ -subgroup of  $\mathfrak{M}$ . Let  $\mathfrak{X}_0$  be a  $S_2$ -subgroup of  $C(J)$  which is contained in  $\mathfrak{M}$ . Thus,  $C(J) \cong \mathfrak{X}_0\mathfrak{B}$ ,  $\mathfrak{X}_0 \cong \mathfrak{H}_0 \times \langle J \rangle$ , and  $\mathfrak{H}_0$  is the central product of 2 quaternion groups.



Since  $C(J)$  contains an element of  $\mathcal{Z}(2)$ , it follows that  $O_2(C(J)) = 1$ . Since  $Z$  centralizes  $\mathfrak{X}_0$ , a  $S_2$ -subgroup of  $C(J)$ , it follows that  $Z \in O_2(C(J))$ , and so  $Z \in Z(O_2(C(J)))$ . Since  $\mathfrak{B} \subseteq C(J)$ , it follows that  $O_2(C(J)) \in \mathcal{N}_{\mathfrak{M}}(\mathfrak{B}; 2)$ . Hence,  $O_2(C(J)) \subseteq O_2(\mathfrak{M}) = \mathfrak{G}$ , since  $\mathfrak{B}$  is a  $S_2$ -subgroup of  $\mathfrak{M}$ . Hence,  $O_2(C(J)) \subseteq \mathfrak{G} \cap \mathfrak{X}_0 = \mathfrak{G}_0 \times \langle J \rangle$ . If  $O_2(C(J))$  is not elementary, then  $\langle Z \rangle \text{ char } O_2(C(J))$ , and so  $C(J) \subseteq \mathfrak{M}$ . Suppose  $O_2(C(J))$  is elementary. Since  $\mathfrak{B}$  is faithfully represented on  $O_2(C(J))$ , it follows that  $|O_2(C(J))| \geq 2^4$ . However,  $O_2(\mathfrak{M})$  contains no elementary subgroup of order  $2^4$  on which  $\mathfrak{B}$  is faithfully represented. This completes the proof of the second assertion of (f).

We turn to the proof of (e). Let  $\mathfrak{Q}$  be a quaternion subgroup of  $\mathfrak{M}$  normalized but not centralized by  $\mathfrak{B}$ . Let  $\mathfrak{B}_0 = \mathfrak{B} \cap C(\mathfrak{Q})$ , so that  $\mathfrak{B}\mathfrak{Q} = \mathfrak{B}_0 \times \mathfrak{B}_1\mathfrak{Q}$ , where  $|\mathfrak{B}_1| = 3$  and  $\mathfrak{B}_1$  is faithfully represented on  $\mathfrak{Q}$ . Let  $\mathfrak{Q}_0 = \mathfrak{Q} \cap \mathfrak{G}$ . By (f),  $\mathfrak{Q}_0 \cong \mathfrak{Q}'$ . Since  $\mathfrak{Q}/\mathfrak{Q}'$  is an irreducible  $\mathfrak{B}$ -group, we may assume by way of contradiction that  $\mathfrak{Q}_0 = \mathfrak{Q}'$ .

Let  $\mathfrak{R}$  be a  $\mathfrak{B}\mathfrak{Q}$ -invariant subgroup of  $Q_2(\mathfrak{M})$  minimal subject to  $[\mathfrak{R}, \mathfrak{Q}] \neq 1$ . Thus,  $\mathfrak{R}$  may be viewed as a  $\mathfrak{B}\mathfrak{Q}/\mathfrak{Q}'$ -group; as such  $\mathfrak{B}_1\mathfrak{Q}/\mathfrak{Q}'$  acts faithfully. Since  $w \leq 4$ , it follows that  $\mathfrak{R}$  is elementary of order  $3^3$  and is centralized by  $\mathfrak{B}_0$ . Thus,  $w = 4$  and  $S_3$ -subgroups of  $\mathfrak{M}$  are of order  $3^5$ . Also,  $\mathfrak{R}$  is incident with an elementary subgroup  $\mathfrak{R}_0$  of  $\mathfrak{M}$  such that  $\langle \mathfrak{B}_0, \mathfrak{R}_0 \rangle$  is elementary of order  $3^4$ .

Let  $\mathfrak{P}$  be a  $S_3$ -subgroup of  $\mathfrak{G}$  containing  $\langle \mathfrak{B}_0, \mathfrak{R}_0 \rangle$  and choose  $\mathfrak{U}$  in  $\mathcal{Z}(\mathfrak{P})$ . Then  $\langle \mathfrak{B}_0, \mathfrak{R}_0 \rangle$  contains an elementary subgroup  $\mathfrak{E}$  of order  $3^3$  which centralizes  $\mathfrak{U}$ . Since  $\mathfrak{E} \subseteq \mathfrak{M}$ , there is an element  $E$  of  $\mathfrak{E}^\#$  such that  $\mathfrak{G} \cap C(E)$  contains a four-group. But then  $E \in \mathfrak{G}_1 \cap \mathfrak{G}_2 \cap \mathfrak{G}_3$ , against (7.2). This contradiction completes the proof of (e) and the lemma.

Throughout the remainder of this section,  $\mathfrak{P}$  denotes a  $S_3$ -subgroup of  $\mathfrak{G}$ .

LEMMA 7.6. *Suppose  $|\mathfrak{P}| > 3^4$ .*

(a) *If  $\mathfrak{P}_0$  is a subgroup of  $\mathfrak{P}$  of index at most 9 and  $\mathfrak{P}_0$  contains an element of  $\mathcal{Z}^*(\mathfrak{P})$ , then  $\mathcal{N}(\mathfrak{P}_0; 2)$  contains only 1.*

(b) *If  $\mathfrak{A}$  is a subgroup of  $\mathfrak{P}$  of type (3, 3) and  $|\mathfrak{P}: C_{\mathfrak{P}}(\mathfrak{A})| \leq 3$ , then  $\mathfrak{A}$  centralizes every element of  $\mathcal{N}(\mathfrak{A}; 2)$ .*

(c) *If  $\mathfrak{A}$  is a subgroup of  $\mathfrak{P}$  of type (3, 3), if  $|\mathfrak{P}: C_{\mathfrak{P}}(\mathfrak{A})| \leq 9$ , and if  $C_{\mathfrak{P}}(\mathfrak{A})$  contains an element of  $\mathcal{Z}^*(\mathfrak{P})$ , then  $\mathfrak{A} \in \mathcal{D}$ .*

(d) *If  $\mathfrak{E}$  is a normal elementary subgroup of  $\mathfrak{P}$  of order 27 and  $|\mathfrak{P}: C_{\mathfrak{P}}(\mathfrak{E})| = 3$ , then  $\mathfrak{A} \in \mathcal{E}(3)$  for each subgroup  $\mathfrak{A}$  of index 3 in  $\mathfrak{E}$ .*

*Proof.* (a) Let  $\mathfrak{B}$  be an element of  $\mathcal{Z}^*(\mathfrak{P})$  with  $\mathfrak{B} \subseteq \mathfrak{P}_0$ . We will show that  $\mathcal{N}(\mathfrak{B}; 2)$  contains only 1. To do this, we first show that if  $\mathfrak{X} \in \mathcal{Z}(3)$ , then  $|C(\mathfrak{X})|$  is odd. Suppose  $J$  is an involution of  $C(\mathfrak{X})$ . By Lemma 5.38,  $C(J)$  contains an element  $\mathfrak{Y}$  of  $\mathcal{Z}(2)$ . Hence,

by Lemmas 7.1 and 7.4,  $\langle \mathfrak{X}, \mathfrak{Y} \rangle$  is nonsolvable, against  $\langle \mathfrak{X}, \mathfrak{Y} \rangle \subseteq C(J)$ . In particular,  $\mathfrak{X}$  does not centralize any quaternion subgroup of  $\mathfrak{G}$ . By Lemma 7.1(iii), it follows that  $\mathfrak{B}$  centralizes every element of  $\mathcal{N}(\mathfrak{B}; 2)$ . Suppose  $K$  is an involution of  $C(\mathfrak{B})$ . Then by Lemma 5.38,  $C(\mathfrak{K})$  contains an element  $\mathfrak{Z}$  of  $\mathcal{U}(2)$ , so by Lemmas 7.1 and 7.4,  $\langle \mathfrak{B}, \mathfrak{Z} \rangle$  is nonsolvable, against  $\langle \mathfrak{B}, \mathfrak{Z} \rangle \subseteq C(K)$ . We conclude that  $|C(\mathfrak{B})|$  is odd, and so  $\mathcal{N}(\mathfrak{B}; 2)$  contains only 1. Since  $\mathcal{N}(\mathfrak{B}_0; 2) \subseteq \mathcal{N}(\mathfrak{B}; 2)$ , (a) follows.

Suppose (b) is false. Let  $\mathfrak{Q}$  be a 2-group normalized by  $\mathfrak{U}$  and minimal subject to  $[\mathfrak{Q}, \mathfrak{U}] \neq 1$ . Then  $\mathfrak{Q} = [\mathfrak{Q}, \mathfrak{U}]$  is either a quaternion group or a four-group, and  $\mathfrak{U} = \mathfrak{U}_0 \times \mathfrak{U}_1$  where  $|\mathfrak{U}_i| = 3$  and  $\mathfrak{U}_0 = C_{\mathfrak{U}}(\mathfrak{Q})$ .

Let  $\mathfrak{C} = C(\mathfrak{U}_0) \cong \langle C_{\mathfrak{B}}(\mathfrak{U}), \mathfrak{Q} \rangle$ . Since  $C_{\mathfrak{B}}(\mathfrak{U})$  is of index at most 3 in  $\mathfrak{B}$ , it follows that  $C_{\mathfrak{B}}(\mathfrak{U})$  contains an element  $\mathfrak{U}$  of  $\mathcal{U}(\mathfrak{B})$ . We argue that  $C_{\mathfrak{B}}(\mathfrak{U})$  contains an element of  $\mathcal{S}_{m_3}(\mathfrak{B})$ . Namely, let

$$\mathfrak{Z} = \Omega_1(Z(C_{\mathfrak{B}}(\mathfrak{U}))) .$$

If  $m(\mathfrak{Z}) \geq 3$ , let  $\mathfrak{B}$  be an element of  $\mathcal{S}_{m_3}(\mathfrak{B})$  which contains  $\mathfrak{Z}$ . Since  $\mathfrak{U} \subseteq \mathfrak{B}$ , we get  $\mathfrak{B} \subseteq C_{\mathfrak{B}}(\mathfrak{U})$ . Suppose  $m(\mathfrak{Z}) \leq 2$ . Then  $\mathfrak{Z} = \mathfrak{U} \triangleleft \mathfrak{B}$ , so by Lemma 0.8.9,  $\mathfrak{U}$  is contained in some element of  $\mathcal{S}_{m_3}(\mathfrak{B})$ . So  $C_{\mathfrak{B}}(\mathfrak{U})$  contains an element of  $\mathcal{S}_{m_3}(\mathfrak{B})$ . By Hypothesis 7.4(v),  $O_3(\mathfrak{C}) = 1$ .

Let  $\mathfrak{P}^*$  be a  $S_3$ -subgroup of  $C$  which contains  $C_{\mathfrak{B}}(\mathfrak{U})$ . Since  $\mathfrak{U}_1$  does not centralize  $O_3(\mathfrak{C}) = \mathfrak{H}$ , it follows that  $\mathfrak{P}^* = \mathfrak{H}C_{\mathfrak{B}}(\mathfrak{U})$  is a  $S_3$ -subgroup of  $\mathfrak{G}$ . Also, since  $\mathfrak{U}_1\mathfrak{H}/\mathfrak{H} \subseteq Z(\mathfrak{P}^*/\mathfrak{H})$ , it follows that  $\mathfrak{U}_1 \subseteq O_{3,3',3}(\mathfrak{C})$ . Hence,  $\mathfrak{Q} \subseteq O_{3,3',3}(\mathfrak{C})$ . Since  $C_{\mathfrak{Q}}(\mathfrak{U}_1)$  is of index 3 in  $\mathfrak{H}$ , it follows that  $[Q_3'(\mathfrak{C}), \mathfrak{U}_1]$  is a quaternion group. Hence,  $\tilde{\mathfrak{C}} = \mathfrak{Q}\mathfrak{P}^*$  is a group. Let  $\tilde{\mathfrak{H}} = O_3(\tilde{\mathfrak{C}})$ . Thus,  $\mathfrak{P}^* = \tilde{\mathfrak{H}}\mathfrak{U}_1$  and  $\tilde{\mathfrak{H}} \cap C(\mathfrak{Q})$  is of index 9 in  $\tilde{\mathfrak{H}}$ , while  $\tilde{\mathfrak{H}} \cap C(\mathfrak{Q}) \triangleleft \mathfrak{P}^*$ . Since  $\mathfrak{Q}'$  centralizes no element of  $\mathcal{U}^*(\mathfrak{P}^*)$ , it follows that  $\tilde{\mathfrak{H}} \cap C(\mathfrak{Q})$  is cyclic. Since  $|\mathfrak{B}| > 3^4$ , so also  $|\mathfrak{P}^*| > 3^4$ , so  $\mathfrak{U}_0$  is a proper subgroup of  $\tilde{\mathfrak{H}} \cap C(\mathfrak{Q}) = \tilde{\mathfrak{U}}_0$ .

Let  $\tilde{\mathfrak{G}}_0 = [\mathfrak{Q}, \tilde{\mathfrak{H}}]$ . By the three subgroups lemma,  $\tilde{\mathfrak{G}}_0$  and  $\tilde{\mathfrak{U}}_0$  commute elementwise. Furthermore, either  $\tilde{\mathfrak{G}}_0$  is elementary of order 9 and  $\tilde{\mathfrak{H}} = \tilde{\mathfrak{G}}_0 \times \tilde{\mathfrak{U}}_0$  or  $\tilde{\mathfrak{G}}_0$  is a non abelian group of order 27 and exponent 3 and  $\tilde{\mathfrak{H}}$  is the central product of  $\tilde{\mathfrak{G}}_0$  and  $\tilde{\mathfrak{U}}_0$ .

Since  $C(\mathfrak{U}_1) \cap \tilde{\mathfrak{H}}$  is of index 3 in  $\tilde{\mathfrak{H}}$ , it follows that  $\mathfrak{U}_1$  centralizes  $\tilde{\mathfrak{U}}_0$ . Set  $\mathfrak{B} = \langle \mathfrak{U}_1, \tilde{\mathfrak{U}}_0 \rangle = \mathfrak{U}_1 \times \tilde{\mathfrak{U}}_0$ , and let  $I$  be the involution of  $\mathfrak{Q}$ . Thus,  $\mathfrak{B} \subseteq C(I)$  and  $C(I)$  contains an element of  $\mathcal{U}(2)$ . Thus,  $C(I)$  contains no element of  $\mathcal{U}^*(3)$ . Since  $\mathfrak{B}$  is of index 9 in  $\mathfrak{P}^*$ , it follows that  $\mathfrak{B}$  is a  $S_3$ -subgroup of  $C(I)$ . Let  $\mathfrak{B}$  be a  $S_{2,3}$ -subgroup of  $C(I)$  which contains  $\mathfrak{B}\mathfrak{Q}$ . Then  $O_3(\mathfrak{B}) = 1$ , so  $\mathfrak{B}$  is faithfully represented on  $O_2(\mathfrak{B})$ . We can thus choose a subgroup  $\mathfrak{B}_0$  of order 3 in  $\mathfrak{B}$  such that  $\tilde{\mathfrak{U}}_0$  is faithfully represented on  $O_2(\mathfrak{B}) \cap C(\mathfrak{B}_0)$ . Let  $\mathfrak{X} = C(\mathfrak{B}_0)$ . Then  $O_3(\mathfrak{X})$  is of odd order by (a). Thus,  $O_{3',3}(\mathfrak{X}) \cap \tilde{\mathfrak{U}}_0 = 1$ , so that  $|O_{3',3}(\mathfrak{X})|_3 \leq 27$ . But  $\tilde{\mathfrak{U}}_0$  is faithfully represented on the Frattini quotient group  $\mathfrak{B}$  of  $O_{3',3}(\mathfrak{X})/O_3(\mathfrak{X})$ . Since  $|\mathfrak{B}| \leq 27$  and  $\tilde{\mathfrak{U}}_0$  is cyclic of order  $\geq 9$ , we have a contradiction. The proof of (b) is complete.

Suppose (c) is false. Let  $\mathfrak{B}$  be a four group in  $\mathfrak{N}(\mathfrak{A})$  which is not centralized by  $\mathfrak{A}$ . Then  $\mathfrak{A} = \mathfrak{A}_0 \times \mathfrak{A}_1$  where  $|\mathfrak{A}_i| = 3$  and  $\mathfrak{A}_0 = C_{\mathfrak{A}}(\mathfrak{B})$ .

Set  $\mathfrak{C} = C(\mathfrak{A}_0)$  and let  $\mathfrak{P}^*$  be a  $S_3$ -subgroup of  $\mathfrak{C}$  which contains  $C_{\mathfrak{P}}(\mathfrak{A})$ . By (a),  $O_3(\mathfrak{C})$  is of odd order, so  $\mathfrak{A}_1\mathfrak{B}$  is faithfully represented on  $O_{3',3}(\mathfrak{C})/O_3(\mathfrak{C})$ . Set  $\mathfrak{G} = \mathfrak{P}^* \cap O_{3',3}(\mathfrak{C})$ . By (B),  $|\mathfrak{P}^*: C_{\mathfrak{G}}(\mathfrak{A}_1)| \geq 9$ . Thus,  $\mathfrak{P}^* = \mathfrak{G}C_{\mathfrak{P}}(\mathfrak{A})$  is a  $S_3$ -subgroup of  $\mathfrak{G}$ , so that  $O_3(\mathfrak{C}) = 1$ .

We may now apply Lemma 5.42 with  $\mathfrak{C}/\mathfrak{G}$  in the role of  $\mathfrak{S}$ ,  $\mathfrak{G}/D(\mathfrak{G})$  in the role of  $\mathfrak{C}$ , and  $\mathfrak{A}_1$  in the role of  $\mathfrak{Z}$ . Let  $\mathfrak{C}_1$  be the inverse image in  $\mathfrak{C}$  of  $[Q_3(\mathfrak{C}), \mathfrak{A}_1]$ . Thus,  $\mathfrak{C}_1 = \mathfrak{G}\Omega$  where  $\Omega$  is either a four-group or is the central product of 2 quaternion groups. Since  $C_{\mathfrak{P}}(\mathfrak{A}_1)$  covers  $\mathfrak{P}^*/\mathfrak{G}$ , it follows that  $\mathfrak{C}_1$  is a minimal subgroup of the group  $\mathfrak{R} = \mathfrak{B}\mathfrak{P}^*$ . Let  $\mathfrak{L} = O_3(\mathfrak{R})$ , so that  $\mathfrak{P}^*/\mathfrak{L}$  is elementary of order 3 or 9. Since  $\mathfrak{B} \subset \mathfrak{C}_1$ , we assume without loss of generality that  $\mathfrak{B} \subseteq \Omega$ .

Since  $\mathfrak{A}_1$  centralizes  $D(\mathfrak{B})$ , so does  $\Omega$ . Thus,  $C(\Omega) \cap \mathfrak{L} \triangleleft \mathfrak{L}$ . Since  $N(\Omega) \cap \mathfrak{R}$  normalizes  $C(\Omega) \cap \mathfrak{L}$ , it follows that  $C(\Omega) \cap \mathfrak{L} \triangleleft \mathfrak{P}^*$ . Since  $\Omega$  centralizes no element of  $\mathcal{Z}^*(3)$ , it follows that  $C(\Omega) \cap \mathfrak{L}$  is cyclic. Naturally,  $\mathfrak{A}_0 \subseteq C(\Omega) \cap \mathfrak{L}$ .

*Case 1.*  $\mathfrak{P}^* = \mathfrak{L}\mathfrak{A}_1$ .

Since  $\mathfrak{A}_1$  normalizes  $\mathfrak{B}$ , it follows that  $\mathfrak{R}_1 = \mathfrak{P}^*\mathfrak{B}$  is a group and that  $\mathfrak{L} = O_3(\mathfrak{R}_1)$ . Let  $\mathfrak{L}_1 = C_{\mathfrak{L}}(\mathfrak{B}) \cong D(\mathfrak{L})$ . Thus,  $\mathfrak{L}_1 \triangleleft \mathfrak{R}_1$  and  $\mathfrak{L}/\mathfrak{L}_1$  is elementary of order 27. Also,  $\mathfrak{L}_1$  is cyclic, since no element of  $\mathcal{Z}^*(3)$  is centralized by  $\mathfrak{B}$ . Since  $\mathfrak{L}/\mathfrak{L}_1$  is a chief factor of  $\mathfrak{R}_1$  or order 27, it follows that  $\mathfrak{L} = \mathfrak{L}_1 \times \mathfrak{L}_2$ , where  $\mathfrak{L}_2 = [\mathfrak{L}, \mathfrak{B}]$  is elementary of order 27,  $\mathfrak{L}_2 \triangleleft \mathfrak{R}_1$ . Let  $V$  be an involution of  $\mathfrak{B}$ . Thus,  $\mathfrak{B} = \langle C(V) \cap \mathfrak{L}_2, \mathfrak{A}_0 \rangle$  is elementary of order 9 and  $|\mathfrak{P}^*: C_{\mathfrak{P}^*}(\mathfrak{B})| = 3$ .

By (b),  $\mathfrak{B}$  centralizes every element of  $\mathfrak{N}(\mathfrak{B}; 2)$ . Since  $C(V)$  contains an element of  $\mathcal{Z}(2)$ , (7.2) is violated.

*Case 2.*  $\mathfrak{P}^* \supset \mathfrak{L}\mathfrak{A}_1$ .

In this case,  $\mathfrak{P}^*/\mathfrak{L}$  is elementary of order 9, so  $\Omega$  is the central product of 2 quaternion groups.

Suppose  $\mathfrak{L}$  is abelian. Then  $\mathfrak{L} = \mathfrak{L}_1 \times \mathfrak{L}_2$  where  $\mathfrak{L}_1 = [\mathfrak{L}, \Omega]$  is elementary of order  $3^4$  and  $\mathfrak{L}_2 = C_{\mathfrak{L}}(\Omega)$  is cyclic. Notice that  $\mathfrak{A}_0 \subseteq \mathfrak{L}_2$ . Since  $C_{\mathfrak{L}}(\mathfrak{B})$  is of index 27 in  $\mathfrak{L}$  by (B), it follows that  $\mathfrak{L} \cap C(\mathfrak{A}_1) \cap C(\mathfrak{B})$  contains a subgroup  $\mathfrak{B}$  of type (3, 3). But then  $|\mathfrak{P}^*: C_{\mathfrak{P}^*}(\mathfrak{B})| \leq 3$ , so by (b),  $\mathfrak{B}$  centralizes every element of  $\mathfrak{N}(\mathfrak{B}; 2)$ . Hence,  $\mathfrak{B}^{\#} \subseteq \mathfrak{G}_1 \cap \mathfrak{G}_2 \cap \mathfrak{G}_3$ , against (7.2). We conclude that  $\mathfrak{L}$  is non abelian.

Since  $\mathfrak{A}_1$  centralizes  $D(\mathfrak{B})$ , so does  $\Omega$ , so  $\mathfrak{L}_2 = C_{\mathfrak{L}}(\Omega) \triangleleft \mathfrak{L}$ . Hence  $\mathfrak{L}_2 \triangleleft \mathfrak{P}^*$ , and  $\mathfrak{L}_2$  is cyclic. Let  $\mathfrak{L}_1 = [\mathfrak{L}, \Omega]$ . Then  $\mathfrak{L}_1/D(\mathfrak{L}_1)$  is elementary of order  $3^4$ ,  $\mathfrak{L}'_1 = D(\mathfrak{L}_1)$  and  $\mathfrak{L}_1/D(\mathfrak{L}_1)$  is a chief factor of  $K$ . Being a chief factor,  $\mathfrak{L}_1/D(\mathfrak{L}_1)$  is centralized by  $\mathfrak{L}$ . Hence,  $[\mathfrak{L}_2, \mathfrak{L}_1] \subseteq D(\mathfrak{L}_1) \subseteq \mathfrak{L}_2$ , so  $[\mathfrak{L}_2, \mathfrak{L}_1, \Omega] = 1$ . Since  $[\Omega, \mathfrak{L}_2] = 1$ , so also  $[\Omega, \mathfrak{L}_2, \mathfrak{L}_1] = 1$ . By the three subgroups lemma,  $[\mathfrak{L}_1, \Omega, \mathfrak{L}_2] = 1$ , that is,  $[\mathfrak{L}_1, \mathfrak{L}_2] = 1$ . Hence,

$D(\mathfrak{L}_1) \subseteq Z(\mathfrak{L}_1)$ . Since  $\mathfrak{L}$  is nonabelian, so is  $\mathfrak{L}_1$ . Since  $\mathfrak{L}_1/D(\mathfrak{L}_1)$  is a chief factor of  $\mathfrak{K}$ ,  $D(\mathfrak{L}_1) = Z(\mathfrak{L}_1) = \mathfrak{L}'_1 \subseteq \mathfrak{L}_2$ , so  $\mathfrak{L}_1$  is extra special of order  $3^5$ .

Now  $\mathfrak{U}_1\mathfrak{B}$  is faithfully represented on  $\mathfrak{L}_1$ . Also,  $|\mathfrak{L}:\mathfrak{L} \cap C(\mathfrak{B})| = |\mathfrak{L}_1:\mathfrak{L}_1 \cap C(\mathfrak{B})| = 3^3$ , by (B). Hence,  $\mathfrak{L}_1 \cap C(\mathfrak{B}) = 3^2$ . This is not the case, since  $\mathfrak{L}_1 \cap C(\mathfrak{B})$  is either extra special or is  $\mathfrak{L}'_1$ . The proof of (c) is complete.

Suppose (d) is false. Let  $\mathfrak{Q}$  be an element of  $\mathfrak{N}(\mathfrak{A}; 3')$  minimal subject to  $[\mathfrak{A}, \mathfrak{Q}] \neq 1$ . Then  $\mathfrak{Q}$  is a  $q$ -group for some prime  $q$ ,  $\mathfrak{Q} = [\mathfrak{Q}, \mathfrak{A}]$ , and  $\mathfrak{A} = \mathfrak{A}_0 \times \mathfrak{A}_1$ , where  $|\mathfrak{A}_i| = 3$  and  $\mathfrak{A}_0 = C_{\mathfrak{A}}(\mathfrak{Q})$ . By (b),  $q \neq 2$ . Let  $\mathfrak{C} = C(\mathfrak{A}_0)$ . Since  $C_{\mathfrak{B}}(\mathfrak{A})$  contains an element of  $\mathcal{S}\mathcal{C}\mathcal{N}_3(\mathfrak{B})$ , it follows from Hypothesis 7.4(v) that  $O_3(\mathfrak{C}) = 1$ . Let  $\mathfrak{B}^*$  be a  $S_3$ -subgroup of  $\mathfrak{C}$  which contains  $C_{\mathfrak{B}}(\mathfrak{A}_0)$ . Since  $|\mathfrak{B}^*:C_{\mathfrak{B}}(\mathfrak{A})| \leq 3$ , so also  $|\mathfrak{B}^*:C_{\mathfrak{B}}(\mathfrak{A}_0)| \leq 3$ , and so  $[\mathfrak{B}^*, \mathfrak{A}, \mathfrak{A}] = 1$ .

Let  $\mathfrak{H}$  be a  $S_{3,q}$ -subgroup of  $\mathfrak{C}$  which contains  $\mathfrak{B}^*$ . Since  $q$  is odd, (B) implies that  $\mathfrak{A} \subseteq O_3(\mathfrak{H})$ . Let  $\mathfrak{H}^*$  be a  $S_{3,q}$ -subgroup of  $\mathfrak{C}$  which contains  $\mathfrak{A}\mathfrak{Q}$ . By Lemma 0.7.5, we get  $\mathfrak{A} \subseteq O_3(\mathfrak{H}^*)$ , so  $\mathfrak{Q} = [\mathfrak{Q}, \mathfrak{A}] \subseteq O_3(\mathfrak{H}^*)$ . This contradiction completes the proof of (d) and the lemma.

LEMMA 7.7. *Assume the following:*

- (a)  $\mathfrak{W}$  is a normal elementary subgroup of  $\mathfrak{B}$ ,  $\mathfrak{A} = A_{\mathfrak{B}}(\mathfrak{W})$ .
- (b)  $\bar{\mathfrak{B}}$  is the image of  $\mathfrak{B}$  in  $\mathfrak{A}$  and  $\bar{\mathfrak{B}}$  is faithfully represented on  $\mathfrak{Q}$ ,  $\mathfrak{Q}$  being a non abelian special 2-subgroup of  $\mathfrak{A}$ .
- (c)  $\bar{\mathfrak{B}}$  contains a subgroup  $\bar{\mathfrak{B}}_0$  of order 3 which centralizes a hyperplane of  $\mathfrak{W}$ .

Then  $\bar{\mathfrak{B}}$  centralizes  $\mathfrak{Q}'$ .

*Proof.* Let  $\mathfrak{Q}_0 = [\bar{\mathfrak{B}}_0, \mathfrak{Q}]$ . Thus,  $\mathfrak{Q}_0$  is a quaternion group, and  $\mathfrak{W} = \mathfrak{W}_0 \times \mathfrak{W}_1$ , where  $\mathfrak{W}_0 = [\mathfrak{Q}_0, \mathfrak{W}]$  is of order 9 and  $\mathfrak{W}_1 = C_{\mathfrak{W}}(\mathfrak{Q}_0)$ . Since  $|C(\mathfrak{W})|$  is odd, some involution  $I$  of  $N(\mathfrak{W})$  maps to the involution of  $\mathfrak{Q}_0$ . Let  $\bar{\mathfrak{B}}_1$  be the normal closure of  $\bar{\mathfrak{B}}_0$  in  $\bar{\mathfrak{B}}$ . Thus,  $\bar{\mathfrak{B}}_1$  centralizes  $\mathfrak{Q}'$ . Let  $\mathfrak{W}_2 = C_{\mathfrak{W}}(\bar{\mathfrak{B}}_1)$  so that  $\mathfrak{Q}'$  is faithfully represented on  $\mathfrak{W}_2$ . Suppose  $\bar{\mathfrak{B}}$  does not centralize  $\mathfrak{Q}'$ . Then by Lemma 4.4 of [17], there is an elementary subgroup  $\mathfrak{W}^*$  of  $\mathfrak{W}_2$  which is of order 27, normal in  $\mathfrak{B}$  and with  $|\mathfrak{B}:C_{\mathfrak{B}}(\mathfrak{W}^*)| = 3$ . Since  $\bar{\mathfrak{B}}_0$  centralizes  $\mathfrak{W}^*$ , it follows that  $\mathfrak{W}^* \cap \mathfrak{W}_1$  is noncyclic. Let  $\mathfrak{B}$  be a subgroup of  $\mathfrak{W}^* \cap \mathfrak{W}_1$  of order 9. With  $\mathfrak{W}^*$  in the role of  $\mathfrak{C}$  in Lemma 7.6(d), we conclude that  $\mathfrak{B} \in \mathcal{E}(3)$ . But now  $C(I)$  contains an element of  $\mathcal{Z}(2)$  and also contains  $\mathfrak{B}$ , against Lemma 7.4. The proof is complete.

LEMMA 7.8. *Suppose that  $\mathfrak{B}$  is of exponent 3, order 81 and that  $|Z(\mathfrak{B})| = 9$ . Then  $N(\mathfrak{B})$  is the unique element of  $\mathcal{MS}(\mathfrak{G})$  which contains  $\mathfrak{B}$ .*

*Proof.* Suppose false. Let  $\mathfrak{S}$  be a solvable subgroup of  $\mathfrak{G}$  which

contains  $\mathfrak{P}$  and is minimal subject to  $\mathfrak{P} \triangleleft \mathfrak{G}$ . Let  $\mathfrak{P}_0 = O_3(\mathfrak{G})$ . Since  $Z(\mathfrak{P}) \subset \mathfrak{P}_0 \subset \mathfrak{P}$ , it follows that  $\mathfrak{P}_0$  is abelian of order 27. Since  $\mathfrak{G}$  is not 3-closed, it follows that  $\mathfrak{G} = \mathfrak{P}\Omega$  where  $\Omega$  is a quaternion group.

Let  $\mathfrak{P}_0 = \mathfrak{P}_1 \times \mathfrak{P}_2$  where  $\mathfrak{P}_1 = C_{\mathfrak{P}_0}(\Omega)$ ,  $\mathfrak{P}_2 = [\mathfrak{P}_0, \Omega]$ . Thus,  $|\mathfrak{P}_i| = 3^i$ ,  $i = 1, 2$ . Let  $\tilde{\mathfrak{P}}_1 = \mathfrak{P}_2 \cap Z(\mathfrak{P})$ . Thus,  $Z(\mathfrak{P}) = \mathfrak{P}_1 \times \tilde{\mathfrak{P}}_1$  and  $\mathfrak{P}' = \tilde{\mathfrak{P}}_1$ . Let  $\Omega' = \langle I \rangle \subseteq N(\mathfrak{P})$ . Write  $N(\mathfrak{P}) = \mathfrak{P}\mathfrak{R}$  where  $\mathfrak{R}$  is a complement to  $\mathfrak{P}$  in  $N(\mathfrak{P})$  which contains  $I$ . Since  $\mathfrak{R}$  normalizes  $\tilde{\mathfrak{P}}_1$ , it follows that  $A_{\mathfrak{G}}(Z(\mathfrak{P}))$  is abelian. Hence,  $Z(\mathfrak{P}) \cap C(I) \triangleleft N(\mathfrak{P})$ . Since  $Z(\mathfrak{P}) \cap C(I) = \tilde{\mathfrak{P}}_1$ , we get that  $N(\mathfrak{P}) \subseteq N(\mathfrak{P}_1)$ . Since  $I \in N(\mathfrak{P})$ , it follows that  $\mathfrak{P}_1$  may be characterized as the only subgroup of  $Z(\mathfrak{P})$  of order 3 which is normal in  $N(\mathfrak{P})$  and is not contained in  $\mathfrak{P}'$ .

Let  $\mathfrak{R}$  be any solvable subgroup of  $\mathfrak{G}$  which contains  $\mathfrak{P}$ . We will show that  $\mathfrak{R} \subseteq N(\mathfrak{P}_1)$ . We may assume that  $\mathfrak{P} \triangleleft \mathfrak{R}$ . Let  $\tilde{\mathfrak{P}}_0 = O_3(\mathfrak{R}) \supset Z(\mathfrak{P})$ . Thus,  $\tilde{\mathfrak{P}}_0$  is abelian of order 27. By our characterization of  $\mathfrak{P}_1$ , it follows that  $\mathfrak{P}_1 \triangleleft \mathfrak{R}$ , that is,  $\mathfrak{R} \subseteq N(\mathfrak{P}_1)$ .

Set  $\mathfrak{M} = N(\mathfrak{P}_1)$ , so that  $\mathfrak{M}$  is the unique element of  $\mathcal{MS}(\mathfrak{G})$  which contains  $\mathfrak{P}$ . Let  $\mathfrak{A}$  be any elementary subgroup of  $\mathfrak{P}$  of order 27. Then  $\mathfrak{P} \subseteq N(\mathfrak{A})$ , so  $N(\mathfrak{A}) \subseteq \mathfrak{M}$ . Now let  $A$  be any element of  $\mathfrak{P}^*$ . We will show that  $C(A) \subseteq \mathfrak{M}$ . This is clear if  $A \in Z(\mathfrak{P})$ . Suppose  $A \notin Z(\mathfrak{P})$ . Then  $C_{\mathfrak{P}}(A) = \mathfrak{A}$  is of order 27 and is abelian. Hence,  $N(\mathfrak{A}) \subseteq \mathfrak{M}$ . This implies that some  $S_3$ -subgroup of  $C(A)$  is contained in  $\mathfrak{M}$ . If  $C(A)$  contains a  $S_3$ -subgroup of  $\mathfrak{G}$ , then  $C(A) \subseteq \mathfrak{M}$ , by uniqueness of  $\mathfrak{M}$ . So suppose that  $\mathfrak{A}$  is a  $S_3$ -subgroup of  $C(A)$ . Then since  $\mathfrak{N}(\mathfrak{A})$  is trivial, we get that  $\mathfrak{A} \triangleleft C(A)$ , so in any case,  $C(A) \subseteq \mathfrak{M}$ .

Let  $\mathfrak{G}$  be any non identity subgroup of  $\mathfrak{P}$ . We will show that  $N(\mathfrak{G}) \subseteq \mathfrak{M}$ . If  $|\mathfrak{G}'| = 3$ , it suffices to show that  $N(\mathfrak{G}') \subseteq \mathfrak{M}$ . If  $|\mathfrak{G}'| \neq 3$ , then  $\mathfrak{G}$  is abelian, since  $|\mathfrak{P}'| = 3$ . This, we may assume that  $\mathfrak{G}$  is abelian. By the preceding paragraph,  $C(\mathfrak{G}) \subseteq \mathfrak{M}$ . Let  $\mathfrak{G}^*$  be a  $S_3$ -subgroup of  $C(\mathfrak{G})$ . Then  $N(\mathfrak{G}) = C(\mathfrak{G}) \cdot (N(\mathfrak{G}) \cap N(\mathfrak{G}^*))$ , so it suffices to show that  $N(\mathfrak{G}^*) \subseteq \mathfrak{M}$ . But  $|\mathfrak{G}^*| \geq 27$ , so  $N(\mathfrak{G}^*) \subseteq \mathfrak{M}$ .

It is a consequence of the preceding results, that if  $\mathfrak{H}$  is a solvable subgroup of  $\mathfrak{G}$  such that  $\mathfrak{H} \cap \mathfrak{P}$  is noncyclic, then  $\mathfrak{H} \subseteq \mathfrak{M}$ .

Let  $\mathfrak{B} = \mathfrak{P} \cap N(\Omega')$  so that  $\mathfrak{B}$  is noncyclic. Hence,  $N(\Omega') \subseteq \mathfrak{M}$ . This is not the case since  $N(\Omega')$  contains an element of  $\mathcal{Z}(2)$ , while  $\mathfrak{M}$  contains an element of  $\mathcal{Z}(3)$ .

### 8. A characterization of $E_2(3)$ .

**THEOREM 8.1.**  $E_2(3)$  is the only simple group  $\mathfrak{G}$  with the following properties:

- (i) 1 is the only 3-signalizer of  $\mathfrak{G}$ .
- (ii) The center of a  $S_3$ -subgroup of  $\mathfrak{G}$  is noncyclic.
- (iii) The normalizer of every nonidentity 3-subgroup of  $\mathfrak{G}$  is solvable.

- (iv) *The centralizer of every involution of  $\mathfrak{G}$  is solvable.*
- (v)  *$S_2$ -subgroup of  $\mathfrak{G}$  contain normal elementary subgroups of order 8.*
- (vi) *If  $\mathfrak{X}$  is a  $S_2$ -subgroup of  $\mathfrak{G}$  and  $\mathfrak{A} \in \mathcal{S}_{\mathfrak{G}}(\mathfrak{X})$ , then  $\mathfrak{N}(\mathfrak{A})$  is trivial.*
- (vii)  *$2 \sim 3$ .*

The proof of Theorem 8.1 is elaborate. I am indebted to J. Tits for helpful discussion.

We first derive some properties of  $E_2(q)$ . We use the notation and calculations of Ree [30]. In addition, we let  $\mathfrak{B} = \mathfrak{U}\mathfrak{S}$ ,  $\mathfrak{N} = \langle \mathfrak{S}, \omega_a, \omega_b \rangle$ .  $F_q$  is the field of  $q = p^n$  elements, and if  $x \in F_q$ , then  $tr(x) = tr_{F_q/F_p}(x) = \sum x^\sigma$ ,  $\sigma$  ranging over all the automorphisms of  $F_q$ . If  $r \in \Sigma$ , then  $\mathfrak{X}_r = \langle x_r(t) \mid t \in F_q \rangle$ .

We need the usual sort of omnibus lemma.

LEMMA 8.1. *Let  $\mathfrak{U}, \mathfrak{B}, \mathfrak{S}, \mathfrak{N}$  denote the subgroups of  $E_2(q)$  given above.*

(i)  *$\mathfrak{B}_0 = \langle \omega_a^2 \omega_a, \omega_b^2 \omega_b \rangle$  is a dihedral group of order 12 and is a complement to  $\mathfrak{S}$  in  $\mathfrak{N}$ .*

(ii)  *$\mathfrak{S}$  is the direct product of two cyclic groups of order  $q - 1$ , with generators  $H_1 = h(\chi_{a,z}), H_2 = h(\chi_{b,z})$ . Here  $z$  is a generator for  $F_q^*$ . If  $W_1 = \omega_a^2 \omega_a, W_2 = \omega_b^2 \omega_b$ , then*

$$\begin{aligned} W_1^{-1}H_1W_1 &= H_1^{-1}, & W_1^{-1}H_2W_1 &= H_1H_2, \\ W_2^{-1}H_1W_2 &= H_1H_2^3, & W_2^{-1}H_2W_2 &= H_2^{-1}, \end{aligned}$$

(iii) *If  $q$  is a power of 3 and  $\nu$  is a nonsquare in  $F_q$ , then*

$$\{x_{3a+2b}(1), x_{2a+b}(1), x_{3a+2b}(1)x_{2a+b}(1), x_{a+b}(1)x_{3a+b}(1), x_{a+b}(1)x_{3a+b}(\nu)\}$$

*is a set of representatives for the conjugacy classes of  $E_2(q)$  of order 3. If  $c \in F_q$  satisfies  $tr(c) = 1$ , then  $\{x_a(1)x_b(1)x_{3a+b}(ec), e = 0, 1, -1\}$  is a set of representatives for the conjugacy classes of elements of  $E_2(q)$  of order 9.  $\mathfrak{U}$  is of exponent 9.*

(iv) *Assume that  $q$  is odd.*

(a) *Let  $\mathfrak{B} = C_{\mathfrak{B}}(\omega_a^2), \mathfrak{N} = C_{\mathfrak{N}}(\omega_a^2)$ . Let  $\mathfrak{C} = C_{E_2(q)}(\omega_a^2)$ . Then  $\mathfrak{C} = \mathfrak{B}\mathfrak{N}\mathfrak{B}$ .  $\mathfrak{C}$  contains a subgroup  $\mathfrak{C}_0 = \mathfrak{C}_1\mathfrak{C}_2$ , where  $\mathfrak{C}_i \cong SL(2, q)$ ,  $i = 1, 2, \mathfrak{C}_1 \cap \mathfrak{C}_2 = \mathbf{Z}(\mathfrak{C}_i) = \langle \omega_a^2 \rangle, \mathfrak{C}_1$  and  $\mathfrak{C}_2$  commute elementwise and  $|\mathfrak{C} : \mathfrak{C}_0| = 2$ . Furthermore,  $\mathfrak{C}_i \triangleleft \mathfrak{C}, i = 1, 2$ .*

(b) *For  $i = 1, 2$ , let  $\alpha_i$  be the isomorphism from  $\mathfrak{C}_i$  to  $SL(2, q)$  induced by  $x_{r_i}(t) \rightarrow \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}, z_{-r_i}(t) \rightarrow \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}$ , where  $r_1 = a, r_2 = 3a + 2b$ . Each element  $X$  in  $\mathfrak{C} - \mathfrak{C}_0$  induces an automorphism  $\varphi_X^{(i)}$  of  $\mathfrak{C}_i$  such that  $\alpha_i \varphi_X^{(i)} \alpha_i^{-1}$  coincides with the automorphism of  $SL(2, q)$*

induced by an element of  $GL(2, q)$  whose determinant is a nonsquare.

(c) There are involutions in  $\mathfrak{C} - \mathfrak{C}_0$ . If  $X$  is an involution in  $\mathfrak{C} - \mathfrak{C}_0$ , and  $q \equiv \varepsilon \pmod{4}$ ,  $\varepsilon = \pm 1$ , then  $C(X) \cap \mathfrak{C}_0$  has order  $2(q + \varepsilon)^2$  and

$$\begin{aligned} C(X) \cap \mathfrak{C}_0 &\cong gp\langle w, x, y, z \mid x^{(q+\varepsilon)/2} \\ &= y^{(q+\varepsilon)/2} = w, w^2 = 1, xy = yx, z^{-1}xz \\ &= x^{-1}, z^{-1}yz = y^{-1}, z^2 = 1 \rangle. \end{aligned}$$

(v) If  $q$  is odd, then  $i(E_2(q)) = 1$ .

*Proof.* The Weyl group of  $G_2$  is dihedral of order 12, so  $w_a w_b$  is of order 6. By (1.8) of [30],  $(\omega_a \omega_b)^6 = h(\chi)$ , for some  $\chi \in X$ . We show that  $\chi = 1$ . It suffices to show that  $\chi(a) = \chi(b) = 1$ , that is,  $\eta_a = \eta_b = 1$ . This follows readily from table (3.4) of [30]. Since  $\omega_a^{-1} \omega_b^2 \omega_a = \omega_b^2 \omega_a^2$ , and  $\omega_b^{-1} \omega_a^2 \omega_b = \omega_a^2 \omega_b^2$ , the elements  $\omega_b^2 \omega_a$  and  $\omega_a^2 \omega_b$  are involutions. We have  $(\omega_b^2 \omega_a)(\omega_a^2 \omega_b) = \omega_b^2 \omega_a^{-1} \omega_b \sim \omega_b^{-1} \omega_a^{-1} = (\omega_a \omega_b)^{-1}$ , proving (i).

It is convenient for calculations to use the following character table:

	a	b	
$\chi_{a,z}$	$z^2$	$z^{-3}$	
$\chi_{b,z}$	$z^{-1}$	$z^2$	.

To determine this character table, we need to compute the values  $u(r)$ ,  $u$ ,  $r \in \Sigma$  (see [30], p. 433). The relevant values of  $u(r)$  are given as follows:

$r$	a	b	
$u$	2	-1	
$a$	-3	2	.

Using this table, we compute the values  $w_r(s)$ , as follows:

$s$	a	b	
$r$	-a	$3a + b$	
$a$	$a + b$	-b	.

(Using the geometric interpretation of  $w_r$ , we can read these results directly from Figure 1 of [30].)

We next compute  $w_r(\chi)$  for  $r = a, b$  and  $\chi = \chi_{a,z}, \chi_{b,z}$ . For example,

$[w_a(\chi_{a,z})](a) = \chi_{a,z}(w_a(a)) = \chi_{a,z}(-a) = \chi_{a,z}(a)^{-1} = z^{-2}$ . Continuing in this fashion, we get the following table of values:

	$a$	$b$
$w_a(\chi_{a,z})$	$z^{-2}$	$z^3$
$w_a(\chi_{b,z})$	$z$	$z^{-1}$
$w_b(\chi_{a,z})$	$z^{-1}$	$z^3$
$w_b(\chi_{b,z})$	$z$	$z^{-2}$

Referring back to the character table, we have

$$w_a(\chi_{a,z}) = \chi_{a,z}^{-1}, w_a(\chi_{b,z}) = \chi_{a,z}\chi_{b,z},$$

$$w_b(\chi_{a,z}) = \chi_{a,z}\chi_{b,z}^3, w_b(\chi_{b,z}) = \chi_{b,z}^{-1}.$$

The map from  $X$  to  $\mathfrak{S}$  induced by  $\chi_{r,z} \rightarrow h(\chi_{r,z})$  is an isomorphism, since  $X$  and  $\mathfrak{S}$  have order  $(q - 1)^2$ . The previous information, together with (1.7) of [30] implies that (ii) holds.

Let  $\mathfrak{U}_1 = \mathfrak{U} \cap \mathfrak{U}^a, \mathfrak{U}_2 = \mathfrak{U} \cap \mathfrak{U}^b$ . By using (3.10) of [30] it is straightforward to verify that  $\mathfrak{U}_1 \cup \mathfrak{U}_2$  is the set of elements of  $\mathfrak{U}$  of order 1 or 3. This then implies easily that every element of  $E_2(q)$  of order 3 is conjugate to an element of  $\mathfrak{U}_1 \cap \mathfrak{U}_2 = \langle \mathfrak{X}_{a+b}, \mathfrak{X}_{2a+b}, \mathfrak{X}_{3a+b}, \mathfrak{X}_{3a+2b} \rangle$ . Since  $\mathfrak{B} = N(\mathfrak{U})$ , it follows from Lemma 14.3.1 of [21] that elements of  $Z(\mathfrak{U})$  are conjugate in  $E_2(q)$  only if they are conjugate in  $\mathfrak{B}$ . Since the action of  $\mathfrak{S}$  on  $Z(\mathfrak{U}) = \langle \mathfrak{X}_{2a+b}, \mathfrak{X}_{3a+2b} \rangle$  is determined by (1.5) of [30] and our character table, it follows that any element of  $E_2(q)^*$  which is conjugate to an element of  $Z(\mathfrak{U})$  is conjugate to exactly one of  $x_{2a+b}(1), x_{3a+2b}(1), x_{2a+b}(1)x_{3a+2b}(1)$ . Furthermore, since the Weyl group permutes transitively the roots of a given length, and since  $2a + b$  and  $3a + 2b$  have different lengths, it follows that every element of the shape  $x_r(t), r \in \Sigma$ , is conjugate to an element of  $Z(\mathfrak{U})$ . Suppose  $x \in \mathfrak{U}_1 \cap \mathfrak{U}_2, x = x_{a+b}(t_1)x_{2a+b}(t_2)x_{3a+b}(t_3)x_{3a+2b}(t_4)$ , and that  $x$  is conjugate to no element of  $Z(\mathfrak{U})$ . Hence, either  $t_1 \neq 0$  or  $t_3 \neq 0$ . Suppose  $t_3 = 0$ . Conjugation by  $x_a(-t_1^{-1}t_2/2)$  enables us to assume that  $t_2 = 0$ . Conjugation by  $\omega_a$  then yields that  $x$  is conjugate to an element of  $Z(\mathfrak{U})$ . Hence,  $t_3 \neq 0$ . Suppose  $t_1 = 0$ . Conjugation by  $x_b(t_3^{-1}t_4)$  enables us to assume that  $t_4 = 0$ . Conjugation by  $\omega_b$  yields that  $x$  is conjugate to an element of  $Z(\mathfrak{U})$ . Hence,  $t_1 t_3 \neq 0$ . Conjugation by  $x_a(-t_1^{-1}t_2/2)x_b(t_3^{-1}t_4)$  enables us to assume that  $t_2 = t_4 = 0$ . Since  $h(\chi_{a,z})x_{a+b}(t_1)h(\chi_{a,z})^{-1} = x_{a+b}(z^{-1}t_1)$ , we may assume that  $t_1 = 1$ . Since  $h(\chi_{a,z}\chi_{b,z})$  centralizes  $x_{a+b}(1)$  and since  $h(\chi_{a,z}\chi_{b,z})x_{3a+b}(t_3)h(\chi_{a,z}\chi_{b,z})^{-1} = x_{3a+b}(t_3z^2)$ , we may assume that  $t_3 = 1$  or  $\nu$ . A direct calculation shows that the centralizer of  $x_{a+b}(1)x_{3a+b}(u)$  does not contain a  $S_3$ -subgroup of  $E_2(q)$  for any  $u$  in  $F_q^*$ , and a further calculation shows that  $x_{a+b}(1)x_{3a+b}(1)$  is not conjugate to  $x_{a+b}(1)x_{3a+b}(\nu)$ ,



completing the proof of the first part of (iii).

If  $tu \neq 0$ , it is easy to verify that  $x_a(t)x_b(u)x$  has order 9 for all  $x$  in  $\mathfrak{U}_1 \cap \mathfrak{U}_2$  and that  $(x_a(t)x_b(u))^3 = (x_a(t)x_b(u)x)^3$ . A calculation shows that  $\mathfrak{G}$  permutes transitively the elements  $x_a(t)x_b(u)$ ,  $tu \in F_q^*$ , so every element of  $E_2(q)$  of order 9 is conjugate to an element of the shape  $x_a(1)x_b(1)x$ , with  $x$  in  $\mathfrak{U}_1 \cap \mathfrak{U}_2$ . Let  $x = x_{a+b}(t_1)x_{2a+b}(t_2)x_{3a+b}(t_3)x_{3a+2b}(t_4)$ . Conjugation by  $x_a(u)$  enables us to assume that  $t_3 = 0$ . A further conjugation by  $x_{a+b}(u_1)x_{3a+b}(u_2)$  enables us to assume that  $t_2 = t_4 = 0$ . Thus, it suffices to show that

$$x_a(1)x_b(1)x_{a+b}(u) \text{ is conjugate to } x_a(1)x_b(1)x_{a+b}(v)$$

if and only if  $tr(u) = tr(v)$ . If  $g$  conjugates the first element into the second then  $g$  centralizes  $(x_a(1)x_b(1))^3$ . A calculation shows that the centralizer of  $(x_a(1)x_b(1))^3$  is  $\mathfrak{U}$ , and a further calculation completes the proof of (iii).

By a direct calculation,  $\tilde{\mathfrak{B}} = \langle \mathfrak{X}_a, \mathfrak{X}_{3a+2b}, \mathfrak{G} \rangle$ ,  $\tilde{\mathfrak{N}} = \langle \mathfrak{G}, \omega_a, (\omega_a \omega_b)^3 \rangle$ . Suppose  $\omega_a^2$  centralizes  $xh\omega x'$ ,  $x \in \mathfrak{U}$ ,  $h \in \mathfrak{G}$ ,  $\omega \in \mathfrak{B}_0$ ,  $x' \in \mathfrak{U}_w$ ,  $w$  being the image of  $\omega$  in the Weyl group. Then the normal form implies that  $x, x', h, \omega \in C(\omega_a^2)$ , so the first assertion of (iv) is proved.

Let  $\mathfrak{C}_1 = \langle \mathfrak{X}_a, \mathfrak{X}_{-a} \rangle$ ,  $\mathfrak{C}_2 = \langle \mathfrak{X}_{3a+2b}, \mathfrak{X}_{-(3a+2b)} \rangle$ , so that  $\mathfrak{C}_1 \cong \mathfrak{C}_2 \cong SL(2, q)$ . Clearly,  $\mathfrak{C}_1$  and  $\mathfrak{C}_2$  commute elementwise. Since  $\chi_{3a+2b, -1} = \chi_{a, -1}$ , it follows that  $\mathfrak{C}_1 \cap \mathfrak{C}_2 = \langle \omega_a^2 \rangle$ , so that  $\mathfrak{C}_0 = \mathfrak{C}_1 \mathfrak{C}_2$  is the central product of  $\mathfrak{C}_1$  and  $\mathfrak{C}_2$ . Setting  $\tilde{\mathfrak{U}} = \mathfrak{U} \cap \tilde{\mathfrak{B}}$ , we have

$$|\tilde{\mathfrak{U}} \cap \tilde{\mathfrak{U}}^{\omega_a}| = |\tilde{\mathfrak{U}} \cap \tilde{\mathfrak{U}}^{\omega_a(\omega_a \omega_b)^3}| = q,$$

and  $\tilde{\mathfrak{U}} \cap \tilde{\mathfrak{U}}^{(\omega_a \omega_b)^3} = 1$ , it follows that  $|\mathfrak{C}| = q^2(q-1)^2(1+2q+q^2)$ . Hence,

$$(*) \quad |\mathfrak{C}: \mathfrak{C}_0| = |\mathfrak{G}: \mathfrak{G} \cap \mathfrak{C}_0| = 2.$$

Since  $\mathfrak{G}$  normalizes  $\mathfrak{X}_r$  for all  $r$  in  $\Sigma$ , (iv) (a) is proved.

We observe that by (1.5) of [30],

$$\begin{aligned} h(\chi_{b,z})x_a(t)h(\chi_{b,z})^{-1} &= x_a(z^{-1}t), \\ h(\chi_{b,z})x_{-a}(t)h(\chi_{b,z})^{-1} &= x_{-a}(zt). \end{aligned}$$

Hence, if  $\eta = \varphi_{h(\chi_{b,z})}^{(1)}$  denotes the automorphism of  $\mathfrak{C}_1$  induced by  $h(\chi_{b,z})^{-1}$ , then  $\alpha_1 \eta \alpha_1^{-1}$  is the automorphism of  $SL(2, q)$  induced by the map

$$\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & z^{-1}t \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 \\ zt & 1 \end{pmatrix}.$$

This automorphism therefore coincides with the automorphism induced by  $\begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix}$ . A similar argument applies to  $\mathfrak{C}_2$ . Since  $\mathfrak{C} - \mathfrak{C}_0$  coincides with the coset  $\mathfrak{C}_0 h(\chi_{b,z})$  whenever  $z$  is not a square of  $F_q^*$ , the proof of (iv)(b) is complete.

Let  $K = (W_1W_2)^3$ . By (i),  $K$  is an involution, and by (ii),  $K$  inverts  $\mathfrak{S}$ . Thus,  $\mathfrak{S}K$  is a set of involutions in  $\mathfrak{C}$ . If  $\mathfrak{S}K \subseteq \mathfrak{C}_0$ , we get  $\mathfrak{S} \subseteq \langle \mathfrak{S}K \rangle \subseteq \mathfrak{C}_0$ , against (\*). So  $\mathfrak{C} - \mathfrak{C}_0$  contains an involution.

We will use (iv)(b) in the proof of (iv)(c). First, suppose  $\varepsilon = -1$ . In this case,  $-1$  is not a square in  $F_q$ . Since  $\mathfrak{C}_1$  and  $\mathfrak{C}_2$  commute elementwise, we assume without loss of generality that for  $i = 1, 2$ ,  $\alpha_i\varphi_X^{(i)}\alpha_i^{-1}$  is the automorphism of  $SL(2, q)$  induced by  $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ . Hence,  $\mathfrak{C}_i \cap \mathcal{C}(X)$  is cyclic of order  $q - 1$ . Since the commutator of  $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$  and  $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$  is  $-I$ , and since  $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$  inverts  $\begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix}$  for all  $x \in F_q^*$ , (iv)(c) follows in this case.

Now suppose  $\varepsilon = 1$ . In this case,  $-1$  is a square in  $F_q$ . Choose  $a, b \in F_q$  such that  $a^2 + b^2 = c$  is a nonsquare. We may assume that for  $i = 1, 2$ ,  $\alpha_i\varphi_X^{(i)}\alpha_i^{-1}$  is the automorphism of  $SL(2, q)$  induced by  $\begin{pmatrix} a & b \\ b & -a \end{pmatrix}$ . A short calculation shows that (iv)(c) holds.

Since  $|\mathcal{C}(\omega_a^2)| = (q(q^2 - 1))^2$ , it follows that  $\mathcal{C}(\omega_a^2)$  contains a  $S_2$ -subgroup of  $E_2(q)$ . Thus, to prove (v), it suffices to show that each involution  $X$  of  $\mathfrak{C}$  is conjugate to  $\omega_a^2$  in  $E_2(q)$ . Since  $E_2(q)$  is simple, Lemma 5.38 (a)(i) implies that  $X$  is conjugate in  $E_2(q)$  to an element of  $\mathfrak{C}_0$ .

Thus, it suffices to show that all involutions of  $\mathfrak{C}_0$  are conjugate in  $E_2(q)$ . Since  $\mathfrak{C}_i$  has just involution for  $i = 1, 2$ , it follows that every involution of  $\mathfrak{C}_0$  different from  $\omega_a^2$  is of the shape  $I_1I_2$  where  $I_i \in \mathfrak{C}_i$  and  $I_i^2 = \omega_a^2$ . Since  $\mathfrak{C}_i$  has just 1 conjugacy class of elements of order 4, it follows that  $\mathfrak{C}_0$  has two conjugacy classes of involutions.

*Case 1.* Every involution of  $\mathfrak{S}$  is in  $\mathfrak{C}_0$ .

By (ii), all involutions of  $\mathfrak{S}$  are fused in  $\mathfrak{N}$ . By the preceding paragraph, (v) follows.

*Case 2.*  $J$  is an involution of  $(\mathfrak{C} - \mathfrak{C}_0) \cap \mathfrak{S}$ .

Set  $I = \omega_a^2, K = (W_1W_2)^3$ , so that  $K$  inverts  $\mathfrak{S}$  and so centralizes  $I$  and  $J$ . Let  $\mathfrak{A} = \langle I, J, K \rangle$ . By (ii), the involutions of  $\mathfrak{A}$  are fused in  $\mathfrak{N}$  as follows:

$$I \sim J \sim IJ, IK \sim JK \sim IJK.$$

It is clear that in  $\mathfrak{C}$  all the involutions of  $\mathfrak{C} - \mathfrak{C}_0$  are conjugate. Let  $\mathfrak{A}_0 = \mathfrak{A} \cap \mathfrak{C}_0$ . Thus,  $\mathfrak{A}_0$  is one of  $\langle I, K \rangle, \langle I, JK \rangle$ . Suppose  $\mathfrak{A}_0 = \langle I, K \rangle$ . Then,  $J, JI, JK, JIK$  are the involutions of  $\mathfrak{A} - \mathfrak{A}_0$ , so are all conjugate in  $\mathfrak{C}$ . Since  $K \in \mathfrak{C}_0$ ,  $K$  and  $KI$  are conjugate in  $\mathfrak{C}_0$ . Thus, all involutions of  $\mathfrak{A}$  are conjugate in  $E_2(q)$ , so (v) follows. Suppose  $\mathfrak{A}_0 = \langle I, JK \rangle$ . Then,  $J, JI, K, KI$  are the involutions of  $\mathfrak{A} - \mathfrak{A}_0$ , so are all conjugate in  $\mathfrak{C}$ . Again all involutions of  $\mathfrak{A}$  are conjugate in  $E_2(q)$ . The proof of (v) is complete.

LEMMA 8.2. (a) Suppose  $\mathfrak{G}$  is a finite group and  $\mathfrak{B}$  is a four-subgroup of  $\mathfrak{G}$ . Suppose also that whenever  $I$  and  $J$  are distinct involutions of  $\mathfrak{B}$ ,  $I$  and  $IJ$  are conjugate in  $C(J)$ . Then  $A(\mathfrak{B}) = \text{Aut}(\mathfrak{B})$ .

(b)  $A_{E_2(3)}(\mathfrak{B}) = \text{Aut}(\mathfrak{B})$  for every four-subgroup  $\mathfrak{B}$  of  $E_2(3)$ .

*Proof.* Choose  $X$  in  $C(J)$  such that  $X^{-1}IX = IJ$ . Thus,

$$X \in N(V) \cap C(J).$$

Replacing the pair  $I, J$  by the pair  $J, I$  we choose  $Y$  in  $C(I)$  with  $Y^{-1}JY = IJ$ . Then  $\mathfrak{X} = \langle XY \rangle$  permutes  $I, J, IJ$  cyclically. Thus,  $\langle X, Y \rangle$  maps onto  $\text{Aut}(\mathfrak{B})$ , proving (a).

Let  $\mathfrak{B}$  be a four-subgroup of  $E_2(3)$  and let  $I, J$  be distinct involutions of  $\mathfrak{B}$ . We will produce  $X$  in  $C(J)$  such that  $X^{-1}IX = IJ$ . We may assume that  $J = \omega_a^2$ , since  $i(E_2(3)) = 1$ . Since  $O_2(C(\omega_a^2))$  is extra special, we are done in case  $I \in O_2(C(\omega_a^2))$ . If  $I \notin O_2(C(\omega_a^2))$ , then  $I$  induces an outer automorphism of both quaternion subgroups of  $O_2(C(\omega_a^2))$ , so again  $X$  is available. Now (b) follows from (a).

We omit the proof that  $\mathfrak{C} = C_{E_2(3)}(\omega_a^2)$  has exactly  $19 + 72$  involutions; namely,  $O_2(\mathfrak{C})$  has exactly 19 involutions, while all involutions of  $\mathfrak{C} - O_2(\mathfrak{C})$  are conjugate in  $\mathfrak{C}$ . Furthermore, it is straightforward to verify that  $\mathfrak{C}$  has exactly 3 conjugacy classes of elementary subgroups of order 8. Representatives  $\mathfrak{C}_1, \mathfrak{C}_2, \mathfrak{C}_3$  for these classes may be chosen so that if  $\mathfrak{X}$  denotes a fixed  $S_2$ -subgroup of  $\mathfrak{C}$ , then  $\mathfrak{C}_i \triangleleft \mathfrak{X}$ , and  $\mathfrak{C}_i \cong O_2(\mathfrak{C})$ ,  $i = 1, 2$ .

We argue that  $\mathfrak{C}_1$  and  $\mathfrak{C}_2$  are not conjugate in  $E_2(3)$ . Suppose  $\mathfrak{C}_1^g = \mathfrak{C}_2$ . Then  $\mathfrak{X}^g$  normalizes  $\mathfrak{C}_2$ , as does  $\mathfrak{X}$ . Then  $\mathfrak{X}^g = \mathfrak{X}^N$  for some  $N$  in  $N(\mathfrak{C}_2)$ . Hence,  $GN^{-1} \in N(\mathfrak{X}) = \mathfrak{X}$ , so  $G \in \mathfrak{X}N \cong N(\mathfrak{C}_2)$ . Since  $\mathfrak{C}_1^g = \mathfrak{C}_2$ , we get  $\mathfrak{C}_1 = \mathfrak{C}_2$ , a contradiction.

Set  $\mathfrak{B} = \mathfrak{C}_1 \cap \mathfrak{C}_2$  so that  $\mathfrak{B}$  is a four-subgroup of  $\mathfrak{X}$  and  $O_2(\mathfrak{X}) \cap C(\mathfrak{B}) = \mathfrak{C}_1\mathfrak{C}_2$  is the direct product of a group of order 2 and a dihedral group of order 8. Let  $\mathfrak{D} = C_{E_2(3)}(\mathfrak{B}) = C_{\mathfrak{X}}(\mathfrak{B})$ , a group of order 32. We omit the proof that  $\mathfrak{D}$  has exactly 4 elementary subgroups of order 8, among which are  $\mathfrak{C}_1$  and  $\mathfrak{C}_2$ . By Lemma 8.2 (b),  $N(\mathfrak{B})$  has an element  $A$  of order 3 which permutes transitively the involutions of  $\mathfrak{B}$ . If  $A$  normalizes both  $\mathfrak{C}_1$  and  $\mathfrak{C}_2$ , then  $A$  normalizes the derived group of  $\mathfrak{C}_1\mathfrak{C}_2$ , that is,  $A$  normalizes  $\langle \omega_a^2 \rangle$ . Since this is not the case, we can choose  $i$  in  $\{1, 2\}$  so that the orbit of  $\mathfrak{C}_i$  under  $\langle A \rangle$  has 3 elements. Since  $\mathfrak{C}_1$  and  $\mathfrak{C}_2$  are in different orbits under  $\langle A \rangle$ , it follows that  $A$  normalizes  $\mathfrak{C}_j$ , where  $\{i, j\} = \{1, 2\}$ .

We omit the proof that  $N(\mathfrak{C}_j) \cap \mathfrak{C}$  permutes transitively the involutions of  $\mathfrak{C}_j - \langle \omega_a^2 \rangle$ . Since  $A$  does not centralize  $\omega_a^2$ , it follows that  $N(\mathfrak{C}_j)$  permutes transitively the involutions of  $\mathfrak{C}_j$ . Thus,  $|N(\mathfrak{C}_j)| = 7 \cdot |N(\mathfrak{C}_j) \cap \mathfrak{C}| = 8 \cdot 24 \cdot 8$ . Hence,

$$\mathfrak{A}_{E_2(3)}(\mathfrak{C}_j) = \text{Aut}(\mathfrak{C}_j).$$

We have proved (a) of the next lemma.

- LEMMA 8.3. (a)  $E_2(3)$  is not an  $N$ -group.  
 (b)  $E_2(3)$  satisfies the hypotheses of Theorem 8.1.

*Proof.* It suffices to verify (b).

By Lemma 8.1 (iv), hypothesis (iv) of Theorem 8.1 is satisfied. By definition of  $\sim$ , so is hypothesis (vii),  $C_{E_2(3)}(\omega_a^2)$  being the relevant solvable group. Hypothesis (ii) is clearly satisfied, since

$$Z(\mathfrak{U}) = \langle \mathfrak{X}_{2a+b}, \mathfrak{X}_{3a+2b} \rangle.$$

Clearly, 1 is the only 2-signalizer of  $C(\omega_a^2)$ , so if  $\mathfrak{X}$  is a  $S_2$ -subgroup of  $C(\omega_a^2)$  and  $\mathfrak{Z}$  is a nonidentity 2'-subgroup of  $E_2(3)$  normalized by  $\mathfrak{X}$ , then  $\omega_a^2$  inverts  $\mathfrak{Z}$ , so  $\mathfrak{Z}$  is abelian. Furthermore,  $\mathfrak{Z}$  is a 3-group, as every  $\{2, 3\}'$ -subgroup of  $E_2(3)$  is cyclic. Since  $\mathfrak{Z}$  is a faithful  $\mathfrak{X}$ -module,  $|\mathfrak{Z}| \geq 3^4$ . Since  $\mathfrak{U}$  has no abelian subgroup of order  $3^5$ , it follows that  $\mathfrak{Z}$  is elementary of order  $3^4$ . It is straightforward to verify that every elementary subgroup of  $\mathfrak{U}$  of order  $3^4$  is conjugate to

$$\langle \mathfrak{X}_{2a+b}, \mathfrak{X}_{3a+b}, \mathfrak{X}_{a+b}, \mathfrak{X}_{3a+2b} \rangle;$$

the normalizer of this last group is  $\mathfrak{B}$ , so does not contain a  $S_2$ -subgroup of  $E_2(3)$ . Thus, 1 is the only 2-signalizer of  $E_2(3)$ . It is trivial to verify that 1 is the only 3-signalizer of  $E_2(3)$ , so hypothesis (i) is verified.

Since  $E_2(3)$  is of order  $2^6 \cdot 3^6 \cdot 7 \cdot 13$ , and since the centralizer of every nonidentity 3-element of  $E_2(3)$  is a 2, 3-group, it is easy to check that hypothesis (iii) is satisfied. Since  $S_2$ -subgroups of  $E_2(3)$  are of order 64, and since (\*\*) holds, hypothesis (v) is satisfied.

Suppose that  $\mathfrak{U} \in \mathcal{S}\mathcal{C}\mathcal{N}_3(\mathfrak{B})$  for a  $S_2$ -subgroup  $\mathfrak{B}$  of  $E_2(3)$ , and  $\mathfrak{B}$  is minimal nontrivial element of  $\mathfrak{N}(\mathfrak{U})$ . Then  $\mathfrak{U}\mathfrak{B}$  is contained in the centralizer of an involution; say  $\mathfrak{U}\mathfrak{B} \subseteq \mathfrak{C} = C(\omega_a^2)$ . But, by Lemma 8.1 (iv),  $\mathfrak{C}$  contains no nontrivial 2'-subgroup  $\mathfrak{B}$  for which  $N_{\mathfrak{C}}(\mathfrak{B})$  contains an elementary subgroup of order  $2^3$ . This contradiction proves that  $\mathfrak{N}(\mathfrak{U}) = \{1\}$ , which is hypothesis (vi). The proof is complete.

The remaining results in this section are proved under the hypothesis that  $\mathfrak{G}$  satisfies the hypothesis of Theorem 8.1.

- LEMMA 8.4. (i)  $\mathfrak{G}$  satisfies Hypothesis 7.4.  
 (ii)  $\mathfrak{G}$  satisfies Hypothesis 7.1 for  $p = 2$  and for  $p = 3$ .

*Proof.* We first show that  $\mathcal{S}\mathcal{C}\mathcal{N}_3(3) \neq \emptyset$ . Suppose false. Let  $\mathfrak{B}$  be a  $S_3$ -subgroup of  $\mathfrak{G}$ . Since  $\mathcal{S}\mathcal{C}\mathcal{N}_3(\mathfrak{B}) = \emptyset$ , it follows that

$\Omega_1(\mathfrak{B}) = \Omega_1(\mathbf{Z}(\mathfrak{B}))$  is of type (3, 3). This implies that every 3-solvable subgroup of  $\mathfrak{G}$  has 3-length at most 1. Since 1 is the only 3-signalizer of  $\mathfrak{G}$ , it follows that  $\mathfrak{B} = C(\Omega_1(\mathbf{Z}(\mathfrak{B})))$ . Hence, 1 is the only element in  $\mathfrak{N}(\Omega_1(\mathbf{Z}(\mathfrak{B})); 3')$ . Thus, if  $\mathfrak{R}$  is a 3-solvable subgroup of  $\mathfrak{G}$  and  $S_3$ -subgroups of  $\mathfrak{R}$  are noncyclic, then  $\mathfrak{R}$  is 3-closed. This implies by definition of  $\sim$  that  $A_{\mathfrak{G}}(\Omega_1(\mathbf{Z}(\mathfrak{B})))$  contains an abelian subgroup of type (4, 2) or an elementary subgroup of order 8. Neither of these possibilities holds in  $\text{Aut}(\Omega_1(\mathbf{Z}(\mathfrak{B})))$ . Hence,  $\mathcal{S}_{\text{inv}_3}(3) \neq \emptyset$ . We have shown that (i), (ii), (iii), (iv) of Hypothesis 7.4 hold. If  $\mathfrak{A} \in \mathcal{S}_{\text{inv}_3}(2)$ , then  $\mathfrak{N}(\mathfrak{A})$  contains only 1 by Hypothesis (vi) of Theorem 8.1. Suppose  $\mathfrak{A} \in \mathcal{S}_{\text{inv}_3}(3)$ , and  $\Omega \in \mathfrak{N}(\mathfrak{A})$ ,  $\Omega \neq 1$ ,  $\Omega$  minimal with these properties. Let  $\mathfrak{B}$  be a  $S_3$ -subgroup of  $N(\mathfrak{A})$ . Since  $\mathbf{Z}(\mathfrak{B})$  is noncyclic, we may choose  $Z$  in  $C(\Omega) \cap \mathbf{Z}(\mathfrak{B})^\#$ . It follows that  $\Omega \subseteq O_3(C(Z))$  against Hypothesis (i) of Theorem 8.1. (i) is proved.

Hypothesis 7.1 follows from Hypothesis 7.4 since if  $p = 2$  or 3 and  $\mathfrak{B} \in \mathcal{Z}^*(p)$ , then  $C(\mathfrak{B})$  contains an element of  $\mathcal{S}_{\text{inv}_3}(p)$ .

In the remainder of this section,  $\mathfrak{B}$  denotes a  $S_3$ -subgroup of  $\mathfrak{G}$ , and  $\mathfrak{B} \in \mathcal{Z}(\mathfrak{B})$ .

Let  $\mathfrak{B}_i, 1 \leq i \leq 4$ , be the subgroups of  $\mathfrak{B}$  of order 3. Let  $\mathfrak{N}_i = N(\mathfrak{B}_i)$ , let  $\mathfrak{D}_i = \mathfrak{B}^{\mathfrak{N}_i}$  and let  $\mathfrak{C}_i = C_{\mathfrak{N}_i}(\mathfrak{D}_i)$ . Since  $3 \in \pi_4$  and  $\mathfrak{B} \subseteq \mathfrak{N}_i$ , we have  $O_3(\mathfrak{N}_i) = 1$ . Hence, by Lemma 5.10,  $\mathfrak{D}_i$  is 3-reducible in  $\mathfrak{N}_i$ . Finally, let  $\mathfrak{E}_i = \mathfrak{N}_i/\mathfrak{C}_i$ . Thus,  $\mathfrak{E}_i$  may be identified with a subgroup of  $\text{Aut}(\mathfrak{D}_i)$ ,  $\mathfrak{E}_i \cong A_{\mathfrak{N}_i}(\mathfrak{D}_i)$ , and as such  $\mathfrak{E}_i$  is a 3-solvable group with no nontrivial normal 3-subgroups. We let  $\mathfrak{R}_i = O^3(\mathfrak{E}_i)$ , so that  $\mathfrak{R}_i$  is that subgroup of  $\mathfrak{E}_i$  generated by the 3-elements of  $\mathfrak{E}_i$ .

The following lemma is important.

LEMMA 8.5. *Suppose for some  $i, 1 \leq i \leq 4$ ,  $\mathfrak{R}_i$  contains an element of order 3 which centralizes a subgroup of  $\mathfrak{D}_i$  of index 3. Then*

- (a)  $|\mathfrak{D}_i| = 27$ .
- (b)  $\mathfrak{R}_i \cong SL(2, 3)$ .
- (c)  $\mathfrak{D}_i = \mathfrak{B}_i \times \mathfrak{C}_i$ , where  $\mathfrak{C}_i \triangleleft \mathfrak{N}_i$ .
- (d)  $\mathfrak{R}_i$  is faithfully and irreducibly represented on  $\mathfrak{C}_i$ .

*Proof.* Let  $\mathfrak{U}$  be the set of 3-elements of  $\mathfrak{N}_i$  which centralize some subgroup of index 3 in  $\mathfrak{D}_i$ . Since  $\mathfrak{D}_i \triangleleft \mathfrak{N}_i$ ,  $\mathfrak{U}$  is an invariant subset of  $\mathfrak{N}_i$ . By hypothesis,  $\mathfrak{U} - \mathfrak{C}_i \neq \emptyset$ .

Let  $\mathfrak{U}^* = \mathfrak{U} \cap \mathfrak{B}$ , and let  $\mathfrak{U}_1 = \langle \mathfrak{U} \mid \mathfrak{U} \in \mathfrak{U}^* \rangle$ . For any subset  $\mathfrak{H}$  of  $\mathfrak{N}_i$ , let  $\bar{\mathfrak{H}} = \mathfrak{H}\mathfrak{C}_i/\mathfrak{C}_i$ . Since  $\mathfrak{E}_i$  is 3-reduced, so is  $\mathfrak{R}_i$ . Furthermore, if  $U \in \mathfrak{U} - \mathfrak{C}_i$ , then  $\bar{U}$  is an exceptional element in the sense of Hall-Higman [26, p. 10], or as we might say, an exceptional element, being the identity on a hyperplane of  $\mathfrak{D}_i$ . (In a perhaps more frequently used terminology,  $\bar{U}$  is a transvection.)

Let  $\mathfrak{H} = O_3(\mathfrak{N}_i \text{ mod } \mathfrak{C}_i)$ , so that  $\bar{\mathfrak{U}}_1$  is faithfully represented on  $\bar{\mathfrak{H}}$ .

By (B),  $\bar{U}_1$  centralizes some  $S_2$ -subgroup of  $\bar{\mathfrak{G}}$ . Let  $\bar{\mathfrak{K}} = [\bar{\mathfrak{G}}, \bar{U}_1]$ . Since  $\bar{\mathfrak{K}}$  is solvable, it follows that  $\bar{\mathfrak{K}}$  is a  $\bar{\mathfrak{G}}$ -invariant 2-group on which  $U_1$  is faithfully represented, and that  $\bar{\mathfrak{K}} = [\bar{\mathfrak{K}}, \bar{U}_1]$ . By Lemma 5.17,  $\bar{\mathfrak{K}}$  is special, since by (B),  $\bar{U}_1$  centralizes every abelian  $\bar{U}_1$ -invariant subgroup of  $\bar{\mathfrak{K}}$ .

It may now be verified that if  $U \in U^* - \mathfrak{C}_i$  and  $\mathfrak{B} = \langle U \rangle$ , then  $[\bar{\mathfrak{K}}, \bar{\mathfrak{B}}]$  is a quaternion group and  $[\bar{\mathfrak{K}}, \bar{\mathfrak{B}}]$  centralizes a subgroup  $\mathfrak{F}$  of index 9 in  $\mathfrak{G}_i$ . Furthermore,  $\mathfrak{D}_i = \mathfrak{C}_i \times \mathfrak{F}$ , where  $\mathfrak{C}_i = [\mathfrak{K}, \mathfrak{B}, \mathfrak{D}_i]$  is of order 9, and  $\mathfrak{C}_{i0} = [\mathfrak{C}_i, \mathfrak{B}]$  is of order 3. Since  $\mathfrak{C}_i$  and  $\mathfrak{F}$  are  $U$ -invariant, and since  $U$  centralizes some hyperplane of  $\mathfrak{D}_i$ , it follows that  $C_{\mathfrak{D}_i}(U) = \mathfrak{C}_{i0} \times \mathfrak{F}$ .

Let  $\bar{\mathfrak{P}}_1 = \bar{\mathfrak{P}} \cap C(\bar{\mathfrak{K}})$  and let  $\bar{\mathfrak{D}} = \bar{\mathfrak{P}}\bar{\mathfrak{K}}$ . Since  $\bar{\mathfrak{D}}$  is faithfully represented on  $\mathfrak{D}_i$ , so is its subgroup  $\bar{\mathfrak{P}}_1\bar{\mathfrak{K}} = \bar{\mathfrak{P}}_1 \times \bar{\mathfrak{K}}$ . Hence, by Lemma 3.7 of [20],  $\bar{\mathfrak{K}}$  is faithfully represented on  $\bar{\mathfrak{D}}_i = C_{\mathfrak{D}_i}(\bar{\mathfrak{P}}_1)$ . Since  $\bar{\mathfrak{P}}/\bar{\mathfrak{P}}_1$  is faithfully represented on  $\bar{\mathfrak{K}}$ , it follows that  $\bar{\mathfrak{D}}/\bar{\mathfrak{P}}_1$  is faithfully represented on  $\bar{\mathfrak{D}}_i$ . By Lemma 7.7,  $\bar{\mathfrak{P}}/\bar{\mathfrak{P}}_1$  centralizes  $\bar{\mathfrak{K}}'\bar{\mathfrak{P}}_1/\bar{\mathfrak{P}}_1$ . Since  $\bar{\mathfrak{K}}\bar{\mathfrak{P}}_1 = \bar{\mathfrak{K}} \times \bar{\mathfrak{P}}_1$ , it follows that  $\bar{\mathfrak{P}}$  centralizes  $\bar{\mathfrak{K}}'$ .

Since  $\bar{U}_1$  is faithfully represented on  $\bar{\mathfrak{K}}/\bar{\mathfrak{K}}'$ , and since each element  $\bar{U}^*$  centralizes a subgroup of  $\bar{\mathfrak{K}}/\bar{\mathfrak{K}}'$  of index 4, it is straightforward to verify that  $\bar{U}_1$  is elementary. It then follows that every element of  $\bar{U}_1^*$  is exceptional (though we don't contend that every element of  $\bar{U}_1$  centralizes a hyperplane of  $\mathfrak{D}_i$ ).

The preceding paragraph, together with  $[\bar{\mathfrak{K}}', \bar{\mathfrak{P}}] = 1$  and Corollary 2 of § 2.6 of [24] imply that  $\bar{U}_1 \subseteq Z(\bar{\mathfrak{P}})$ . Returning to  $\mathfrak{B}$ , we see that  $[\bar{\mathfrak{K}}, \bar{\mathfrak{B}}]$  is  $\bar{\mathfrak{P}}$ -invariant. This in turn implies that  $\mathfrak{F}$  is  $\mathfrak{P}$ -invariant. If  $|\mathfrak{F}| \geq 9$ , then  $\mathfrak{F}$  contains an element of  $\mathcal{U}^*(3)$  and Lemma 7.4 is violated. Hence,  $|\mathfrak{F}| < 9$ . Since  $\mathfrak{B}_i \subseteq \mathfrak{F}$ , we see that (a) and (c) hold. By construction, (b) and (d) follow. The proof is complete.

$J$  denotes the subset of  $\{1, 2, 3, 4\}$  whose elements satisfy the hypothesis of Lemma 8.5.

LEMMA 8.6. *Let  $i \in J$  and let  $\mathfrak{A}$  be a subgroup of  $\mathfrak{C}_i$  of order 3. Let  $\mathfrak{N} = N(\mathfrak{A})$ , let  $\bar{\mathfrak{N}}$  be the normal closure of  $\mathfrak{B}_i$  in  $\mathfrak{N}$  and  $\bar{\mathfrak{N}}_1$  be the normal closure of  $\mathfrak{D}_i$  in  $\mathfrak{N}$ . Then*

- (a)  $[\bar{\mathfrak{N}}, \bar{\mathfrak{N}}_1] = 1$ .
- (b)  $[\bar{\mathfrak{N}}_1, \bar{\mathfrak{N}}_1] \subseteq \mathfrak{A}$ .

*Proof.* Let  $\mathfrak{A}^* = \mathfrak{A} \times \mathfrak{B}_i$ . Since the subgroups of  $\mathfrak{C}_i$  of order 3 are permuted transitively in  $\mathfrak{N}_i$ , it follows that  $C_{\mathfrak{N}_i}(\mathfrak{A}^*)$  contains a  $S_3$ -subgroup  $\mathfrak{P}^*$  of  $\mathfrak{G}$ . Thus,  $\mathfrak{B}_i$  is contained in the center of a  $S_3$ -subgroup of  $\mathfrak{N}$ , namely  $\mathfrak{P}^*$ . By Lemma 5.10,  $\bar{\mathfrak{N}}$  is 3-reducible in  $\mathfrak{N}$ . Since  $\mathfrak{P}^* \subseteq C_{\mathfrak{G}}(\mathfrak{A})$  and  $3 \in \pi_i$ , we have  $O_3(C_{\mathfrak{G}}(\mathfrak{A})) = 1$ , which implies that  $O_3(C_{\mathfrak{G}}(\mathfrak{A})/\mathfrak{A}) = 1$ . Since  $\mathfrak{D}_i/\mathfrak{A} \subseteq Z(\mathfrak{P}^*/\mathfrak{A})$ , we conclude that

$$\mathfrak{D}_i/\mathfrak{A} \cong O_3(C_{\mathfrak{G}}(\mathfrak{A})/\mathfrak{A}) \cong O_3(\mathfrak{N}/\mathfrak{A}),$$

and so  $\mathfrak{D}_i \subseteq C_{\mathfrak{N}}(\tilde{\mathfrak{N}})$ , by 3-reducibility of  $\tilde{\mathfrak{N}}$  in  $\mathfrak{N}$ . Since  $C_{\mathfrak{N}}(\tilde{\mathfrak{N}}) \triangleleft \mathfrak{N}$ , we have  $[\tilde{\mathfrak{N}}, \tilde{\mathfrak{N}}_i] = 1$ . Since  $\mathfrak{D}_i/\mathfrak{A} \subseteq Z(O_3(\mathfrak{N}/\mathfrak{A}))$ , we also have  $[\tilde{\mathfrak{N}}_i, \tilde{\mathfrak{N}}_i] \subseteq \mathfrak{A}$ .

We now set  $\mathfrak{D} = \langle \mathfrak{D}_1, \mathfrak{D}_2, \mathfrak{D}_3, \mathfrak{D}_4 \rangle$ . Since  $\mathfrak{B} \subseteq Z(\mathfrak{B})$ , it is clear that  $\mathfrak{B} \subseteq Z(\mathfrak{D})$  and that  $\mathfrak{D} \triangleleft \mathfrak{B}$ .

HYPOTHESIS 8.1. (i)

$$\begin{aligned} \mathfrak{B} &\subseteq \mathfrak{M} \in \mathcal{MS}(\mathfrak{G}), \mathfrak{B} = \Omega_1(Z(\mathfrak{B}))^{\mathfrak{M}}, \\ \mathfrak{C} &= C_{\mathfrak{M}}(\mathfrak{B}), \mathfrak{B}^* = V(ccl_{\mathfrak{G}}(\mathfrak{D}); \mathfrak{B}). \end{aligned}$$

(ii)  $\mathfrak{B}^* \not\subseteq \mathfrak{C}$ .

The long argument to follow is carried out under Hypothesis 8.1.

Choose  $G$  in  $\mathfrak{G}$  so that  $\mathfrak{D}^G \subseteq \mathfrak{B}$  but  $\mathfrak{D}^G \not\subseteq \mathfrak{C}$ . The element  $G$  plays a passive but important role. If  $\mathfrak{H}$  is any subset of  $\mathfrak{G}$ , we set  $\mathfrak{H}^{\circ} = \mathfrak{H}^G$ , while if  $\mathfrak{H}$  is any subset of  $\mathfrak{M}$ , we set  $\tilde{\mathfrak{H}} = \mathfrak{H}\mathfrak{C}/\mathfrak{C}$ .

Let  $\mathfrak{R}$  be any subgroup of  $O_3(\overline{\mathfrak{M}})$  which admits  $\mathfrak{D}^{\circ}$  and is minimal subject to  $[\overline{\mathfrak{D}}^{\circ}, \mathfrak{R}] \neq 1$ . (Notice that  $\mathfrak{R}$  is available.) Let  $N = N(\mathfrak{R}) = \{i \mid 1 \leq i \leq 4, \overline{\mathfrak{D}}_i \text{ does not centralize } C_{\mathfrak{R}}(\overline{\mathfrak{B}}_i)\}$ . We argue that  $N \neq \emptyset$ . This is clear if  $\overline{\mathfrak{B}}^{\circ}$  centralizes  $\mathfrak{R}$ , so we may assume that  $[\overline{\mathfrak{B}}^{\circ}, \mathfrak{R}] \neq 1$ . Since  $\mathfrak{B}^{\circ}$  is noncyclic, it follows that  $\mathfrak{R} = \langle \mathfrak{R} \cap C(\mathfrak{B}_i) \mid 1 \leq i \leq 4 \rangle$ , so we can choose  $i$  such that  $\overline{\mathfrak{D}}^{\circ}$  does not centralize  $C_{\mathfrak{R}}(\overline{\mathfrak{B}}_i)$ . Minimality of  $\mathfrak{R}$  guarantees that  $\mathfrak{R} = C_{\mathfrak{R}}(\overline{\mathfrak{B}}_i)$ . Thus,  $\mathfrak{B}^{\circ}$  does not centralize  $C_{\mathfrak{R}}(\overline{\mathfrak{B}}_i)$ . Since  $\mathfrak{B}^{\circ} \subseteq \mathfrak{D}_i$ , we have  $i \in N(\mathfrak{R})$ . In the following discussion,  $\mathfrak{R}$  is a fixed subgroup of  $O_3(\overline{\mathfrak{M}})$  which admits  $\mathfrak{D}^{\circ}$  and is minimal subject to  $[\overline{\mathfrak{D}}^{\circ}, \mathfrak{R}] \neq 1$ , and  $j$  is a fixed element of  $N(\mathfrak{R})$ . As already observed,  $\overline{\mathfrak{B}}_j$  centralizes  $\mathfrak{R}$ .

Let  $\tilde{\mathfrak{Q}}$  be a  $\mathfrak{D}_j$ -subgroup of  $\mathfrak{R}$  minimal subject to  $[\overline{\mathfrak{D}}_j, \tilde{\mathfrak{Q}}] \neq 1$ . Let  $\mathfrak{D}_0 = \ker(\mathfrak{D}_j \rightarrow \text{Aut}(\tilde{\mathfrak{Q}}))$ , so that  $|\mathfrak{D}_j : \mathfrak{D}_0| = 3$  and  $\mathfrak{B}_j \subseteq \mathfrak{D}_0$ .

Since  $\tilde{\mathfrak{Q}}$  is faithfully represented on  $\mathfrak{B}$ , Lemma 3.7 of [18] implies that  $\tilde{\mathfrak{Q}}$  faithfully represented on  $C_{\mathfrak{B}}(\mathfrak{D}_0)$ . Since  $\mathfrak{D}_j$  does not centralize  $C_{\mathfrak{B}}(\mathfrak{D}_0)$ , we may choose  $V$  in  $\mathfrak{D}_{\mathfrak{B}}(\mathfrak{D}_0) - C(\mathfrak{D}_j)$ . Then

$$V \in C(\mathfrak{B}_j) \subseteq N(\mathfrak{B}_j) = \mathfrak{R}_j.$$

Thus,  $GVG^{-1}$  is a 3-element of  $\mathfrak{R}_j - \mathfrak{C}_j$  which centralizes a subgroup of  $\mathfrak{D}_j$  of index 3. By Lemma 8.5,

$$(8.1) \quad j \in J, |\mathfrak{D}_j| = 27, \dots$$

This implies that  $|C_{\mathfrak{B}}(\mathfrak{D}_0) : C_{\mathfrak{B}}(\mathfrak{D}_j)| = 3$ , which in turn implies that  $\tilde{\mathfrak{Q}}$  is a quaternion group.

Since  $\tilde{\mathfrak{Q}}$  is a quaternion group, the following assertions hold:

(a)  $\mathfrak{D}$  centralizes a  $S_2$ -subgroup of  $O_3(\overline{\mathfrak{M}})$ .

(b)  $\mathfrak{D}$  centralizes every abelian subgroup of  $O_3(\overline{\mathfrak{M}})$  which  $\mathfrak{D}$  normalizes.

(c) If  $\overline{\mathfrak{F}}$  is the normal closure in  $\mathfrak{P}$  of  $\mathfrak{D}$ , then  $[\overline{\mathfrak{F}}, O_3(\overline{\mathfrak{M}})] = \overline{\mathfrak{E}}$  is a special 2-group whose derived group is centralized by  $\overline{\mathfrak{F}}$ . Namely, if either (a) or (b) were false, we could find  $\mathfrak{R}$  such that  $\mathfrak{R}$  contains no quaternion group. Since this is not the case, (a) and (b) hold. Now (c) follows from Lemma 5.17, together with the solvability of  $O_3(\overline{\mathfrak{M}})$ . We retain the previous notation and continue the argument.

Let  $\mathfrak{A} = [C_{\mathfrak{B}}(\mathfrak{D}_i), \mathfrak{D}_i]$ . Thus,  $\mathfrak{A}$  is a subgroup of  $\mathfrak{E}_i$  of order 3.

Let  $\mathfrak{B} = \mathfrak{B}_0 \times \mathfrak{B}_1$ , where  $\mathfrak{B}_0 = C_{\mathfrak{B}}(\tilde{\mathfrak{Q}})$ ,  $\mathfrak{B}_1 = [\mathfrak{B}, \tilde{\mathfrak{Q}}]$ . Since  $\mathfrak{A} \subseteq \mathfrak{B}$ , we have  $\mathfrak{B} \subseteq N(\mathfrak{A}) = \mathfrak{A}$ , so that  $[\mathfrak{B}, \mathfrak{B}_i] \subseteq \mathfrak{B}_i^{\mathfrak{A}}$ . By Lemma 8.6,  $[\mathfrak{B}, \mathfrak{B}_i, \mathfrak{D}_i] = 1$ . This implies that  $\tilde{\mathfrak{Q}}$  centralizes  $[\mathfrak{B}, \mathfrak{B}_i]$ , which in turn implies that  $[\mathfrak{B}, \mathfrak{B}_i] \subseteq \mathfrak{B}_0$ . Hence,  $\mathfrak{B}_i$  centralizes  $\mathfrak{B}_1$ . Hence,  $\mathfrak{D}_i$  centralizes  $[\mathfrak{B}_1, \mathfrak{D}_i]$ . As  $\tilde{\mathfrak{Q}}$  normalizes  $[\mathfrak{B}_1, \mathfrak{D}_i]$ , it follows that  $\tilde{\mathfrak{Q}}$  centralizes  $[\mathfrak{B}_1, \mathfrak{D}_i]$ . By definition of  $\mathfrak{B}_1$ , we get  $[\mathfrak{B}_1, \mathfrak{D}_i] = 1$ . Hence,  $[\mathfrak{B}_1, \mathfrak{D}_i] = \mathfrak{A}$  and  $|\mathfrak{B}_1| = 9$ .

Suppose  $\overline{\mathfrak{F}}$  centralizes  $\overline{\mathfrak{E}}$ . Since  $\tilde{\mathfrak{Q}} \subseteq \overline{\mathfrak{E}}$ , it follows that  $\overline{\mathfrak{F}}$  centralizes  $\tilde{\mathfrak{Q}}$ , a group of order 2. Hence,  $\overline{\mathfrak{F}}$  normalizes  $\mathfrak{B}_0 = C_{\mathfrak{B}}(\tilde{\mathfrak{Q}})$ . Since the inverse image of  $\tilde{\mathfrak{Q}}$  in  $\mathfrak{M}$  contains an involution, it follows that  $\mathfrak{B}_0$  contains no element of  $\mathcal{U}^*(3)$ . But  $\mathfrak{B}_0 \triangleleft \mathfrak{P}$ , so the only possibility is that  $\mathfrak{B}_0$  is cyclic. Since  $Z(\mathfrak{P})$  is non cyclic, we get  $|\mathfrak{B}_0| = 3$ .

Suppose  $\overline{\mathfrak{F}}$  does not centralize  $\overline{\mathfrak{E}}$ . Let  $\overline{\mathfrak{E}}_1 = [\overline{\mathfrak{E}}, \overline{\mathfrak{F}}]$ , and let  $\mathfrak{B}$  be a subgroup of  $\mathfrak{B}$  which admits  $\overline{\mathfrak{F}}\overline{\mathfrak{E}}$  and is minimal subject to  $[\overline{\mathfrak{E}}_1, \mathfrak{B}] \neq 1$ . Since  $\overline{\mathfrak{F}}$  centralizes  $\overline{\mathfrak{E}}$ , it follows that  $\overline{\mathfrak{F}}$  centralizes  $\mathfrak{B}$ ; so  $\mathfrak{D}$  centralizes  $\mathfrak{B}$ . Hence,  $\mathfrak{B} \subseteq \mathfrak{B}_0 \times \mathfrak{A}$ . By Lemma 4.4 of [19],  $\mathfrak{B}$  contains a subgroup  $\mathfrak{B}_0$  of order 27 such that  $\mathfrak{B}_0 \triangleleft \mathfrak{P}$ ,  $|\mathfrak{P} : C_{\mathfrak{P}}(\mathfrak{B}_0)| = 3$ . Since  $|\mathfrak{A}| = 3$ , it follows that  $\mathfrak{B}_0 \cap \mathfrak{A}$  is noncyclic. Let  $\mathfrak{B}_1$  be a subgroup of  $\mathfrak{B}_0 \cap \mathfrak{A}$  of order 9. Since  $|\mathfrak{P}|$  is clearly larger than  $3^4$ , we conclude from Lemma 7.6 (d) that  $\mathfrak{B}_1 \in \mathcal{E}(3)$ . Let  $I$  be an involution in the inverse image of  $\tilde{\mathfrak{Q}}$  in  $\mathfrak{M}$ ; then  $I$  centralizes  $\mathfrak{B}_0$ , so centralizes  $\mathfrak{B}_1$ . Hence, by Lemma 7.4,  $C(I)$  is nonsolvable. This contradiction shows that  $[\overline{\mathfrak{F}}, \overline{\mathfrak{E}}] = 1$ . Hence,  $|\mathfrak{B}_0| = 3$ , an important equality.

Since  $\mathfrak{M} \in \mathcal{MS}(G)$ , it follows that  $\mathfrak{M} = N(\mathfrak{B}_0)$ , so that  $\mathfrak{M} = \mathfrak{N}_i$  for some  $i$ ,  $1 \leq i \leq 4$ . Thus,  $i \in J$ ,  $\mathfrak{B}_0 = \mathfrak{B}_i$ ,  $\mathfrak{B}_1 = \mathfrak{E}_i$ ,  $\mathfrak{B} = \mathfrak{D}_i$ ,  $\mathfrak{E} = \mathfrak{E}_i$ .

Let  $\mathfrak{P}_0 = \mathfrak{P} \cap \mathfrak{E}_i$ ,  $\mathfrak{N}_0 = N_{\mathfrak{M}_i}(\mathfrak{P}_0)$ , so that  $\mathfrak{N}_0\mathfrak{E}_i = \mathfrak{N}_i$ . Let  $\mathfrak{D}_0$  be a  $S_2$ -subgroup of  $\mathfrak{N}_0$  permutable with  $\mathfrak{P}$ . Let  $\mathfrak{S} = \mathfrak{P}\mathfrak{D}_0 \cap C(\mathfrak{B}_i)$ , and set  $\mathfrak{Q} = \mathfrak{S} \cap \mathfrak{D}_0$ . Then  $\mathfrak{S} = \mathfrak{P}\mathfrak{Q}$ .

Since  $i \in J$ , Lemma 8.5 implies that a  $S_{2,3}$ -subgroup of  $\mathfrak{N}_i/\mathfrak{E}_i$  is not 3-closed. Since  $\mathfrak{P}\mathfrak{D}_0$  is not 3-closed, neither is  $\mathfrak{S}$ , since  $|\mathfrak{P}\mathfrak{D}_0 : \mathfrak{S}| \leq 2$ . Let  $I$  be an involution of  $\mathfrak{Q}$ . Suppose  $\mathfrak{D}_i \cap C(I) \supset \mathfrak{B}_i$ . Then  $\mathfrak{D}_i \cap C(I)$  contains a subgroup  $\tilde{\mathfrak{D}}$  of order 9 with  $\tilde{\mathfrak{D}} \supset \mathfrak{B}_i$ . Since  $\mathfrak{S}$  permutes transitively the subgroups of  $\mathfrak{E}_i$  of order 3, it follows that  $\tilde{\mathfrak{D}}$  is central



in some  $S_3$ -subgroup of  $\mathfrak{C}$ , that is,  $I$  centralizes an element of  $\mathcal{U}(3)$ . This is not the case, since  $C(I)$  contains an element of  $\mathcal{U}(2)$ . This contradiction forces  $\mathfrak{D}_i \cap C(I) = \mathfrak{B}_i$  for all involutions  $I$  of  $\mathfrak{Q}$ . Since  $\mathfrak{B} \not\triangleleft \mathfrak{C}$ ,  $\mathfrak{Q}$  is not cyclic. Thus,  $\mathfrak{Q}$  is a quaternion group. Also,  $\mathfrak{F}_0 = \mathcal{O}_3(\mathfrak{C}) = \mathfrak{C} \cap \mathfrak{C}_i$  and  $\mathfrak{C}/\mathfrak{F}_0 \cong SL(2, 3)$ . In particular,  $\mathfrak{B}_j \subseteq \mathfrak{F}_0$ , while  $\mathfrak{D}_j \not\subseteq \mathfrak{F}_0$ .

Since  $j \in J$ , it follows that  $[C_{\mathfrak{F}_0}(\mathfrak{B}_j), \mathfrak{D}_j] \subseteq \mathfrak{C}_j$ . Since  $\mathfrak{B}_j \subseteq \mathfrak{F}_0$  and  $\mathfrak{D}_j \not\subseteq \mathfrak{F}_0$ , it follows that  $\mathfrak{C}_j \not\subseteq \mathfrak{F}_0$ . Hence,  $\mathfrak{C}_j \cap \mathfrak{F}_0 = \mathfrak{A}$ . This implies that

$$(8.2) \quad [C_{\mathfrak{F}_0}(\mathfrak{B}_j), \mathfrak{D}_j] = \mathfrak{A}.$$

For any subset  $\mathfrak{I}$  of  $\mathfrak{C}$ , let  $\bar{\mathfrak{I}} = \mathfrak{I}\mathfrak{C}_i/\mathfrak{C}_i$ . It is important to show that

$$(8.3) \quad C_{\bar{\mathfrak{F}}_0}(\bar{\mathfrak{B}}_j) = C_{\mathfrak{F}_0}(\mathfrak{B}_j)/\mathfrak{C}_i.$$

Namely, suppose  $P$  in  $\mathfrak{F}_0$  satisfies  $[\mathfrak{B}_j, P] \subseteq \mathfrak{C}_i$ . Now  $\mathfrak{C}_i = \mathfrak{A} \times \mathfrak{A}^*$ , where  $\mathfrak{A}$  and  $\mathfrak{A}^*$  are of order 3 and  $\mathfrak{A} \subseteq \mathfrak{C}_j$ . We may apply Lemma 8.6 to  $\mathfrak{A}$ . Since  $P \in \mathfrak{A}$ , we get  $[\mathfrak{B}_j, P, \mathfrak{D}_j] = 1$ . Hence,  $[\mathfrak{B}_j, P] \subseteq \mathfrak{C}_i \cap C(\mathfrak{D}_j) = \mathfrak{A}$ . Consider the group  $\mathfrak{B}_j \times \mathfrak{A}$ , which is normalized by the 3-element  $P$ . Since  $\mathfrak{N}_j$  permutes transitively the subgroups of  $\mathfrak{C}_j$  of order 3, it follows that  $\mathfrak{B}_j \times \mathfrak{A}$  is in the center of some  $S_3$ -subgroup of  $\mathfrak{C}$ . Hence,  $A_{\mathfrak{G}}(\mathfrak{B}_j \times \mathfrak{A})$  is a 3'-group, so  $P$  centralizes  $\mathfrak{B}_j \times \mathfrak{A}$ . We have proved (8.3).

Since  $\mathfrak{A} \subseteq \mathfrak{C}_i$ , so also  $\mathfrak{F}_0 \subseteq \mathfrak{A}$ . Hence

$$(8.4) \quad [\mathfrak{F}_0, \mathfrak{D}_j, \mathfrak{D}_j] \subseteq \mathfrak{A},$$

by Lemma 8.6 (b) applied to  $\mathfrak{A}$ . We will use this fact several times.

We next show that

$$(8.5) \quad C_{\mathfrak{F}_0}(\mathfrak{Q}') = C_{\mathfrak{F}_0}(\mathfrak{Q}).$$

Since  $\mathfrak{Q}\mathfrak{C}_i$  is a Frobenius group, it suffices to show that  $C_{\bar{\mathfrak{F}}_0}(\mathfrak{Q}') = C_{\bar{\mathfrak{F}}_0}(\mathfrak{Q})$ . Let  $\mathcal{C}$  be part of a chief series of  $\mathfrak{C}$  from  $\mathfrak{F}_0$  to 1, one of whose terms is  $\mathfrak{C}_i$ . If  $\mathfrak{F}$  is a chief factor of  $\mathcal{C}$ , it suffices to show that if  $\mathfrak{Q}'$  centralizes  $\mathfrak{F}$ , so does  $\mathfrak{Q}$ . If this were not the case, then elements of  $\mathfrak{D}_j - \mathfrak{F}_0$  would have minimal polynomial  $(x - 1)^3$  on  $\mathfrak{F}$ , against (8.4). Thus, (8.5) holds.

Suppose  $\mathfrak{Q}'$  centralizes  $\bar{\mathfrak{F}}_0$ . Let  $\mathfrak{F}_1 = C_{\mathfrak{F}_0}(\mathfrak{Q}')$ . We get  $\mathfrak{F}_0 = \mathfrak{F}_1\mathfrak{C}_i$ ,  $\mathfrak{F}_1 \cap \mathfrak{C}_i = 1$ . Since  $\mathfrak{C}_i$  is an irreducible  $\mathfrak{Q}$ -module, we have  $\mathfrak{F}_0 = \mathfrak{F}_1 \times \mathfrak{C}_i$ . If  $\mathfrak{F}_1$  is not cyclic, then  $\mathfrak{Q}'$  centralizes an element of  $\mathcal{U}^*(3)$ , which is not the case, since  $\mathfrak{Q}'$  centralizes an element of  $\mathcal{U}(2)$ . Thus,  $\mathfrak{F}_1$  is cyclic. Clearly,  $\mathfrak{F}_1 \neq 1$ , since  $\mathfrak{B}_j \subseteq \mathfrak{F}_1$ . If

$$|\mathfrak{F}_1| > 3, \text{ then } \mathcal{O}^1(\mathfrak{F}_1) \triangleleft \langle \mathfrak{N}_i, \mathfrak{N}_j \rangle,$$

while it is trivial that  $\langle \mathfrak{N}_i, \mathfrak{N}_j \rangle$  is non solvable. Hence,  $\mathfrak{F}_1 = \mathfrak{B}_j$ . But then Lemma 7.8 is violated.

Let  $\mathfrak{P}_2 = [\mathfrak{P}_0, \Omega]$ . By the preceding paragraph,

$$(8.6) \quad \overline{\mathfrak{P}}_2 \neq 1 .$$

We will show that

$$(8.7) \quad \overline{\mathfrak{P}}_2 \text{ is of exponent 3 and class at most 2 .}$$

Let  $\mathfrak{R}_1 = [\mathfrak{P}_2, \mathfrak{D}_j]$ . Since  $\mathfrak{P}_2 \subseteq \mathfrak{R}^*$ , Lemma 8.6 (b) implies that  $[\mathfrak{R}_1, \mathfrak{R}_1] \subseteq \mathfrak{R}^*$ , so that  $\overline{\mathfrak{R}}_1$  is abelian. Since  $\mathfrak{D}_j$  is elementary so is  $\overline{\mathfrak{R}}_1$ . Thus  $\overline{\mathfrak{R}}_1$  is a normal elementary subgroup of  $\overline{\mathfrak{P}}_2$ . Let  $Q$  be an element of  $\Omega$  of order 4, and set  $\mathfrak{R}_2 = \mathfrak{R}_1^Q$ . We argue that  $\overline{\mathfrak{R}}_1 \overline{\mathfrak{R}}_2 = \overline{\mathfrak{P}}_2$ . To see this, observe that since  $\Omega'$  inverts  $\mathfrak{P}_2/D(\mathfrak{P}_2)$ , and since the minimal polynomial of each element of  $\mathfrak{D}_j$  on  $\overline{\mathfrak{P}}_2/D(\overline{\mathfrak{P}}_2)$  is a divisor of  $(x - 1)^2$ , it follows that  $\mathfrak{R}_1, \mathfrak{R}_2$  map onto subspaces of  $\overline{\mathfrak{P}}_2/D(\overline{\mathfrak{P}}_2)$  which generate  $\overline{\mathfrak{P}}_2/D(\overline{\mathfrak{P}}_2)$ , so our assertion follows. Since  $\overline{\mathfrak{R}}_1, \overline{\mathfrak{R}}_2$  are normal elementary subgroups of  $\overline{\mathfrak{P}}_2$ , (8.7) holds. Since we now have  $D(\overline{\mathfrak{P}}_2) = [\overline{\mathfrak{R}}_1, \overline{\mathfrak{R}}_2] \subseteq \overline{\mathfrak{R}}_1$ , and since  $\overline{\mathfrak{D}}_j$  centralizes  $\overline{\mathfrak{R}}_1$ , it follows that

$$(8.8) \quad \Omega \text{ centralizes } D(\overline{\mathfrak{P}}_2) .$$

Since  $\Omega$  has no fixed points on  $\overline{\mathfrak{P}}_2/D(\overline{\mathfrak{P}}_2)$ , it follows from (8.4) that

$$(8.9) \quad \begin{array}{l} \Omega \text{ operates on } \overline{\mathfrak{P}}_2/D(\overline{\mathfrak{P}}_2) \text{ as a multiple } d \text{ of the} \\ \text{faithful irreducible } \Omega\text{-representation .} \end{array}$$

In particular,

$$(8.10) \quad |\overline{\mathfrak{P}}_2 : D(\overline{\mathfrak{P}}_2)| = 3^{2d} .$$

Let  $B$  a generator for  $\mathfrak{P}_j$ , and for any element  $S$  of  $\mathfrak{S}$ , let  $S]$  be the mapping of  $\mathfrak{P}_2$  into itself which sends  $P$  to  $[P, S]$ . We may view  $B]$  in more than one way. Since  $\Omega$  centralizes  $\mathfrak{P}_0/\mathfrak{P}_2$ , we have  $B = CU$ , where  $C \in C_{\mathfrak{P}_0}(\Omega)$  and  $U \in \mathfrak{P}_2$ . Since  $U \in \mathfrak{P}_2$ ,  $B]$  and  $C]$  induce the same mapping from  $\mathfrak{P}_2/D(\mathfrak{P}_2)$  to itself. In particular,  $[\mathfrak{P}_2, B]D(\mathfrak{P}_2)$  admits  $\Omega$ . By Lemma 8.6 applied to  $\mathfrak{R}^*$ , we have  $[\mathfrak{P}_2, B, \mathfrak{D}_j] = 1$ . This implies that  $\Omega$  centralizes  $[\mathfrak{P}_2, B]D(\mathfrak{P}_2)/D(\mathfrak{P}_2)$ , so by construction of  $\mathfrak{P}_2$ , we have

$$(8.11) \quad [\mathfrak{P}_2, B] \subseteq D(\mathfrak{P}_2) .$$

Hence,  $[\mathfrak{P}_2, C] \subseteq D(\mathfrak{P}_2)$ . Since  $C$  centralizes  $\Omega$  and  $\Omega$  centralizes  $D(\overline{\mathfrak{P}}_2)$ , we conclude that  $C$  centralizes  $\overline{\mathfrak{P}}_2$ , by the three subgroups lemma. Hence,  $B$  and  $U$  induce the same automorphism of  $\overline{\mathfrak{P}}_2$ .

By Lemma 8.6 applied to  $\mathfrak{R}^*$ ,  $B$  centralizes the normal closure of  $\mathfrak{D}_j$  in  $\mathfrak{R}^*$ . Hence,  $C_{\mathfrak{P}_2}(B) \cong [\mathfrak{P}_2, \mathfrak{D}_j]\mathfrak{C}_i\mathfrak{P}'_2$ . We will show that

$$(8.12) \quad C_{\mathfrak{P}_2}(B) = [\mathfrak{P}_2, \mathfrak{D}_j]\mathfrak{C}_i\mathfrak{P}'_2 ,$$

$$(8.13) \quad |\overline{C_{\mathfrak{P}_2}(B)} : D(\overline{\mathfrak{P}}_2)| = 3^d .$$

Let  $\mathfrak{X}_1$  be the set of fixed points of  $\mathfrak{D}_j$  on  $\overline{\mathfrak{P}}_2/D(\overline{\mathfrak{P}}_2)$ , let

$$\mathfrak{X}_2 = [\overline{\mathfrak{D}_j}, \overline{\mathfrak{P}}_2]D(\overline{\mathfrak{P}}_2)/D(\overline{\mathfrak{P}}_2), \text{ and let } W_3 = \overline{C_{\mathfrak{P}_2}(B)}/D(\overline{\mathfrak{P}}_2).$$

From (8.2) and (8.3), we get that  $\mathfrak{X}_3 \subseteq \mathfrak{X}_1$ . By Lemma 8.6 (a) applied to  $\mathfrak{N}$ , we get  $\mathfrak{X}_2 \subseteq \mathfrak{X}_3$ . By (8.10) and Lemma 5.2, it follows that  $|\mathfrak{X}_1| \leq 3^d$ . Using (8.10) once again, we get  $|\mathfrak{X}_2| \geq 3^d$ . Since  $\mathfrak{X}_2 \subseteq \mathfrak{X}_3 \subseteq \mathfrak{X}_1$ , it follows that  $\mathfrak{X}_1 = \mathfrak{X}_2 = \mathfrak{X}_3$  is of order  $3^d$ . This yields (8.12) and (8.13).

Let  $\mathfrak{B}^* = \mathfrak{B}_j^{\mathfrak{P}}$ . Since  $\mathfrak{P}$  centralizes  $\mathfrak{U}$ , we have  $\mathfrak{B}^* \subseteq \mathfrak{B}_j^{\mathfrak{U}}$ . By Lemma 8.6,  $\mathfrak{B}^*$  and  $\mathfrak{D}_j^{\mathfrak{P}}$  commute elementwise. Set  $\mathfrak{P}_3 = [\mathfrak{P}_2, \mathfrak{B}^*]\mathfrak{C}_i \triangleleft \mathfrak{P}$ . Since  $B$  centralizes  $\overline{\mathfrak{P}}_2/D(\overline{\mathfrak{P}}_2)$ , (8.8) implies that  $\Omega$  normalizes  $\mathfrak{P}_3$ . Thus,  $\mathfrak{P}_3 \triangleleft \mathfrak{C}$ . Since  $\mathfrak{D}_j^{\mathfrak{P}}$  and  $\mathfrak{B}^*$  commute elementwise,  $[\mathfrak{P}_2, \mathfrak{D}_j]$  centralizes  $\mathfrak{P}_3$ . Hence,  $[\mathfrak{P}_2, \mathfrak{D}_j]^q$  centralizes  $\mathfrak{P}_3^q = \mathfrak{P}_3$ ,  $Q$  being an element of  $\Omega - \Omega'$ . Since  $\mathfrak{P}_2 = [\mathfrak{P}_2, \mathfrak{D}_j][\mathfrak{P}_2, \mathfrak{D}_j]^q$ , it follows that  $\mathfrak{P}_2$  centralizes  $\mathfrak{P}_3$ .

Let  $\tilde{\mathfrak{P}}_3 = \mathfrak{P}_3 \cap C(\Omega)$ . Thus,  $\mathfrak{P}_3 = \tilde{\mathfrak{P}}_3 \times \mathfrak{C}_i$ . Clearly,  $N_{\mathfrak{C}}(\Omega)$  normalizes  $\tilde{\mathfrak{P}}_3$ ; so does  $\mathfrak{P}_2$  since  $\mathfrak{P}_2$  centralizes  $\mathfrak{P}_3$ . Since  $\mathfrak{C} = \mathfrak{P}_2 N_{\mathfrak{C}}(\Omega)$ , we have  $\tilde{\mathfrak{P}}_3 \triangleleft \mathfrak{C}$ . Since  $\Omega$  contains an involution, no subgroup of  $\tilde{\mathfrak{P}}_3$  is in  $\mathcal{Z}^*(\mathfrak{P})$ . Hence,  $\tilde{\mathfrak{P}}_3$  is cyclic. Since  $\mathfrak{P}_3$  is isomorphic to a subgroup of  $\overline{\mathfrak{P}}_2$ , it follows that  $\tilde{\mathfrak{P}}_3$  is of order 1 or 3.

Suppose  $[\mathfrak{P}_2, B] \subseteq \mathfrak{C}_i$ . Then (8.3) forces  $[\mathfrak{P}_2, B] = 1$ . This violates (8.6), (8.10), (8.13). Thus,  $\tilde{\mathfrak{P}}_3$  is of order 3 and  $\mathfrak{P}_3 = [\mathfrak{P}_2, B]\mathfrak{C}_i$ , and  $[\mathfrak{P}_2, B]$  is of order 3. Now (8.10) and (8.13) yield that  $d = 1$ .

Suppose  $D(\overline{\mathfrak{P}}_2) = 1$ . Then by (8.11),  $B$  centralizes  $\overline{\mathfrak{P}}_2$ . This conflicts with (8.10) and (8.13). Hence,  $D(\overline{\mathfrak{P}}_2) \neq 1$ , so that

$$(8.14) \quad |\mathfrak{P}_2| = 3^5.$$

Since  $\mathfrak{B}_i \tilde{\mathfrak{P}}_3$  is a normal subgroup of  $\mathfrak{C}$  centralized by  $\Omega$ , we get  $\mathfrak{B}_i = \tilde{\mathfrak{P}}_3$ , as  $\Omega$  centralizes no element of  $\mathcal{Z}^*(3)$ . Hence,

$$(8.15) \quad Z(\mathfrak{P}_2) = \mathfrak{D}_i.$$

Since  $\mathfrak{P}_2$  is the normal closure of  $[\mathfrak{P}_2, \mathfrak{D}_j]$  in  $\mathfrak{C}$ , and since  $\mathfrak{D}_j^{\mathfrak{U}}$  is of exponent 3, it follows that  $\mathfrak{P}_2$  is generated by elements of order 3. Since  $\mathfrak{P}_2$  is of class 2, it follows that

$$(8.16) \quad \mathfrak{P}_2 \text{ is of exponent } 3.$$

Since  $\mathfrak{B}_i \subset \mathfrak{P}_2$ , the group  $\mathfrak{P}_2/\mathfrak{B}_i$  is of order  $3^4$  and is inverted by the involution of  $\Omega$ . Hence,  $\mathfrak{P}'_2 \subseteq \mathfrak{B}_i$ . Since  $\mathfrak{P}_2$  is non abelian it follows that

$$(8.17) \quad \mathfrak{P}'_2 = \mathfrak{B}_i.$$

We next show that  $B \in \mathfrak{P}_2$ . Namely,  $\mathfrak{B} = CU$ , so that  $[C, U] = [C, CU] = [C, B]$ . As we have already seen,  $C$  centralizes  $\overline{\mathfrak{P}}_2$ , that is,  $[C, U] \in \mathfrak{C}_i$ . Since  $[C, U] = [C, B]$ , (8.3) implies that  $[C, B] = 1$ . Since

we have also shown that  $[\mathfrak{F}_2, B]$  has order  $3^d = 3$ , it follows that  $U$  is not in  $Z(\mathfrak{F}_2)$ . Since  $\mathfrak{F}_2/\mathfrak{F}'_2$  is an irreducible  $\Omega$ -module, it follows that  $C$  centralizes  $\mathfrak{F}_2$ , as  $C$  centralizes an element of  $\mathfrak{F}_2$  (namely,  $U$ ) which does not map into  $\mathfrak{F}'_2$ . Since  $C$  and  $U$  commute, and since  $B$  and  $U$  have order 1 or 3, it follows that  $C$  has order 1 or 3. If  $C \notin \mathfrak{F}_2$ , then  $\Omega_1(C_{\mathfrak{F}_0}(\mathfrak{F}_2)) \cap C(\Omega)$  is noncyclic, so that  $\Omega$  centralizes an element of  $\mathcal{U}^*(3)$ . Since this is not the case, we conclude that

$$(8.18) \quad B \in \mathfrak{F}_2 .$$

We will next show that  $\mathfrak{F}_0 = \mathfrak{F}_2$ .

Since  $B \in \mathfrak{F}_2$ ,  $[\mathfrak{F}_0, B]$  is a subgroup of  $\mathfrak{F}_2$  centralized by  $\mathfrak{D}_j$ . Suppose  $[\mathfrak{F}_0, B] \not\subseteq \mathfrak{F}'_2 (= \mathfrak{F}_i)$ . Let  $B_i$  be a generator for  $\mathfrak{F}_i$ ,  $A$  a generator for  $\mathfrak{X}$ . Choose  $P$  in  $\mathfrak{F}_0$  so that  $[P, B] = B_i^a A^b$  with  $b \neq 0$ . Clearly,  $a \neq 0$ , since  $A_{\mathfrak{G}}(\langle B, A \rangle)$  is a 3'-group. Since  $[\mathfrak{F}_2, B] = \mathfrak{F}'_2 = \mathfrak{F}_i$ , we may choose  $P_2$  in  $\mathfrak{F}_2$  so that  $[P_2, B] = B_i^{-a}$ . Then  $[PP_2, B] = A^b$ , which is impossible. Hence,  $[\mathfrak{F}_0, B] = \mathfrak{F}'_2$ . Hence,  $[\mathfrak{F}_1, \mathfrak{F}_2] = 1$ , by the three subgroups lemma. Here  $\mathfrak{F}_1 = \mathfrak{F}_0 \cap C(\Omega)$ . Hence,  $\mathfrak{F}_1 \triangleleft \mathfrak{E}$ , so  $\mathfrak{F}_1$  is cyclic, as  $\Omega$  centralizes no element of  $\mathcal{U}^*(3)$ . If  $|\mathfrak{F}_1| > 3$ , it is easy to verify that  $\mathcal{U}^1(\mathfrak{F}_1) \triangleleft \langle \mathfrak{N}_i, \mathfrak{N}_j \rangle$  against the nonsolvability of  $\langle \mathfrak{N}_i, \mathfrak{N}_j \rangle$ . Hence,  $\mathfrak{F}_1$  is of order 3, so that  $\mathfrak{F}_1 = \mathfrak{F}_i$ . Hence,

$$(8.19) \quad \mathfrak{F}_0 = \mathfrak{F}_2 \text{ is of order } 3^5 ,$$

$$(8.20) \quad \mathfrak{F} \text{ is of order } 3^6 .$$

With the preceding information at our disposal, we turn our attention to  $\mathfrak{M} = \mathfrak{N}_i$  once again. Let  $\tilde{\mathfrak{E}}$  be a  $S_{(2,3)'}-$ subgroup of  $\mathfrak{M}$  permutable with  $\mathfrak{F}$ . Then  $\tilde{\mathfrak{E}}$  centralizes  $\mathfrak{D}_i$ , for otherwise  $2^3 \cdot 3 \cdot 13$  divides  $A_{\mathfrak{M}}(\mathfrak{D}_i)$ , forcing nonsolvability of  $A_{\mathfrak{M}}(\mathfrak{D}_i)$ . Since  $|O_2(\mathfrak{M}) : \mathfrak{D}_i| \leq 9$ ,  $\tilde{\mathfrak{E}}$  also centralizes  $O_3(\mathfrak{M})/\mathfrak{D}_i$ . Hence,  $\tilde{\mathfrak{E}}$  centralizes  $O_3(\mathfrak{M})$ , or equivalently,

$$(8.21) \quad \mathfrak{M} \text{ is a } 2, 3\text{-group} .$$

Let  $\mathfrak{T}$  be a  $S_2$ -subgroup of  $\mathfrak{N}_i$  containing  $\Omega$ . Since  $\mathfrak{E}_i \in \mathcal{U}^*(\mathfrak{F})$ , no element of  $\mathfrak{T}^\#$  centralizes  $\mathfrak{E}_i$ . Thus,

$$(8.22) \quad \mathfrak{F}_0 = C(\mathfrak{E}_i) .$$

Since  $\mathfrak{F}_i \subseteq \mathfrak{F}$ , it follows that  $C(\mathfrak{F}) \subseteq \mathfrak{N}_i = \mathfrak{M}$ . By Lemma 7.4,  $|C(\mathfrak{F})|$  is odd. From (8.21) we conclude that

$$(8.23) \quad \mathfrak{F} = C(\mathfrak{F}) .$$

By construction,  $\mathfrak{D}_i \subseteq \mathfrak{F}_0$ . Suppose  $j_0$  is an index such that  $\mathfrak{D}_{j_0} \not\subseteq \mathfrak{F}_0$ . Then  $[\mathfrak{D}_i, \mathfrak{D}_{j_0}] \neq 1$ . In this case, two applications of Lemma 8.5 imply that  $|\mathfrak{D}_i| = |\mathfrak{D}_{j_0}| = 27$  and that  $[\mathfrak{F}, \mathfrak{D}_{j_0}]$  is of order 3. Hence,  $[\mathfrak{F}, \mathfrak{D}_{j_0}] =$

$[\mathfrak{D}_i, \mathfrak{D}_{j_0}]$  so that  $\mathfrak{D}_{j_0}$  centralizes  $\mathfrak{F}_0/\mathfrak{D}_i$ . This contradicts (8.6) and (8.19). Hence, no such  $j_0$  exists, that is,

$$(8.24) \quad \mathfrak{D} \subseteq \mathfrak{F}_0 .$$

Our previous information shows that  $Z(\mathfrak{F}) = \mathfrak{B}$ . Hence,  $N(\mathfrak{F})$  normalizes  $\mathfrak{B}$ , so permutes the groups  $\mathfrak{B}^{N(\mathfrak{B}_k)}$  among themselves,  $1 \leq k \leq 4$ . By definition of  $\mathfrak{D}$ , we get

$$(8.25) \quad N(\mathfrak{F}) \subseteq N(\mathfrak{D}) .$$

Suppose  $\mathfrak{D} \triangleleft \mathfrak{F}$ . Since  $\mathfrak{D} \triangleleft \mathfrak{F}$ , it follows that  $\mathfrak{D} \triangleleft \langle \mathfrak{F}, \mathfrak{F} \rangle$ . We can choose  $N$  in  $N(\mathfrak{D})$  so that  $\mathfrak{F}^N = \mathfrak{F}$ . Hence,  $\mathfrak{D}^N = \mathfrak{D}$  and  $\mathfrak{F}^{GN} = \mathfrak{F}$ . Let  $H = GN$ . Then  $\mathfrak{D} = \mathfrak{D}^H$  and  $H \in N(\mathfrak{F})$ . By (8.25), we get  $\mathfrak{D} = \mathfrak{D}^H = \mathfrak{D}$ . This conflicts with (8.24), since by construction  $\mathfrak{D} \not\subseteq \mathfrak{F}_0$ . Thus,

$$(8.26) \quad \mathfrak{D} \not\triangleleft \mathfrak{F} .$$

Suppose  $\mathfrak{F}^*$  is a  $S_3$ -subgroup of  $\mathfrak{N}_i$  and that  $\mathfrak{F}^* \subseteq \mathfrak{N}_i$ . Thus,  $Z(\mathfrak{F}^*) \subseteq \mathfrak{D}_i \cap \mathfrak{D}_j = \mathfrak{A}$ . This is impossible since  $|\mathfrak{A}| = 3$ ,  $|Z(\mathfrak{F}^*)| = 9$ . We conclude that

$$(8.27) \quad \mathfrak{N}_i \cap \mathfrak{N}_j \text{ contains no } S_3\text{-subgroup of } \mathfrak{G} .$$

Since  $\mathfrak{D} \not\triangleleft \mathfrak{F}$ , (8.20) implies that  $|\mathfrak{D}| \leq 3^4$ . Suppose  $|\mathfrak{D}| \leq 3^3$ . Then  $\mathfrak{D} \supseteq \mathfrak{D}_i$  implies  $|\mathfrak{D}| = 3^3$  and  $\mathfrak{D} = \mathfrak{D}_i$ . Thus, for each  $k$ ,  $1 \leq k \leq 4$ , we have  $\mathfrak{B} \subseteq \mathfrak{D}_k \subseteq \mathfrak{D}_i$ . If  $\mathfrak{B} = \mathfrak{D}_k$ , then  $\mathfrak{N}_k$  normalizes  $C(\mathfrak{B})$ . By (8.23), we have  $\mathfrak{F} \triangleleft \mathfrak{N}_k$ , so by (8.25), we have  $\mathfrak{N}_k \subseteq N(\mathfrak{D})$ . Thus, if  $|\mathfrak{D}| = 3^3$ , then  $\langle \mathfrak{N}_1, \mathfrak{N}_2, \mathfrak{N}_3, \mathfrak{N}_4 \rangle \subseteq N(\mathfrak{D})$ . Since  $\mathfrak{A} \subseteq Z(\mathfrak{F})$ , it follows that  $\mathfrak{A} = \mathfrak{B}_k$  for some  $k$ . Hence,  $N(\mathfrak{A}) \subseteq \mathfrak{N}_i$ . But  $\mathfrak{A}$  is a subgroup of  $\mathfrak{D}_j$ , so there is a  $S_3$ -subgroup  $\mathfrak{F}^*$  of  $\mathfrak{N}_j$  which contains  $\mathfrak{A}$  in its center. This violates (8.27). Hence,  $|\mathfrak{D}| = 3^4$ . Since  $\mathfrak{F}_0 \supset \mathfrak{D} \supset \mathfrak{D}_i$ , we conclude that

$$(8.28) \quad \mathfrak{D} \text{ is elementary of order } 3^4 .$$

Let  $\mathfrak{C}$  be any subgroup of  $\mathfrak{D}_j$  which is of order 3 and is not contained in  $\mathfrak{F}_0$ . Thus,  $\mathfrak{D}_j = \mathfrak{C} \times \mathfrak{B}_j \times \mathfrak{A}$ . Since  $\mathfrak{D} \subseteq \mathfrak{F}$ , it follows that  $\mathfrak{D} \subseteq C_{\mathfrak{F}}(\mathfrak{D}_j) = \mathfrak{C} \cdot C_{\mathfrak{F}_0}(\mathfrak{D}_j)$ , and as we have already shown,  $C_{\mathfrak{F}_0}(\mathfrak{B}_j) = \mathfrak{D}_i \mathfrak{B}_j$  (that is,  $\mathfrak{B}_j \not\subseteq Z(\mathfrak{F}_0)$ ). Since  $\mathfrak{D}_j$  does not centralize  $\mathfrak{C}$ , it follows that  $C_{\mathfrak{F}_0}(\mathfrak{D}_j) = \mathfrak{B}_j \times \mathfrak{B}_i \times \mathfrak{A}$ . Since  $\mathfrak{D} \not\triangleleft \mathfrak{F}$ , it follows that  $N_{\mathfrak{F}}(\mathfrak{D}) = \mathfrak{D} \cdot \mathfrak{C}$ . Choose  $P$  in  $\mathfrak{F}_0 - N_{\mathfrak{F}}(\mathfrak{D})$ . Since

$$[P, \mathfrak{B}_j, \mathfrak{B}_i \mathfrak{A}] \subseteq \mathfrak{B}_i \subseteq \mathfrak{D} ,$$

it follows that  $[P, C] \in \mathfrak{D}$ , where  $C$  is a generator for  $\mathfrak{C}$ . Hence,  $[P, C] = DE_i$  with  $E_i$  in  $\mathfrak{C} - \mathfrak{A}$  and  $D$  in  $\mathfrak{D} \cap \mathfrak{F}_0$ . Hence,  $[P, C, C] = [DE_i, C] = [E_i, C]$  is a generator for  $\mathfrak{A}$ . This is a subtle and important

bit of information, since it shows that the  $\mathbb{C}$  module  $\mathfrak{P}_0/\mathfrak{P}'_0$  has an indecomposable constituent of dimension 3. Thus,

$$(8.29) \quad \begin{array}{l} \text{the indecomposable direct factors of } \mathfrak{P}_0/\mathfrak{P}'_0 \\ \text{as } \mathbb{C}\text{-modules are of dimensions 1 and 3.} \end{array}$$

We note

$$(8.30) \quad \mathfrak{P} = V(\text{ecl}_{\mathbb{C}}(\mathfrak{D}); \mathfrak{P}) .$$

Namely,  $\mathfrak{Q}$  does not normalize  $\mathfrak{D}$ ,  $\mathfrak{D}^{\mathfrak{Q}} = \mathfrak{P}_0$ . Since  $\mathfrak{P} = \langle \mathfrak{P}_0, \mathfrak{D} \rangle$ , (8.30) holds. This fact has an important consequence. Namely, if  $\mathfrak{P} \subseteq \mathfrak{M}^* \in \mathcal{MS}(\mathbb{G})$  and  $\mathfrak{P} \not\triangleleft \mathfrak{M}^*$ , then  $\mathfrak{M}^*$  satisfies Hypothesis 8.1. If this were not so, then  $\mathfrak{P} \subseteq C_{\mathfrak{M}^*}(\Omega_1(\mathbf{Z}(\mathfrak{P})))^{\mathfrak{M}^*}$ . Now (8.23) implies that  $\mathfrak{P} \triangleleft \mathfrak{M}^*$ . Thus,

$$(8.31) \quad \begin{array}{l} \text{if } \mathfrak{P} \subseteq \mathfrak{M}^* \in \mathcal{MS}(\mathbb{G}), \text{ then either } \mathfrak{P} \triangleleft \mathfrak{M}^* \\ \text{or } \mathfrak{M}^* \text{ satisfies Hypothesis 8.1.} \end{array}$$

Let  $\tilde{\mathfrak{M}}$  be an element of  $\mathcal{MS}(\mathbb{G})$  which contains  $N(\mathfrak{A}^{\cdot})$  and let  $\mathfrak{P}_0 = \mathcal{O}_3(\tilde{\mathfrak{M}})$ . We argue that

$$(8.32) \quad \mathfrak{P} \not\triangleleft N(\mathfrak{A}^{\cdot}) .$$

Namely,  $\mathfrak{P} \subseteq N(\mathfrak{A}^{\cdot})$ . Also,  $N(\mathfrak{A}^{\cdot})$  contains a  $S_3$ -subgroup of  $\mathfrak{R}_i$ . By (8.27), this implies that  $N(\mathfrak{A}^{\cdot})$  has more than one  $S_3$ -subgroup so (8.32) holds. By (8.31), it follows that  $|\mathfrak{P}_0| = 3^3$  and that  $\tilde{\mathfrak{M}} = N(\mathfrak{X})$ , where  $\mathfrak{X}$  is some subgroup of  $\mathbf{Z}(\mathfrak{P})$  of order 3. Clearly,  $\mathfrak{X} \neq \mathfrak{P}_i$ , since  $N(\mathfrak{A}^{\cdot}) \not\subseteq \tilde{\mathfrak{M}}$ . On the other hand, if  $I$  is the involution of  $\mathfrak{D}$ , then  $I \in \tilde{\mathfrak{M}}$ . Since  $\mathfrak{A}^{\cdot}$  and  $\mathfrak{P}_i$  are the only subgroups of  $\mathbf{Z}(\mathfrak{P})$  of order 3 which are normalized by  $I$ , it follows that  $\mathfrak{X} = \mathfrak{A}^{\cdot}$ .

Since  $\mathfrak{M} \neq \tilde{\mathfrak{M}}$ , so also  $\mathfrak{P}_0 \neq \tilde{\mathfrak{P}}_0$ . Hence, (8.20) implies that  $\mathfrak{P}_0 \cap \tilde{\mathfrak{P}}_0$  is of order  $3^4$ . Now (8.24) implies that  $\mathfrak{D} \subseteq \mathfrak{P}_0 \cap \tilde{\mathfrak{P}}_0$ , so by (8.28), we have

$$(8.33) \quad \mathfrak{D} = \mathfrak{P}_0 \cap \tilde{\mathfrak{P}}_0 .$$

Since  $\tilde{\mathfrak{M}} = N(\mathfrak{A}^{\cdot})$ , we have  $\mathfrak{D}^{\cdot} \subseteq \mathcal{O}_3(\tilde{\mathfrak{M}})$ . Hence,

$$(8.34) \quad \tilde{\mathfrak{P}}_0 = \langle \mathfrak{D}, \mathfrak{D}^{\cdot} \rangle .$$

Since  $I$  inverts  $\mathfrak{A}^{\cdot}$ , we have  $I \in \tilde{\mathfrak{M}}$ . Let  $\tilde{\mathfrak{Q}}$  be a  $S_2$ -subgroup of  $\mathcal{O}^{3'}(\tilde{\mathfrak{M}})$  which is normalized by  $I$ . Thus,  $\tilde{\mathfrak{Q}}$  is a quaternion group, and by (8.22) (with  $\tilde{\mathfrak{M}}$  in the role of  $\mathfrak{M}$ ), we get that

$$(8.35) \quad \tilde{\mathfrak{Q}}\langle I \rangle \text{ is a } S_2\text{-subgroup of } \tilde{\mathfrak{M}} .$$

Let  $J$  be the involution of  $\tilde{\mathfrak{Q}}$  and set

$$(8.36) \quad \mathfrak{G} = \langle I, J \rangle .$$

Thus,  $\mathfrak{G}$  is a four-group and  $\mathfrak{G} \subseteq N(\mathfrak{P})$ . Since  $\mathfrak{B}_i = Z(\mathfrak{P}) \cap C(I)$ , it follows that  $\mathfrak{G} \subseteq \mathfrak{M}$ . Hence,

$$(8.37) \quad \mathfrak{P} \triangleleft \mathfrak{P}\mathfrak{G} = \mathfrak{M} \cap \tilde{\mathfrak{M}}.$$

Notice that by (8.22), we have  $\mathfrak{M} = \mathfrak{E}\langle J \rangle$ . Consider  $C_{\mathfrak{M}}(I) = C_{\mathfrak{E}}(I)\langle J \rangle$ . By (8.5), it follows that  $\mathfrak{Q} \triangleleft C_{\mathfrak{E}}(I)$ . Thus,  $J$  normalizes  $\mathfrak{Q}$  so that

$$(8.38) \quad \mathfrak{Q}\langle J \rangle \text{ is a } S_2\text{-subgroup of } \mathfrak{M}.$$

Let  $Q$  be an element of  $\mathfrak{Q}$  of order 4 which normalizes  $\mathfrak{G}$  and let  $\tilde{Q}$  be an element of  $\tilde{\mathfrak{Q}}$  of order 4 which normalizes  $\mathfrak{G}$ . By (8.22), it follows that  $N_{\mathfrak{M}}(\mathfrak{Q})/N_{\mathfrak{P}_0}(\mathfrak{Q}) \cong GL(2, 3)$ . Similarly for  $\tilde{\mathfrak{M}}$ . Hence, neither  $Q$  nor  $\tilde{Q}$  centralizes  $\mathfrak{G}$ , that is,

$$(8.39) \quad \langle Q, \mathfrak{G} \rangle \text{ and } \langle \tilde{Q}, \mathfrak{G} \rangle \text{ are dihedral groups of order } 8.$$

We set

$$(8.40) \quad I_1 = JQ, \quad I_2 = IQ.$$

Thus,  $I_1$  and  $I_2$  are involutions and

$$(8.41) \quad \begin{aligned} I_1 J I_1 &= J I, & I_1 I I_1 &= I, \\ I_2 J I_2 &= J, & I_2 I I_2 &= I J. \end{aligned}$$

Finally, we get

$$(8.42) \quad \mathfrak{P}_0 = \langle I_1, I_2 \rangle \subseteq N(\mathfrak{G}).$$

We next show that  $\mathfrak{P}_0$  is complemented in  $\mathfrak{M}$ . It is clear from the structure of  $\mathfrak{M} = \mathfrak{N}_i$  that  $C_{\mathfrak{M}}(I)$  covers  $\mathfrak{N}_i/\mathfrak{C}_i = \mathfrak{N}_i/\mathfrak{P}_0$  and that  $C_{\mathfrak{M}}(I) \cap \mathfrak{P}_0 = \mathfrak{B}_0$ . Hence  $\mathfrak{M}$  will split over  $\mathfrak{P}_0$  if  $C_{\mathfrak{M}}(I)$  splits over  $\mathfrak{B}_i$ . Since  $\mathfrak{B}_i$  is an abelian 3-group, this occurs if and only if a  $S_3$ -subgroup of  $C_{\mathfrak{M}}(I)$  splits over  $\mathfrak{B}_i$ , hence, if and only if  $C_{\mathfrak{P}}(I)$  is elementary of order  $3^2$ . Regarding  $I$  as an element of  $\tilde{\mathfrak{M}}$ , we know from the structure of this group that  $C_{\mathfrak{P}}(I) = C_{\tilde{\mathfrak{P}}_0}(I)$ . But  $\tilde{\mathfrak{P}}_0$  has exponent 3, by (8.16) and (8.19). Since the structure of  $\mathfrak{M}$  implies that  $|C_{\mathfrak{P}}(I)| = 3^2$ , we have proved that

$$(8.43) \quad \mathfrak{M} \text{ splits over } \mathfrak{P}_0; \tilde{\mathfrak{M}} \text{ splits over } \tilde{\mathfrak{P}}_0.$$

We define

$$(8.44) \quad \mathfrak{x}_6 = \mathfrak{U}, \mathfrak{x}_5 = \mathfrak{B}_i, \mathfrak{x}_4 = \mathfrak{U}^q, \mathfrak{x}_3 = \tilde{\mathfrak{x}}_3^q.$$

Since  $\langle \mathfrak{x}_4, \mathfrak{x}_5, \mathfrak{x}_6 \rangle = \mathfrak{B}^{\mathfrak{M}}$  and  $\langle \mathfrak{x}_3, \mathfrak{x}_5, \mathfrak{x}_6 \rangle = \mathfrak{B}^{\tilde{\mathfrak{M}}}$ , (8.28) implies that

$$(8.45) \quad \mathfrak{D} = \langle \mathfrak{x}_3, \mathfrak{x}_4, \mathfrak{x}_5, \mathfrak{x}_6 \rangle.$$

We set

$$(8.46) \quad \mathfrak{x}_1 = \mathfrak{x}_3^q, \mathfrak{x}_2 = \mathfrak{x}_4^{\tilde{q}}.$$

It then follows that

$$(8.47) \quad \begin{aligned} &\mathfrak{x}_{i+1} \cdots \mathfrak{x}_6 \text{ is a subgroup of } \mathfrak{P} \text{ of} \\ &\text{order } 3^{6-i}, i = 0, \dots, 5. \end{aligned}$$

Further, by construction,

$$(8.48) \quad \mathfrak{S} \text{ normalizes } \mathfrak{x}_i, 1 \leq i \leq 6.$$

We now set up a 6 by 2 array whose  $(i, j)$  entry is  $\mathfrak{x}_i^{j'}$ , in case  $\mathfrak{x}_i^{j'} \subseteq \mathfrak{P}$ , and is - otherwise.

$$(8.49) \quad \begin{array}{c|cc} & I_1 & I_2 \\ \hline \mathfrak{x}_1 & \mathfrak{x}_3 & - \\ \mathfrak{x}_2 & - & \mathfrak{x}_4 \\ \mathfrak{x}_3 & \mathfrak{x}_1 & \mathfrak{x}_5 \\ \mathfrak{x}_4 & \mathfrak{x}_6 & \mathfrak{x}_2 \\ \mathfrak{x}_5 & \mathfrak{x}_5 & \mathfrak{x}_3 \\ \mathfrak{x}_6 & \mathfrak{x}_4 & \mathfrak{x}_6 \end{array} .$$

We will eventually determine  $\mathfrak{M}$  and  $\tilde{\mathfrak{M}}$  in terms of generators and relations. To do this, a number of choices must be made, and some care is required to guarantee that these choices are possible. We have already chosen the groups  $\mathfrak{x}_i, 1 \leq i \leq 6$ , each of order 3 and each normalized by  $\mathfrak{S}$ . Since  $\mathfrak{x}_1 \not\subseteq \tilde{\mathfrak{P}}_0$ , and since  $\mathfrak{x}_1$  centralizes  $J$ , we have  $\mathfrak{x}_1 \subseteq C_{\mathfrak{M}}(J) = \tilde{\mathfrak{Q}}\langle I \rangle \mathfrak{x}_1 \mathfrak{x}_6$ . Hence,  $\mathfrak{x}_1$  normalizes  $\tilde{\mathfrak{Q}}$ . We therefore may choose a generator  $X_1$  of  $\mathfrak{x}_1$  such that  $X_1 \tilde{Q}$  has order 3. Namely, let  $X_1$  be any generator for  $\mathfrak{x}_1$ . Then  $(X_1 \tilde{Q})^3 \in \langle J \rangle$ , so either  $(X_1 \tilde{Q})^3 = 1$  or  $(X_1 \tilde{Q})^3 = J$ . Since  $\tilde{Q}J = \tilde{Q}^{-1}$ , in the second case we get  $(X_1 \tilde{Q}^{-1})^3 = 1$ , or equivalently,  $(\tilde{Q} X_1^{-1})^3 = 1$ , or equivalently,  $(X_1^{-1} \tilde{Q})^3 = 1$ . Thus, we may assume that

$$(8.50) \quad (X_1 \tilde{Q})^3 = 1.$$

For the same reason, we may choose a generator  $X_2$  for  $\mathfrak{x}_2$  such that

$$(8.51) \quad (X_2 Q)^3 = 1.$$

We set  $X_3 = X_1^q, X_4 = X_2^{\tilde{q}}, X_5 = X_3^{\tilde{q}}, X_6 = X_4^q$ . Notice that

$$(8.52) \quad \langle X_i \rangle = \mathfrak{x}_i, 1 \leq i \leq 6.$$

It is now convenient to draw up a table listing the action of  $\mathfrak{S}$  on each  $\mathfrak{x}_i$ . This information is available since we know the action





$ab \neq 0$ . It follows from (8.29) that  $[X_1, X_2, X_2]$  does not lie in  $\mathfrak{X}_5$ , so  $d \neq 0$ . By symmetry,  $c \neq 0$ . To determine the values  $a$  through  $f$  explicitly, we make use of the following identities:

$$\begin{aligned} [AB, C] &= [A, C][A, C, B][B, C] \\ [A, BC] &= [A, C][A, B][A, B, C] \\ [A^{-1}, C] &= [A, C, A^{-1}]^{-1}[A, C]^{-1} \\ [A, B^{-1}] &= [A, B, B^{-1}]^{-1}[A, B]^{-1} \\ [A^{-1}, B^{-1}] &= [A, B^{-1}, A^{-1}]^{-1}[A, B^{-1}]^{-1}. \end{aligned}$$

Since  $X_2Q$  has order 3, we have

$$Q^{-1}X_2Q = IQX_2Q = IX_2^{-1}Q^{-1}X_2^{-1} = X_2^{-1}QX_2^{-1}.$$

Using this relation, conjugate (8.59) by  $Q$ , to obtain

$$[X_3, X_2^{-1}QX_2^{-1}] = X_1^{-c}X_6^dX_5^eX_4^{-f}.$$

Since  $X_2$  and  $X_3$  commute, we have

$$[X_3, X_2^{-1}QX_2^{-1}] = [X_3, QX_2^{-1}].$$

By the preceding identities,  $[X_3, QX_2^{-1}] = [X_3, Q][X_3, Q, X_2^{-1}]$ . Now

$$[X_3, Q] = X_3^{-1}Q^{-1}X_3Q = X_3^{-1}X_1^{-1},$$

so that

$$[X_3, QX_2^{-1}] = X_3^{-1}X_1^{-1}[X_3^{-1}X_1^{-1}, X_2^{-1}].$$

Since  $\mathfrak{X}_2$  and  $\mathfrak{X}_3$  commute, we have  $[X_3^{-1}X_1^{-1}, X_2^{-1}] = [X_1^{-1}, X_2^{-1}]$ . Now by the preceding identities, we have

$$\begin{aligned} [X_1^{-1}, X_2^{-1}] &= [X_1, X_2^{-1}, X_1^{-1}]^{-1}[X_1, X_2^{-1}]^{-1} \\ &= [[X_1, X_2, X_2^{-1}]^{-1}[X_1, X_2]^{-1}, X_1^{-1}]^{-1} \\ &\quad \times ([X_1, X_2, X_2^{-1}]^{-1}[X_1, X_2]^{-1})^{-1}. \end{aligned}$$

Since  $[X_1, X_2, X_2^{-1}] \in \mathfrak{C}_i$ , it follows that

$$[[X_1, X_2, X_2^{-1}]^{-1}[X_1, X_2]^{-1}, X_1^{-1}]^{-1} = [[X_1, X_2]^{-1}, X_1^{-1}]^{-1}.$$

We get that  $[X_1^{-1}, X_2^{-1}] = X_5^{ac}X_3^cX_4^dX_5^eX_6^{f+bd}$ . Since  $X_3^{-1}X_1^{-1} = X_1^{-1}X_3^{-1}X_5^{-a}$ , we see that  $[X_3, X_2^{-1}QX_2^{-1}] = X_1^{-1}X_3^{-1}X_5^{-a+ac}X_3^cX_4^dX_5^eX_6^{f+bd}$ . This gives us the following equations:  $c = 1, d = -f, f + bd = d$ . Conjugating (8.59) successively by  $I, J, IJ$  and using the fact that  $d \neq 0$  yield the values  $b = -1, a = e, d = -f$ . No more information is forthcoming from  $\mathfrak{M}$ , so we conjugate (8.59) by  $\tilde{Q}$  and work in  $\tilde{\mathfrak{M}}$ . We state the result of these calculations:

$$(8.60) \quad a = -1, b = -1, c = 1, d = -1, e = -1, f = 1 .$$

Let  $\mathfrak{R}^* = \langle \mathfrak{X}_2, \mathfrak{X}_5 \rangle$  and note that  $\mathfrak{R}^* = C_{\mathfrak{F}}(I)$ . By construction,  $\mathfrak{X}_2 \subseteq O_3(\tilde{\mathfrak{M}})$  and  $\mathfrak{X}_3 \subseteq Z(O_3(\tilde{\mathfrak{M}}))$ . Hence,

$$C_{\mathfrak{F}}(\mathfrak{X}_2) = C_{\mathfrak{F}}(\mathfrak{R}^*) = \langle X_2, X_3, X_5, X_6 \rangle ,$$

so that  $|\mathfrak{F}:C_{\mathfrak{F}}(\mathfrak{R}^*)| = 9$ . With  $\mathfrak{R}^*$  in the role of  $\mathfrak{A}$  in Lemma 7.6 (c), it follows that  $\mathfrak{R}^*$  centralizes every abelian subgroup of  $\mathfrak{N}(\mathfrak{R}^*; 2)$ .

Since  $O^3(\mathfrak{M}) \cap C(I) = \mathfrak{R}^*\mathfrak{Q}$ , it follows that  $\mathfrak{Q}$  is normalized by  $\mathfrak{R}^*$  but is not centralized by  $\mathfrak{R}^*$ . Let  $\mathfrak{X}^*$  be a  $S_{2,3}$ -subgroup of  $C(I)$  which contains  $\mathfrak{R}^*\mathfrak{Q}$ . Then  $\mathfrak{X}^*$  contains an element  $\mathfrak{Z}(2)$ . Let  $\mathfrak{X}_2$  be a  $S_2$ -subgroup of  $\mathfrak{X}^*$  which contains  $\mathfrak{Q}$ . By Lemma 7.5, there is an element  $\mathfrak{M}_1$  of  $\mathcal{MS}(\mathfrak{G})$  such that  $(\mathfrak{R}^*, \mathfrak{X}^*, \mathfrak{X}_2, \mathfrak{R} = O_2(\mathfrak{M}_1), \mathfrak{M}_1)$  satisfies all parts of Lemma 7.5 with  $\mathfrak{R}^*$  in the role of  $\mathfrak{B}$ ,  $\mathfrak{R}$  in the role of  $\mathfrak{S}$ ,  $\mathfrak{M}_1$  in the role of  $\mathfrak{M}$ . Since by (e) of Lemma 7.5,  $\mathfrak{Q} \subseteq \mathfrak{R}$ , it follows that  $\mathfrak{M}_1 = C(I)$ . Hence,  $J \in \mathfrak{M}_1$ .

The next task is shown that

$$(8.61) \quad N(\mathfrak{D}) \subseteq N(\mathfrak{F}) .$$

By our preceding results,  $\mathfrak{D} \triangleleft \mathfrak{F}$ . It is straightforward to verify that  $N_{\mathfrak{M}}(\mathfrak{D}) \subseteq N_{\mathfrak{M}}(\mathfrak{F})$ . Let  $\mathfrak{M}^* \in \mathcal{MS}(\mathfrak{G})$  with  $N(\mathfrak{D}) \subseteq \mathfrak{M}^*$ . If  $\mathfrak{F} \triangleleft \mathfrak{M}^*$ , we have our desired containment. Otherwise,  $\mathfrak{M}^*$  satisfies Hypothesis 8.1. Hence,  $N(\mathfrak{D}) = N_{\mathfrak{M}^*}(\mathfrak{D}) \subseteq N_{\mathfrak{M}^*}(\mathfrak{F}) \subseteq N(\mathfrak{F})$ , as desired.

We next show that

$$(8.62) \quad \text{if } X \in \mathfrak{X}_2^\#, Y \in \mathfrak{X}_5^\#, \text{ then } |C(XY)|_3 = 3^4 .$$

Let  $Z = XY$ . Let  $X^* = X^{\bar{q}}, Y^* = Y^{\bar{q}}, Z^* = X^*Y^*$ . Then  $X^* \in \mathfrak{X}_4, Y^* \in \mathfrak{X}_3$ , so it suffices to show that  $\mathfrak{D}$  is a  $S_3$ -subgroup of  $C(Z^*)$ . Suppose false. Let  $\tilde{\mathfrak{D}}$  be a  $S_3$ -subgroup of  $C(\mathfrak{X}^*)$  which contains  $\mathfrak{D}$ , and let  $\mathfrak{D} \subseteq \mathfrak{D}^* \subseteq \tilde{\mathfrak{D}}$ , with  $|\mathfrak{D}^*:\mathfrak{D}| = 3$ . Then  $\mathfrak{D}^* \subseteq N(\mathfrak{D}) \subseteq N(\mathfrak{F})$ , so  $\mathfrak{D}^* \subseteq \mathfrak{F}$ . However,  $\mathfrak{D} = C_{\mathfrak{F}}(Z^*)$ . Notice that we have shown that  $\mathfrak{D} \triangleleft C(Z^*)$ . Namely,  $\mathfrak{D}$  is a  $S_3$ -subgroup of  $C(Z^*)$ , and since  $\mathfrak{D} \in \mathcal{S}_{3,3}(P)$ , we have  $O_3(C(Z^*)) = 1$  so that

$$(8.63) \quad \mathfrak{D} \text{ is a normal } S_3\text{-subgroup of } C(Z^*) .$$

Retaining the preceding notation we will show that  $\langle I \rangle = C_{\mathfrak{R}}(Z)$ . Suppose false. Since  $\langle I \rangle = C_{\mathfrak{R}}(\mathfrak{R}^*)$ , it follows that  $1 \neq [C_{\mathfrak{R}}(Z), \mathfrak{R}^*]$ . This violates the fact that  $C(Z)$  is 3-closed by (8.63).

We next observe that  $\mathfrak{X}_2 \sim \mathfrak{X}_5$  so that  $|C(\mathfrak{X}_2)|_2 = |C(\mathfrak{X}_5)|_2 = 8$ . These equalities together with the preceding paragraph show that  $\mathfrak{R}$  is extra special of width 2 and that

$$(8.64) \quad \mathfrak{R} \text{ is the central product of quaternion groups } \mathfrak{Q}, \mathfrak{Q}_1, \text{ where } \mathfrak{Q} = C_{\mathfrak{R}}(\mathfrak{X}_5), \mathfrak{Q}_1 = C_{\mathfrak{R}}(\mathfrak{X}_2) .$$

This choice of notation conforms with our previous definition of  $\mathfrak{Q}$ .

Since  $\mathfrak{R}^*$  maps onto a  $S_3$ -subgroup of  $\mathfrak{A}_{\mathfrak{G}}(\mathfrak{R})$ , it follows that  $\mathfrak{R}^*\mathfrak{R} \triangleleft \mathfrak{M}_1$ . Let

$$(8.65) \quad \mathfrak{Z} = N_{\mathfrak{M}_1}(\mathfrak{R}^*) ,$$

so that  $\mathfrak{Z} \cap \mathfrak{R} = \langle I \rangle$ , and  $\mathfrak{M}_1 = \mathfrak{R}\mathfrak{Z}$ .  $\mathfrak{Z}$  acts as a permutation group on the subgroups of  $\mathfrak{R}^*$  of order 3. By the previous arguments,  $\mathfrak{x}_2$  and  $\mathfrak{x}_3$  are permuted among themselves. Let

$$(8.66) \quad \mathfrak{Z}^* = N_{\mathfrak{Z}}(\mathfrak{x}_2) = N_{\mathfrak{Z}}(\mathfrak{x}_3)$$

so that  $|\mathfrak{Z}:\mathfrak{Z}^*| \leq 2$ . Also, if  $L \in \mathfrak{Z}^*$  and  $L$  centralizes  $\mathfrak{x}_3$ , then  $L \in \mathfrak{R}^*\langle I \rangle$ . Hence,  $\mathfrak{Z}^* = \mathfrak{R}^*\mathfrak{Z}$ , and  $|\mathfrak{M}_1:\mathfrak{R}\mathfrak{Z}^*| \leq 2$ . Let  $\mathfrak{Z}_2$  be a  $S_2$ -subgroup of  $\mathfrak{Z}$  which contains  $\mathfrak{Z}$ . Thus,  $|\mathfrak{Z}_2| = 4$  or  $8$ .

We must now show that

$$(8.67) \quad \mathfrak{Z} = \mathfrak{Z}^* .$$

Suppose false. Since  $\mathfrak{R}^{*\tilde{q}} = \langle \mathfrak{x}_3, \mathfrak{x}_4 \rangle$ , it follows that  $\mathfrak{Z}^{\tilde{q}}$  normalizes  $\langle \mathfrak{x}_3, \mathfrak{x}_4 \rangle$ . From (8.63), we conclude that  $\mathfrak{D} \text{ char } C(\mathfrak{x}_3\mathfrak{x}_4)$ . Hence,  $N(\mathfrak{x}_3\mathfrak{x}_4) \subseteq N(\mathfrak{D})$ . Now by (8.61), we have  $N(\mathfrak{D}) \subseteq N(\mathfrak{P})$ . Thus  $\mathfrak{Z}^{\tilde{q}}$  normalizes  $\mathfrak{P}$ .

It is a straightforward consequence of (8.55) through (8.60) that  $\mathfrak{P}_0 \cup \tilde{\mathfrak{P}}_0$  is the set of elements of  $\mathfrak{P}$  of order at most 3. Hence,  $\mathfrak{P}$  contains exactly  $3^6 - 2 \cdot 3^5 + 3^4 = 4 \cdot 3^4$  elements of order 9. Thus, some involution  $I_0$  of  $\mathfrak{Z}_2^{\tilde{q}}$  centralizes an element  $P$  of  $\mathfrak{P}$  of order 9. It is clear from (8.53) that  $I_0 \in \mathfrak{Z}$ .

If  $X \in \mathfrak{x}_3^*\mathfrak{x}_6^*$ , we will show that  $C(X) \subseteq N(\mathfrak{P})$ . Suppose false. Let  $\mathfrak{M}^* \in \mathcal{MS}(\mathfrak{G})$  with  $C(X) \subseteq \mathfrak{M}^*$ . We may apply all the preceding results to  $\mathfrak{M}^*$  in place of  $\mathfrak{M}$  and conclude that  $O_3(\mathfrak{M}^*)$  is of exponent 3 and order  $3^5$ . However,  $\mathfrak{P}_0$  and  $\tilde{\mathfrak{P}}_0$  are the only subgroups of  $\mathfrak{P}$  meeting these conditions, so  $C(X) \subseteq \mathfrak{M}$  or  $C(X) \subseteq \tilde{\mathfrak{M}}$ , from which the desired containment is obvious. In particular,

$$(8.68) \quad C(P^3) \subseteq N(\mathfrak{P}) .$$

Let  $\mathfrak{N}_2$  be a  $S_2$ -subgroup of  $N(\mathfrak{P})$  which contains  $\mathfrak{Z}_2^{\tilde{q}}$ . By (8.23)  $\mathfrak{N}_2$  is faithfully represented on  $\mathfrak{B} = Z(\mathfrak{P}) = \langle \mathfrak{x}_5, \mathfrak{x}_6 \rangle$ . It is clear that  $\text{Aut}(\mathfrak{P})$  is a 2, 3-group, so we conclude that

$$(8.69) \quad N(\mathfrak{P}) = \mathfrak{P}\mathfrak{Z}_2^{\tilde{q}} .$$

It now follows from (8.68), (8.69), and (8.53) that

$$(8.70) \quad C(P^3) = \mathfrak{P}\langle I_0 \rangle .$$

By hypothesis,  $C(I_0)$  is solvable. Let  $\mathfrak{F} = O_2(C(I_0))$ . Suppose  $\langle P \rangle$  acts faithfully on  $\mathfrak{F}$ . Then  $m(\mathfrak{F}) \geq 6$ , since  $P$  has order 9. But  $\mathfrak{M}_1$

contains a  $S_2$ -subgroup of  $\mathfrak{G}$ , and since  $S_2$ -subgroups of  $\mathfrak{M}_1$  are extensions of  $\mathfrak{R}$  by a 4 group, it follows that every 2-subgroup of  $\mathfrak{G}$  is generated by 4-elements (naturally, this uses the action of the 4-group on  $\mathfrak{R}$ ). Hence,  $P^3$  centralizes  $\mathfrak{F}$ . By (8.70), we get  $\mathfrak{F} = \langle I_0 \rangle$ .

By Lemma 5.38 (a)(ii),  $C(I_0)$  contains an element  $\mathfrak{U}$  of  $\mathcal{Z}(2)$ . Since  $O_2(C(I_0)) = \langle I_0 \rangle$ , we get that  $O_{2,2'}(C(I_0)) = \langle I_0 \rangle \times O_{2'}(C(I_0))$ , so that by Lemma 7.1,  $\mathfrak{U}$  centralizes  $O_{2,2'}(C(I_0))$ . But  $C(I_0)$  is solvable, so that  $O_{2,2'}(C(I_0))$  contains its centralizer. Thus,  $\mathfrak{U} \subseteq O_{2,2'}(C(I_0))$ , an absurdity. This contradiction establishes (8.67). Notice that (8.67) is equivalent to

$$(8.71) \quad \mathfrak{M}_1 = \mathfrak{R}\mathfrak{R}^*\langle J \rangle.$$

Since  $J$  inverts  $\mathfrak{R}^*$ , it follows that  $\mathfrak{Q}\langle J \rangle$  and  $\mathfrak{Q}_1\langle J \rangle$  are both isomorphic to  $S_2$ -subgroups of  $GL(2, 3)$ . This implies that

$$(8.72) \quad C_{\mathfrak{M}_1}(J) \text{ is elementary of order } 8.$$

The hard work is now completed. We may now determine the Weyl group. Recall that  $I_1 = JQ, I_2 = I\tilde{Q}$ , so that  $I_1$  and  $I_2$  are involutions. Let  $W = I_1I_2$ . Thus  $W^3$  centralizes  $\mathfrak{G}$ . Since  $W$  centralizes no element of  $\mathfrak{G}^*$ ,  $W^3$  is not in  $\mathfrak{G}^*$ . Since  $W^3 \in O(\mathfrak{G}) \subseteq \mathfrak{M}_1$ , and since the structure of  $C_{\mathfrak{M}_1}(J)$  is given in (8.72), it follows that  $W^6 = 1$ , so that  $W$  is of order 3 or 6.

From (8.49), we get that  $\mathfrak{X}_1^{W^3} = \mathfrak{X}_1^2 \neq \mathfrak{X}_1$ , and conclude that  $W$  is of order 6. Thus,

$$(8.73) \quad W_0 = W^3 \text{ is an involution in the center of } \langle I_1, I_2 \rangle = \mathfrak{B}_0.$$

We argue that

$$(8.74) \quad \mathfrak{P} \cap \mathfrak{P}^{W_0} = 1.$$

Since  $\mathfrak{P} \cap \mathfrak{P}^{W_0}$  is normalized by  $\mathfrak{G}$  and by  $W_0$ , (8.53) implies that if  $\mathfrak{P}^* = \mathfrak{P} \cap \mathfrak{P}^{W_0}$ , then

$$\mathfrak{P}^* = (\mathfrak{P}^* \cap \langle \mathfrak{x}_1, \mathfrak{x}_6 \rangle)(\mathfrak{P}^* \cap \langle \mathfrak{x}_2, \mathfrak{x}_5 \rangle)(\mathfrak{P}^* \cap \langle \mathfrak{x}_3, \mathfrak{x}_4 \rangle).$$

If  $X \in \mathfrak{X}_3^*\mathfrak{X}_4^*$ , we know that  $C(X) \subseteq N(\mathfrak{D})$ . This fact, coupled with (8.49) implies that  $\mathfrak{P}^* = 1$ , so that (8.74) holds.

Let  $\mathfrak{B} = \mathfrak{P}\mathfrak{G}$ . (No confusion with previous notation is to be feared.) We then get that  $\mathfrak{M} = \mathfrak{B} \cup \mathfrak{B}I_1\mathfrak{B}, \tilde{\mathfrak{M}} = \mathfrak{B} \cup \mathfrak{B}I_2\mathfrak{B}$ . Hence, (8.49) implies that conditions (i') and (iv) of Théorème 1 of [40] are satisfied. Hence,  $\mathfrak{B}\mathfrak{B}_0\mathfrak{B} = \mathfrak{G}_0$  is a group and if we let  $\mathfrak{B}_X$  be the largest subset of  $\mathfrak{B}$  such that  $\mathfrak{B}_X^X \subseteq \mathfrak{P}^{W_0}$ , it follows easily from (8.74) that each element of  $\mathfrak{G}_0$  has a unique representation of the shape  $BXB_X, B \in \mathfrak{B}, X \in \mathfrak{B}_0, B_X \in \mathfrak{B}_X$ . Thus,  $|\mathfrak{G}_0| = |E_2(3)|$ , by an easy calculation. Hence, (8.41), (8.50), (8.51), (8.53), (8.54), (8.57), (8.58), (8.59), (8.60), (8.73) determine the multiplication table of  $\mathfrak{G}_0$ . Thus, if  $\mathfrak{G}^*$  is any group which satisfies

the hypothesis of Theorem 8.1 and also satisfies Hypothesis 8.1, it follows that  $\mathbb{G}^*$  contains a subgroup isomorphic to  $\mathbb{G}_0$ . Since we may take  $\mathbb{G}^* = E_2(3)$ , it follows that  $\mathbb{G}_0 \cong E_2(3)$ , and so  $i(\mathbb{G}_0) = 1$ . Clearly,  $\mathbb{G}_0$  contains  $\mathfrak{M}_1$ , so that  $\mathbb{G}_0$  contains the centralizer of each of its involutions. Hence,  $i(\mathbb{G}) = 1$ , by Lemma 5.35.

Since  $E_2(3)$  does not satisfy  $E_{7,13}$  (by Sylow's theorem), it follows from Lemma 5.35 that  $\mathbb{G}_0 = \mathbb{G} \cong E_2(3)$ .

The remaining lemmas are proved under the following hypothesis:

**HYPOTHESIS 8.2.** Whenever  $\mathfrak{F} \subseteq \mathfrak{M} \in \mathcal{MS}(\mathbb{G})$  and  $\mathfrak{B} = \Omega_1(\mathbf{Z}(\mathfrak{F}))^{\mathfrak{M}}$ , then  $V(\text{ccl}_{\mathbb{G}}(\mathfrak{D}); \mathfrak{F}) \subseteq C_{\mathfrak{M}}(\mathfrak{B})$ .

We must derive a contradiction from this hypothesis. When this is done, the proof of Theorem 8.1 will be complete.

**LEMMA 8.7.** *If  $\mathfrak{X}$  is a 2, 3-subgroup of  $\mathbb{G}$  and  $\mathfrak{X}_3$  is a  $S_3$ -subgroup of  $\mathfrak{X}$ , then  $V(\text{ccl}_{\mathbb{G}}(\mathfrak{D}); \mathfrak{X}_3) \triangleleft \mathfrak{X}$ .*

*Proof.* We assume without loss of generality that  $\mathfrak{X}_3 \subseteq \mathfrak{F}$ . First, suppose  $\mathfrak{X}_3 = \mathfrak{F}$ . Let  $\mathfrak{X} \subseteq \mathfrak{M} \in \mathcal{MS}(\mathbb{G})$ , and let  $\mathfrak{X}^*$  be a  $S_{2,3}$ -subgroup of  $\mathfrak{M}$  containing  $\mathfrak{X}$ . Let  $\mathfrak{B} = \Omega_1(\mathbf{Z}(\mathfrak{F}))^{\mathfrak{M}}$ ,  $\mathbb{C} = C_{\mathfrak{M}}(\mathfrak{B})$ . As  $\mathbb{C} \triangleleft \mathfrak{M}$ ,  $\mathbb{C} \cap \mathfrak{X}^*$  is a  $S_{2,3}$ -subgroup of  $\mathbb{C}$ . By Hypothesis 8.2,  $\mathfrak{B}^* \subseteq \mathbb{C} \cap \mathfrak{X}^*$ , where  $\mathfrak{B}^* = \mathfrak{B}(\text{ccl}_{\mathbb{G}}(\mathfrak{D}); \mathfrak{F})$ . Since  $\mathfrak{B} \subseteq \mathfrak{B}$ , Lemmas 7.4 and 5.38 imply that  $|\mathbb{C} \cap \mathfrak{X}^*|$  is odd. Hence,  $\mathbb{C} \cap \mathfrak{X}^* \triangleleft \mathfrak{X}^*$  implies  $\mathfrak{B}^* = V(\text{ccl}_{\mathbb{G}}(\mathfrak{D}); \mathbb{C} \cap \mathfrak{X}^*) \triangleleft \mathfrak{X}^*$ .

We may now assume that  $\mathfrak{X}_3 \subset \mathfrak{F}$ . We proceed by induction on  $|\mathfrak{F}|/|\mathfrak{X}_3|$ . Let  $\mathfrak{B}^* = V(\text{ccl}_{\mathbb{G}}(\mathfrak{D}); \mathfrak{X}_3)$ . As  $\mathfrak{B}^*$  is generated by conjugates of  $\mathfrak{B}$ , it follows that  $\mathfrak{B}^*$  centralizes  $O_2(\mathfrak{X})$ . Hence, if  $\mathfrak{B}^* \neq 1$ , then  $O_2(\mathfrak{X}) = 1$ , so that  $O_{2,3}(\mathfrak{X}) = O_3(\mathfrak{X})$ . If  $\mathfrak{B}^* = 1$ , the lemma is trivial, so suppose  $\mathfrak{B}^* \neq 1$ . In particular,  $O_3(\mathfrak{X}) \neq 1$ . If  $\mathfrak{X}_3$  is not a  $S_3$ -subgroup of  $N(O_3(\mathfrak{X}))$ , let  $\mathfrak{X}^*$  be a  $S_{2,3}$ -subgroup of  $N(O_3(\mathfrak{X}))$  containing  $\mathfrak{X}$ , and let  $\mathfrak{X}_3^*$  be a  $S_3$ -subgroup of  $\mathfrak{X}^*$  which contains  $\mathfrak{X}_3$ . Then  $V(\text{ccl}_{\mathbb{G}}(\mathfrak{D}); \mathfrak{X}_3^*) \triangleleft \mathfrak{X}^*$ . In particular,  $[\mathfrak{B}^*, \mathfrak{X}]$  is a 3-group, so  $\mathfrak{B}^* \triangleleft \mathfrak{X}$ . Hence, we may assume that  $\mathfrak{X}_3$  is a  $S_3$ -subgroup of  $N(O_3(\mathfrak{X}))$ .

Let  $\mathfrak{B}_0 = \Omega_1(\mathbf{Z}(O_3(\mathfrak{X})))$ , so that  $\mathfrak{B} \subseteq \mathfrak{B}_0$ . Since  $|C_7(\mathfrak{B}_0)|$  is odd, it follows that  $O_3(\mathfrak{X}) = C_{\mathfrak{X}}(\mathfrak{B}_0)$ . Suppose  $\mathfrak{B}^* \not\subseteq O_2(\mathfrak{X})$ . Choose  $G$  in  $\mathbb{G}$  so that  $\mathfrak{D}^G \subseteq \mathfrak{B}^*$  but  $\mathfrak{D}^G \not\subseteq O_3(\mathfrak{X})$ , and for any subset  $\mathfrak{S}$  of  $\mathbb{G}$ , let  $\mathfrak{S}^\bullet = \mathfrak{S}^G$ .

It is a straightforward consequence of Hypothesis 8.2 that  $\mathfrak{D}^\bullet = 1$ .

As  $\mathfrak{D}^\bullet$  acts nontrivially on  $\mathbf{Q}_3^i(\mathfrak{X})$ , we let  $\mathfrak{D}$  be a  $\mathfrak{D}^\bullet$ -invariant subgroup of  $\mathbf{Q}_3^i(\mathfrak{X})$  minimal subject to  $[\mathfrak{D}^\bullet, \mathfrak{D}] \neq 1$ . Let  $\mathfrak{D}_i = C_{\mathfrak{D}}(\mathfrak{D})$ , so that  $|\mathfrak{D}^\bullet : \mathfrak{D}_i| = 3$ . Thus,  $\mathfrak{B}_0 = C_{\mathfrak{B}_0}(\mathfrak{D}_i)$  is invariant under  $\mathfrak{D}^\bullet$  and  $\mathfrak{D}$ .

Let  $N = \{i \mid 1 \leq i \leq 4, \mathfrak{B}_i \subseteq \mathfrak{D}_i, \mathfrak{D}_i \not\subseteq \mathfrak{D}_i\}$ . If  $\mathfrak{B} \subseteq \mathfrak{D}_i$ , then it is obvious that  $N \neq \emptyset$ . If  $\mathfrak{B} \subseteq \mathfrak{D}_i$ , then no  $\mathfrak{D}_i$  is contained in  $\mathfrak{D}_i, 1 \leq i \leq 4$ . Since  $\mathfrak{B} \cap \mathfrak{D}_i$  is of order 3 in this case, we again conclude that  $N \neq \emptyset$ . Choose  $i \in N$ . Thus,  $\mathfrak{D}^\bullet = \langle \mathfrak{D}_i, \mathfrak{D}_i \rangle$  and  $|\mathfrak{D}_i : \mathfrak{D}_i \cap \mathfrak{D}_i| =$

3. Since  $\Omega$  is faithfully represented on  $\tilde{\mathfrak{W}}_0$ , it follows that  $[\mathfrak{D}; \tilde{\mathfrak{W}}_0] \neq 1$ . By Lemma 8.5,  $[\Omega, \tilde{\mathfrak{W}}_0]$  is of order 9 and is not centralized by  $\mathfrak{D}^\cdot$ . Since  $\mathfrak{B} \subseteq \mathfrak{W}_0$ , so also  $\mathfrak{B} \subseteq \tilde{\mathfrak{W}}_0$ . Hence,  $\Omega$  centralizes some  $B$  of  $\mathfrak{B}^\#$ , so if  $\Omega = \Omega^*/C_{\mathfrak{X}}(\mathfrak{W}_0)$ , then  $\langle \mathfrak{D}^\cdot, \Omega^* \rangle \subseteq C(B)$ . By the preceding argument,  $[\Omega^*, \mathfrak{D}^\cdot]$  is a 3-group, violating the nontrivial action of  $\mathfrak{D}^\cdot$  on  $\Omega$ . Thus,  $\mathfrak{B}^* \subseteq O_3(\mathfrak{X})$ , and so  $\mathfrak{B}^* \triangleleft \mathfrak{X}$ , completing the proof of this lemma.

For the remainder of this section, we let

$$\mathfrak{B} = V(ccl_{\mathfrak{G}}(\mathfrak{D}); \mathfrak{B}), \mathfrak{N} = N(\mathfrak{B}) .$$

LEMMA 8.8. (i)  $\mathfrak{N}$  contains no element of  $\mathcal{F}(2)$ . (See Definition 2.9.)

(ii) If  $\mathfrak{X}_0$  is any 2-subgroup of  $\mathfrak{N}$ , then  $A_{\mathfrak{N}}(\mathfrak{X}_0)$  does not contain a subgroup of type (3, 3).

(iii) If  $\mathfrak{C}$  is any subgroup of  $Z(\mathfrak{B})$  of type (3, 3), then  $(\mathfrak{N}, \mathfrak{C}) \in \mathcal{N}$  for all  $\mathfrak{N} \in \mathcal{Z}(2)$ . (See Definition 7.2).

(iv) If  $\mathfrak{X}_1$  is an abelian 2-subgroup of  $\mathfrak{N}$ , the  $A_{\mathfrak{N}}(\mathfrak{X}_1)$  is a 3'-group.

*Proof.* We first prove (iii). We invoke Lemma 7.4, so that (iii) will hold if we can show that  $\mathfrak{C}$  centralizes every element of  $\mathfrak{N}(\mathfrak{C}; 2)$ . Suppose  $\Omega \in \mathfrak{N}(\mathfrak{C}; 2)$  is minimal subject to  $[\Omega, \mathfrak{C}] \neq 1$ . Let  $\mathfrak{C}_0 = C_{\mathfrak{C}}(\Omega) \neq 1$ . Let  $\mathfrak{X}$  be a  $S_{2,3}$ -subgroup of  $C(\mathfrak{C}_0)$  containing  $\mathfrak{C}\Omega$ . Since  $C(\mathfrak{C}_0) \cong \mathfrak{B}$ , it follows that if  $\mathfrak{X}_0$  is a  $S_{2,3}$ -subgroup of  $C(\mathfrak{C}_0)$  containing  $\mathfrak{B}$ , then  $\mathfrak{B} \subseteq O_3(\mathfrak{X}_0)$ . By Lemma 0.7.5, we have  $\mathfrak{C} \subseteq O_3(\mathfrak{X})$ , so that  $[\Omega, \mathfrak{C}] \subseteq \Omega \cap O_3(\mathfrak{X}) = 1$ . (iii) is proved.

Let  $\mathfrak{X} \in \mathcal{F}(2)$ ,  $\mathfrak{X} \subseteq \mathfrak{N}$ . We may assume that  $\mathfrak{X}$  is a noncyclic abelian group of order 8. Since  $\mathfrak{B} \subseteq Z(\mathfrak{B})$ ,  $Z(\mathfrak{B})$  is noncyclic. Hence,  $\mathfrak{X}$  contains an involution  $I$  such that  $C(I) \cap Z(\mathfrak{B})$  is noncyclic. Thus,  $C(I)$  contains an element of  $\mathcal{Z}(2)$  and also a subgroup  $\mathfrak{C}$  of  $Z(\mathfrak{B})$  of type (3, 3). By hypothesis,  $C(I)$  is solvable, in violation of (iii). (i) is proved.

(ii) is a straightforward consequence of (i).

To prove (iv), let  $\mathfrak{X}_1$  be an abelian 2-subgroup of  $\mathfrak{N}$  minimal subject to  $3 \mid |A_{\mathfrak{N}}(\mathfrak{X}_1)|$ . Thus,  $\mathfrak{X}_1$  is a four-group, and the involutions of  $\mathfrak{X}_1$  are all  $\mathfrak{N}$ -conjugate. Thus, (iii) implies that  $C(I) \cap Z(\mathfrak{B})$  is cyclic for all  $I \in \mathfrak{X}_1^\#$ . This implies that  $|\Omega_1(Z(\mathfrak{B}))| \leq 3^2$ . Since the reverse inequality holds by (B), we find that  $Z(\mathfrak{B}) \cap \mathfrak{X}_1$  is cyclic. This is not the case, since  $\mathfrak{B} \subseteq Z(\mathfrak{B}) \cap \mathfrak{X}_1$ . (iv) is proved.

LEMMA 8.9. If  $\mathfrak{X}$  is a 2, 3-subgroup of  $\mathfrak{G}$  and  $\mathfrak{X}$  contains a conjugate of  $\mathfrak{B}$ , then  $\mathfrak{X}$  is contained in conjugate of  $\mathfrak{N}$ .

*Proof.* We assume without loss of generality that  $\mathfrak{X}$  is a maximal

2, 3-subgroup of  $\mathfrak{G}$ , and that  $\mathfrak{B} \subseteq \mathfrak{X}$ . Since  $\mathfrak{B}$  centralizes  $O_2(\mathfrak{X})$ , it follows that  $O_2(\mathfrak{X}) = 1$ , and so  $O_3(\mathfrak{X}) \neq 1$ . Let  $\mathfrak{X}_3$  be a  $S_3$ -subgroup of  $\mathfrak{X}$ . By maximality of  $\mathfrak{X}$ ,  $\mathfrak{X}_3$  is a  $S_3$ -subgroup of  $N(O_3(\mathfrak{X}))$ . We assume without loss of generality that  $\mathfrak{X}_3 \subseteq \mathfrak{P}$ . This implies that  $\mathfrak{B} \subseteq Z(O_3(\mathfrak{X}))$ .

If  $\mathfrak{X}$  contains a conjugate of  $\mathfrak{D}$ , we are done by Lemma 8.7. We therefore suppose that for each  $G$  in  $\mathfrak{G}$ ,  $\mathfrak{D}^G \not\subseteq \mathfrak{X}$ .

Suppose  $1 \leq i \leq 4$ , and  $\mathfrak{D}_i \cap O_3(\mathfrak{X}) = \mathfrak{D}_i \cap \mathfrak{X}_3$ . We conclude that  $\mathfrak{D}_i \subseteq O_3(\mathfrak{X})$ . Since  $\mathfrak{D} \not\subseteq \mathfrak{X}_3$ , we may choose  $i$  with  $1 \leq i \leq 4$  such that  $\mathfrak{D}_i \cap O_3(\mathfrak{X}) \subset \mathfrak{D}_i \cap \mathfrak{X}_3$ . Set  $\mathfrak{F} = \mathfrak{D}_i \cap O_3(\mathfrak{X})$ ,  $\mathfrak{F}^* = \mathfrak{D}_i \cap \mathfrak{X}_3$ . The index  $i$  is fixed in the following discussion. We note that  $\mathfrak{F}$  and  $\mathfrak{F}^*$  are normal elementary subgroups of  $\mathfrak{X}_3$ .

Let  $\mathfrak{Q}$  be a  $\mathfrak{F}^*$ -invariant subgroup of  $Q_3(\mathfrak{X})$  minimal subject to  $[\mathfrak{F}^*, \mathfrak{Q}] \neq 1$ . Thus,  $\mathfrak{F}^*$  acts irreducibly on the Frattini quotient group of  $\mathfrak{Q}$ . We remark that  $\mathfrak{Q}$  is available, since  $O_2(\mathfrak{X}) = 1$ .

Let  $\mathfrak{B}_0 = \Omega_1(Z(O_3(\mathfrak{X})))$ , so that  $\mathfrak{B} \subseteq \mathfrak{B}_0$ .

Choose  $Q \in \mathfrak{Q} - \mathfrak{Q}'$ . We will show that  $\mathfrak{B}^Q \cap C(\mathfrak{D}_i) = 1$ . Suppose false, and that  $B$  in  $\mathfrak{B}^Q$  satisfies  $B^Q \in C(\mathfrak{D}_i)$ . Hence, for  $D$  in  $\mathfrak{F}^*$  ( $\subseteq \mathfrak{D}_i$ ), we have  $B^{Q^D} = B^Q$ , or  $B^{Q^D Q^{-1}} = B$ . Hence,  $QDQ^{-1}D^{-1}$  centralizes  $B$  for each  $D$  in  $\mathfrak{F}^*$ . This implies that  $\mathfrak{Q}$  centralizes  $B$ . Apply Lemma 8.7 to  $C(B)$  and conclude that if  $\mathfrak{Q} = \mathfrak{Q}^*/O_3(\mathfrak{X})$ , then  $[\mathfrak{Q}^*, \mathfrak{F}^*]$  is a 3-group. As this violates the nontrivial action of  $\mathfrak{F}^*$  on  $\mathfrak{Q}$ , the assertion follows.

Since  $\mathfrak{B}^Q \subseteq O_3(\mathfrak{X}) \subseteq \mathfrak{X}_3 \subseteq \mathfrak{P}$ , we have  $\mathfrak{B}^Q \subseteq N(\mathfrak{D}_i)$ . Since  $\mathfrak{D}_i$  is 3-reducible in  $N(\mathfrak{D}_i)$ , it follows that  $\mathfrak{B}^Q$  is faithfully represented on  $\mathfrak{B} = O_3(N(\mathfrak{D}_i)/C(\mathfrak{D}_i))$ . On the other hand, if  $B \in \mathfrak{B}^Q$ , then  $[C(B)^Q, \mathfrak{B}^Q, \mathfrak{B}^Q] = 1$ . This implies that  $\mathfrak{B}^Q$  centralizes every 2'-subgroup of  $\mathfrak{B}$  which  $\mathfrak{B}^Q$  normalizes. Thus, there is a 2-subgroup  $\mathfrak{X}_0$  of  $N(\mathfrak{D}_i)$  such that  $A_{N(\mathfrak{D}_i)}(\mathfrak{X}_0)$  contains a subgroup of type (3, 3). This violates Lemma 8.8 by  $D_{2,3}$  in  $N(\mathfrak{D}_i)$ . The proof is complete.

LEMMA 8.10. *If  $\mathfrak{C}$  is any subgroup of  $\mathfrak{G}$  of type (3, 3), then  $\mathfrak{C}$  centralizes every abelian subgroup in  $\mathfrak{N}(\mathfrak{C}; 2)$ .*

*Proof.* Suppose  $\mathfrak{Q}$  is a four-group in  $\mathfrak{N}(\mathfrak{C}; 2)$  with  $[\mathfrak{Q}, \mathfrak{C}] \neq 1$ . Let  $\mathfrak{C}_0 = C_{\mathfrak{G}}(\mathfrak{Q})$ . Let  $\mathfrak{X}$  be a  $S_{2,3}$ -subgroup of  $C(\mathfrak{C}_0)$  which contains  $\mathfrak{C}\mathfrak{Q}$ . By Lemma 8.9,  $\mathfrak{X}^G \subseteq \mathfrak{N}$  for some  $G$  in  $\mathfrak{G}$ . Lemma 8.8 (iv) is violated.

LEMMA 8.11. *Hypothesis 7.2 is satisfied with  $p = 2$ . Furthermore,  $\mathfrak{M}$  has the following properties:*

- (i)  $S_3$ -subgroups of  $\mathfrak{M}$  are noncyclic.
- (ii)  $\mathfrak{M}$  is a 2, 3-group.
- (iii)  $\mathfrak{M}$  contains no elementary subgroup of order 27.
- (iv)  $m(\mathfrak{M}_0) \leq 2$  for every 3-subgroup  $\mathfrak{M}_0$  of  $\mathfrak{M}$ .



*Proof.* Let  $\mathfrak{X}$  be a 2, 3-subgroup of  $\mathfrak{G}$  which contains elements of  $\mathcal{S}(2)$  and  $\mathcal{S}(3)$ ;  $\mathfrak{X}$  is available by hypothesis (vii) of Theorem 8.1. We assume without loss of generality that  $\mathfrak{X}$  is a maximal 2, 3-subgroup of  $\mathfrak{G}$ . By Lemma 8.8 (i),  $\mathfrak{X}$  is contained in no conjugate of  $\mathfrak{N}$ . By Lemma 8.9,  $\mathfrak{X}$  contains no conjugate of  $\mathfrak{B}$ . This fact together with maximality of  $\mathfrak{X}$  implies that  $O_3(\mathfrak{X}) = 1$ .

Let  $\mathfrak{C}$  be a subgroup of  $\mathfrak{X}$  of type (3, 3) and let  $\mathfrak{X}_3$  be a  $S_3$ -subgroup of  $\mathfrak{X}$  containing  $\mathfrak{C}$ . By Lemma 8.10,  $\mathfrak{C}$  centralizes  $Z(O_2(\mathfrak{X}))$ . Hence,  $\Omega_1(\mathfrak{X}_3)$  centralizes  $Z(O_2(\mathfrak{X}))$ . By Lemma 8.9, each  $S_{2,3}$ -subgroup of  $N(\Omega_1(\mathfrak{X}_3))$  is contained in a conjugate of  $\mathfrak{N}$ . Hence,  $\mathfrak{X}_2$  centralizes  $Z(O_2(\mathfrak{X}))$  by Lemma 8.8 (iv). By hypothesis (iv) of Theorem 8.1,  $\mathfrak{X} \cdot C(Z(O_2(\mathfrak{X})))$  is solvable, so by maximality of  $\mathfrak{X}$ , we conclude that  $\mathfrak{X}$  is a  $S_{2,3}$ -subgroup of  $\mathfrak{X}C(Z(O_2(\mathfrak{X})))$ . Hence, we can choose a  $S_3$ -subgroup  $\mathfrak{P}_2$  of  $\mathfrak{G}$  such that  $\mathfrak{P}_2 \cap \mathfrak{X} = \mathfrak{X}_2$  is a  $S_2$ -subgroup of  $\mathfrak{X}$ , and be guaranteed that  $Z(\mathfrak{P}_2) \subseteq Z(O_2(\mathfrak{X}))$ . Hence,  $\mathfrak{X} \subseteq C(Z(\mathfrak{P}_2))$ , so by maximality of  $\mathfrak{X}$ , we have  $\mathfrak{X}_2 = \mathfrak{P}_2$ .

By Lemma 7.4,  $\Omega_1(Z(O_2(\mathfrak{X}))) = \Omega_1(Z(\mathfrak{X}_2))$  is of order 2 and

$$N(\Omega_1(Z(\mathfrak{X}_2))) = \mathfrak{M} \in \mathcal{MS}(G).$$

By construction,  $\mathfrak{X}_3 \subseteq \mathfrak{M}$ , so (i) is satisfied. By Lemma 7.5,  $O_2(\mathfrak{M}) = \mathfrak{G}$  is of symplectic type with  $w \leq 4$ . (ii) is an easy consequence of this fact together with (i).

Suppose  $\mathfrak{C}$  is an elementary subgroup of  $\mathfrak{M}$  of order 27. Clearly, the width of  $\mathfrak{G}$  is at least 3. By Lemma 7.5, no element of  $\mathfrak{C}^\#$  centralizes any four-subgroup of  $\mathfrak{G}$ . This is obviously impossible.

It remains to prove (iv). By Lemma 7.5 (c),  $\mathfrak{M}_0$  is isomorphic to a subgroup  $\mathfrak{M}_1$  of  $(Z_3 \wr Z_3) \times Z_3$ . By Lemma 8.11 (iii), the intersection of  $\mathfrak{M}_1$  with the normal abelian subgroup  $\mathfrak{A}$  such that  $m(\mathfrak{A}) = 4$  in  $(Z_3 \sim Z_3) \times Z_3$ , is of order at most  $3^2$ . It follows that  $\mathfrak{M}_0$  is either trivial, abelian of type (3), (3, 3) or  $(3^2, 3)$ , or non abelian of order  $3^3$ . In all cases,  $m(\mathfrak{M}_0) \leq 2$ . The proof is complete.

Let  $\mathfrak{C}$  be a subgroup of  $\mathfrak{M}$  of type (3, 3), let  $\mathfrak{X}_3$  be a  $S_3$ -subgroup of  $\mathfrak{M}$  containing  $\mathfrak{C}$ , and let  $\mathfrak{G}_1 = [\mathfrak{G}, \mathfrak{C}]$ , where  $\mathfrak{G} = O_2(\mathfrak{M})$ . Let  $I$  be the involution of  $\mathfrak{G}'$ . Choose  $C$  in  $\mathfrak{C}^\#$  so that  $C_{\mathfrak{G}_1}(C) = \mathfrak{Q}$  is not centralized by  $\mathfrak{C}$ . We may assume that  $C_{\mathfrak{M}}(C) \subseteq \mathfrak{N}$ , since replacing  $\mathfrak{M}$  by a suitable conjugate guarantees this. Let  $\mathfrak{X}$  be a  $S_{2,3}$ -subgroup of  $\mathfrak{N}$  containing  $\mathfrak{C}\mathfrak{Q}$ . This notation is fixed throughout the concluding argument.

LEMMA 8.12. (i)  $\mathfrak{Q}$  is a quaternion group.

(ii)  $\mathfrak{N}$  is a 2, 3-group.

(iii)  $\mathfrak{N} \in \mathcal{MS}(\mathfrak{G})$  and  $\mathfrak{N}$  is the only element of  $\mathcal{MS}(\mathfrak{G})$  which contains  $\mathfrak{B}$ .

(iv)  $\mathfrak{N}$  is the only element of  $\mathcal{MS}(\mathfrak{G})$  which contains  $\mathfrak{C}$ .

*Proof.* Since  $\mathfrak{H}_1$  is extra special, so is  $\Omega$ . Since  $\Omega \subseteq O^3(\mathfrak{M})$ , Lemma 8.8 and Lemma 5.27 imply (i)

By (i) and Lemma 8.8,  $\Omega$  is a  $S_2$ -subgroup of  $O^3(\mathfrak{M})$ . Clearly, since  $\mathfrak{M}$  is a 2, 3-group,  $N_{\mathfrak{M}}(\Omega)$  is a 2, 3-group. Since  $\mathbb{C}\Omega \subseteq O^3(\mathfrak{M})$ , it follows that  $\Omega \subseteq O^3(\mathfrak{M})'$ . Hence,  $\Omega$  has a normal complement  $\mathfrak{R}$  in  $O^3(\mathfrak{M})'$ . To prove (ii), it suffices to show that  $\mathfrak{R}$  is a 3-group. Let  $\mathfrak{R}_0$  be a  $S_3$ -subgroup of  $\mathfrak{R}$  normalized by  $\Omega$ . Then  $I$  inverts  $\mathfrak{R}_0$  since  $\mathfrak{M}$  is a 2, 3-group. Choose  $\mathfrak{F}$  char  $O_3(\mathfrak{R})$  with  $\ker(\mathfrak{R} \rightarrow \text{Aut}(\mathfrak{F}))$  a 3-group, and with  $\mathfrak{F}$  of exponent 3. Such an  $\mathfrak{F}$  is available by Lemma 5.18 and 0.3.6. As  $\Omega$  is nonabelian,  $\mathfrak{R}_0$  is noncyclic. It follows readily that  $I$  centralizes a subgroup of  $\mathfrak{F}/D(\mathfrak{F})$  of order 27. This implies that  $C_{\mathfrak{F}}(I)$  contains an elementary subgroup of order 27, in violation of Lemma 8.11. (ii) is proved.

Let  $\mathfrak{P} \subseteq \mathfrak{N}_1 \in \mathcal{MS}(\mathfrak{G})$ . By Hypothesis 8.2, it suffices to show that  $C_{\mathfrak{N}_1}(\tilde{\mathfrak{B}}) \subseteq \mathfrak{N}$ , where  $\tilde{\mathfrak{B}} = \Omega_1(\mathbf{Z}(\mathfrak{P}))^{\mathfrak{N}_1}$ . Since  $C_{\mathfrak{N}_1}(\tilde{\mathfrak{B}}) \subseteq N(\Omega_1(\mathbf{Z}(\mathfrak{P})))$ , and since  $I \in N(\mathfrak{P}) \subseteq N(\Omega_1(\mathbf{Z}(\mathfrak{P})))$ , we may replace  $\mathfrak{N}_1$  by an element of  $\mathcal{MS}(\mathfrak{G})$  which contains  $N(\Omega_1(\mathbf{Z}(\mathfrak{P})))$  and so assume that  $I \in \mathfrak{N}_1$ .

Let  $\mathfrak{B}$  be a  $S_3$ -subgroup of  $\mathfrak{N}_1$  which contains  $I$ . By Lemma 8.8 (i) and Lemma 8.9, it follows that  $\mathfrak{B}$  has a normal 2-complement  $\mathfrak{R}$ . Since  $\mathfrak{M}$  is a 2, 3-group,  $I$  inverts  $\mathfrak{R}$ . Suppose by way of contradiction that  $\mathfrak{R} \neq 1$ . Since  $O_3(\mathfrak{N}_1) = 1$ ,  $\mathfrak{R}\langle I \rangle$  is faithfully represented as automorphisms of  $O_3(\mathfrak{N}_1)$ . By Lemma 8.11 (iv), the only possibility is that  $|\mathfrak{R}| = 5$ , that  $C(\mathfrak{R}) \cap O_3(\mathfrak{N}_1) \cong D(O_3(\mathfrak{N}_1))$  and that  $O_3(\mathfrak{N}_1)/C(\mathfrak{R}) \cap O_3(\mathfrak{N}_1)$  is elementary of order  $3^4$ . Since  $\mathfrak{R}$  is an  $S$ -subgroup of  $\mathfrak{N}_1$ , it follows that  $\mathfrak{R}O_3(\mathfrak{N}_1)/O_3(\mathfrak{N}_1)$  is a chief factor of  $\mathfrak{N}_1$ . Hence,  $I \notin \mathfrak{N}_1'$ . This implies that  $O_3(\mathfrak{N}_1) = \mathfrak{P}$ , so that  $\mathfrak{N}_1 \subseteq N(\mathfrak{P}) = \mathfrak{N}$ . Hence,  $\mathfrak{N}_1 = \mathfrak{N}$ . This is absurd since  $I \in \mathfrak{N}'$ . This contradiction forces  $\mathfrak{R} = 1$ , that is,  $\mathfrak{N}_1$  is a 2, 3-group.

Since  $|C(\Omega_1(\mathbf{Z}(\mathfrak{P})))|$  is odd, it follows that  $C_{\mathfrak{N}_1}(\tilde{\mathfrak{B}}) = C_{\mathfrak{P}}(\tilde{\mathfrak{B}}) \subseteq \mathfrak{P} \subseteq \mathfrak{N}$ . Thus, (iii) holds.

We turn to (iv). Let  $\mathcal{S} = \{\mathfrak{P}_0 \mid \text{(i) } \mathfrak{P}_0 \text{ is a 3-subgroup of } N, \text{(ii) } \mathfrak{P}_0 \cong \mathfrak{B}^N \text{ for some } N \text{ in } \mathfrak{N}, \text{(iii) } \mathfrak{P}_0 \text{ is contained in a solvable subgroup of } \mathfrak{G} \text{ which is not contained in } \mathfrak{N}\}$ . Suppose by way of contradiction that  $\mathcal{S} \neq \emptyset$ . Choose  $\mathfrak{P}_0$  in  $\mathcal{S}$  with  $|\mathfrak{P}_0|$  maximal. We assume without loss of generality that  $\mathfrak{P}_0 \subseteq \mathfrak{P}$ . Let  $\mathfrak{R}$  be a solvable subgroup of  $\mathfrak{G}$  which contains  $\mathfrak{P}_0$  and is minimal subject to  $\mathfrak{R} \not\subseteq \mathfrak{N}$ . Since  $\mathfrak{P} \in \mathcal{S}$ , it follows that  $\mathfrak{P}_0 \subset \mathfrak{P}$ , so maximality of  $|\mathfrak{P}_0|$  forces  $N_{\mathfrak{P}}(\mathfrak{P}_0) \in \mathcal{S}$ . In particular,  $N(\mathfrak{P}_0) \subseteq \mathfrak{N}$ . This implies that  $\mathfrak{P}_0$  is a  $S_p$ -subgroup of  $\mathfrak{R}$ . Minimality of  $\mathfrak{R}$  yields that  $\mathfrak{R} = \mathfrak{P}_0\mathfrak{R}_1$ , where  $\mathfrak{R}_1$  is a  $q$ -group for some prime  $q \neq 3$ .

Since  $\mathfrak{B}^N \subseteq \mathfrak{P}_0$  for some  $N$  in  $\mathfrak{N}$ , it follows that  $O_3(\mathfrak{R}) \subseteq \mathfrak{N}$ , as  $O_3(\mathfrak{R})$  is generated by its subgroups  $O_3(\mathfrak{R}) \cap C(B)$ ,  $B \in (\mathfrak{B}^N)^\#$ .

Suppose  $q = 2$ . Then by Lemma 8.9,  $\mathfrak{R} \subseteq \mathfrak{N}^G$  for some  $G$  in  $\mathfrak{G}$ . Hence  $\mathfrak{P}_0 \subseteq \mathfrak{N}^G$ . Let  $\mathfrak{P}^*$  be a  $S_3$ -subgroup of  $\mathfrak{N} \cap \mathfrak{N}^G$  which contains  $\mathfrak{P}_0$ .

Maximality of  $|\mathfrak{P}_0|$  forces  $\mathfrak{P}_0 = \mathfrak{P}^*$ . But then since  $N(\mathfrak{P}_0) \subseteq \mathfrak{N}$ , we get that  $\mathfrak{P}_0$  is a  $S_3$ -subgroup of  $\mathfrak{N}^G$ . This is absurd. Hence,  $q \neq 2$ .

It is a consequence of [43] that  $\mathfrak{K} = O_3(\mathfrak{K})\mathfrak{A}_1\mathfrak{A}_2$ , where

$$\mathfrak{A}_1 = C_{\mathfrak{K}}(Z(\mathfrak{P}_0)), \mathfrak{A}_2 = N_{\mathfrak{K}}(J(\mathfrak{P}_0)).$$

Maximality of  $|\mathfrak{P}_0|$  forces  $N(Z(\mathfrak{P}_0)) \subseteq \mathfrak{N}$ ,  $N(J(\mathfrak{P}_0)) \subseteq \mathfrak{N}$ , so  $\mathfrak{K} \subseteq \mathfrak{N}$ . This establishes (iv).

We may now complete the proof of Theorem 8.1. Choose  $C_1$  in  $\mathfrak{C}^\#$ . Then  $C(C_1) \cong \mathfrak{B}$ , so that  $C(C_1) \subseteq \mathfrak{N}$ . Hence,  $\mathfrak{G} \subseteq \mathfrak{N}$ , in violation of Lemma 8.8 (ii).

### 9. A characterization of $S_4(3)$ .

**THEOREM 9.1.**  *$S_4(3)$  is the only simple group  $\mathfrak{G}$  with the following properties:*

- (i)  $\mathfrak{G}$  contains an elementary subgroup of order 27.
- (ii) If  $\mathfrak{P}$  is a  $S_3$ -subgroup of  $\mathfrak{G}$  and  $\mathfrak{A} \in \mathcal{S}_{m_3}(\mathfrak{P})$ , then  $\mathfrak{N}(\mathfrak{A})$  is trivial.
- (iii) The center of a  $S_3$ -subgroup of  $\mathfrak{G}$  is cyclic.
- (iv) The normalizer of every nonidentity 3-subgroup of  $\mathfrak{G}$  is solvable.
- (v)  $S_2$ -subgroups of  $\mathfrak{G}$  contain normal elementary subgroups of order 8.
- (vi) If  $\mathfrak{X}$  is a  $S_2$ -subgroup of  $\mathfrak{G}$  and  $\mathfrak{B} \in \mathcal{S}_{m_2}(\mathfrak{X})$ , then  $\mathfrak{N}(\mathfrak{B})$  is trivial.
- (vii) The centralizer of every involution of  $\mathfrak{G}$  is solvable.
- (viii)  $2 \sim 3$ . (See Definition 2.9.).

After careful translation, it can be shown that Dickson [12] lists several properties of  $S_4(3)$ . Namely,  $A(4, 3)$  is Dickson's notation for  $S_4(3)$  (pp. 89-100). Now in § 194 (pp. 109-191), Dickson sets  $FO(m, p^n) = O'(m, p^n)$  (for  $m$  odd), so by § 189 (pp. 179-183),  $A(4, 3) \cong FO(5, 3) \cong S_4(3)$ . Thus, by § 270 (pp. 292-293),  $S_4(3)$  has a subgroup of index 27 which is a split extension of an elementary group of order 16 by  $A_5$ . So  $S_4(3)$  is not an  $N$ -group. That  $S_4(3)$  satisfies the hypothesis of Theorem 9.1 is left as an exercise. We remark that (viii) holds for  $S_4(3)$ , the centralizers of suitable involutions exhibiting  $2 \sim 3$ .

Throughout most of this section the following notation is used:  $\mathfrak{P}$  is a  $S_3$ -subgroup of  $\mathfrak{G}$ ,

$$\mathfrak{Z} = \Omega_1(Z(\mathfrak{P})), \mathfrak{N} = N(\mathfrak{Z}), \mathfrak{G} = O_3(\mathfrak{N}).$$

By hypothesis (iii),  $|\mathfrak{Z}| = 3$ , and by hypothesis (ii),  $O_3(\mathfrak{N}) = 1$ . By hypothesis (iv),  $\mathfrak{N}$  is solvable, so by Lemma 1.2.3 of [26],  $C_{\mathfrak{N}}(\mathfrak{G}) = Z(\mathfrak{G})$ .

Clearly,  $C_{\mathfrak{N}}(\mathfrak{G}) = C(\mathfrak{G})$ .

We remark that  $\mathfrak{G}$  satisfies Hypothesis 7.4 and also satisfies Hypothesis 7.1 for  $p = 2$  and for  $p = 3$ .

**HYPOTHESIS 9.1.**  $\mathfrak{G}$  is the central product of a cyclic group and a nonabelian group of order 27 and exponent 3.

**LEMMA 9.1.** *Assume that Hypothesis 9.1 is satisfied. Then*

- (i)  $|\mathfrak{P}:\mathfrak{G}| = 3$ .
- (ii)  $|\mathfrak{G}| = 27$ .
- (iii)  $O^3(\mathfrak{N})/\mathfrak{G} \cong SL(2, 3)$ .

*Proof.* We remark that  $GL(2, 3)$  contains no noncyclic abelian subgroup of order 8.

As  $\mathfrak{N}/\mathfrak{G}$  is faithfully represented on  $\Omega_1(\mathfrak{G})/D(\Omega_1(\mathfrak{G}))$ , it follows that  $\mathfrak{N}$  is a 2, 3-group, and  $\mathfrak{N}/\mathfrak{G}$  is isomorphic to a subgroup of  $GL(2, 3)$ . As  $\mathfrak{G}$  contains no elementary subgroup of order 27 and  $\mathfrak{P}$  does, (i) holds.

Let  $\mathfrak{A}$  be a normal elementary subgroup of  $\mathfrak{P}$  of order 27. Since  $|\mathfrak{P}:\mathfrak{G}| = 3$ , it follows that  $O^3(\mathfrak{N})/\mathfrak{G} \cong SL(2, 3)$ , yielding (iii). Let  $\mathfrak{Q}$  be a  $S_2$ -subgroup of  $O^3(\mathfrak{N})$  so that  $\mathfrak{Q}$  is a quaternion group. Let  $\mathfrak{A}_0 = \mathfrak{A} \cap \mathfrak{G}$ . Let  $I$  be the involution of  $\mathfrak{Q}$ . As  $I$  inverts every element of  $\Omega_1(\mathfrak{G})/\mathfrak{G}$ , it follows that  $I$  normalizes  $\mathfrak{A}_0$ . Since  $I$  also normalizes  $\mathfrak{P}$ , it follows that  $I$  normalizes  $C_{\mathfrak{P}}(\mathfrak{A}_0) = \langle \mathfrak{A}, Z(\mathfrak{G}) \rangle$ . Hence,  $I$  normalizes  $\Omega_1(C_{\mathfrak{P}}(\mathfrak{A}_0)) = \mathfrak{A}$ . Since  $I$  centralizes the factor  $O^3(\mathfrak{N})/O^3(\mathfrak{N})'\mathfrak{G} \cong \mathfrak{P}/\mathfrak{G}$ , it follows that  $\mathfrak{A} - \mathfrak{A}_0$  contains an element  $A_1$  such that  $A_1^I = A_1$ . Since  $I$  also centralizes  $\mathfrak{G}$ , it follows that  $C_{\mathfrak{A}}(I) = \langle A_1 \rangle \times \mathfrak{G} = \mathfrak{A}_1$ . Also,  $C_{\mathfrak{P}}(I) = \mathfrak{A}_1\mathfrak{G}_1$ , where  $\mathfrak{G}_1 = Z(\mathfrak{G})$ , and it is clear that  $C_{\mathfrak{P}}(I)$  is a  $S_3$ -subgroup of  $C_{\mathfrak{N}}(I)$ .

Suppose  $|\mathfrak{G}_1| > 3$ . Thus,  $|\mathfrak{P}| > 3^4$ , so Lemma 7.6 is at our disposal. If  $G \in \mathfrak{G}$  and  $\mathfrak{Z}_1^G \subseteq \mathfrak{P}$ , then  $\mathcal{C}^1(\mathfrak{Z}_1^G)$  centralizes  $\mathfrak{G}$ , and so  $\Omega_1(\mathfrak{Z}_1^G) = \Omega_1(\mathfrak{Z}_1) = \mathfrak{G}$ , so that  $G \in \mathfrak{N}$ ,  $\mathfrak{Z}_1^G = \mathfrak{Z}_1$ . We may therefore apply Theorem 14.4.2 of [21] and conclude that  $\mathfrak{P} \subseteq \mathfrak{N}'$ . Since  $\text{Aut}(\mathfrak{Z}_1)$  is abelian, this implies that  $\mathfrak{Z}_1 = Z(\mathfrak{P})$ . We may therefore appeal to Lemma 7.6 (d) and conclude that if  $\mathfrak{A}^*$  is any subgroup of  $\mathfrak{A}$  of type (3, 3), then  $\mathfrak{A}^*$  centralizes every element of  $\mathcal{N}(\mathfrak{A}^*; 2)$ . Taking  $\mathfrak{A}^* = \mathfrak{A}_1$ , Lemma 7.4 is violated. This completes the proof of (ii).

**LEMMA 9.2.** *Assume that Hypothesis 9.1 is satisfied. Let  $\mathfrak{A} \in \text{Soc}_2(\mathfrak{P})$  and let  $I$  be an involution of  $\mathfrak{N}$ . Then*

- (i)  $S_2$ -subgroups of  $\mathfrak{N}$  are quaternion.
- (ii) If  $\mathfrak{A}_0 = C_{\mathfrak{N}}(I)$ , then
  - (a)  $|\mathfrak{A}_0| = 9$ .
  - (b)  $\mathfrak{A}_0$  contains a subgroup  $\mathfrak{A}_1$  of order 3 such that  $C(\mathfrak{A}_1) \not\subseteq \mathfrak{N}$ .

(iii) If  $\mathfrak{M} = C(I)$ , then  $O_2(\mathfrak{M})$  is extra special of width 2,  $O_2(\mathfrak{M}) = 1$ , and  $|\mathfrak{M}:O_2(\mathfrak{M})|_2 = 2$ .

(iv)  $A_{\mathfrak{G}}(\mathfrak{A}) \cong \Sigma_4$ .

*Proof.* Let  $\mathfrak{Q}$  be a  $S_2$ -subgroup of  $O^{3'}(\mathfrak{N})$ . By Lemma 9.1 (i),  $\mathfrak{Q}$  is a quaternion group. It clearly suffices to prove the lemma on the assumption that  $I$  is the involution of  $\mathfrak{Q}$ .

By Lemma 9.1 and hypothesis (i) of Theorem 9.1, the group  $\mathfrak{F}$  is  $Z_3 \wr Z_3$ . Hence,  $\mathfrak{A}$  char  $\mathfrak{F}$ . Since  $I$  normalizes  $\mathfrak{F}$ , it therefore normalizes  $\mathfrak{A}$ . This implies (ii)(a), since  $I$  centralizes  $Z(\mathfrak{G})$  and  $O^{3'}(\mathfrak{N})/O^{3'}(\mathfrak{N})'$ .

Clearly,  $\mathfrak{M}$  contains an element of  $\mathcal{U}(2)$ . It is equally clear from (B) and Lemma 9.1 that

(\*) if  $\mathfrak{X}$  is any noncyclic subgroup of  $\mathfrak{A}$ , then  $\mathfrak{X}$  centralizes every abelian subgroup of  $\mathfrak{N}(\mathfrak{X}; 2)$ .

Let  $\mathfrak{X}$  be a  $S_{2,3}$ -subgroup of  $\mathfrak{M}$  which contains  $\langle \mathfrak{A}_0, \mathfrak{Q} \rangle$ . Let  $\mathfrak{X}_2$  be a  $S_2$ -subgroup of  $\mathfrak{X}$  which contains  $\mathfrak{Q}$ . We may apply Lemma 7.5 with  $\mathfrak{A}_0$  in the role of  $\mathfrak{B}$ . Thus, there is an element  $\tilde{\mathfrak{M}}$  of  $\mathcal{MS}(\mathfrak{G})$  satisfying the conclusions of Lemma 7.5. By Lemma 7.5(e), we get  $\mathfrak{Q} \subseteq O_2(\tilde{\mathfrak{M}})$ . Hence,  $\langle I \rangle \triangleleft \tilde{\mathfrak{M}}$ , so  $\tilde{\mathfrak{M}} = C(I) = \mathfrak{M}$ . Since  $\langle I \rangle$  is a  $S_2$ -subgroup of  $C(\mathfrak{A}_0)$ , it follows that

(9.1)  $O_2(\mathfrak{M})$  is extra special of width  $w = 2, 3$ , or  $4$ .

Thus, (ii)(b) holds.

Again, let  $\mathfrak{X}$  be a noncyclic subgroup of  $\mathfrak{A}$ . Suppose that  $|C(\mathfrak{X}) \cap N(\mathfrak{A})|$  is even. Then of course  $|\mathfrak{X}| = 9$ , as  $\mathfrak{A}$  is a self-centralizing subgroup of  $\mathfrak{G}$ . Let  $J$  be an involution of  $C(\mathfrak{X}) \cap N(\mathfrak{A})$ . Then (\*) and Lemma 7.5 yield that  $J$  and  $I$  are conjugate in  $\mathfrak{G}$ . Since  $\mathfrak{X}$  is faithfully represented on  $O_2(C(J))$ , we can choose a subgroup  $\mathfrak{Y}$  of  $\mathfrak{X}$  of order 3 such that

(9.2)  $\mathfrak{X}$  does not centralize  $C(\mathfrak{Y}) \cap O_2(C(J))$ .

Thus,  $C(\mathfrak{Y})$  is not 3-closed. Thus,  $\mathfrak{A}$  is not a  $S_3$ -subgroup of  $C(\mathfrak{Y})$ . This implies that

(9.3)  $C(\mathfrak{Y})$  contains a  $S_3$ -subgroup of  $\mathfrak{G}$ .

Let  $\tilde{\mathfrak{F}}$  be a  $S_3$ -subgroup of  $C(\mathfrak{Y})$  which contains  $\mathfrak{A}$ . Thus  $\langle \tilde{\mathfrak{F}}, J \rangle \subseteq C(\mathfrak{Y})$ . Thus,  $J$  normalizes both  $\mathfrak{A}$  and  $O_3(C(\mathfrak{Y}))$ , so  $J$  normalizes  $\langle \mathfrak{A}, O_3(C(\mathfrak{Y})) \rangle$ . Thus, Lemma 9.1 yields that

(9.4) if  $\mathfrak{X}$  is any noncyclic subgroup of  $\mathfrak{A}$ , then each involution of  $C(\mathfrak{X}) \cap N(\mathfrak{A})$  normalizes some  $S_3$ -subgroup of  $N(\mathfrak{A})$ .

By (9.3) with the pair  $(\mathfrak{A}_1, \mathfrak{A}_0)$  in the role of  $(\mathfrak{Y}, \mathfrak{X})$ , we conclude that  $C(\mathfrak{A}_1)$  contains a  $S_3$ -subgroup  $\mathfrak{P}^*$  of  $\mathfrak{G}$  with  $\mathfrak{A} \subset \mathfrak{P}^*$ . Hence,  $N(\mathfrak{A})$  is not 3-closed, since  $\mathfrak{P}$  and  $\mathfrak{P}^*$  are distinct  $S_3$ -subgroups of  $N(\mathfrak{A})$ .

Set  $\tilde{\mathfrak{N}} = N(\mathfrak{A})$ . Clearly,

$$\mathfrak{A} = O_3(\tilde{\mathfrak{N}}) = C(\mathfrak{A}), 1 = O_3(\tilde{\mathfrak{N}}), I \in \tilde{\mathfrak{N}}.$$

Suppose  $13 \mid |\tilde{\mathfrak{N}}|$ . Since  $I \in \tilde{\mathfrak{N}}$ , it follows that  $I$  centralizes a  $S_{13}$ -subgroup of  $\tilde{\mathfrak{N}}$ , since the nonidentity 13-elements of  $GL(3, 3)$  are nonreal. However,  $13 \nmid |\mathfrak{M}|$ , since  $4 \geq w$ . Hence,  $\tilde{\mathfrak{N}}$  is a 2, 3-group.

Let  $\mathfrak{X}_0$  be a  $S_2$ -subgroup of  $O_{3,2}(\tilde{\mathfrak{N}})$ , and let  $\mathfrak{R} = N_{\tilde{\mathfrak{N}}}(\mathfrak{X}_0)$ . Thus,  $\tilde{\mathfrak{N}} = \mathfrak{A}\mathfrak{R}$ ,  $\mathfrak{A} \cap \mathfrak{R} = 1$ , so that  $\mathfrak{R} \cong A_{\mathfrak{G}}(\mathfrak{A})$ . Suppose  $J$  is an involution of  $\mathfrak{X}_0$  and  $\mathfrak{A} \cap C(J) = \mathfrak{X}$  is noncyclic. Thus,  $|\mathfrak{X}| = 9$ . By (9.4),  $J$  normalizes some  $S_3$ -subgroup  $\tilde{\mathfrak{P}}$  of  $\tilde{\mathfrak{N}}$ . Since  $\tilde{\mathfrak{P}} \supset \mathfrak{A}$ ,  $\tilde{\mathfrak{P}} \cap \mathfrak{R}$  is of order 3. Hence,  $[\tilde{\mathfrak{P}} \cap \mathfrak{R}, J] \subseteq \mathfrak{X}_0 \cap \tilde{\mathfrak{P}} = 1$ , so that  $\tilde{\mathfrak{P}} \cap \mathfrak{R}$  centralizes  $J$ . Hence,  $\tilde{\mathfrak{P}} \cap \mathfrak{R}$  normalizes  $\mathfrak{X}$ , that is,  $\mathfrak{X} \triangleleft \tilde{\mathfrak{P}}$ . Hence,  $\mathfrak{X} \in \mathcal{Z}(3)$ . Thus,  $J$  centralizes elements of  $\mathcal{Z}(3)$  and  $\mathcal{Z}(2)$ . This violates the solvability of  $C(J)$ . Hence,

(9.5) no element of  $\mathfrak{X}_0^\ddagger$  centralizes a noncyclic subgroup of  $\mathfrak{A}$ .

Since  $|Z(\mathfrak{P})| = 3$ ,  $\mathfrak{P} \cap \mathfrak{R}$  is indecomposable on  $\mathfrak{A}$ . Hence,  $\mathfrak{X}_0$  acts on  $\mathfrak{A}$  as a multiple of the sum of the  $\mathfrak{R}$ -conjugates of a fixed  $F_3$ -irreducible representation  $\rho$ . If  $\mathfrak{X}_0$  is nonabelian, then 2 divides  $\deg \rho$ . Hence, 2 divides  $3 = m(\mathfrak{A})$ , a contradiction. Hence,  $\mathfrak{X}_0$  is abelian. If  $\mathfrak{X}_0$  is not elementary, then  $\deg \rho \neq 1$  or 3. So  $\deg \rho = 2$ , which again gives a contradiction. Hence,  $\mathfrak{X}_0$  is elementary. Now (9.5) implies that  $|\mathfrak{X}_0| \leq 4$ , so we must have equality, since  $\mathfrak{X}_0 = O_2(\mathfrak{R})$  and  $|\mathfrak{P} \cap \mathfrak{R}| = 3$ . Since  $I \in \mathfrak{N} \cap N(\mathfrak{P})$ , it follows that  $\mathfrak{R} \cong \Sigma_4$ , which establishes (iv) and also (i).

It remains to show that  $w = 2$  and that  $|\mathfrak{M}: O_2(\mathfrak{M})|_2 = 2$ , since by (9.1) we know that as  $O_2(\mathfrak{M})$  is extra special.

Suppose  $\mathfrak{A}$  is any subgroup of  $\mathfrak{A}$  of order 3 which is conjugate to  $\mathfrak{Z}$ ,  $\mathfrak{A}^*_{\mathfrak{G}} \mathfrak{Z}$ . We contend that  $\mathfrak{A}^*_{N(\mathfrak{A})} \mathfrak{Z}$ . Namely, let  $\mathfrak{P}^*$  be a  $S_3$ -subgroup of  $C(\mathfrak{A}^*)$  which contains  $\mathfrak{A}$ . Then  $\mathfrak{P}$  and  $\mathfrak{P}^*$  both normalize  $\mathfrak{A}$ , since  $|\mathfrak{P}^*: \mathfrak{A}| = 3$ . We may thus choose  $N$  in  $N(\mathfrak{A})$  such that  $\mathfrak{P}^{*N} = \mathfrak{P}$ ; since  $\mathfrak{Z} = Z(\mathfrak{P})$ , we necessarily have  $\mathfrak{A}^{*N} = \mathfrak{Z}$ , as desired.

Since  $\mathfrak{P} \langle I \rangle$  normalizes  $\mathfrak{Z}$ , we obtain all  $N(\mathfrak{A})$ -conjugates of  $\mathfrak{Z}$  by transformation with elements of  $\mathfrak{X}_0$ . We will show that  $\mathfrak{Z}$  and  $\mathfrak{A}_1$  are the only  $N(\mathfrak{A})$ -conjugates of  $\mathfrak{Z}$  which are in  $\mathfrak{A}_0$ . If  $K \in \mathfrak{X}_0^\ddagger$  and  $\mathfrak{Z}^K \subseteq \mathfrak{A}_0$ , then since no element of  $\mathfrak{X}_0^\ddagger$  normalizes  $\mathfrak{Z}$ , we conclude that  $K$  normalizes  $\mathfrak{A}_0$ . It is clear that  $\mathfrak{X}_0$  does not normalize  $\mathfrak{A}_0$ , so our assertion follows.

It is an immediate consequence of the preceding paragraph that  $w = 2$ . That is, only  $\mathfrak{Z}$  and  $\mathfrak{A}_1$  centralize elements of  $O_2(\mathfrak{M}) - \langle I \rangle$ . Since  $N(\mathfrak{A}_0) \subseteq N(\mathfrak{A})$ , we have  $|\mathfrak{M}: O_2(\mathfrak{M})|_2 = 2$ , and the proof is complete.

We now change notation somewhat in order to conform with more standard notation. Let  $\mathfrak{B}_1 = N(\mathfrak{B})$ ,  $\mathfrak{B}_2 = N(\mathfrak{A})$ , and let  $\mathfrak{B} = \mathfrak{P}\langle I \rangle$ ,  $\mathfrak{G} = \langle I \rangle$ . Let  $\mathfrak{Q}_1$  be a  $S_2$ -subgroup of  $\mathfrak{B}_1$  which contains  $I$ , and let  $\mathfrak{Q}_2$  be a  $S_2$ -subgroup of  $\mathfrak{B}_2$  which contains  $I$ . Thus,  $\mathfrak{Q}_1$  is a quaternion group and  $\mathfrak{Q}_2$  is a dihedral group of order 8. Let  $\mathfrak{X}_2 = \mathfrak{Q}_2 \cap \mathcal{O}^{3'}(\mathfrak{B}_2)$ , so that  $\mathfrak{X}_2$  is a four-group.

Let  $\mathfrak{C}_2 = N_{\mathfrak{B}_2}(\mathfrak{X}_2)$ . Thus,  $\mathfrak{C}_2$  is a complement to  $\mathfrak{A}$  in  $\mathfrak{B}_2$  and  $\mathfrak{C}_2 \cong \Sigma_4$ . Let  $\mathfrak{X}_1 = \mathfrak{P} \cap \mathfrak{C}_2$ , so that  $\mathfrak{X}_1$  is of order 3 and is inverted by  $I$ . Since  $\mathcal{O}_3(\mathfrak{B}_1)$  contains all elements of  $\mathfrak{P}$  which are inverted by  $I$ , we have  $\mathfrak{X}_1 \subseteq \mathcal{O}_3(\mathfrak{B}_1)$ . Since  $\mathfrak{Q}_1$  permutes transitively the subgroups of  $\mathcal{O}_3(\mathfrak{B}_1)/\mathfrak{B}$  of order 3, we may choose  $Q$  in  $\mathfrak{Q}_1$  so that  $\mathfrak{X}_2 = \mathfrak{X}_1^Q$  lies in  $\mathfrak{A}$ . Thus,  $I$  inverts  $\mathfrak{X}_2$ , since  $Q$  centralizes  $I$ . Let  $\langle J \rangle = \mathfrak{X}_2 \cap C(I)$ , that is, let  $J$  be a generator for  $Z(\mathfrak{Q}_2)$ . Since  $\mathfrak{A} \cap C(I)$  is of order 9,  $\mathfrak{X}_2$  is the only subgroup of  $\mathfrak{A}$  of order 3 which is inverted by  $I$ , so that  $\mathfrak{X}_2^J = \mathfrak{X}_2$ . Let  $\mathfrak{X}_4 = \mathfrak{B}$  and set  $\mathfrak{X}_3 = \mathfrak{X}_1^J$ . We may now draw up the following table:

	$J$	$Q$
$\mathfrak{X}_1$	—	$\mathfrak{X}_2$
$\mathfrak{X}_2$	$\mathfrak{X}_2$	$\mathfrak{X}_1$
$\mathfrak{X}_3$	$\mathfrak{X}_4$	—
$\mathfrak{X}_4$	$\mathfrak{X}_3$	$\mathfrak{X}_4$ .

Let  $X_i$  be a generator for  $\mathfrak{X}_i$ , so that we have the following table:

	$I$
$X_1$	$X_1^{-1}$
$X_2$	$X_2^{-1}$
$X_3$	$X_3$
$X_4$	$X_4$ .

Let  $\mathfrak{N} = \langle J, Q \rangle$ . Since  $\mathfrak{N} \subseteq \mathfrak{M}$ , the structure of  $\mathfrak{N}$  may be easily determined. Let  $\mathfrak{Q}_1, \mathfrak{Q}_1^*$  be the quaternion subgroups of  $\mathcal{O}_3(\mathfrak{M})$ ,  $\mathfrak{Q}_1$  being as above. As  $J$  normalizes  $\langle \mathfrak{X}_3, \mathfrak{X}_4 \rangle$  and  $\langle \mathfrak{X}_3, \mathfrak{X}_4 \rangle$  is a  $S_3$ -subgroup of  $\mathfrak{M}$ , it follows that  $\mathfrak{Q}_1^J = \mathfrak{Q}_1^*$ . Hence,  $(JQ)^2 = JQJQ$  is an involution distinct from  $I$ . This means that  $\mathfrak{N}/\langle I \rangle$  is a dihedral group of order 8 with involutory generators  $J\langle I \rangle, Q\langle I \rangle$ .

Finally, notice that  $\mathfrak{B} = \mathfrak{P}\langle I \rangle = N(\mathfrak{P})$ .

Since  $(JX_1)^3 = 1$  and since  $(\mathfrak{Q}X_3)^3 \in \langle I \rangle$ , it is straightforward to deduce from the first table that  $\mathfrak{B}\mathfrak{N}\mathfrak{B} = \mathfrak{G}_1$  is a group. We will determine the multiplication table of  $\mathfrak{G}_1$ . First, we assume without

loss of generality that  $(QX_3)^3 = 1$ , since replacement of  $Q$  by  $Q^{-1} = QI$  will achieve this if  $(QX_3)^3 = I$ . Since  $I$  inverts  $X_1$  and centralizes  $X_3$ , it follows easily that  $I$  neither inverts nor centralizes  $[X_1, X_3]$ . Thus, we may choose  $X_2, X_4$  as generators for  $\mathfrak{x}_2, \mathfrak{x}_4$  respectively such that

$$(9.6) \quad [X_1, X_3] = X_2X_4 .$$

By construction,  $\mathfrak{x}_4 = \mathfrak{z} = \mathbf{Z}(\mathfrak{P})$ , so to complete the determination of  $\mathfrak{B}$ , we must compute  $[X_1, X_2]$ . Conjugation of (9.6) by  $I$  yields  $[X_1^{-1}, X_3] = X_2^{-1}X_4$ , from which we find easily that  $[X_1, X_2] = X_4$ .

Let  $X_1^Q = X_2^a$ . Since  $(QX_3)^3 = 1$ , an easy calculation (conjugation of (9.6) by  $Q$ ) shows that  $a = +1$ . Since  $J \in \mathfrak{X}_2 \triangleleft \mathfrak{C}_2$ , it follows that  $C_{\mathfrak{M}}(J)$  has order 3. Since  $J$  normalizes but does not invert  $\langle X_3, X_4 \rangle$ , it follows that  $C_{\mathfrak{M}}(J) \subseteq \langle X_3, X_4 \rangle$ . Hence,  $X_2^J = X_2^{-1}$ . Let  $X_3^J = X_4^b$ . Since  $(X_1J)^3 = 1$ , an easy calculation (conjugation of (9.6) by  $J$ ) shows that  $b = -1$ .

Set  $W_0 = (JQ^2)$ . We argue that

$$(9.7) \quad \mathfrak{P} \cap \mathfrak{P}^{W_0} = 1 .$$

Suppose by way of contradiction that 9.7 is false. Since  $W_0$  is an involution,  $\mathfrak{D} = \mathfrak{P} \cap \mathfrak{P}^{W_0}$  is normalized by  $W_0$ . Since  $W_0 \in \mathbf{Z}(\mathfrak{N})$ , it follows that  $\mathfrak{G} = \langle I \rangle$  also normalizes  $\mathfrak{D}$ . Since  $C_{\mathfrak{P}}(I) = \langle X_3, X_4 \rangle$  and since  $W_0 \in \mathfrak{M}$ , it follows from the construction of  $\mathfrak{N}$  that  $I$  inverts  $\mathfrak{D}$ . Thus,  $\mathfrak{D} \subset O_3(\mathfrak{B}_1)$ , since  $O_3(\mathfrak{B}_1)$  contains all the elements of  $\mathfrak{P}$  which are inverted by  $I$ . As  $\mathfrak{D}$  is abelian, and as  $I$  centralizes  $\mathfrak{x}_4 = \mathbf{Z}(O_3(\mathfrak{B}_1))$ , it follows that  $|\mathfrak{D}| = 3$ . There are exactly 4 subgroups of  $O_3(\mathfrak{B}_1)$  of order 3 which are inverted by  $I$ ; they are all of the shape  $\mathfrak{x}_2^{Q^*}$  for some  $Q^*$  in  $\mathfrak{Q}_1$ . Since  $I$  normalizes  $\mathfrak{x}_2$  and since  $\mathfrak{Q}_1 = \langle Q \rangle \cup \langle Q^{X_3} \rangle \cup \langle Q^{X_3^{-1}} \rangle$ , we may assume that  $\mathfrak{D} = \mathfrak{x}_2^{Q^*}$ , where  $Q^*$  is one of  $1, Q, X_3^{-1}QX_3, X_3QX_3^{-1}$ . Since  $W_0$  normalizes  $\mathfrak{D}$ , we get that  $Q^*W_0Q^{*-1} \in N(\mathfrak{x}_2)$ . Since  $Q^* \in \mathfrak{M}$  and  $W_0 \in O_2(\mathfrak{M})$ , we get that  $Q^*W_0Q^{*-1} \in N(\mathfrak{x}_2) \cap O_2(\mathfrak{M}) = \mathfrak{Z}$ , say. Since  $I$  inverts  $X_2$ , it follows that  $I \notin D(\mathfrak{Z})$ . Thus,  $\mathfrak{Z}$  is elementary. But  $1$  and  $\langle I \rangle$  are the only elementary subgroups of  $O_2(\mathfrak{M})$  which admit  $\langle X_3, X_4 \rangle$ , so  $Q^*W_0Q^{*-1} = \langle I \rangle$ . This is not the case, since  $Q^* \in \mathfrak{M}$ ,  $I \in \mathbf{Z}(\mathfrak{M})$ , and  $I \neq W_0$ . This proves (9.7).

Set  $\mathfrak{B} = \{1, Q, J, QJ, JQ, QJQ, JQJ, W_0\}$ , a set of representatives for the cosets of  $\mathfrak{G}$  in  $\mathfrak{N}$ . For each  $W$  in  $\mathfrak{B}$ , let  $\mathfrak{B}_W = \langle \mathfrak{x}_i \mid 1 \leq i \leq 4, \mathfrak{x}_i^{W^{-1}} \subseteq \mathfrak{P}^{W_0} \rangle$ . It follows that condition (iii) of Théorème I of [36] is satisfied, so by that theorem, so is condition (ii), that is, if  $W_1, W_2 \in \mathfrak{B}$  and  $BW_1B = BW_2B$ , then  $W_1 = W_2$ . In view of our preceding information, we conclude that each element of  $\mathfrak{G}_1$  has a normal form of the shape  $PHWP'$ , where  $P \in \mathfrak{P}, H \in \mathfrak{G}, W \in \mathfrak{B}, P' \in \mathfrak{B}_W$ . Furthermore, it is clear that the normal forms for  $PHWP'J$  and  $PHWP'Q$  are determined by our information. This implies immediately that if  $\mathfrak{G}^*$  is any group which satisfies the hypothesis of Theorem 9.1 and Hy-



pothesis 9.1, then  $\mathfrak{G}^*$  contains a subgroup  $\mathfrak{G}_1^*$  isomorphic to  $\mathfrak{G}_1$ . Taking  $\mathfrak{G}^* = S_4(3)$ , a comparison of orders yields  $\mathfrak{G}_1 \cong S_4(3)$ . In particular,  $i(\mathfrak{G}_1) = 2$  and  $I, W_0$  are representatives for the two classes of involutions of  $\mathfrak{G}_1$ . Since  $\mathfrak{M} = \langle Q_1, J, X_3, X_4 \rangle$ , we have  $\mathfrak{M} \subseteq \mathfrak{G}_1$ . We will show that  $C(W_0) \subseteq \mathfrak{G}_1$ . Let  $\mathfrak{R} = O_2(C_{\mathfrak{G}_1}(W_0))$ . Then  $\mathfrak{R}$  is elementary of order  $2^4$  and  $\mathfrak{R}$  is characteristic in a  $S_2$ -subgroup of  $C_{\mathfrak{G}_1}(W_0)$ . Thus, it suffices to show that  $N(\mathfrak{R}) = N_{\mathfrak{G}_1}(\mathfrak{R})$ . Since  $N_{\mathfrak{G}_1}(\mathfrak{R})$  is an extension of  $\mathfrak{R}$  by  $A_5$  and since  $\mathfrak{R} = C(\mathfrak{R})$ , it follows that  $A_{\mathfrak{G}_1}(\mathfrak{R})$  is a subgroup of  $\text{Aut}(\mathfrak{R}) = L_4(2)$  which contains a subgroup isomorphic to  $A_5$  and has  $S_2$ -subgroups of order 4. Hence,  $A_{\mathfrak{G}_1}(\mathfrak{R}) = A_{\mathfrak{G}_1}(\mathfrak{R}) \cong A_5$ . Hence,  $\mathfrak{G}_1$  contains the centralizer of each of its involutions. By Lemma 5.35,  $\mathfrak{G} = \mathfrak{G}_1$ . Thus, Theorem 9.1 is proved in case Hypothesis 9.1 is satisfied.

We now revert to our previous notation.

**HYPOTHESIS 9.2.**  $\mathfrak{G}$  is of symplectic type and width  $w \geq 2$ .

**HYPOTHESIS 9.3.** (i)  $\mathfrak{G}$  is extra special of order  $3^5$ .

(ii)  $|\mathfrak{P}| = 3^6$ .

(iii)  $\mathfrak{Z}$  is not weakly closed in  $\mathfrak{G}$ .

Lemmas 9.3 through 9.10 are all proved under Hypothesis 9.2. [Notice that Hypothesis 9.3 trivially implies Hypothesis 9.2.

**LEMMA 9.3.** (i)  $C(\mathfrak{Z})$  does not contain a four-group.

(ii) If  $\mathfrak{Q}$  is any abelian 2-subgroup of  $\mathfrak{R}$ , then  $A_{\mathfrak{R}}(\mathfrak{Q})$  is a 2-group.

(iii) If  $\mathfrak{A}$  is any subgroup of  $\mathfrak{G}$  of type  $(3, 3)$  which contains  $\mathfrak{Z}$ , then  $|C(\mathfrak{A})|$  is odd.

(iv) If  $\mathfrak{A}$  is any subgroup of  $\mathfrak{G}$  of type  $(3, 3)$  which contains  $\mathfrak{Z}$ , then  $\mathfrak{A} \in \mathcal{E}(3)$ .

*Proof.* Clearly, (i) implies (ii), and (iii) implies (i). Suppose (iv) holds, but  $I$  is an involution in  $C(\mathfrak{A})$ . By Lemma 5.37,  $C(I)$  contains an element of  $\mathcal{Z}(2)$ . By Lemma 7.4,  $C(I)$  is nonsolvable. Hence, (iv) implies (iii). To complete the proof of the lemma, it suffices to prove (iv). However, (iv) is a consequence of Lemma 7.2.

**LEMMA 9.4.** Suppose  $B \in \Omega_1(\mathfrak{G}) - \mathfrak{Z}$  and  $\mathfrak{G}_0 = C_{\mathfrak{G}}(B)$ . Then

$$C(\mathfrak{G}_0) = Z(\mathfrak{G}_0) = \langle B \rangle \times Z(\mathfrak{G}) .$$

*Proof.* Since  $\mathfrak{Z} \subset \mathfrak{G}_0$ , it follows that  $C(\mathfrak{G}_0) = C_{\mathfrak{R}}(\mathfrak{G}_0)$ . Since a  $S_3$ -subgroup of  $\mathfrak{R}$  is faithfully represented on  $\mathfrak{G}$ , it follows that  $C(\mathfrak{G}_0)$  is a 3-group. It suffices to show that  $C(\mathfrak{G}_0) \subseteq \mathfrak{G}$ . Suppose false and

$C \in C(\mathfrak{H}_0)$ ,  $C \notin \mathfrak{H}$ . We may assume that  $C^3 \in \mathfrak{H}$ . In this case,  $\langle C \rangle / \langle C^3 \rangle$  is faithfully represented on  $Q_3(\mathfrak{N})$  and by Lemma 5.30, it follows that  $[Q_3(\mathfrak{N}), \langle C \rangle] = \tilde{\mathfrak{D}}$  is a quaternion group. Let  $\mathfrak{D}$  be a subgroup of  $\mathfrak{N}$  incident with  $\tilde{\mathfrak{D}}$ . Clearly,  $\mathfrak{H} = C_{\mathfrak{H}}(\mathfrak{D})[\mathfrak{D}, \mathfrak{H}]$  and  $C_{\mathfrak{H}}(\mathfrak{D})$  commutes elementwise with  $[\mathfrak{D}, \mathfrak{H}]$ . By Lemma 9.3 (iii),  $\mathfrak{D}'$  centralizes no noncyclic subgroup of  $\mathfrak{H}$ . It follows that  $C_{\mathfrak{H}}(\mathfrak{D}) = Z(\mathfrak{H})$  is cyclic. However,  $w \geq 2$  and  $C$  centralizes  $\mathfrak{H}_0$ .

LEMMA 9.5. *Hypothesis 9.3 is not satisfied.*

*Proof.* Suppose false.

Let  $\mathfrak{X} = \{\mathfrak{Z}_1 \mid \mathfrak{Z}_1 \subseteq \mathfrak{H}, \mathfrak{Z}_1 \sim \mathfrak{Z}, \mathfrak{Z}_1 \neq \mathfrak{Z}\}$ . By Hypothesis 9.3 (iii),  $\mathfrak{X} \neq \emptyset$ . Since  $\mathfrak{H} \triangleleft \mathfrak{N}$ ,  $\mathfrak{X}$  is invariant in  $\mathfrak{N}$ . Choose  $\mathfrak{Z}_1 \in \mathfrak{X}$  such that  $C_{\mathfrak{H}}(\mathfrak{Z}_1)$  is a  $S_3$ -subgroup of  $C_{\mathfrak{N}}(\mathfrak{Z}_1)$ . Let  $\mathfrak{P}_0 = C_{\mathfrak{H}}(\mathfrak{Z}_1)$ .

If  $\mathfrak{P}_0 = C_{\mathfrak{H}}(\mathfrak{Z}_1)$ , then  $\mathfrak{Z}$  char  $\mathfrak{P}_0$ . This is impossible since  $\mathfrak{P}_0$  is not a  $S_3$ -subgroup of  $C(\mathfrak{Z}_1)$ . Hence,  $|\mathfrak{P}_0| = 3^5$ .

Let  $\mathfrak{D} = \langle \mathfrak{Z}, \mathfrak{Z}_1 \rangle$ , so that  $\mathfrak{D} \subseteq Z(\mathfrak{P}_0)$ . If  $\mathfrak{D} \subset Z(\mathfrak{P}_0)$ , then choose  $Z \in Z(\mathfrak{P}_0) - \mathfrak{H}$ , so that  $Z$  centralizes a 3-dimensional subspace of  $\mathfrak{H}/\mathfrak{H}'$ . This implies that some involution of  $O'(\mathfrak{N})$  has a noncyclic fixed point set on  $\mathfrak{H}$ , in violation of Lemma 9.3 (iii). Hence,  $\mathfrak{D} = Z(\mathfrak{P}_0)$ .

Let  $\mathfrak{P}^*$  be a  $S_3$ -subgroup of  $C(\mathfrak{Z}_1)$  which contains  $\mathfrak{P}_0$ . Thus,  $\langle \mathfrak{P}, \mathfrak{P}^* \rangle \subseteq N(\mathfrak{P}_0) \subseteq N(\mathfrak{D})$ , so  $O'(A_{\mathfrak{G}}(\mathfrak{D})) \cong SL(2, 3)$ .

By Lemma 9.3,  $|C(\mathfrak{D})|$  is odd. Since  $\mathfrak{P}_0$  is a  $S_3$ -subgroup of  $C(\mathfrak{D})$  and since  $O_3(C(\mathfrak{D})) = 1$ , it follows that  $\mathfrak{P}_0 = C(\mathfrak{D})$ . Hence,  $N(\mathfrak{P}_0) = N(\mathfrak{D})$ . Let  $\mathfrak{M} = N(\mathfrak{D})$ .

Let  $\mathfrak{Q}$  be a  $S_2$ -subgroup of  $O'(\mathfrak{M})$ . Thus,  $\mathfrak{Q}$  is a quaternion group. Let  $J$  be the involution of  $\mathfrak{Q}$ . Let  $\mathfrak{Q}^*$  be a  $S_2$ -subgroup of  $O'(\mathfrak{M})$ . Thus  $\mathfrak{Q}^*$  is a quaternion group. Let  $I$  be the involution of  $\mathfrak{Q}^*$ . Since  $J$  inverts  $\mathfrak{H}/\mathfrak{H}'$ ,  $J \in \mathfrak{M}$ . Since  $I$  inverts  $\mathfrak{D}$ ,  $I \in \mathfrak{N}$ . We assume without loss of generality that  $I$  normalizes  $\mathfrak{Q}$  and  $J$  normalizes  $\mathfrak{Q}^*$ .

Since  $J$  neither inverts nor centralizes  $\mathfrak{D}$ , it is clear that  $A_{\mathfrak{G}}(\mathfrak{D}) \cong GL(2, 3)$  and so  $\langle J, \mathfrak{Q}^* \rangle$  is isomorphic to a  $S_2$ -subgroup of  $GL(2, 3)$ . Let  $Q^*$  be an element of  $\mathfrak{Q}^*$  of order 4 which is inverted by  $J$ .

We will show that  $\langle I, \mathfrak{Q} \rangle$  is isomorphic to a  $S_2$ -subgroup of  $GL(2, 3)$ . Since  $I \in \mathfrak{N}$ , we need only prove that  $I$  inverts  $\mathfrak{P}_0/\mathfrak{P}_0 \cap \mathfrak{H}$ . Suppose false. Then  $I$  centralizes  $\mathfrak{P}_0/\mathfrak{P}_0 \cap \mathfrak{H}$ . We know that  $\mathfrak{D} \subseteq \mathfrak{P}_0$ , because  $\mathfrak{M}$  operates irreducibly on  $\mathfrak{D}$  and  $\mathfrak{D}$  contains  $\mathfrak{Z} = (\mathfrak{P}_0 \cap \mathfrak{H})'$ . Since  $\mathfrak{Q}^*$  is faithfully represented on  $\mathfrak{P}_0$ , there must be a 2-dimensional subspace of  $\mathfrak{P}_0/D(\mathfrak{P}_0)$  which  $I$  inverts and  $\mathfrak{Q}^*$  leaves invariant. Since  $I$  centralizes  $\mathfrak{P}_0/\mathfrak{P}_0 \cap \mathfrak{H}$  and  $|\mathfrak{P}_0| = 3^5$ , we conclude that  $\mathfrak{D} = \mathfrak{P}_0' = D(\mathfrak{P}_0)$ , and that  $\mathfrak{P}_0 \cap \mathfrak{H}/\mathfrak{D}$  is the subspace of  $\mathfrak{P}_0/\mathfrak{P}_0'$  inverted by  $I$ . This forces  $I$  to invert both  $\mathfrak{P}_0 \cap \mathfrak{H}/\mathfrak{D}$  and  $\mathfrak{D}$ , that is, to invert  $\mathfrak{P}_0 \cap \mathfrak{H}$ . So  $\mathfrak{P}_0 \cap \mathfrak{H}$  is abelian, which is false.

Let  $Q$  be an element of  $\mathfrak{Q}$  of order 4 which is inverted by  $I$ .

Set  $\mathfrak{x}_6 = \mathfrak{B}$ . Since  $J$  normalizes  $\mathfrak{D}$  and centralizes  $\mathfrak{x}_6$ , we can choose an element  $X_5$  of  $\mathfrak{D}$  of order 3 such that  $X_5^J = X_5^{-1}$ . Let  $\mathfrak{x}_5 = \langle X_5 \rangle$ ,  $\mathfrak{x}_4 = \mathfrak{x}_5^q$ ,  $X_4 = X_5^q$ . Then we have relations  $X_5^I = X_5^{-1}$ ,  $X_5^J = X_5^{-1}$ ,  $X_4^I = X_4$ ,  $X_4^J = \mathfrak{x}_4^{-1}$ . Suppose  $[X_4, X_5] \neq 1$ . The following argument is designed to exclude this possibility.

Let  $[X_4, X_5] = X_6$  so that  $X_6$  is a generator for  $\mathfrak{x}_6$ . Since  $X_4 \notin \mathfrak{P}_0$ , it follows that  $\mathfrak{P}_0 \cap C(I)$  is of order 3 with generator  $X_3$ , say. Thus,  $\langle X_3, X_4 \rangle = C_{\mathfrak{P}_0}(I)$  is of order 9, so that  $[X_3, X_4] = 1$ . As  $\mathfrak{S}$  contains all the elements of  $\mathfrak{P}$  which are centralized by  $I$ , we have  $X_3 \in H$ . Since  $\langle X_3 \rangle = C_{\mathfrak{P}_0}(I)$ , it follows that  $J$  normalizes  $\langle X_3 \rangle$ , so that  $X_3^J = X_3^{-1}$ , as  $J$  inverts  $\mathfrak{S}/\mathfrak{B}$ . Let  $X_2 = X_3^q$ . Since  $[X_3, X_4] = 1$ , so also  $[X_2, X_4^I] = 1$ . But  $X_4^I = X_5^{q^2} = X_5^{-1}$ , so  $X_2$  centralizes  $X_5$ , that is,  $X_2 \in \mathfrak{P}_0$ . Let  $X_1 = X_2^{q^2}$ , and let  $\mathfrak{x}_i = \langle \mathfrak{x}_i \rangle$ ,  $1 \leq i \leq 6$ . We obtain the following data:

Table 1			Table 2		
	$J$	$I$		$Q$	$Q^*$
$X_1$	$X_1$	$X_1^{-1}$	$X_1$	—	$X_2^{-1}$
$X_2$	$X_2^{-1}$	$X_2^{-1}$	$X_2$	$X_3^{-1}$	$X_1$
$X_3$	$X_3^{-1}$	$X_3$	$X_3$	$X_2$	$X_3$
$X_4$	$X_4^{-1}$	$X_4$	$X_4$	$X_5^{-1}$	—
$X_5$	$X_5^{-1}$	$X_5^{-1}$	$X_5$	$X_4$	$X_6^{-a}$
$X_6$	$X_6$	$X_6^{-1}$	$X_6$	$X_6$	$X_5^a$ .

Here  $a^2 = 1$ , and the last two entries in Table 2 are at our disposal since  $Q^*$  normalizes  $\langle I, J \rangle$  and since  $\mathfrak{x}_5, \mathfrak{x}_6$  are the only subgroups of  $\mathfrak{D}$  of order 3 which admit  $\langle I, J \rangle$ . In addition we have the following commutation relations:

$$[X_i, X_6] = 1, 1 \leq i \leq 6, [X_4, X_5] = X_6,$$

$$[X_i, X_5] = 1, 1 \leq i \leq 3, [X_3, X_4] = [X_2, X_4] = 1.$$

Furthermore,  $[X_2, X_3] = X_6^b$ , so by Table 2, we get  $[X_1, X_3] = X_5^{ab}$ . Here  $b^2 = 1$ , for if  $b = 0$ , we get  $X_3 \in Z(\mathfrak{S})$ , which is not the case. The as yet undetermined commutation relations are:

$$[X_1, X_4] = X_2^x X_3^y X_5^z X_6^t, \quad [X_1, X_2] = X_3^c X_5^d X_6^e.$$

Use Table 1 and conjugate the second relation by  $J$ , obtaining  $e = bc$ . Then conjugation by  $I$  yields  $d = abc$ . Conjugation of the first relation by  $J$  yields  $t = xyb + z$ . Conjugation of the first relation by  $I$  yields  $y = cx$ .

Assume  $c \neq 0$ . Then

$$\mathfrak{P}'_0 = \langle X_3, X_5, X_6 \rangle, [\mathfrak{P}'_0, \mathfrak{P}_0] = \langle X_5, X_6 \rangle = \mathfrak{D} = Z(\mathfrak{P}_0).$$

We see that  $\mathfrak{P}'_0$  is elementary abelian. If  $A \in \mathfrak{P}_0, B \in \mathfrak{P}'_0$ , then  $(AB)^3 = A^3 B^{A^2+A+1}$ . But  $cl(\mathfrak{P}_0) = 3$  and so  $B^{A^2+A+1} = B^3 [B, A]^3 = 1$ . Hence,

there is a map  $\varphi$  of  $\mathfrak{P}_0/\mathfrak{P}'_0$  given by  $\varphi(A\mathfrak{P}'_0) = A^3$ .

Clearly,  $\varphi(X_1\mathfrak{P}'_0) = 1$ . But  $\mathfrak{M}$  operates as  $GL(2, 3)$  on  $\mathfrak{D}$ . Since  $|\mathfrak{P}_0/\mathfrak{P}'_0| = 3^2$ , this forces  $\mathfrak{M}$  to operate as  $GL(2, 3)$  on  $\mathfrak{P}_0/\mathfrak{P}'_0$ . In particular, the four subgroups of  $\mathfrak{P}_0/\mathfrak{P}'_0$  of order 3 are all conjugate under  $\mathfrak{M}$ . Hence,  $\varphi(A\mathfrak{P}'_0) = 1$  for all  $A$  in  $\mathfrak{P}_0$  and  $\mathfrak{P}_0$  is of exponent 3.

By [21, p. 324], the order of the Burnside group of exponent 3 on 2 generators is 27. Since  $\mathfrak{P}_0$  must be a homomorphic image of this group, we get a contradiction, as  $|\mathfrak{P}_0| = 3^5 > 27$ . So  $c = 0$ .

Since  $c = 0$ , so also  $d = e = y = 0$ . Since  $y = 0$ , we also have  $t = z$ . Conjugation of the first relation by  $Q$  yields  $[Q^{-1}X_1Q, X_5^{-1}] = X_3^{-z}X_4X_6^z$ . Now  $C(J) \cap O^3(\mathfrak{N}) = \langle X_1, X_6, \mathfrak{D} \rangle$ , so  $C(J) \cap O^3(\mathfrak{N})$  is 2-closed, that is,  $X_1$  normalizes  $\mathfrak{D}$ . Hence,  $(QX_1)^3 = J^u, u = 0$  or 1. Hence,  $Q^{-1}X_1Q = JX_1^{-1}Q^{-1}X_1^{-1}J^u$ . The previous commutation relation now yields  $x = 0$ .

Since  $x = 0$ , it follows that  $X_4$  centralizes  $\mathfrak{P}_0/\mathfrak{D}$ . Hence,  $\mathfrak{D}^*$  is forced to centralize  $\mathfrak{P}_0/\mathfrak{D}$ . This is not the case, since  $\mathfrak{D} \subseteq \mathfrak{P}'_0$ . We conclude that  $[X_4, X_5] = 1$ .

Since  $I$  centralizes  $X_4$ , it follows that  $\langle X_4, X_5, X_6 \rangle = \mathfrak{E} \triangleleft \mathfrak{M}$ . Namely,  $\mathfrak{D} \triangleleft \mathfrak{M}$ , so we need to show that  $\mathfrak{E}/\mathfrak{D} \triangleleft \mathfrak{M}/\mathfrak{D}$ . Since  $X_4$  centralizes  $\mathfrak{D}$ , we have  $X_4 \in \mathfrak{P}_0$ . Since  $\mathfrak{P}_0/\mathfrak{D}$  is of order 27 and admits  $\mathfrak{D}^*$  as a group of automorphisms, it follows that  $\mathfrak{E}/\mathfrak{D} = C_{\mathfrak{P}_0/\mathfrak{D}}(I) \triangleleft \mathfrak{M}/\mathfrak{D}$ . Thus,  $\langle \mathfrak{M}, Q \rangle \subseteq N(\mathfrak{E})$ . Since  $\langle \mathfrak{P}, Q \rangle = O^3(\mathfrak{N})$ , it follows that both  $\mathfrak{M}$  and  $O^3(\mathfrak{N})$  are subgroups of  $N(\mathfrak{E})$ .

Let  $\mathfrak{E}^* = O_3(N(\mathfrak{E}))$ . Thus,

$$\mathfrak{E} \subseteq \mathfrak{E}^* \subseteq O_3(\mathfrak{M}) \cap O_3(\mathfrak{N}) = \mathfrak{P}_0 \cap \mathfrak{H}.$$

Suppose  $\mathfrak{E}^* = \mathfrak{P}_0 \cap \mathfrak{H}$ . Then  $\mathfrak{B} = \mathfrak{E}^{*'} \triangleleft N(\mathfrak{E})$ , against  $O^3(\mathfrak{N}) \subset N(\mathfrak{E})$ . Hence,  $\mathfrak{E}^* \subset \mathfrak{P}_0 \cap \mathfrak{H}$ . Since  $|\mathfrak{E}| = 3^3$  and  $|\mathfrak{P}_0 \cap \mathfrak{H}| = 3^4$ , it follows that  $\mathfrak{E}^* = \mathfrak{E}$ . Thus,  $N(\mathfrak{E})/\mathfrak{E}$  is isomorphic to a subgroup of  $\text{Aut}(\mathfrak{E})$  which (a) is solvable, (b) contains a  $S_3$ -subgroup of  $\text{Aut}(\mathfrak{E})$ , (c) is 3-reduced. There are no such groups. The proof of the lemma is complete.

LEMMA 9.6. *Let  $\mathfrak{B}$  be a subgroup of  $\mathfrak{H}$  of type (3, 3). Then  $\mathfrak{B} \in \mathcal{D}$ . (See Definition 7.3.)*

*Proof.* We first show that if  $B \in \mathfrak{B}$ , then

$$(9.8) \quad \text{for some } N \text{ in } \mathfrak{N}, B \text{ centralizes an element of } \mathcal{U}(\mathfrak{P}^N).$$

Let

$$\mathfrak{H}_1 = \Omega_1(\mathfrak{H}), \mathfrak{B} = \mathfrak{H}_1/D(\mathfrak{H}_1), \mathfrak{B}_0 = C_{\mathfrak{B}}(\mathfrak{B}).$$

Suppose  $|\mathfrak{B}_0| > 3$ . Then  $\mathfrak{B}_0 = \mathfrak{B}/D(\mathfrak{H}_1)$  and every subgroup of  $\mathfrak{B}$  which contains  $D(\mathfrak{H}_1)$  is normal in  $\mathfrak{B}$ . Let  $\mathfrak{B}_1 = \mathfrak{B} \cap C(B)$  so that  $|\mathfrak{B}:\mathfrak{B}_1| \leq 3$ .

Since  $|\mathfrak{B}_0| > 3$ , so also  $|\mathfrak{B}_1| \geq 9$ . Thus,  $B$  centralizes an element of  $\mathcal{Z}(\mathfrak{B})$  in this case. We may assume that  $|\mathfrak{B}_0| = 3$ .

Suppose  $\mathfrak{N}/\mathfrak{G}$  has a normal subgroup  $\mathfrak{N}/\mathfrak{G} = \mathfrak{X}$  of odd order  $\neq 1$ . Let  $k$  be a field of characteristic 3 which contains all  $|\mathfrak{X}|^{\text{th}}$  roots of 1. Let  $\tilde{\mathfrak{B}} = k \otimes_{F_3} \mathfrak{B}$ . Thus,  $\tilde{\mathfrak{B}}$  admits  $\mathfrak{N}/\mathfrak{G}$  and  $k \otimes \mathfrak{B}_0$  is the set of all fixed points of  $\mathfrak{B}/\mathfrak{G}$  on  $\tilde{\mathfrak{B}}$ . Let  $\tilde{\mathfrak{B}} = \bigoplus_{\rho} \tilde{\mathfrak{B}}(\rho)$ , where  $\tilde{\mathfrak{B}}(\rho)$  is the largest  $\mathfrak{X}$ -submodule of  $\tilde{\mathfrak{B}}$  on which  $\mathfrak{X}$  acts as a multiple of the irreducible representation  $\rho$ . Since  $\tilde{\mathfrak{B}}$  inherits the non singular symplectic structure of  $\mathfrak{B}$ , it follows that  $\rho$  and  $\rho^*$  appear with the same multiplicity in  $\tilde{\mathfrak{B}}$ ,  $\rho^*$  denoting the contragredient representation of  $\rho$ . Since  $|\mathfrak{B}|$  is odd,  $\tilde{\mathfrak{B}}(\rho)$  and  $\tilde{\mathfrak{B}}(\rho^*)$  are not conjugate under  $\mathfrak{B}$ . Hence,  $\mathfrak{B}_0$  is not 1-dimensional in this case.

We may now assume that

$$(9.9) \quad F(\mathfrak{N}/\mathfrak{G}) \text{ is a 2-group .}$$

If  $\mathfrak{G} = \mathfrak{B}$ , then (9.8) is obvious, so suppose  $\mathfrak{G} \subset \mathfrak{B}$ . Set  $\mathfrak{N}^* = C(\mathfrak{B})$ , so that  $|\mathfrak{N}:\mathfrak{N}^*| \leq 2$ . By Lemma 9.3 (i), together with (9.9), we conclude that  $\mathfrak{N}$  is a 2, 3-group, and that a  $S_2$  subgroup of  $\mathfrak{N}^*$  is quaternion. Hence,  $|\mathfrak{B}:\mathfrak{G}| = 3$ . Since  $|\mathfrak{B}_0| = 3$ , we get that the width of  $\mathfrak{G}$  is 1, against Hypothesis 9.2. Thus, (9.8) holds.

Suppose  $\mathfrak{B} \notin \mathcal{S}$ . Then  $\mathfrak{N}(\mathfrak{B}; 2)$  contains a four-group  $\mathfrak{Q}$  which is not centralized by  $\mathfrak{B}$ . Hence,  $[\mathfrak{B}, \mathfrak{Q}] = \mathfrak{Q}$ , and  $\mathfrak{B}_0 = C_{\mathfrak{B}}(\mathfrak{Q})$  is of order 3.

Let  $\mathfrak{C} = C(\mathfrak{B}_0)$ ,  $\mathfrak{G}_0 = C_{\mathfrak{B}}(\mathfrak{B}_0)$ . By Lemma 7.2 applied to  $\langle \mathfrak{B}_0, \mathfrak{B} \rangle$ , it follows that  $\mathfrak{B}$  centralizes  $O_3(\mathfrak{C})$ . Hence,  $[O_3(\mathfrak{C}), \mathfrak{G}_0] \subseteq \mathfrak{G} \cap O_3(\mathfrak{C}) = 1$ . This implies that  $O_3(\mathfrak{C}) = 1$  by Lemma 9.4.

Let  $\mathfrak{B}_0$  be a  $S_3$ -subgroup of  $\mathfrak{C}$  containing  $\mathfrak{G}_0$  and let  $\mathfrak{B}^*$  be a  $S_3$ -subgroup of  $\mathfrak{C}$  containing  $\mathfrak{B}_0$ . Then  $\mathfrak{B}^* = \mathfrak{B}^{\mathfrak{G}}$ , so that with  $\mathfrak{B}^* = \mathfrak{B}^{\mathfrak{G}}$ , it follows that  $\mathfrak{B}^* \subseteq Z(O_3(\mathfrak{C}))$ . Let  $\mathfrak{B} = \mathfrak{B}^{*\mathfrak{C}}$ , so that  $\mathfrak{B}$  is 3-reducible in  $\mathfrak{C}$ . Set  $\mathfrak{C}_1 = C_{\mathfrak{C}}(\mathfrak{B})$ . We argue that  $\mathfrak{C}_1 \cap \mathfrak{Q} = 1$ . If not, then  $\mathfrak{Q} \subseteq C(\mathfrak{B})$ , as  $\mathfrak{Q}$  is an irreducible  $\mathfrak{B}$ -module. Hence,  $\mathfrak{Q} \subseteq C(\mathfrak{B}^*)$ , against Lemma 9.3 (i). Hence,

$$(9.10) \quad \mathfrak{C}_1 \cap \mathfrak{Q} = 1 .$$

By (B), elements of  $\mathfrak{B} - \mathfrak{B}_0$  have minimal polynomial  $(x - 1)^3$  on  $\mathfrak{B}$ .

We next argue that  $\mathfrak{B} \subseteq \mathfrak{C}_1$ . If not, then  $\mathfrak{G}_0$  contains an extra special subgroup of width  $w - 1$  disjoint from  $\mathfrak{C}_1$ . We get that  $m(\mathfrak{B}) \geq 2 \cdot 3^{w-1}$ . Since  $m(\mathfrak{B} \cap \mathfrak{G}^{\mathfrak{G}}) \leq w + 1$ , we have  $m(\mathfrak{B}/\mathfrak{B} \cap \mathfrak{G}^{\mathfrak{G}}) \geq 2 \cdot 3^{w-1} - w - 1$ . By [32], it follows that  $w^2 \geq 2 \cdot 3^{w-1} - w - 1$ . This is false for  $w \geq 3$ , so  $w = 2$ . Thus,  $C(\mathfrak{B})/\mathfrak{G}$  is isomorphic to a subgroup of  $GL(4, 3)$  which (a) is solvable, (b) is 3-reduced, (c) has an elementary subgroup of order 27. There are no such groups. We conclude that  $\mathfrak{B} \subseteq \mathfrak{C}_1$ .

Since  $\mathfrak{B} \subseteq \mathfrak{C}_1$ , we have  $\mathfrak{B} \subseteq \mathfrak{N}$ , so that  $[\mathfrak{B}, \mathfrak{B}, \mathfrak{B}] \subseteq \mathfrak{B}$ . Hence by (B),

$$(9.11) \quad [\mathfrak{B}, \mathfrak{B}, \mathfrak{B}] = \mathfrak{Z} .$$

Since  $\mathfrak{B} \subseteq Z(\mathbb{C}_1)$ , we get

$$(9.12) \quad \mathfrak{Z} \subseteq Z(\mathbb{C}_1) ,$$

which implies that

$$(9.13) \quad \mathbb{C}_1 \subseteq \mathfrak{N} .$$

By (9.13), we get  $[\mathbb{C}_1, \mathfrak{B}, \mathfrak{B}] \subseteq \mathfrak{Z}$ , and in particular,

$$(9.14) \quad [O_3(\mathbb{C}), \mathfrak{B}, \mathfrak{B}] = \mathfrak{Z} ,$$

equality holding by (9.11) and the obvious containment  $\mathfrak{B} \subseteq O_3(\mathbb{C})$ .

Now (9.14) and (9.11) yield

$$(9.15) \quad O_3(\mathbb{C}) = \mathfrak{B}_1 \times \mathfrak{B}_2 ,$$

where

$$(9.16) \quad \mathfrak{Z} \subset \mathfrak{B}_1 = [\mathfrak{Q}, O_3(\mathbb{C})], \text{ and } \mathfrak{B}_1 \text{ is elementary of order } 27 ,$$

$$(9.17) \quad \mathfrak{B}_2 = C(\mathfrak{Q}) \cap O_3(\mathbb{C}) .$$

Let  $\mathbb{C}_2 = O_{3'}(\mathbb{C} \text{ mod } \mathbb{C}_1)$ ,  $\mathbb{C}_3 = \mathbb{C}_2 Z(\mathfrak{H})$ . Thus,  $\mathbb{C}_1 Z(\mathfrak{H})$  contains a  $S_3$ -subgroup of  $\mathbb{C}_3$ . Thus,  $Z(\mathfrak{H})$  is normal in a  $S_3$ -subgroup of  $\mathbb{C}_3$ . By Lemma 5.22, we get  $Z(\mathfrak{H}) \subseteq O_3(\mathbb{C}_3)$ . Hence,  $Z(\mathfrak{H}) \subseteq O_3(\mathbb{C})$ . From (9.16), we conclude that  $Z(\mathfrak{H}) = \mathfrak{Z}$ , that is,

$$(9.18) \quad \mathfrak{H} \text{ is extra special .}$$

We argue that  $O_{3,3'}(\mathbb{C})$  does not centralize  $Z(O_3(\mathbb{C}))$ . If it does, then since  $\mathfrak{Z} \subseteq Z(O_3(\mathbb{C}))$ , it follows that  $O_{3,3'}(\mathbb{C}) \subseteq \mathfrak{N}$ , so  $[O_{3,3'}(\mathbb{C}), \mathfrak{H}_0] \subseteq \mathfrak{H}$ , which implies that  $\mathfrak{H}_0 \subseteq O_3(\mathbb{C})$ , which in turn gives  $\mathfrak{Q} = [\mathfrak{B}, \mathfrak{Q}] \subseteq O_3(\mathbb{C})$ . Since  $\mathfrak{B}_0 \subseteq Z(\mathbb{C})$ , it follows that

$$[Z(O_3(\mathbb{C})), O_{3,3'}(\mathbb{C})] \text{ and } C(O_{3,3'}(\mathbb{C})) \cap Z(O_3(\mathbb{C}))$$

are disjoint nontrivial normal abelian subgroups of  $\mathbb{C}$ . In particular, if  $\mathfrak{P}_0$  is a  $S_3$ -subgroup of  $\mathbb{C}$  containing  $\mathfrak{H}_0$ , then  $Z(\mathfrak{P}_0)$  is not cyclic. By Lemma 9.4, we get that  $\Omega_1(Z(\mathfrak{P}_0)) = \mathfrak{B}_0 \times \mathfrak{Z}$ , and in particular,  $\mathfrak{P}_0 \subseteq \mathfrak{N}$ .

Since  $\mathfrak{P}_0 \subseteq \mathfrak{N}$ , we get that  $[\mathfrak{P}_0, \mathfrak{B}, \mathfrak{B}] \subseteq \mathfrak{Z}$ . Thus, if  $B \in \mathfrak{B}$ , the minimal polynomial of  $B$  on the Frattini quotient group of  $O_{3,3',3}(\mathbb{C})/O_{3,3'}(\mathbb{C})$  divides  $(x - 1)^2$ . By (B), it follows that  $\mathfrak{Q}$  centralizes  $O_{3,3',3}(\mathbb{C})/O_{3,3'}(\mathbb{C})$ , and so  $\mathfrak{Q} \subseteq O_{3,3'}(\mathbb{C})$ .

Let  $\mathfrak{R} = \langle \mathfrak{Q}, \mathfrak{H}_0 \rangle \subseteq \mathbb{C}$ , let  $\mathfrak{R}_0 = C_{\mathfrak{R}}(O_3(\mathbb{C}))$  and for any subset  $\mathfrak{S}$  of  $\mathfrak{R}$ , let  $\bar{\mathfrak{S}} = \mathfrak{S}\mathfrak{R}_0/\mathfrak{R}_0$ .

We argue that  $\bar{\mathfrak{Q}} \subseteq O_3(\bar{\mathfrak{R}})$ . Namely,  $\mathfrak{Q} \subseteq O_{3,3'}(\mathbb{C})$ , and so  $\mathfrak{Q} \subseteq O_{3,3'}(\mathfrak{R})$ ,

Thus, it suffices to show that  $[O_3(\mathfrak{K}), \mathfrak{Q}] \subseteq \mathfrak{R}_0$ . But

$$[O_3(\mathfrak{K}), \mathfrak{Q}] \subseteq O_3(\mathfrak{K}) \cap O_{3,3'}(\mathbb{C}) \subseteq O_3(\mathfrak{K}) \cap O_3(\mathbb{C}),$$

and so

$$\begin{aligned} [O_3(\mathfrak{K}), \mathfrak{Q}] &= [O_3(\mathfrak{K}), \mathfrak{Q}, \mathfrak{Q}] \subseteq [O_3(\mathfrak{K}) \cap O_3(\mathbb{C}), \mathfrak{Q}] \subseteq [O_3(\mathbb{C}), \mathfrak{Q}] \\ &= \mathfrak{B}_1 \subseteq Z(O_3(\mathbb{C})), \end{aligned}$$

whence  $[O_3(\mathfrak{K}), \mathfrak{Q}] \subseteq \mathfrak{K} \cap Z(O_3(\mathbb{C})) \subseteq \mathfrak{R}_0$ .

*Case 1.*  $\langle \overline{\mathfrak{Q}}, \overline{\mathfrak{Q}^H} \rangle$  is abelian for all  $H \in \mathfrak{G}_0$ .

Since  $[\overline{\mathfrak{Q}^{\mathfrak{H}_0}}, \mathfrak{B}]$  admits the abelian group  $\mathfrak{H}_0$ , and since  $\mathfrak{Q} \subseteq [\mathfrak{Q}^{\mathfrak{H}_0}, \mathfrak{B}]$ , it follows that  $[\overline{\mathfrak{Q}^{\mathfrak{H}_0}}, \mathfrak{B}] = \overline{\mathfrak{Q}^{\mathfrak{H}_0}}$ . Since  $\mathfrak{B}_2 = C(\mathfrak{Q}) \cap O_3(\mathbb{C})$  admits the abelian group  $\overline{\mathfrak{Q}^{\mathfrak{H}_0}}$ , (B) implies that  $\overline{\mathfrak{Q}^{\mathfrak{H}_0}}$  centralizes  $\mathfrak{B}_2$ . Hence,  $\overline{\mathfrak{Q}^{\mathfrak{H}_0}}$  is isomorphic to an elementary 2-subgroup of  $\text{Aut}(\mathfrak{B}_1)$ . Since  $[\overline{\mathfrak{Q}^{\mathfrak{H}_0}}, \mathfrak{B}] = \overline{\mathfrak{Q}^{\mathfrak{H}_0}}$ , we get that  $\overline{\mathfrak{Q}} = \overline{\mathfrak{Q}^{\mathfrak{H}_0}}$ , so that  $\overline{\mathfrak{Q}}$  is a  $S_2$ -subgroup of  $\mathfrak{K}$ .

Let  $\mathfrak{K}_1 = O_3(\mathfrak{K} \text{ mod } \mathfrak{R}_0)$ . Thus,  $\mathfrak{K}_1 \cap \mathfrak{G}_0$  is of index 3 in  $\mathfrak{G}_0$  and  $\langle \mathfrak{K}_1 \cap \mathfrak{G}_0, \mathfrak{B} \rangle = \mathfrak{G}_0$ . Since  $|\mathfrak{K}_0|$  is odd, it follows that  $\mathfrak{Q}$  is a  $S_2$ -subgroup of  $\mathfrak{K}$ . Let  $\mathfrak{L} = \mathfrak{K}_1 O_3(\mathbb{C})$  and let  $\mathfrak{L}_3$  be a  $S_2$ -subgroup of  $\mathfrak{L}$  which contains  $\mathfrak{K}_1 \cap \mathfrak{G}_0$  and is normalized by  $\mathfrak{G}_0$ . Since  $|\mathfrak{L}|$  is odd, it follows that  $S_2$ -subgroups of  $N(\mathfrak{L}_3) \cap \mathfrak{L}\mathfrak{K}$  are four-groups. If  $\mathfrak{Z} \subseteq D(\mathfrak{L}_3)$ , then by (B),  $S_2$ -subgroups of  $N(\mathfrak{L}_3) \cap \mathfrak{L}\mathfrak{K}$  centralize  $\mathfrak{L}_3$ . This is not the case, as  $\mathfrak{Q}$  does not centralize  $\mathfrak{B}_1$ . Hence,  $\mathfrak{Z} \not\subseteq D(\mathfrak{L}_3)$ . In particular,  $\mathfrak{Z} \not\subseteq D(\mathfrak{K}_1 \cap \mathfrak{G}_0)$ . But  $\mathfrak{K}_1 \cap \mathfrak{G}_0$  is of index 3 in  $\mathfrak{G}_0$ . Since  $\mathfrak{G}$  is extra special, it follows that  $w = 2$ . Clearly,  $\mathfrak{G} \subset \mathfrak{F}$ , since  $O_3(\mathbb{C})$  contains an elementary subgroup of order  $3^4$ . On the other hand, Lemma 9.3 implies that  $|\mathfrak{F} : \mathfrak{G}| \leq 3$ . Hence,  $|\mathfrak{F} : \mathfrak{G}| = 3$ , and  $|\mathfrak{F}| = 3^6$ . Since  $\mathfrak{B}_0$  is obviously not conjugate to  $\mathfrak{Z}$ , it follows that  $O_3(\mathbb{C})$  is elementary of order  $3^4$  and  $\mathfrak{F}_0 = O_3(\mathbb{C})\mathfrak{G}_0$ ,  $|\mathfrak{G}_0 \cap O_3(\mathbb{C})| = 27$ . Clearly,  $O_3(\mathbb{C}) \text{ char } \mathfrak{F}_0$ , since  $O_3(\mathbb{C})$  is the only elementary subgroup of its order in  $\mathfrak{F}_0$ .

Let  $\mathfrak{M} = N(O_3(\mathbb{C}))$  so that  $\mathfrak{M}$  contains a  $S_3$ -subgroup  $\mathfrak{F}$  of  $\mathfrak{G}$  with  $\mathfrak{F} \supset \mathfrak{F}_0$ . Since  $\mathfrak{Z} = Z(\mathfrak{F}_0) \cap \mathfrak{F}_0 \text{ char } \mathfrak{F}_0$ , we have  $\mathfrak{Z} \triangleleft \mathfrak{F}$ . In particular,  $\mathfrak{G} \subset \mathfrak{M}$ . We therefore assume without loss of generality that  $\mathfrak{F} = \mathfrak{F}$ .

It is clear that  $O_3(\mathbb{C}) = O_3(\mathfrak{M})$  and that  $\mathfrak{M}$  is a 2, 3-group. It is equally clear that  $l_3(\mathfrak{M}) = 2$ , so that  $\mathfrak{Q} \subseteq O_{3,2}(\mathfrak{M})$ . Hence,  $\mathfrak{B}$  is a  $S_3$ -subgroup of  $N(\mathfrak{Q}) \cap \mathfrak{M}$ , so we can choose a subgroup  $\mathfrak{B}_1$  of  $\mathfrak{B}$  of order 3 such that  $\mathfrak{B} = \mathfrak{B}_0 \times \mathfrak{B}_1$  and such that  $\mathfrak{B}_1$  normalizes some  $S_2$ -subgroup  $\mathfrak{X}_0$  of  $O_{3,2}(\mathfrak{M})$ . Let  $\mathfrak{A} = N(\mathfrak{X}_0) \cap \mathfrak{F}$ . Thus,  $\mathfrak{A}$  is elementary of order 9,  $\mathfrak{B}_1 \subset \mathfrak{A}$  and  $\mathfrak{F} = \mathfrak{A}O_3(\mathfrak{M})$ ,  $\mathfrak{A} \cap O_3(\mathfrak{M}) = 1$ . Since  $O_3(\mathfrak{M}) \cap C(\mathfrak{B}_1)$  is of order 9, it follows that  $C(\mathfrak{B}_1) \cap \mathfrak{F}$  is of order  $3^4$ . Hence,  $C(\mathfrak{B}_1) \cap \mathfrak{F} = C(\mathfrak{B}_1) \cap \mathfrak{G}$ ; since  $\mathfrak{A}$  is elementary we get  $\mathfrak{A} \subset \mathfrak{G}$ .

We now choose  $\mathfrak{A}_1$  of order 3 in  $\mathfrak{A}$  so that  $\mathfrak{A}$  does not centralize  $C_{\mathfrak{X}_0}(\mathfrak{A}_1)$ . Let  $\mathfrak{X}_1 = [C_{\mathfrak{X}_0}(\mathfrak{A}_1), \mathfrak{A}]$ . Thus,  $\mathfrak{X}_1$  is faithfully represented on  $C(\mathfrak{A}_1) \cap O_3(\mathfrak{M}) = \mathfrak{R}$ . It is straightforward to verify that  $|\mathfrak{R}| = 9$  and

that  $\mathfrak{X}_1$  is a quaternion group. Hence,  $\mathfrak{R} = [O_3(\mathfrak{M}), \mathfrak{X}_1]$ , so  $\mathfrak{R} \subseteq \mathfrak{G}$ . Since  $\mathfrak{Z} \subset \mathfrak{R}$ , it follows that  $\mathfrak{Z}$  is not weakly closed in  $\mathfrak{G}$ . As this violates Lemma 9.5, we conclude that Case 1 does not hold.

*Case 2.* There is an element  $H$  of  $\mathfrak{G}_0$  such that  $\langle \overline{\mathfrak{D}}, \overline{\mathfrak{D}}^H \rangle$  is nonabelian.

Set  $\mathfrak{W} = \langle \mathfrak{W}_1, \mathfrak{W}_1^H \rangle$ , so that  $\langle \mathfrak{D}, \mathfrak{D}^H \rangle$  normalizes  $\widetilde{\mathfrak{W}}$  and centralizes  $O_3(\mathbb{C})/\widetilde{\mathfrak{W}}$ . Since  $\mathfrak{W}_1 \cap \mathfrak{G} \supset \mathfrak{Z}$ , it follows that  $|\mathfrak{W}_1 \cap \mathfrak{W}_1^H| \geq 9$ . Clearly,  $\mathfrak{W}_1 \neq \mathfrak{W}_1^H$ , since  $\langle \mathfrak{D}, \mathfrak{D}^H \rangle$  is nonabelian. Hence,  $\widetilde{\mathfrak{W}}$  is elementary of order  $3^4$ . Since  $\langle \overline{\mathfrak{D}}, \overline{\mathfrak{D}}^H \rangle$  is injected into  $\text{Aut}(\widetilde{\mathfrak{W}})$  under the restriction map, it follows readily that  $\langle \overline{\mathfrak{D}}, \overline{\mathfrak{D}}^H \rangle$  is the central product of two quaternion groups, each of which necessarily admits  $\mathfrak{B}$ . In particular,  $\langle \overline{\mathfrak{D}}, \overline{\mathfrak{D}}^H \rangle$  is of order 2 and inverts  $\overline{\mathfrak{W}_1} \cap \mathfrak{G}$ . Since no involution of  $\mathfrak{G}$  centralizes  $\overline{\mathfrak{W}_1} \cap \mathfrak{G}$ , it follows that  $\overline{\mathfrak{D}}^{\mathfrak{G}_0}$  is extra special of order 32. Hence,  $[\overline{\mathfrak{D}}^{\mathfrak{G}_0}, O_3(\mathbb{C})]$  is elementary of order  $3^4$ . This implies that  $O_3(\mathbb{C})$  contains  $[\overline{\mathfrak{D}}^{\mathfrak{G}_0}, O_3(\mathbb{C})] \times \mathfrak{B}_0$ , an elementary subgroup of order  $3^5$ . Hence,  $w \geq 3$ .

Write  $O_3(\mathbb{C}) = \mathfrak{x} \times \mathfrak{y}$ , where

$$\mathfrak{x} = [O_3(\mathbb{C}), \overline{\mathfrak{D}}^{\mathfrak{G}_0}] \quad \text{and} \quad \mathfrak{y} = O_3(\mathbb{C}) \cap C(\overline{\mathfrak{D}}^{\mathfrak{G}_0}).$$

Thus,  $\mathfrak{G}_0$  normalizes both  $\mathfrak{x}$  and  $\mathfrak{y}$ . Suppose  $Y \in \mathfrak{y} \cap \mathfrak{G}$ . Then

$$[Y, \mathfrak{G}_0] \subseteq \mathfrak{Z} \cap \mathfrak{y} = 1, \quad \text{so} \quad Y \in Z(\mathfrak{G}_0) = \mathfrak{Z} \times \mathfrak{B}_0.$$

Hence,  $\mathfrak{y} \cap \mathfrak{G} = \mathfrak{B}_0$ . Since  $[\mathfrak{y}, \mathfrak{G}_0] \subseteq \mathfrak{G}$ , it follows that  $[\mathfrak{y}, \mathfrak{G}_0] \subseteq \mathfrak{B}_0$ . Since  $\overline{\mathfrak{D}}^{\mathfrak{G}_0}$  is absolutely irreducible on  $\mathfrak{x}$ , it follows that

$$[C_{\mathfrak{G}_0}(\overline{\mathfrak{D}}^{\mathfrak{G}_0}), O_3(\mathbb{C})] \subseteq \mathfrak{B}_0, \quad \text{so} \quad C_{\mathfrak{G}_0}(\overline{\mathfrak{D}}^{\mathfrak{G}_0}) \subseteq O_3(\mathbb{C}),$$

since  $O_3(\mathbb{C}) = O_3(\mathbb{C} \bmod \mathfrak{B}_0)$ .

Clearly,  $|\mathfrak{G}_0 : C_{\mathfrak{G}_0}(\overline{\mathfrak{D}}^{\mathfrak{G}_0})| = 3^a$ ,  $a = 1$  or  $2$ , since  $\overline{\mathfrak{D}}^{\mathfrak{G}_0}$  is extra special of order 32. If  $a = 1$ , then  $\mathfrak{G}_0 \cap O_3(\mathbb{C})$  is of index 9 in  $\mathfrak{G}$ , so is nonabelian since  $w \geq 3$ . This is impossible, since  $\mathfrak{Z} \not\subseteq D(O_3(\mathbb{C}))$ .

Suppose  $a = 2$ . Set  $\mathfrak{A} = \mathfrak{G}_0 \cap O_3(\mathbb{C})$ . Since  $\mathfrak{A}$  is abelian,  $w = 3$ . Thus,  $\mathfrak{A} \in \text{Sen}(\mathfrak{G})$ . Let  $\mathfrak{A}_1 = \mathfrak{x} \cap \mathfrak{A}$ , so that  $27 \geq |\mathfrak{A}_1| \geq 9$ . Suppose  $|\mathfrak{A}_1| = 9$ . Let  $\mathfrak{A}_2$  be a complement to  $\mathfrak{A}_1$  in  $\mathfrak{x}$ , so that  $|\mathfrak{A}_2| = 9$ , and  $\mathfrak{A}_2 \cap \mathfrak{G} = 1$ . Since  $\mathfrak{A}_2$  centralizes  $\mathfrak{A}$ , we get  $[\mathfrak{G}, \mathfrak{A}_2] \subseteq \mathfrak{G} \cap C(\mathfrak{A}) = \mathfrak{A}$ , so that  $[\mathfrak{G}, \mathfrak{A}_2, \mathfrak{A}_2] = 1$ . Thus,  $[Q_3^*(\mathfrak{N}), \mathfrak{A}_2]$  is a 2-group on which  $\mathfrak{A}_2$  is faithfully represented. This violates Lemma 9.3. Hence,  $\mathfrak{A}_1$  is of order  $3^3$ , so that  $\mathfrak{A} = \mathfrak{A}_1 \times \mathfrak{B}_0$ . Suppose  $\mathfrak{B}_0 \subset \mathfrak{y}$ . Let  $\mathfrak{y}_1$  be a subgroup of  $\mathfrak{y}$  of order 9 which contains  $\mathfrak{B}_0$ . Then  $\mathfrak{x}\mathfrak{y}_1$  is abelian of order  $3^5$ , and  $[\mathfrak{G}, \mathfrak{x}\mathfrak{y}_1] \subseteq \mathfrak{G} \cap C(\mathfrak{A}) = \mathfrak{A}$ , so that  $[\mathfrak{G}, \mathfrak{x}\mathfrak{y}_1, \mathfrak{x}\mathfrak{y}_1] = 1$ . It follows that  $[Q_3^*(\mathfrak{N}), \mathfrak{x}\mathfrak{y}_1]$  is a 2-group on which  $\mathfrak{x}\mathfrak{y}_1/\mathfrak{A}$  is faithfully represented. This again violates Lemma 9.3, so  $\mathfrak{y} = \mathfrak{B}_0$ .



Since  $\mathfrak{U} = \mathfrak{B}_0$  and  $a = 2$ ,  $O_3(\mathbb{C})$  is elementary of order  $3^5$  and  $|\mathfrak{B}_0| = 3^7$ . Lemma 5.2 implies that if  $U$  is any element of  $\mathbb{C}/O_3(\mathbb{C})$  of order 3, then  $C(U) \cap O_3(\mathbb{C})$  is of order at most  $3^3$ .

Suppose by way of contradiction that  $\mathfrak{U}$  is an elementary subgroup of  $\mathfrak{B}_0$  of order  $3^5$  which is distinct from  $O_3(\mathbb{C})$ . By the previous paragraph, we conclude that  $\mathfrak{U} \cap O_3(\mathbb{C})$  is of order  $3^3$ , and that if  $U \in \mathfrak{U} - O_3(\mathbb{C})$ , then  $O_3(\mathbb{C}) \cap C(U) = O_3(\mathbb{C}) \cap \mathfrak{U}$ . Let  $\mathfrak{U}_0$  be a complement to  $\mathfrak{U} \cap O_3(\mathbb{C})$  in  $\mathfrak{U}$ . Thus,  $\mathfrak{U}_0$  is faithfully represented on  $Q_3^*(\mathbb{R})$ , the central product of two quaternion groups. Let  $\mathfrak{R}$  be a quaternion subgroup of  $Q_3^*(\mathbb{R})$ , and let  $\mathfrak{U}_1 = C(\mathfrak{R}) \cap \mathfrak{U}_0$ . Thus,  $\mathfrak{U}_1$  is of order 3. By Lemma 3.7 of [20],  $\mathfrak{R}$  is faithfully represented on  $O_3(\mathbb{C}) \cap C(\mathfrak{U}_1)$ . This is absurd, since  $\mathfrak{U}_0$  centralizes  $O_3(\mathbb{C}) \cap C(\mathfrak{U}_1)$ . We conclude that  $O_3(\mathbb{C})$  is the only elementary subgroup of its order in  $\mathfrak{B}_0$ .

Since  $|\mathfrak{B}_0| = |\mathfrak{G}| = 3^7$  and since  $\mathfrak{B}_0$  is obviously not extra special it follows that  $\mathfrak{B}_0$  is not a  $S_3$ -subgroup of  $\mathfrak{G}$ . Hence,  $\mathfrak{B}_0$  is not a  $S_3$ -subgroup of  $N(O_3(\mathbb{C}))$ . Hence,  $A_{\mathfrak{G}}(O_3(\mathbb{C}))$  is a solvable subgroup of  $GL(5, 3)$  with  $S_3$ -subgroups of order at least 27 and with no nonidentity normal 3-subgroups. There are no such groups. The proof of Lemma 9.6 is complete.

LEMMA 9.7. *Every involution in  $\mathfrak{R}$  centralizes  $\mathfrak{Z}$ .*

*Proof.* Suppose false. Let  $\mathfrak{G}_1 = O_1(\mathfrak{G})$ , so that  $\mathfrak{G}_1$  is extra special of exponent 3 and width  $w \geq 2$ . Let  $\mathfrak{G}_0 = C_{\mathfrak{G}_1}(I)$  and let  $\mathfrak{G}_2$  be the set of elements of  $\mathfrak{G}_1$  inverted by  $I$ . Here  $I$  is an involution of  $\mathfrak{R}$  which does not centralize  $\mathfrak{Z}$ . Since  $\mathfrak{Z} \subseteq \mathfrak{G}_2$ , it follows that  $\mathfrak{G}_0$  is abelian. Since  $I$  centralizes  $[H, H']$  for all  $H, H'$  in  $\mathfrak{G}_2$ , it follows that  $\langle \mathfrak{G}_2 \rangle$  is abelian. Hence,  $\mathfrak{G}_2 = \langle \mathfrak{G}_2 \rangle$ . As is well known,  $\mathfrak{G}_1 = \mathfrak{G}_0 \mathfrak{G}_2$  and  $\mathfrak{G}_0 \cap \mathfrak{G}_2 = 1$ . Hence,  $\mathfrak{G}_2$  is elementary of order  $3^{w+1}$  and  $\mathfrak{G}_0$  is elementary of order  $3^w$ .

By Lemmas 7.5 and 9.6, there is a subgroup  $\mathfrak{M}$  in  $\mathcal{MS}(\mathfrak{G})$  with  $\mathfrak{G}_0 \subseteq \mathfrak{M}$  such that  $\mathfrak{M}$  satisfies Hypothesis 7.2 and  $p = 2$ . Let  $w_1$  be the width of  $O_2(\mathfrak{M})$ . Hence,  $w \leq w_1 \leq 4$ , the first inequality holding since  $\mathfrak{G}_0$  is faithfully represented on  $O_2(\mathfrak{M})$ , the second inequality holding by Lemma 7.5.

Suppose  $w \geq 3$ . Hence,  $w_1 \geq 3$ . If  $H \in \mathfrak{G}_0^*$  and  $C(H) \cap O_2(\mathfrak{M})$  contains a four-subgroup  $\mathfrak{B}$  containing  $l_1(Z(O_2(\mathfrak{M})))$ , then by Lemma 7.2, both  $\langle H, \mathfrak{B} \rangle$  and  $\mathfrak{B}$  satisfy the hypothesis of Lemma 7.4, so  $C(H)$  is nonsolvable. This is impossible, so  $H$  is not available. This implies that  $w = 2$ .

Suppose  $\mathfrak{G} = \mathfrak{B}$ . In this case, if  $H$  is any element of order 3 in  $\mathfrak{G}$ , then  $\mathfrak{Z} \text{ char } C_{\mathfrak{G}}(H)$ . This implies immediately that  $\mathfrak{Z}$  is weakly closed in  $\mathfrak{B}$ , which in turn implies that  $\mathfrak{R}$  contains the centralizer of each of its nonidentity 3-elements. This implies that  $O_2(\mathfrak{M}) \subseteq \mathfrak{R}$ , which

is not the case. Hence  $\mathfrak{H} \subset \mathfrak{P}$ .

Since every 3, 5-subgroup of  $S_4(3)$  is either a 3-group or a 5-group, it follows from the preceding paragraph that  $\mathfrak{N}$  is a 2, 3-group,  $S_4(3)$  being a 2, 3, 5-group. By Lemma 9.3, it then follows that  $O^{3'}(\mathfrak{N})/\mathfrak{H} \cong SL(2, 3)$ . Furthermore, if  $J$  is an involution of  $O^{3'}(\mathfrak{N})$ , then  $C_{\mathfrak{H}}(J) \triangleleft \mathfrak{N}$ . It follows that  $J$  inverts  $\mathfrak{H}/Z(\mathfrak{H})$ .

If  $I$  centralizes  $O^{3'}(\mathfrak{N})/O^{3'}(\mathfrak{N})'$ , we conclude that  $I$  centralizes  $O^{3'}(\mathfrak{N})/\mathfrak{H}$ . But  $C(I) \subseteq \mathfrak{M}$ , so in particular,  $C_{\mathfrak{N}}(I) \subseteq \mathfrak{M}$ . Since  $I$  centralizes  $O^{3'}(\mathfrak{N})/\mathfrak{H}$ , it follows that  $I$  centralizes a  $S_2$ -subgroup  $\mathfrak{Q}$  of  $O^{3'}(\mathfrak{N})$ . Hence,  $\mathfrak{Q}$  normalizes  $\mathfrak{H}_0$ . Hence,  $\mathfrak{Q}\mathfrak{H}_0$  is of index 3 in  $C(I) \cap O^{3'}(\mathfrak{N})$ . Let  $\mathfrak{Q}' = \langle J \rangle$ . By the preceding paragraph,  $\mathfrak{Q}$  is faithfully represented on  $\mathfrak{H}_0$ . Thus,  $\mathfrak{D} = C(I) \cap N(\mathfrak{Q}) \cap O^{3'}(\mathfrak{N}) \cong SL(2, 3)$  and  $\mathfrak{D}$  is faithfully represented on  $\mathfrak{H}_0$ .

Since  $\mathfrak{H}_0$  is faithfully represented on  $O_2(\mathfrak{M})$ , so is  $\mathfrak{H}_0\mathfrak{D}$ . Since  $\mathfrak{H}_0 \cap \mathfrak{D} = 1$ ,  $S_3$ -subgroups of  $\mathfrak{H}_0\mathfrak{D}$  are of exponent 3. Since the four subgroups of  $\mathfrak{H}_0$  of order 3 are permuted transitively by  $\mathfrak{D}$ , it follows that  $w_1 \geq 4$ . Hence,  $w_1 = 4$  and  $O_2(\mathfrak{M})$  is extra special. Let  $\mathfrak{P}_0$  be a  $S_3$ -subgroup of  $\mathfrak{H}_0\mathfrak{D}$ . We can choose  $P$  in  $\mathfrak{P}_0 - \mathfrak{H}_0$  such that  $C(P) \cap O_2(\mathfrak{M})$  contains a four-group. Since  $C_{\mathfrak{N}}(P)$  clearly contains an element of  $\mathcal{Z}(3)$ , Lemma 7.4 is violated. We conclude that  $I$  does not centralize  $O^{3'}(\mathfrak{N})/O^{3'}(\mathfrak{N})'$ .

Since  $\text{Aut}(Z(\mathfrak{H}))$  is abelian, the preceding paragraph implies that  $Z(\mathfrak{H}) = Z(\mathfrak{P})$ .

Since  $O_2(\mathfrak{M}) \not\subseteq \mathfrak{N}$ , we can choose  $H$  in  $\mathfrak{H}_0^*$  such that  $C(H) \not\subseteq \mathfrak{N}$ .

Let  $|Z(\mathfrak{H})| = 3^a$ , and suppose  $a \geq 2$ . Let  $\mathfrak{P}$  be a  $S_3$ -subgroup of  $C_{\mathfrak{N}}(H)$ . Thus,  $Z(\mathfrak{H}) \subseteq Z(\mathfrak{P})$ , and  $\mathfrak{Z} = \sigma^{a-1}(Z(\mathfrak{P})) \text{ char } \mathfrak{P}$ , whence  $\mathfrak{P}$  is a  $S_3$ -subgroup of  $C(H)$ . By Lemma 7.2 applied to  $\langle H, \mathfrak{Z} \rangle$ , it follows that  $\mathfrak{Z}$  centralizes  $O_3(C(H))$ , and so

$$[O_3(C(H)), C_{\mathfrak{H}}(H)] \subseteq \mathfrak{H} \cap O_3(C(H)) = 1.$$

By Lemma 9.4, we have  $O_3(C(H)) = 1$ . Let  $\mathfrak{P}_1 = O_3(C(H)) \subseteq \mathfrak{P}$ . Thus,  $Z(\mathfrak{H}) \subseteq Z(\mathfrak{P}_1)$ , and we get  $\mathfrak{Z} = \sigma^{a-1}(Z(\mathfrak{P}_1))$ , whence  $C(H) \subseteq \mathfrak{N}$ . This contradiction forces  $a = 1$ ,  $Z(\mathfrak{H}) = \mathfrak{Z}$ ,  $|\mathfrak{P}| = 3^6$ .

Throughout the remainder of this lemma, the following notation is used:  $\mathfrak{Q}$  is a  $S_2$ -subgroup of  $O^{3'}(\mathfrak{N})$  normalized by  $I$ . Since  $\mathfrak{Q}$  is a quaternion group, our preceding information implies the existence of an element  $Q$  in  $\mathfrak{Q}$  of order 4 such that  $IQI = Q^{-1}$ . Let  $J = Q^2$ . Thus,  $J$  centralizes  $\mathfrak{Z}$  and inverts  $\mathfrak{H}/\mathfrak{Z}$ .

We argue that  $\mathfrak{G}$  is not 3-normal. Namely, for some  $H$  in  $\mathfrak{H}_0^*$ , we have  $\mathfrak{C} = C(H) \not\subseteq \mathfrak{N}$ . If  $|\mathfrak{C}|_3 = |\mathfrak{G}|_3$ , then  $\langle H \rangle$  is a conjugate of  $\mathfrak{Z}$  contained in  $\mathfrak{H}$ , and we are done. Otherwise, it is clear that  $\mathfrak{C} \cap \mathfrak{N}$  contains a  $S_3$ -subgroup of  $\mathfrak{C}$  and since  $\mathfrak{Z} \subseteq \mathfrak{C}$ ,  $O_3(\mathfrak{C})$  contains at least two conjugates of  $\mathfrak{Z}$ . As  $O_3(\mathfrak{C}) \subseteq \mathfrak{N}$ , we again are done.

We next argue that  $\mathfrak{Z}$  is not weakly closed in  $\mathfrak{H}$ . Choose  $G$  in

$\mathfrak{G}$  such that  $\mathfrak{Z}_1 = \mathfrak{Z}^\sigma \subseteq \mathfrak{P}$  and  $\mathfrak{Z} \neq \mathfrak{Z}_1$ . If  $\mathfrak{Z}_1 \subseteq \mathfrak{H}$ , we are done. Otherwise, let  $\mathfrak{P}_0 = C_{\mathfrak{P}}(\mathfrak{Z}_1) = \mathfrak{Z}_1 \times C_{\mathfrak{H}}(\mathfrak{Z}_1)$ . Since  $\mathfrak{P}_0$  is not a  $S_3$ -subgroup of  $\mathfrak{G}$  but  $\mathfrak{P}_0$  is a  $S_3$ -subgroup of  $C_{\mathfrak{P}}(\mathfrak{Z}_1)$ , it follows that  $\mathfrak{Z}$  ch/ar  $\mathfrak{P}_0$ . This implies that  $\mathfrak{P}_0$  is elementary. Clearly,  $27 \leq |\mathfrak{P}_0| \leq 81$ , since  $w = 2$ . We assume without loss of generality that  $\mathfrak{P} \cap \mathfrak{P}^\sigma = \mathfrak{P}_0$ . If  $\mathfrak{Z} \subseteq \mathfrak{H}^\sigma$ , we are done, so we may assume that  $\mathfrak{Z} \not\subseteq \mathfrak{H}^\sigma$ , which yields  $\mathfrak{P}_0 = \mathfrak{Z} \times (\mathfrak{P}_0 \cap \mathfrak{H}^\sigma)$ . In particular,  $\mathfrak{H} \cap \mathfrak{H}^\sigma \neq 1$ ; let  $\mathfrak{B} = \mathfrak{H} \cap \mathfrak{H}^\sigma$ , a group of order at least 3. Suppose  $|\mathfrak{B}| = 3$ . Let  $\mathfrak{R} = C(B)$ , so that  $|\mathfrak{R} \cap \mathfrak{P}| = |\mathfrak{R} \cap \mathfrak{P}^\sigma| = 3^5$ . If  $|\mathfrak{R}|_3 = 3^6$ , we are done, so we may assume that  $|\mathfrak{R}|_3 = 3^5$ . In this case, we see that  $|O_3(\mathfrak{R})| = 3^4$ , which implies that  $|O_3(\mathfrak{R}) \cap \mathfrak{H}| \geq 27$ ,  $|O_3(\mathfrak{R}) \cap \mathfrak{H}^\sigma| \geq 27$ . Hence,  $|\mathfrak{H} \cap \mathfrak{H}^\sigma| \geq 9$ , contrary to assumption. Thus, we may assume that  $|\mathfrak{B}| = 9$ . We may also assume that  $\mathfrak{B}$  contains no conjugate of  $\mathfrak{Z}$ . We have  $\mathfrak{P}_0 = \mathfrak{Z} \times \mathfrak{Z}_1 \times \mathfrak{B}$ . We argue that  $\mathfrak{P}_0 \triangleleft \langle \mathfrak{P}, \mathfrak{P}^\sigma \rangle$ . Namely, let  $\mathfrak{P}_0 \subset \mathfrak{P}_1 \subset \mathfrak{P}$ . If  $\mathfrak{P}_0$  char  $\mathfrak{P}_1$ , then clearly  $\mathfrak{P}$  normalizes  $\mathfrak{P}_0$ . Suppose  $\mathfrak{P}_0$  ch/ar  $\mathfrak{P}_1$ . Then  $\mathfrak{P}_1 = \mathfrak{P}_0 \mathfrak{P}_0^*$ , where  $\mathfrak{P}_0^*$  is elementary of order  $3^4$ . Hence,  $Z(\mathfrak{P}_1) = \mathfrak{P}_0 \cap \mathfrak{P}_0^*$  is of order 27. This implies that  $\mathfrak{P}_1$  is of order 3. Since  $\mathfrak{P}_1 \cap \mathfrak{H}$  is nonabelian, we have  $\mathfrak{P}_1 = \mathfrak{Z}$ . Thus,  $\mathfrak{Z}_1$  centralizes  $(\mathfrak{P}_1 \cap \mathfrak{H})/\mathfrak{Z}$ . This is not the case, since involutions of  $O^p(\mathfrak{R})$  invert  $\mathfrak{H}/\mathfrak{Z}$ , so that the action of  $\mathfrak{Z}_1$  on  $\mathfrak{H}/\mathfrak{Z}$  is given either by  $J_3 \oplus J_1$  or by  $J_2 \oplus J_2$ . By symmetry, we have  $\mathfrak{P}_0 \triangleleft \langle \mathfrak{P}, \mathfrak{P}^\sigma \rangle$ . It is easy to verify that  $O_3(N(\mathfrak{P}_0))$  is of order  $3^5$ , which implies that  $|\mathfrak{H} \cap \mathfrak{H}^\sigma| \geq 27$ , the desired contradiction.

Since all parts of Hypothesis 9.3 are satisfied, Lemma 9.5 is violated. The proof of the lemma is complete.

LEMMA 9.8.  $\mathfrak{R}$  is the only element of  $\mathcal{MS}(\mathfrak{G})$  which contains  $\mathfrak{H}$ .

*Proof.* Suppose false. Choose  $\mathfrak{R} \in \mathcal{MS}(\mathfrak{G})$  so that  $\mathfrak{H} \subseteq \mathfrak{R} \not\subseteq \mathfrak{R}$ , and with this restriction, minimize  $|\mathfrak{R}|$ . Since  $\mathfrak{N}(\mathfrak{H})$  contains only 1, Lemma 0.7.6 implies that  $l_3(\mathfrak{R}) \leq 2$ . If  $l_3(\mathfrak{R}) = 1$ , then  $\mathfrak{Z} \triangleleft \mathfrak{R}$ , contrary to assumption. Hence,  $l_3(\mathfrak{R}) = 2$ ; and  $\mathfrak{R}$  is a 3,  $p$ -group for some prime  $p$ . Furthermore,  $\mathfrak{H}$  acts irreducibly on  $O_{3,p}(\mathfrak{R})/D(O_{3,p}(\mathfrak{R} \text{ mod } O_3(\mathfrak{R})))$ . Let  $\mathfrak{H}_0 = \mathfrak{H} \cap O_3(\mathfrak{R})$ ,  $\mathfrak{B} = \Omega_1(Z(O_3(\mathfrak{R})))$ . By Lemma 9.4,  $\mathfrak{B} \subseteq \mathfrak{H}$ , so  $|\mathfrak{B}| \leq 9$ . Thus,  $O_{3,p}(\mathfrak{R})/O_3(\mathfrak{R})$  is a quaternion group whose involution inverts  $\mathfrak{B}$ . Since  $\mathfrak{Z} \subset \mathfrak{B}$ , Lemma 9.7 is violated. The proof is complete.

LEMMA 9.9. Every involution  $I$  of  $\mathfrak{R}$  inverts  $\mathfrak{H}/Z(\mathfrak{H})$ .

*Proof.* By Lemma 9.7,  $I$  centralizes  $\mathfrak{Z}$ , so centralizes  $Z(\mathfrak{H})$ . If the lemma is false, then  $C_{\mathfrak{H}}(I)$  contains a subgroup  $\mathfrak{A}$  of type  $(3, 3)$  with  $\mathfrak{A} \supset \mathfrak{Z}$ . This violates Lemma 9.3 (iii).

LEMMA 9.10. If  $\mathfrak{A} \in \mathcal{A}_4(\mathfrak{P})$ , then  $\mathfrak{R}$  is the only element of  $\mathcal{MS}(\mathfrak{G})$  which contains  $\mathfrak{A}$ .

*Proof.* As in  $O$ , let  $\mathcal{A}_1 = \{\mathfrak{A} \mid \mathfrak{A} \text{ is a 3-subgroup of } \mathfrak{N}, \mathfrak{A} \text{ contains an element of } \mathcal{S}\mathcal{E}\mathcal{N}_3(\mathfrak{B}^N) \text{ for some } N \text{ in } \mathfrak{N}, \mathcal{A}_{n+1} = \{\mathfrak{A} \mid \mathfrak{A} \text{ is a 3-subgroup of } \mathfrak{N}, \mathfrak{A} \text{ contains a subgroup } \mathfrak{B} \text{ of type } (3, 3), C_{\mathfrak{M}}(B) \text{ contains an element of } \mathcal{A}_n \text{ for all } B \text{ in } \mathfrak{B}\}$ . Among all  $\mathfrak{A} \in \mathcal{A}_1$  which violate the conclusion of the lemma, maximize  $|\mathfrak{A} \cap \mathfrak{G}|$ , and with this restriction, maximize  $|\mathfrak{A}|$ . By Lemma 9.8,  $\mathfrak{G} \not\subseteq \mathfrak{A}$ . Let  $\mathfrak{M} \in \mathcal{MS}(\mathfrak{G})$  with  $\mathfrak{A} \subseteq \mathfrak{M}$ ,  $\mathfrak{M} \neq \mathfrak{N}$ . By maximality of  $|\mathfrak{A}|$ , it follows that  $\mathfrak{A}$  is a  $S_3$ -subgroup of  $\mathfrak{M}$ . We can therefore choose a prime  $q$  and a  $q$ -subgroup  $\mathfrak{Q}$  of  $\mathfrak{M}$  permutable with  $\mathfrak{A}$  such that  $\mathfrak{Z} = \mathfrak{A}\mathfrak{Q}$  is not contained in  $\mathfrak{N}$ . Let  $\mathfrak{Q}$  be minimal with these properties. By Lemma 0.7.6,  $l_3(\mathfrak{Z}) \leq 2$ .

We first show that  $O_q(\mathfrak{Z}) \subseteq \mathfrak{N}$ . Suppose  $\mathfrak{A} \cap \mathfrak{G}$  is noncyclic. Let  $\mathfrak{B}$  be a subgroup of  $\mathfrak{A} \cap \mathfrak{G}$  of type  $(3, 3)$ . It suffices to show that  $C(B) \subseteq \mathfrak{N}$  for all  $B \in \mathfrak{B}^*$ . Suppose false. Then maximality of  $|\mathfrak{A} \cap \mathfrak{G}|$  yields  $|\mathfrak{G} : \mathfrak{A} \cap \mathfrak{G}| \leq 3$ . In this case,  $\mathcal{N}(\mathfrak{A} \cap \mathfrak{G}) = 1$ , so  $O_q(\mathfrak{Z}) = 1$ . Thus, we may assume that  $\mathfrak{A} \cap \mathfrak{G}$  is cyclic. Since  $w \geq 2$ , it follows that if  $P$  is any element of  $\mathfrak{B}$  of order 3, then  $C_{\mathfrak{G}}(P)$  is noncyclic. Hence, every subgroup of  $\mathfrak{N}$  of type  $(3, 3)$  is in  $\mathcal{A}_1$ . Since  $\mathfrak{A}$  contains a subgroup of type  $(3, 3)$ , maximality of  $|\mathfrak{A} \cap \mathfrak{G}|$  implies that  $C(A) \subseteq \mathfrak{N}$  for all elements  $A$  of  $\mathfrak{A}$  of order 3. Thus, in all cases, we have  $O_q(\mathfrak{Z}) \subseteq \mathfrak{N}$ .

By minimality of  $\mathfrak{Q}$ ,  $O_{q,3}(\mathfrak{Z}) = O_q(\mathfrak{Z}) \times O_3(\mathfrak{Z})$ . Since  $l_3(\mathfrak{Z}) \leq 2$ , it follows that  $l_3(\mathfrak{Z}) = 2$ , by maximality of  $|\mathfrak{A}|$  and the structure of  $O_{q,3}(\mathfrak{Z})$ . Since  $D(\mathfrak{Q})$  is permutable with  $\mathfrak{A}$ , we get  $D(\mathfrak{Q}) \subseteq \mathfrak{N}$ , by minimality of  $\mathfrak{Q}$ .

Clearly,  $\mathfrak{A}$  is a  $S_3$ -subgroup of  $N(O_3(\mathfrak{Z}))$ . Hence,  $\mathfrak{Z} \subseteq Z(O_3(\mathfrak{Z}))$ . Since  $\mathfrak{Q}O_3(\mathfrak{Z}) \triangleleft \mathfrak{Z}$ , and since  $\mathfrak{A}$  is a  $S_3$ -subgroup of  $N(O_3(\mathfrak{Z}))$ , and since  $\mathfrak{G} \not\subseteq \mathfrak{A}$ , it follows that  $\mathfrak{G} \cap \mathfrak{A}$  acts nontrivially on  $Q_3(\mathfrak{Z})$ , but trivially on every proper  $\mathfrak{A}$ -invariant subgroup of  $Q_3(\mathfrak{Z})$ . Since  $D(\mathfrak{Q})$  centralizes  $\mathfrak{Z}$ , it follows that  $D(\mathfrak{Q})$  centralizes  $\mathfrak{B} = \mathfrak{Z}^{\mathfrak{G}}$ .

If  $q \geq 5$ , then maximality of  $\mathfrak{A}$  and Theorem 1 of [39] imply that  $\mathfrak{A} = \mathfrak{B}$ , against Lemma 9.8. Hence,  $q = 2$ . We may apply Theorem 1 of [39] once again and conclude that  $D(\mathfrak{Q}) \neq 1$ . By Lemma 9.9, each element of  $D(\mathfrak{Q})^*$  inverts  $\mathfrak{G}/\mathfrak{Z}(\mathfrak{G})$ . Since  $Z(\mathfrak{G})$  is a normal cyclic subgroup of  $\mathfrak{B}$ , it follows that  $\mathfrak{A} \cap Z(\mathfrak{G}) \subseteq O_3(\mathfrak{Z})$ . Since  $\mathfrak{A} \cap \mathfrak{G} \not\subseteq O_3(\mathfrak{Z})$ , choose  $H \in \mathfrak{A} \cap \mathfrak{G} - O_3(\mathfrak{Z})$ . Let  $I$  be the element in  $D(\mathfrak{Q})^*$ . Then  $H^I = H^{-1}H_0$  with  $H_0$  in  $Z(\mathfrak{G})$ . Since  $H_0 \in \mathfrak{A}$ , it follows that  $[H, I]$  is contained in  $\mathfrak{A} \cap \mathfrak{G} \cap O_{3,q}(\mathfrak{Z}) \subseteq \mathfrak{A} \cap O_3(\mathfrak{Z})$ . This violates the fact that  $\mathfrak{A} \cap \mathfrak{G} \not\subseteq O_3(\mathfrak{Z})$ . The proof is complete.

It is now easy to show that Hypothesis 9.2 is not satisfied. Otherwise,  $\mathfrak{N}$  contains a four-subgroup  $\mathfrak{T}$ . But by Lemma 9.9, each element of  $\mathfrak{T}^*$  inverts  $\mathfrak{G}/Z(\mathfrak{G})$ . This is not possible, since  $\mathfrak{G} = \langle C_{\mathfrak{G}}(J) \mid J \in \mathfrak{T}^* \rangle$ .

The remaining lemmas in this section are proved on the hypothesis

that  $\mathfrak{G}$  contains a noncyclic characteristic abelian subgroup.

Among all noncyclic normal elementary subgroups of  $\mathfrak{N}$ , let  $\mathfrak{E}$  be minimal. Thus,  $\mathfrak{E}/\mathfrak{Z}$  is a chief factor of  $\mathfrak{N}$ . Let  $\mathcal{E} : \mathfrak{E} \supset \mathfrak{Z} \supset 1$ . We will show that  $A_{\mathfrak{G}}(\mathcal{E}) = A(\mathcal{E})$ . First, suppose  $\mathfrak{E}$  is not 3-reducible in  $\mathfrak{N}$ . Let  $\mathfrak{X} = O_3(\mathfrak{N} \text{ mod } C(\mathfrak{E}))$ . Since  $\mathfrak{E}/\mathfrak{Z}$  is a chief factor of  $\mathfrak{N}$ , we have  $[\mathfrak{X}, \mathfrak{E}] = \mathfrak{Z}$ , and  $\mathfrak{Z} = C_{\mathfrak{E}}(\mathfrak{X})$ . These equalities imply immediately that  $\mathfrak{X}$  maps onto  $A(\mathcal{E})$ . Suppose  $\mathfrak{E}$  is 3-reducible in  $\mathfrak{N}$ . Let  $\mathfrak{X} = O_3(\mathfrak{N} \text{ mod } C(\mathfrak{E}))$ . Then  $\mathfrak{Z} = C_{\mathfrak{E}}(\mathfrak{X})$  and  $[\mathfrak{X}, \mathfrak{E}]$  admits  $N_{\mathfrak{N}}(\mathfrak{X}) = \mathfrak{N}$ . Since  $A_{\mathfrak{X}}(\mathfrak{E})$  is a 3'-group it follows that  $[\mathfrak{X}, \mathfrak{E}]$  is a normal subgroup of  $\mathfrak{N}$  disjoint from  $\mathfrak{Z}$ . Hence,  $[\mathfrak{X}, \mathfrak{E}] = 1$ , since  $\mathfrak{Z}$  is the only minimal normal subgroup of  $\mathfrak{N}$ . Since  $\mathfrak{E}$  is 3-reducible in  $\mathfrak{N}$  and  $O_3(\mathfrak{N} \text{ mod } C(\mathfrak{E}))$ , it follows that  $\mathfrak{E} \subseteq Z(\mathfrak{N})$ . This is absurd since  $\mathfrak{Z}$  is the only minimal normal subgroup of  $\mathfrak{N}$ . Thus,  $A_{\mathfrak{G}}(\mathcal{E}) = A(\mathcal{E})$ .

Throughout the remainder of this section, the following notation is used:  $\mathfrak{F}, \mathfrak{Z}, \mathfrak{N}$  are as before, and  $\mathfrak{E}$  is a noncyclic normal elementary subgroup of  $\mathfrak{N}$  such that  $\mathfrak{E}/\mathfrak{Z}$  is a chief factor of  $\mathfrak{N}$ . Also,  $\mathcal{E} : \mathfrak{E} \supset \mathfrak{Z} \supset 1$ .

LEMMA 9.11. (i) If  $\mathfrak{S}$  is a 2, 3-subgroup of  $\mathfrak{G}$  and  $\mathfrak{S}$  contains an element  $\mathfrak{A}$  of  $\mathcal{E}(3)$ , then  $O_2(\mathfrak{S}) = 1$ .

(ii) If  $\mathfrak{A} \in \mathcal{E}(3)$ , then  $|C(\mathfrak{A})|$  is odd.

*Proof.* (i) Suppose  $I$  is an involution in  $O_2(\mathfrak{S})$ . Since  $\mathfrak{A}$  centralizes  $O_2(\mathfrak{S})$ , Lemmas 7.4 and 5.38 imply that  $C(I)$  is nonsolvable.

(ii) Suppose  $I$  is an involution of  $C(\mathfrak{A})$ . Then  $\mathfrak{A} \times \langle I \rangle$  violates (i).

LEMMA 9.12. (i) If  $I$  is an involution of  $C(\mathfrak{Z})$ , then  $I$  inverts  $\mathfrak{E}/\mathfrak{Z}$ .

(ii)  $C(\mathfrak{Z})$  contains no four-group.

(iii) If  $\mathfrak{X}$  is an abelian 2-subgroup of  $\mathfrak{N}$ , then  $A_{\mathfrak{N}}(\mathfrak{X})$  is a 2-group.

*Proof.* (i) is a consequence of Lemmas 9.11 and 7.3, and (ii), (iii) are consequences of (i).

LEMMA 9.13.  $\mathfrak{N}$  does not contain a noncyclic abelian subgroup of order 8.

*Proof.* Suppose false. Let  $\mathfrak{Q}_3^*$  be a  $S_2$ -subgroup of  $\mathfrak{N}$  permutable with  $\mathfrak{F}$ , and let  $\mathfrak{N}_0 = \mathfrak{F}\mathfrak{Q}_3^*$ . Let  $\mathfrak{Q}_0 = \mathfrak{Q}_3^* \cap O^3(\mathfrak{N}_0)$ . Thus,  $\mathfrak{Q}_0$  is either a quaternion group or  $\mathfrak{Q}_0 = 1$ . Let  $\mathfrak{Q}$  be a subgroup of  $\mathfrak{Q}_3^*$  which contains  $\mathfrak{Q}_0$ , is permutable with  $\mathfrak{F}$ , contains a noncyclic abelian subgroup of order 8, and is minimal with these properties. Let  $\mathfrak{N}_1 = \mathfrak{F}\mathfrak{Q}$ . Thus,  $\mathfrak{Q}$  is abelian of type (2, 4) if and only if every 2, 3-subgroup of  $\mathfrak{N}$  is 3-closed. If  $\mathfrak{Q}_0 \neq 1$ , then  $|\mathfrak{Q}| = 2^4$  and  $\mathfrak{Q}$  is the central product of a group of order 2 and  $\mathfrak{Q}_0$  or  $\mathfrak{Q}$  is the central product of a cyclic group of order 4 and  $\mathfrak{Q}_0$ . Let  $\mathfrak{F}/\mathfrak{Z}$  be a chief factor of  $\mathfrak{N}_1$

with  $\mathfrak{F} \subseteq \mathfrak{C}$ . Let  $\mathfrak{P}_0 = O_3(\mathfrak{N}_1)$ ,  $\mathfrak{P}_1 = C_{\mathfrak{P}_0}(\mathfrak{F})$ . Since  $A_{\mathfrak{P}_1}(\mathfrak{C}) = A(\mathfrak{C})$ , so also  $A_{\mathfrak{P}_1}(\mathfrak{C}_0) = A(\mathfrak{C}_0)$  where  $\mathfrak{C}_0: \mathfrak{F} \supset \mathfrak{Z} \supset 1$ . Hence,  $\mathfrak{P}_0/\mathfrak{P}_1$  is also a chief factor of  $\mathfrak{N}_1$  with the same order as  $\mathfrak{F}/\mathfrak{Z}$ . If  $\Omega' = 1$ , then

$$(9.19) \quad \mathfrak{P} \triangleleft \mathfrak{N}_1, \Omega \text{ is of type } (4, 2), \text{ and } |\mathfrak{F}:\mathfrak{Z}| = 9.$$

Suppose  $\Omega' \neq 1$ . If  $\Omega = \Omega_0 \times \Omega_1$ , where  $|\Omega_1| = 2$ , then

$$(9.20) \quad \mathfrak{N}_1/\mathfrak{P}_0 \cong SL(2, 3) \times Z_2, \text{ and } |\mathfrak{F}:\mathfrak{Z}| = 9.$$

Suppose  $\Omega$  is the central product of  $\Omega_0$  and a cyclic group of order 4. Then

$$(9.21) \quad \begin{aligned} \mathfrak{N}_1/\mathfrak{P}_0 \text{ is the central product of} \\ SL(2, 3) \text{ and } Z_4, \text{ and } |\mathfrak{F}:\mathfrak{Z}| = 3^4. \end{aligned}$$

By Lemmas 5.41 and 9.12, (9.19), (9.20), (9.21) exhaust all possibilities.

It is clear from Lemma 9.12 that

$$(9.22) \quad \text{if (9.19) holds, a } S_{2,3}\text{-subgroup of } \mathfrak{N} \text{ is 3-closed.}$$

We next will show that

$$(9.23) \quad \text{every subgroup of } \mathfrak{F} \text{ of order 9 is in } \mathcal{S}.$$

To see this, let  $\mathfrak{F}_0$  be a subgroup of  $\mathfrak{F}$  of order 9. If  $\mathfrak{Z} \subset \mathfrak{F}_0$ , then  $\mathfrak{F}_0 \in \mathcal{E}(3) \subseteq \mathcal{S}$ . Thus, we may assume that  $\mathfrak{F}_0 \cap \mathfrak{Z} = 1$ . Let  $\mathfrak{X}$  be an abelian subgroup in  $\mathfrak{N}(\mathfrak{F}_0; 2)$  and assume by way of contradiction that  $[\mathfrak{X}, \mathfrak{F}_0] \neq 1$ . We may assume that  $\mathfrak{X}$  is a four-group. Let  $\mathfrak{F}_1 = \mathfrak{F}_0 \cap C(\mathfrak{X})$ , a group of order 3. Let  $\mathfrak{C} = C(\mathfrak{F}_1) \cong \langle \mathfrak{F}, \mathfrak{X} \rangle$ . Since  $m(\mathfrak{F}) \geq 3$  and  $\mathfrak{F} \triangleleft \mathfrak{P}$ , there is  $\mathfrak{A} \in \mathcal{SBN}_3(\mathfrak{P})$  with  $\mathfrak{F} \subseteq \mathfrak{A}$ . Hence,  $\mathfrak{A} \subseteq \mathfrak{C} = C(\mathfrak{F}_1)$  implies  $O_3(\mathfrak{C}) = 1$  by hypothesis (ii) of Theorem 9.1.

By Lemma 5.5,  $\mathfrak{Z} \subseteq O_3(\mathfrak{C})$ . Let  $\mathfrak{B} = \Omega_1(\mathbf{Z}(O_3(\mathfrak{C})))$ , and let  $\mathfrak{P}^*$  be a  $S_3$ -subgroup of  $\mathfrak{C}$ . Let  $\mathfrak{P}^{\sigma}$  be a  $S_3$ -subgroup of  $\mathfrak{C}$  which contains  $\mathfrak{P}^*$ . Then  $\mathfrak{Z}^{\sigma} \subset \mathfrak{P}^*$ , so  $\mathfrak{Z}^{\sigma} \subseteq \mathfrak{B}$ . By Lemma 9.12 (iii),  $\mathfrak{X}$  is faithfully represented on  $\mathfrak{B}$ . Hence, if  $F \in \mathfrak{F}_0 - \mathfrak{F} \cap C(\mathfrak{X})$ , then the minimal polynomial of  $F$  on  $\mathfrak{B}$  is  $(x - 1)^3$ . On the other hand,  $\mathfrak{Z}$  centralizes  $\mathfrak{B}$ . Since  $\mathfrak{C} \triangleleft \mathfrak{N}$ , the minimal polynomial of  $F$  on  $\mathfrak{B}$  is a divisor of  $(x - 1)^2$ . This contradiction establishes (9.23).

Since  $\Omega$  contains an abelian subgroup of type (2, 4), we can choose an involution  $I$  of  $\Omega$  such that  $\mathfrak{F}_0 = C_{\mathfrak{F}}(I)$  is noncyclic. By Lemmas 7.4 and 5.38,  $C(I)$  contains no element of  $\mathcal{E}(3)$ . Hence,  $I$  inverts  $\mathfrak{Z}$ . Thus, in cases (9.19), (9.20) respectively, we have

$$(9.19)'-(9.20)' \quad \mathfrak{F} = \mathfrak{F}_0 \times \mathfrak{Z}.$$

In case (9.21), we have

$$(9.21)' \quad |\mathfrak{F}_0| = |C_{\mathfrak{P}_0/\mathfrak{P}_1}(I)| = 9.$$

Thus, in case (9.21), we have  $|C_{\mathfrak{F}_0}(I)| \geq 3^4$ .

Let  $\mathfrak{Z}$  be a  $S_{2,3}$ -subgroup of  $C(I)$  which contains  $\mathfrak{F}_0$ . Since  $\mathfrak{F}_0 \in \mathcal{D}$  and since  $C(I)$  contains an element of  $\mathcal{U}(2)$ , there is an element  $\mathfrak{M}$  of  $\mathcal{M}\mathcal{C}(G)$  which satisfies all the conclusions of Lemma 7.5, contains  $\mathfrak{F}_0$  and contains a  $S_2$ -subgroup of  $\mathfrak{Z}$ . By Lemma 7.5 (f),  $I \in O_2(\mathfrak{M})$ .

We will show that

$$(9.24) \quad C_{\mathfrak{M}_1}(I) \subseteq \mathfrak{M} .$$

By Lemma 7.5 (f), it suffices to show that  $\mathfrak{M}$  contains an  $S_2$ -subgroup of  $C(I)$ . By construction,  $\mathfrak{M}$  contains an  $S_2$ -subgroup of  $\mathfrak{Z}$ , which is an  $S_{2,3}$ -subgroup of  $C(I)$ . This proves (9.24).

Suppose (9.19) holds. In this case, we have (9.22). Also, (9.19) implies that every element of  $\mathfrak{G}$  of order 3 centralizes an element of  $\mathcal{E}(3)$ . Let  $\mathfrak{F}_1$  be a subgroup of  $\mathfrak{F}_0$  of order 3 such that

$$[O_2(\mathfrak{M}) \cap C(\mathfrak{F}_1), \mathfrak{F}_0] = \Omega^* \neq 1 .$$

Thus,  $\Omega^*$  is a quaternion group and a  $S_{2,3}$ -subgroup of  $C(\mathfrak{F}_1)$  is not 3-closed. By (9.22),  $\mathfrak{F}_1 \simeq \mathfrak{B}$ . Since  $|C_{\mathfrak{M}_1}(\mathfrak{F}_1)|_3 = |\mathfrak{B}|/3$ , it follows that  $C_{\mathfrak{M}_1}(\mathfrak{F}_1)$  contains a  $S_3$ -subgroup of  $C(\mathfrak{F}_1)$ . Let  $\mathfrak{B}^*$  be a  $S_3$ -subgroup of  $C_{\mathfrak{M}_1}(\mathfrak{F}_1)$ . Since  $C(\mathfrak{F}_1)$  contains an element of  $\mathcal{L}_{ev_3}(\mathfrak{B})$ , it follows that  $O_3(\mathbb{C}) = 1$ , where  $\mathbb{C} = C(\mathfrak{F}_1)$ . Let  $\mathfrak{R} = O_3(\mathbb{C})/\mathfrak{F}_1$ . Thus,  $\Omega^* \langle F \rangle$  is faithfully represented on  $Z(\mathfrak{R})$  for each  $F$  in  $\mathfrak{F}_0 - \mathfrak{F}_1$ . But  $[\mathfrak{R}, F] \subseteq \langle \mathfrak{B}, \mathfrak{F}_1 \rangle / \mathfrak{F}_1$ , so  $\Omega^*$  centralizes a subgroup of  $O_3(\mathfrak{R})$  of index 9.

Suppose (9.19) holds and  $O_3(\mathbb{C}) \cap C(I)$  is noncyclic, where  $I$  is the involution of  $\Omega^*$ . In this case, since  $\mathfrak{F}_0$  centralizes  $I$  and  $O_3(\mathbb{C}) \cap \mathfrak{F}_0 = \mathfrak{F}_1$ , it follows that  $\mathbb{C} \cap \mathfrak{M}$  contains a subgroup of order 27 and exponent 3. Since every element of  $\mathfrak{G}$  of order 3 centralizes an element of  $\mathcal{E}(3)$ , it follows that a  $S_3$ -subgroup of  $\mathfrak{M}$  is nonabelian of order 27 and the width of  $O_2(\mathfrak{M})$  is 3. Since  $|O_3(\mathbb{C}) : O_3(\mathbb{C}) \cap C(I)| = 9$ , it follows that  $|O_3(\mathbb{C})| \leq 3^4$ . Since  $O_3(\mathbb{C}) \cap C(I)$  is assumed noncyclic, and since  $m(Z(O_3(\mathbb{C}))) \geq 3$ , it follows that  $O_3(\mathbb{C})$  is elementary of order  $3^4$ . Since  $\Omega^* \subseteq \mathbb{C}$ , and since  $\langle I \rangle = O_2(\mathfrak{M}) \cap C(\mathfrak{F}_0)$ , it follows that  $O_3(\mathbb{C})$  is of index 3 in  $\mathfrak{B}^*$ . Hence,  $|\mathfrak{B}| = 3^6$ , since  $\mathfrak{B}^*$  is of index 3 in some  $S_3$ -subgroup of  $\mathfrak{G}$ .

Since  $|\mathfrak{B}^*| = 3^5$ , we have  $\mathfrak{B}^* = O_3(\mathbb{C})\mathfrak{F}_0$ .

We argue that  $C_{\mathfrak{B}^*}(F)$  is of index 9 in  $\mathfrak{B}^*$  for every  $F$  in  $\mathfrak{F}_0 - \mathfrak{F}_1$ . This assertion is equivalent to the assertion that  $O_3(\mathbb{C}) \cap C(F)$  is of order 9, since  $\mathfrak{B}^* = O_3(\mathbb{C}) \langle F \rangle$ . Now  $O_3(\mathbb{C}) = \mathfrak{U}_1 \times \mathfrak{U}_2$ , where  $\mathfrak{U}_1 = C(I) \cap O_3(\mathbb{C})$ ,  $\mathfrak{U}_2$  is inverted by  $I$ , and  $|\mathfrak{U}_1| = |\mathfrak{U}_2| = 9$ . Since  $\mathfrak{U}_i \triangleleft \mathfrak{B}^* \Omega^*$ ,  $i = 1, 2$ , we must show that  $F$  does not centralize either  $\mathfrak{U}_1$  or  $\mathfrak{U}_2$ . It is obvious that  $F$  does not centralize  $\mathfrak{U}_2$ . If  $F$  centralizes  $\mathfrak{U}_1$ , then  $\langle \mathfrak{U}_1, F \rangle$  is elementary of order 27 and is contained in  $\mathfrak{M}$ , whereas we already know that  $S_3$ -subgroups of  $\mathfrak{M}$  are nonabelian of order 27. So

$$|\mathfrak{P}^*: C_{\mathfrak{P}^*}(F)| = 9.$$

Since  $C_{\mathfrak{P}^*}(F)$  is of index 9 in  $\mathfrak{P}^*$  for every  $F$  in  $\mathfrak{F}_0 - \mathfrak{F}_1$ , it follows that  $O_3(\mathbb{C}) \text{ char } \mathfrak{P}^*$ . Thus,  $N(O_3(\mathbb{C}))$  contains a  $S_3$ -subgroup of  $\mathbb{C}$  and  $S_{2,3}$ -subgroup of  $N(O_3(\mathbb{C}))$  are not 3-closed. This implies that if  $\tilde{\mathfrak{P}}$  is a  $S_3$ -subgroup of  $N(O_3(\mathbb{C}))$ , then  $O_3(\mathbb{C})$  is not characteristic in  $\tilde{\mathfrak{P}}$ . More explicitly,  $N(O_3(\mathbb{C})) \cap N(\tilde{\mathfrak{P}})$  does not contain a noncyclic abelian subgroup of order 8, while  $N(\tilde{\mathfrak{P}})$  does. Let  $\mathfrak{A}$  be an elementary subgroup of  $\mathfrak{P}$  of order  $3^4$  with  $\mathfrak{A} \neq O_3(\mathbb{C})$ . If  $\mathfrak{A} \cap O_3(\mathbb{C})$  is of order 9, then  $\tilde{\mathfrak{P}} = \mathfrak{A}O_3(\mathbb{C})$  and  $Z(\tilde{\mathfrak{P}})$  is not cyclic. Hence,  $\mathfrak{A} \cap O_3(\mathbb{C})$  is of order 27. Hence,  $\tilde{\mathfrak{P}} = \mathfrak{P}^*\mathfrak{A}$ , and it follows that  $N(O_3(\mathbb{C})) \cap C(I)$  contains  $S_3$ -subgroups of order  $3^4$ . Furthermore, every subgroup of  $\tilde{\mathfrak{P}}$  of order 3 centralizes an element of  $\mathcal{E}(\mathfrak{B})$ . Since the width of  $O_2(\mathfrak{M})$  is 3, it follows that a  $S_3$ -subgroup of  $\mathfrak{M}$  is of the shape  $Z_3 \wr Z_3$ . But we have already shown that  $S_3$ -subgroups of  $\mathfrak{M}$  are of order 27.

Suppose (9.19) holds and  $O_3(\mathbb{C}) \cap C(I)$  is cyclic. Since  $S_3$ -subgroups of  $\mathfrak{M}$  are of exponent 3 or 9, it follows that  $|O_3(\mathbb{C}) \cap C(I)| = 3$  or 9. Hence,  $|O_3(\mathbb{C})| \leq 3^4$ , so  $O_3(\mathbb{C})$  is abelian. Hence,  $\mathfrak{P}^* = O_3(\mathbb{C})\mathfrak{F}_0$ . Since elements of  $\mathfrak{F}_0 - \mathfrak{F}_1$  have quadratic minimal polynomial of  $O_3(\mathbb{C})$ , it follows that  $\sigma^1(\mathfrak{P}^*) = \sigma^1(O_3(\mathbb{C})) = \sigma^1(O_3(\mathbb{C}) \cap C(I))$ . Hence,  $\sigma^1(\mathfrak{P}^*) = 1$ , since otherwise  $\sigma^1(\mathfrak{P}^*)$  is conjugate to  $\mathfrak{B}$ , against (9.22). Hence,  $O_3(\mathbb{C})$  is elementary of order 27.

Since  $|O_3(\mathbb{C})| = 27$ , we get  $|\mathfrak{P}^*| = 3^4$ ,  $|\mathfrak{P}| = 3^5$ . Since (9.19) holds,  $\Omega$  is of type (4, 2) and  $\Omega$  normalizes  $\mathfrak{P}$ . Let  $\tilde{\Omega} = \Omega \cap C(\mathfrak{B})$ . Thus,  $\tilde{\Omega}$  is cyclic of order 4, by Lemma 9.12 (ii). Also, the involution  $Q$  of  $\tilde{\Omega}$  inverts  $\mathfrak{F}/\mathfrak{B}$ , so inverts  $\mathfrak{P}/\mathfrak{B}$ . Hence,  $\mathfrak{P}/\mathfrak{B}$  is elementary of order  $3^4$  and is the direct sum of  $\mathbb{C}/\mathfrak{B}$  and another irreducible  $\Omega$ -module. This implies that  $\mathfrak{P}$  is of exponent 3 and is extra special. Thus, for each  $P$  in  $\mathfrak{P}$ ,  $\mathfrak{B} \text{ char } C_{\mathfrak{P}}(P)$ . This implies that  $\mathfrak{B}$  is weakly closed in  $\mathfrak{P}$ . But turning back to  $\mathbb{C}$ , it follows that  $\Omega^*$  does not normalize  $\mathfrak{B}$ , so  $\mathfrak{B}$  is not weakly closed in  $\mathfrak{P}$ . This contradiction shows that (9.19) does not hold.

Suppose (9.20) holds. By (9.24) it follows that  $\Omega\mathfrak{F}_0 \subseteq \mathfrak{M}$ . Hence, the width of  $O_2(\mathfrak{M})$  is four. Hence,  $C_{\mathfrak{P}_0}(I) = \mathfrak{F}_0$ . Thus,  $C_{\mathfrak{M}_1}(I)$  contains a  $S_3$ -subgroup  $\tilde{\mathfrak{P}}$  which is a nonabelian group of order 27 and exponent 3. This is not the case, since  $C(P) \cap O_2(\mathfrak{M})$  contains no four-subgroup for any element  $P$  of  $\tilde{\mathfrak{P}}^*$ .

Suppose (9.21) holds. By (9.24) and (9.21)', it follows that  $S_3$ -subgroups of  $\mathfrak{M}$  are of order at least  $3^4$ . Hence, the width of  $O_2(\mathfrak{M})$  is four, and  $C(I) \cap \mathfrak{P}_0$  contains no subgroup of order 27 and exponent 3, and of course  $C(I) \cap \mathfrak{P}_0$  is of exponent 9. This is absurd, since  $S_3$ -subgroups of  $\text{Aut}(O_2(\mathfrak{M}))$  contain subgroups of index and exponent 3. This completes the proof of this lemma.

LEMMA 9.14.  $N(J(\mathfrak{P}))$  does not contain a noncyclic abelian subgroup



of order 8.

*Proof.* First, suppose that  $J(\mathfrak{P})$  is not elementary. Then  $\mathfrak{B} = Z(J(\mathfrak{P})) \cap D(J(\mathfrak{P})) \neq 1$ . If  $\mathfrak{B}$  is cyclic, then  $\Omega_1(\mathfrak{B}) = \mathfrak{Z} \text{ char } N(J(\mathfrak{P}))$ , so  $N(J(\mathfrak{P})) \subseteq \mathfrak{N}$ , and this lemma follows from Lemma 9.13. We may assume that  $\mathfrak{B}$  is noncyclic. Let  $\mathfrak{B}_1$  be a noncyclic elementary subgroup of order 9. We will show that  $\mathfrak{B}_1 \in \mathcal{E}(3)$ . Choose  $\mathfrak{Q} \in \mathcal{N}(\mathfrak{B}_1; 3')$ , minimal subject to  $[\mathfrak{Q}, \mathfrak{B}_1] \neq 1$ . Let  $\mathfrak{B}_0 = C_{\mathfrak{B}_1}(\mathfrak{Q})$  so that  $|\mathfrak{B}_0| = 3$ . Let  $\mathfrak{C} = C(\mathfrak{B}_0)$ , and let  $\mathfrak{P}^*$  be a  $S_3$ -subgroup of  $\mathfrak{C}$  which contains  $J(\mathfrak{P})$ . Hence,  $J(\mathfrak{P}) = J(\mathfrak{P}^*)$ . Let  $\mathfrak{P} = \mathfrak{P}^g$  be a  $S_3$ -subgroup of  $\mathfrak{G}$  which contains  $\mathfrak{P}^*$ .

Since  $\mathfrak{C}$  contains an element of  $\mathcal{S}_{\text{inv}_3}(\mathfrak{P})$ , it follows that  $O_3(\mathfrak{C}) = 1$ . Since  $\mathfrak{B}_1 \subseteq Z(J(\mathfrak{P})) = Z(J(\mathfrak{P}^*))$ , we have  $[O_3(\mathfrak{C}), \mathfrak{B}_1] \subseteq J(\mathfrak{P})$  and  $[O_3(\mathfrak{C}), \mathfrak{B}_1, \mathfrak{B}_1] = 1$ . It follows that  $\mathfrak{Q}$  is a quaternion group. Let  $\mathfrak{B} = \langle \mathfrak{B}_0 \rangle^{\mathfrak{C}}$ . Thus,  $\mathfrak{B}$  is a normal elementary 3-subgroup of  $\mathfrak{C}$ , and by Lemma 5.10,  $\mathfrak{B}$  is 3-reducible in  $\mathfrak{C}$ . It is a straightforward consequence of Lemma 5.2 that  $\mathfrak{B} \subseteq J(\mathfrak{P}^*)$ . Thus,  $\mathfrak{Q}$  centralizes  $\mathfrak{B}$ , as  $\mathfrak{B}_1$  centralizes  $\mathfrak{B}$ . Thus, it follows that  $\mathfrak{B}_1 \not\subseteq O_3(\mathfrak{N}_1)$ , where  $\mathfrak{N}_1$  is a  $S_{2,3}$ -subgroup of  $\mathfrak{N}^g$  which contains  $\mathfrak{P}^g$ . By Lemma 9.12,  $|\mathfrak{P}^g : O_3(\mathfrak{N}_1)| \leq 3$ . Thus,  $\mathfrak{B}_1 \not\subseteq D(\mathfrak{P}^g)$ . This is absurd, since  $\mathfrak{B}_1 \subseteq D(J(\mathfrak{P}))$ , and  $J(\mathfrak{P}) = J(\mathfrak{P}^*) = J(\mathfrak{P}^g)$ .

It is an immediate consequence of the preceding argument and Lemmas 7.4 and 5.38 that if  $D(J(\mathfrak{P})) \neq 1$ , then this lemma holds.

Assume now that  $J(\mathfrak{P})$  is elementary. To complete the proof of the lemma, it suffices to show that each subgroup of  $J(\mathfrak{P})$  of order 9 is in  $\mathcal{E}(3)$ . Suppose false, and  $\mathfrak{B} \subseteq J(\mathfrak{P})$ ,  $|\mathfrak{B}| = 9$ ,  $\mathfrak{B} \notin \mathcal{E}(3)$ . Let  $\mathfrak{X}$  be an element of  $\mathcal{N}(\mathfrak{B}; 3')$  minimal subject to  $[\mathfrak{B}, \mathfrak{X}] \neq 1$ . Let  $\mathfrak{B}_0 = \mathfrak{B} \cap C(\mathfrak{X})$ , so that  $|\mathfrak{B}_0| = 3$ .

Let  $\mathfrak{C} = C(\mathfrak{B}_0) \cong \langle J(\mathfrak{P}), \mathfrak{X} \rangle$ . Let  $\mathfrak{P}^*$  be a  $S_3$ -subgroup of  $\mathfrak{C}$  which contains  $J(\mathfrak{P})$ , and let  $\mathfrak{P}^g$  be a  $S_3$ -subgroup of  $\mathfrak{G}$  which contains  $\mathfrak{P}^*$ . Hence,  $J(\mathfrak{P}) = J(\mathfrak{P}^g)$ . Since  $J(\mathfrak{P}^g) = J(\mathfrak{P})^g$ , we get that  $G \in N(J(\mathfrak{P}))$ . Replacing  $\mathfrak{B}$  by  $\mathfrak{B}^{g^{-1}}$  and  $\mathfrak{X}$  by  $\mathfrak{X}^{g^{-1}}$ , we assume without loss of generality that  $\mathfrak{P}^* \subseteq \mathfrak{P}$ .

Since  $\mathfrak{P}^*$  contains an element of  $\mathcal{S}\mathcal{C}\mathcal{N}_3(\mathfrak{P})$ , it follows that  $O_3(\mathfrak{C}) = 1$ . Hence,  $\mathfrak{B} \subseteq O_3(\mathfrak{C})$ . Let  $\mathfrak{B} = \mathfrak{B}^{\mathfrak{C}}$ , so that  $\mathfrak{B}$  is a normal elementary subgroup of  $\mathfrak{C}$ . Since  $\mathfrak{B}$  is 3-reducible in  $\mathfrak{C}$ , it follows that  $\mathfrak{B} \subseteq J(\mathfrak{P})$ . Hence,  $\mathfrak{X}$  centralizes  $\mathfrak{B}$ . In particular,  $\mathfrak{X}$  centralizes  $\mathfrak{B}$ .

Let  $\tilde{\mathfrak{B}} = J(\mathfrak{P}) \cap O_3(\mathfrak{C})$ . Thus,  $\mathfrak{B} \not\subseteq \tilde{\mathfrak{B}}$ , and  $J(\mathfrak{P})/\tilde{\mathfrak{B}}$  acts faithfully on  $O_{3,3'}(\mathfrak{C})/O_3(\mathfrak{C})$ . Let  $\mathfrak{R} = [O_{3,3'}(\mathfrak{C}), J(\mathfrak{P})]O_3(\mathfrak{C})$ . Since

$$[O_3(\mathfrak{C}), J(\mathfrak{P}), J(\mathfrak{P})] = 1,$$

it follows that  $\bar{\mathfrak{R}} = \mathfrak{R}/O_3(\mathfrak{C})$  is a 2-group, and that  $J(\mathfrak{P})$  centralizes every characteristic abelian subgroup of  $\bar{\mathfrak{R}}$ . Since  $J(\mathfrak{P})$  centralizes  $\mathfrak{B}$ , so does  $\mathfrak{R}$ . By Lemma 9.12 (ii),  $\mathfrak{R}$  contains no four-group. So  $\mathfrak{R}$  is a quaternion group and  $J(\mathfrak{P})/\tilde{\mathfrak{B}}$  is of order 3, whence  $J(\mathfrak{P}) = \tilde{\mathfrak{B}}\mathfrak{B}$ , and so  $\mathfrak{R} =$

$[O_{3,3}(\mathbb{C}), \mathfrak{B}]O_3(\mathbb{C})$ . Since  $J(\mathfrak{P})O_3(\mathbb{C})/O_3(\mathbb{C})$  is of order 3, it follows that  $J(\mathfrak{P}) \subseteq O_{3,3',3}(\mathbb{C})$ , and so  $\mathfrak{X} \subseteq O_{3,3'}(\mathbb{C})$ , whence  $\mathfrak{X} \subseteq \mathfrak{R}$ , and so  $\mathfrak{R} = \mathfrak{X}O_3(\mathbb{C})$ , and  $\mathfrak{X} \cong \bar{\mathfrak{R}}$  is a quaternion group. Note that  $\mathfrak{X}$  is permutable with  $\mathfrak{P}^*$ , as  $\mathfrak{P}^*$  normalizes  $[O_{3,3'}(\mathbb{C}), J(\mathfrak{P})]$ .

Enlarge  $\mathfrak{P}^*\mathfrak{X}$  to a  $S_{2,3}$ -subgroup  $\mathfrak{N}_0$  of  $\mathfrak{N}$ , and let  $\mathfrak{N}_1 = O^*(\mathfrak{N}_0)$ . Thus,  $\mathfrak{N}_1 = \mathfrak{X}\tilde{\mathfrak{P}}$ , where  $\tilde{\mathfrak{P}} = \mathfrak{P}^N$  for some  $N$  in  $\mathfrak{N}$ . Since  $J(\mathfrak{P}) = J(\mathfrak{P}^N) = J(\mathfrak{P})^N$ , replacing  $\mathfrak{B}$  by  $\mathfrak{B}^{N^{-1}}$  and  $\mathfrak{X}$  by  $\mathfrak{X}^{N^{-1}}$ , we assume without loss of generality that  $\tilde{\mathfrak{P}} = \mathfrak{P}$  is permutable with  $\mathfrak{X}$ .

Let  $\mathfrak{W}_1$  be a subgroup of  $\mathfrak{B}$  of order 3 different from  $\mathfrak{W}_0$ . Let  $\mathfrak{P}_0 = O_3(\mathfrak{N}_1)$ . Thus,  $\mathfrak{P} = \mathfrak{P}_0\mathfrak{W}_1$  and  $\mathfrak{W}_1\mathfrak{X}$  is a complement to  $\mathfrak{P}_0$  in  $\mathfrak{N}_1$ . Let  $I$  be the involution of  $\mathfrak{X}$ , let  $T$  be an element of  $\mathfrak{X}$  of order 4, let  $\mathfrak{R} = J(\mathfrak{P}) \cap \mathfrak{P}_0$ , and let  $\mathfrak{Z} = \langle \mathfrak{R}, \mathfrak{R}^T \rangle$ . Since  $T^2 = I$  normalizes  $\mathfrak{P}$ , and  $J(\mathfrak{P})$  char  $\mathfrak{P}$ , it follows that  $T^2$  normalizes  $\mathfrak{R}$ . Hence,  $T$  normalizes  $\mathfrak{Z}$ . Of course,  $\mathfrak{W}_1$  also normalizes  $\mathfrak{Z}$ , since  $[\mathfrak{W}_1, \mathfrak{Z}] \subseteq \mathfrak{R} \subseteq \mathfrak{Z}$ . Since  $\mathfrak{R} \triangleleft \mathfrak{P}_0$ , it follows that  $\mathfrak{Z} \triangleleft \mathfrak{P}_0$ , so  $\mathfrak{Z} \triangleleft \mathfrak{N}_1$ . Since

$$\mathfrak{Z}' = [\mathfrak{R}, \mathfrak{R}^T] \subseteq \mathfrak{R} \cap \mathfrak{R}^T \subseteq \mathfrak{R} \subseteq J(\mathfrak{P}),$$

$J(\mathfrak{P})$  centralizes  $\mathfrak{Z}'$ . Since  $\mathfrak{W}_1 \subseteq J(\mathfrak{P})$ , it follows that  $\mathfrak{X}$  centralizes  $\mathfrak{Z}'$ . Hence,  $\mathfrak{Z}' \subseteq \mathfrak{Z}$ , as otherwise  $I$  centralizes an element of  $\mathcal{Z}(3)$ .

Clearly,  $\mathfrak{Z}$  is of exponent 3, being of class at most 2 and being generated by its elementary subgroups. The definition of  $J(\mathfrak{P})$  forces  $\mathfrak{R} = C_{\mathfrak{Z}}(\mathfrak{R})$ . Hence,  $Z(\mathfrak{Z}) = \mathfrak{Z} = \mathfrak{Z}'$ , so that  $\mathfrak{Z}$  is extra special, while  $\mathfrak{R} \in \mathcal{L}_{ev}(\mathfrak{Z})$ . The width of  $\mathfrak{Z}$  is at least 2, since otherwise, Hypothesis 9.1 would be satisfied.

Now  $I$  centralizes  $\mathfrak{Z}$  and normalizes  $\mathfrak{R}$ . We argue that  $I$  inverts  $\mathfrak{R}/\mathfrak{Z}$ . Suppose false and  $\mathfrak{F}$  is a subgroup of  $\mathfrak{R}$  of order 9 which contains  $\mathfrak{Z}$  and is centralized by  $I$ . Since  $A_{\mathfrak{N}_1}(\tilde{\mathcal{E}}) = A(\tilde{\mathcal{E}})$ , where  $\tilde{\mathcal{E}}: \mathfrak{R} \supset \mathfrak{Z} \supset 1$ , it follows from Lemma 5.5 that  $\mathfrak{F} \in \mathcal{E}(3)$ . Thus,  $C(I)$  is nonsolvable by Lemmas 7.4 and 5.38. This contradiction shows that  $I$  inverts  $\mathfrak{R}/\mathfrak{Z}$ . Hence  $I$  inverts  $\mathfrak{Z}/\mathfrak{Z}$ .

We next show that  $\mathfrak{X}$  centralizes  $\mathfrak{P}_0/\mathfrak{Z}$ . This is clear, since  $[\mathfrak{P}_0, \mathfrak{W}_1] \subseteq \mathfrak{R} \subseteq \mathfrak{Z}$ , so that  $\mathfrak{W}_1$  centralizes  $\mathfrak{P}_0/\mathfrak{Z}$ .

Since  $I$  inverts  $\mathfrak{Z}/\mathfrak{Z}$ , it follows that if  $\tilde{\mathfrak{W}}$  is any subgroup of  $J(\mathfrak{P})$  of order 9, and  $\tilde{\mathfrak{X}}$  is any element of  $\mathcal{N}(\tilde{\mathfrak{W}}; 3')$  which is minimal subject to  $[\tilde{\mathfrak{W}}, \tilde{\mathfrak{X}}] \neq 1$ , then  $\tilde{\mathfrak{X}}$  is a quaternion group and  $\tilde{\mathfrak{W}} \cap C(\tilde{\mathfrak{X}}) \sim \mathfrak{Z}$ . In particular,  $\mathfrak{W} \in \mathcal{S}$ .

Let  $\mathfrak{M}$  be the subgroup given by Lemma 7.5 which contains a  $S_2$ -subgroup of  $C(I)$  and contains  $\mathfrak{W}$ . Then Lemma 9.12 (ii) implies that  $O_2(\mathfrak{M})$  is extra special, and that  $\langle I \rangle = O_2(\mathfrak{M})'$ . Hence,  $C_{\mathfrak{N}_1}(I) \subseteq \mathfrak{M}$ .

If the width of  $O_2(\mathfrak{M})$  is 2, then  $\mathfrak{W} = C_{\mathfrak{P}}(I)$  is a  $S_3$ -subgroup of  $C_{\mathfrak{N}_1}(I)$  so  $\mathfrak{Z} = \mathfrak{P}_0$  is extra special. Since  $\mathfrak{P}_0 = O_3(\mathfrak{N}_1)$ , it follows that Hypothesis 9.2 is satisfied, an excluded case. Hence, the width of  $O_2(\mathfrak{M})$  is 3 or 4. On the other hand, every element of  $(C(I) \cap O_3(\mathfrak{N}_1))\mathfrak{W}$  of order 3

centralizes an element of  $\mathcal{E}(3)$ , so  $C_{\mathfrak{M}_1}(I)$  contains no subgroup of exponent 3 and order 27. Hence,  $C_{\mathfrak{M}_0}(I)$  is cyclic of order 3 or 9. If  $C_{\mathfrak{M}_3}(I) = 3$ , Hypothesis 9.2 is satisfied, an excluded case. Hence,  $C_{\mathfrak{M}_0}(I)$  is cyclic of order 9. This means that if  $\mathfrak{M}_1$  is any subgroup of  $\mathfrak{M}$  of order 3 such that  $C(\mathfrak{M}_1) \cap O_2(\mathfrak{M}) \supset \langle I \rangle$ , then  $\mathfrak{M}_1 = \langle P^3 \rangle$  for some  $P$  in  $\mathfrak{M}$ . This is absurd, since we get that  $\mathfrak{M} \cong \mathcal{O}^1(\mathfrak{F})$  for some  $S_3$ -subgroup  $\mathfrak{F}$  of  $\mathfrak{M}$ , while  $\mathfrak{F}$  is isomorphic to a subgroup of  $Z_3 \times (Z_3 \wr Z_3)$ . The proof is complete.

LEMMA 9.15. *If  $\mathfrak{A} \in \mathcal{E}(3)$  and  $\mathfrak{B}$  is a noncyclic abelian subgroup of order 8, then  $\langle \mathfrak{A}, \mathfrak{B} \rangle$  is nonsolvable.*

*Proof.* Suppose false. Let  $\mathcal{S}$  be the set of all 2, 3-subgroups  $\mathfrak{C}$  of  $\mathfrak{G}$  such that

- (i)  $\mathfrak{C}$  contains an element of  $\mathcal{E}(3)$ .
- (ii)  $\mathfrak{C}/O_3(\mathfrak{C})$  satisfies the hypothesis of Lemma 5.41.

Thus,  $\mathcal{S} \neq \emptyset$ .

If  $\mathfrak{C}_1$  and  $\mathfrak{C}_2$  are elements of  $\mathcal{S}$ , we say that  $\mathfrak{C}_1 \ll \mathfrak{C}_2$  if and only if either  $|\mathfrak{C}_1|_3 < |\mathfrak{C}_2|_3$  or  $\mathfrak{C}_1 = \mathfrak{C}_2$ .

Let  $\mathfrak{C}$  be a maximal element of  $\mathcal{S}$  under  $\ll$ . Let  $\mathfrak{C}_p$  be a  $S_p$ -subgroup of  $\mathfrak{C}$ ,  $p = 2, 3$ . Since  $\mathfrak{C}$  contains an element of  $\mathcal{E}(3)$ , it follows from Lemma 9.11 (ii) that  $O_2(\mathfrak{C}) = 1$ .

Replacing  $\mathfrak{C}$  by a conjugate if necessary, we assume that  $\mathfrak{C}_3 \cong \mathfrak{F}$ . By Lemma 5.41,  $\mathfrak{C}$  has 2-length 1. If  $\mathfrak{C}'_2 = 1$ , then

$$\mathfrak{C} = C_{\mathfrak{C}}(Z(\mathfrak{C}_3)) \cdot N_{\mathfrak{C}}(J(\mathfrak{C}_3))$$

by Theorem 1 of [43]. Since  $Z(\mathfrak{F}) \subseteq Z(\mathfrak{C}_3)$ ,  $S_2$ -subgroups of  $C_{\mathfrak{C}}(Z(\mathfrak{C}_3))$  are cyclic. Thus,  $C_{\mathfrak{C}}(Z(\mathfrak{C}_3)) \subseteq N_{\mathfrak{C}}(\mathfrak{C}_3) \subseteq N_{\mathfrak{C}}(J(\mathfrak{C}_3))$ , so  $J(\mathfrak{C}_3) \triangleleft \mathfrak{C}$ . Maximality of  $\mathfrak{C}$  forces  $\mathfrak{C}_3 = \mathfrak{F}$ , against Lemma 9.14. Hence,  $\mathfrak{C}'_2 \neq 1$ .

Suppose  $\mathfrak{C}_2$  is extra special of width at least 2. By Lemma 5.52, it follows that  $\mathfrak{C} = C_{\mathfrak{C}}(Z(O_3(\mathfrak{C})))N_{\mathfrak{C}}(J(\mathfrak{C}_3))$ . Thus, maximality of  $\mathfrak{C}$  together with Lemmas 9.13 and 9.14 imply that neither  $C_{\mathfrak{C}}(Z(O_3(\mathfrak{C})))$  nor  $N_{\mathfrak{C}}(J(\mathfrak{C}_3))$  contains a noncyclic abelian subgroup of order 8. Let  $\mathfrak{X}_0 = C_{\mathfrak{C}}(Z(O_3(\mathfrak{C}))) \cap \mathfrak{C}_2$ ,  $\mathfrak{X}_1 = N(J(\mathfrak{C}_3)) \cap \mathfrak{C}_2$ .

Since  $\mathfrak{C}_2 = \mathfrak{X}_0\mathfrak{X}_1$  and  $\mathfrak{X}_i$  has no noncyclic abelian subgroup of order 8, the width of  $\mathfrak{C}_2$  is 2, and  $4 \leq |\mathfrak{X}_i| \leq 8$ ,  $i = 0, 1$ .

Suppose  $\mathfrak{C}_3 \cdot C_{\mathfrak{C}}(Z(O_3(\mathfrak{C})))$  is 3-closed. Then  $\mathfrak{X}_0$  normalizes  $\mathfrak{C}_3$ , so normalizes  $J(\mathfrak{C}_3)$ . This yields  $\mathfrak{X}_0 \subseteq \mathfrak{X}_1$ , which is not the case. Thus,  $\mathfrak{C}_3 \cdot C_{\mathfrak{C}}(Z(O_3(\mathfrak{C})))$  is not 3-closed. Since  $\mathfrak{X}_0 \triangleleft \mathfrak{C}_2$ , it follows that  $\mathfrak{X}_0$  is a quaternion group. Suppose  $N_{\mathfrak{C}}(J(\mathfrak{C}_3))$  is 3-closed. Since  $Z(\mathfrak{C}_3) \subseteq Z(O_3(\mathfrak{C}))$ , it follows that  $Z(\mathfrak{C}_3) \triangleleft \mathfrak{C}$ . Maximality of  $|\mathfrak{C}|_3$  forces  $\mathfrak{C}_3 = \mathfrak{F}$ . This violates Lemma 9.13. Thus,  $N_{\mathfrak{C}}(J(\mathfrak{C}_3))$  is not 3-closed. Since  $\mathfrak{C}'_2 \subseteq N_{\mathfrak{C}}(\mathfrak{C}_3) \subseteq N_{\mathfrak{C}}(J(\mathfrak{C}_3))$ , it follows that  $\mathfrak{X}_1$  is also a quaternion group. Since  $l_2(\mathfrak{C}) = 1$ ,  $\mathfrak{C}_2$  is the central product of  $\mathfrak{X}_0$  and  $\mathfrak{X}_1$ .

Let  $\mathfrak{P} = \mathfrak{P}_1, \dots, \mathfrak{P}_r$  be all the  $S_3$ -subgroups of  $\mathfrak{G}$  which contain  $\mathfrak{S}_3$ , and let  $\mathfrak{Z}_i = \Omega_1(\mathbf{Z}(\mathfrak{P}_i))$ . Thus,  $\mathfrak{Z}_i \subseteq \mathbf{Z}(\mathbf{O}_3(\mathfrak{S}))$  for all  $i$ , so that  $\mathfrak{X}_0$  centralizes each  $\mathfrak{Z}_i$ . Let  $\langle T \rangle = \mathfrak{X}_0 \cap \mathfrak{X}_1$  so that  $T$  is an involution which centralizes each  $\mathfrak{Z}_i$ . Also,  $\mathfrak{X}_0 \subseteq C(\mathfrak{Z}_i)$  for each  $i$ , so for each  $i$ ,  $\mathfrak{X}_1 \not\subseteq C(\mathfrak{Z}_i)$ .

Suppose  $\mathfrak{S}_3 = \mathfrak{P}$ . Since  $\mathfrak{S}'_2 = \mathfrak{X}'_0$  centralizes  $\mathbf{Z}(\mathbf{O}_3(\mathfrak{S}))$ , it follows that  $\mathfrak{Z} = \Omega_1(\mathbf{Z}(\mathbf{O}_3(\mathfrak{S})))$ ; otherwise,  $\mathfrak{S}'_2$  centralizes an element of  $\mathcal{U}(\mathfrak{S})$ . Hence,  $\mathfrak{Z} \triangleleft \mathfrak{S}$ , against Lemma 9.13. We conclude that  $\mathfrak{S}_3 \subset \mathfrak{P}$ .

Enlarge  $\mathfrak{S}_3\mathfrak{X}_1$  to a  $S_{2,3}$ -subgroup of  $N(J(\mathfrak{S}_3))$  and enlarge this subgroup to a maximal 2, 3-subgroup  $\mathfrak{Q}$  of  $\mathfrak{G}$ . Let  $\mathfrak{X} = \mathbf{O}^{3'}(\mathfrak{Q})$ . Since  $|\mathfrak{X}|_3 > |\mathfrak{S}|_3$ , it follows that  $\mathfrak{X}$  contains no noncyclic abelian subgroup of order 8. Since  $\mathfrak{X} \cong \mathbf{O}^{3'}(\mathfrak{S}_3\mathfrak{X}_1) = \mathfrak{S}_3\mathfrak{X}_1$ , it follows that  $\mathfrak{X}_1$  is a  $S_2$ -subgroup of  $\mathfrak{X}$ . Let  $\mathfrak{X}_3$  be a  $S_3$ -subgroup of  $\mathfrak{X}$  which contains  $\mathfrak{S}_3$ . Thus,  $\mathfrak{X}_3 \subseteq \mathfrak{P}_i$  for some  $i$ .

Let  $\mathfrak{W}$  be the normal closure of  $\mathfrak{Z}_i$  in  $\mathfrak{X}$ . Thus,  $C_{\mathfrak{Q}}(\mathfrak{W})$  contains  $T$ . Since  $\mathfrak{X}_0 \subseteq C(\mathfrak{Z}_i)$ , it follows that  $C_{\mathfrak{Q}}(\mathfrak{W}) \cap \mathfrak{X}_1 = \langle T \rangle$ , so  $S_2$ -subgroups of  $A_{\mathfrak{Q}}(\mathfrak{W})$  are four-groups. It follows that  $J(\mathfrak{X}_3) \triangleleft \mathfrak{X}$ . Hence,  $\mathfrak{X}_3 = \mathfrak{P}_i$  and so  $T$  centralizes an element of  $\mathcal{U}(\mathfrak{P}_i)$ . Thus, by Lemmas 7.1 (i) and 7.4,  $C(T)$  is nonsolvable. This contradiction shows that  $\mathfrak{S}_2$  is not extra special of width  $\geq 2$ .

Suppose  $\mathfrak{S}_2$  is the central product of a quaternion group and a cyclic group of order 4. If  $J(\mathfrak{S}_3) \subseteq \mathbf{O}_3(\mathfrak{S})$ , then again  $\mathfrak{S}_3 = \mathfrak{P}$  and Lemma 9.14 is violated. Hence,  $J(\mathfrak{S}_3) \not\subseteq \mathbf{O}_3(\mathfrak{S})$ . If  $\mathfrak{S}'_3$  centralizes  $\mathbf{Z}(\mathbf{O}_3(\mathfrak{S}))$ , then we get  $\mathfrak{S} = C_{\mathfrak{S}}(\mathbf{Z}(\mathfrak{S}_3))N_{\mathfrak{S}}(J(\mathfrak{S}_3))$ , so that either  $\mathbf{Z}(\mathfrak{S}_3)$  or  $J(\mathfrak{S}_3)$  is normal in  $\mathfrak{S}$ . Both these possibilities are excluded by Lemmas 9.13 and 9.14, so we may assume that  $[\mathfrak{S}'_3, \mathbf{Z}(\mathbf{O}_3(\mathfrak{S}))] = \mathfrak{W} \neq 1$ . Let  $\mathfrak{X}$  be a minimal normal subgroup of  $\mathfrak{S}$  with  $\mathfrak{X} \subseteq \mathfrak{W}$ . Thus,  $\mathfrak{S}_2$  is faithfully represented on  $\mathfrak{X}$ . Since  $|\mathfrak{S}_3: \mathbf{O}_3(\mathfrak{S})| = 3$  and  $J(\mathfrak{S}_3) \not\subseteq \mathbf{O}_3(\mathfrak{S})$ , it follows that elements of  $\mathfrak{S}_3 - \mathbf{O}_3(\mathfrak{S})$  centralize a hyperplane of  $\mathfrak{X}$ . This is not the case, since  $|\mathfrak{X}| = 3^4$ . Thus,  $\mathfrak{S}_2$  is not the central product of a quaternion group and a cyclic group of order 4.

By Lemma 5.41 and maximality of  $\mathfrak{S}$  under  $\ll$ , it follows that  $\mathfrak{S}_2$  is either the direct product of a quaternion group and a group of order 2 or  $\mathfrak{S}_2$  is special with  $|\mathfrak{S}'_2| = 4$ . Let  $\mathfrak{W} = \mathbf{Z}(\mathfrak{S}_2)$ , so that in both cases,  $\mathfrak{W}$  is a four-group. We will exploit  $\mathfrak{W}$  by showing that  $\mathfrak{S}_3 = \mathfrak{P}$ , that is, by showing that  $\mathfrak{S}_3$  is a  $S_3$ -subgroup of  $\mathfrak{G}$ . Suppose by way of contradiction that  $\mathfrak{S}_3 \subset \mathfrak{P}$ .

We argue that  $\mathfrak{W}$  normalizes  $J(\mathfrak{S}_3)$ . For if this is not the case, then  $\mathfrak{W}$  centralizes  $\mathbf{Z}(\mathbf{O}_3(\mathfrak{S}))$ , against Lemma 9.12 (ii).

Since  $\mathfrak{W}$  normalizes  $J(\mathfrak{S}_3)$ , we may enlarge  $\mathfrak{S}_3\mathfrak{W}$  to a  $S_{2,3}$ -subgroup  $\mathfrak{X}$  of  $N(J(\mathfrak{S}_3))$ . Since  $\mathfrak{S}_3$  is not a  $S_3$ -subgroup of  $\mathfrak{X}$ ,  $\mathfrak{X}$  does not contain a noncyclic abelian subgroup of order 8.

Let  $\mathfrak{X}_p$  be a  $S_p$ -subgroup of  $\mathfrak{X}$ ,  $p = 2, 3$ , with  $\mathfrak{W} \subseteq \mathfrak{X}_3$ ,  $\mathfrak{S}_3 \subset \mathfrak{X}_3$ .

*Case 1.*  $O_3(\mathfrak{C})\mathfrak{B}/O_3(\mathfrak{C}) \cong Z(\mathfrak{C}/O_3(\mathfrak{C}))$ .

Let  $\tilde{\mathfrak{C}}_3$  be a maximal element of  $\mathcal{N}_Q(\mathfrak{B}; 3)$  with  $\mathfrak{C}_3 \subseteq \tilde{\mathfrak{C}}_3$ . Suppose  $\mathfrak{C}_3 \subset \tilde{\mathfrak{C}}_3$ . Choose  $\mathfrak{C}_3^*$  in  $\mathcal{N}_Q(\mathfrak{B}; 3)$  so that  $|\mathfrak{C}_3^* : \mathfrak{C}_3| = 3$ , and let  $\mathfrak{B}_0 = C_{\mathfrak{B}}(\mathfrak{C}_3^*/\mathfrak{C}_3)$ . Hence,  $[\mathfrak{C}_3^*, \mathfrak{B}_0] = [\mathfrak{C}_3, \mathfrak{B}_0]$  is normal in  $\mathfrak{C}$  and in  $\mathfrak{C}_3^*$ . Maximality of  $\mathfrak{C}$  in  $\mathcal{S}$  forces  $[\mathfrak{C}_3, \mathfrak{B}_0] = 1$ , against  $O_2(\mathfrak{C}) = 1$ . Thus,  $\mathfrak{C}_3$  is a maximal element of  $\mathcal{N}_Q(\mathfrak{B}; 3)$ . In particular,  $\mathfrak{B}$  is not 3-closed. Hence,  $O_3(\mathfrak{B})$  is of index 3 in  $\mathfrak{B}$  and  $O_3(\mathfrak{B}) \subseteq \mathfrak{C}_3$ . Hence,  $O_3(\mathfrak{B}) = \mathfrak{C}_3$ . If  $\mathfrak{B} = \mathfrak{L}_2$ , then  $[\mathfrak{B}, \mathfrak{C}_3] \triangleleft \mathfrak{B}$  so that  $\langle \mathfrak{C}, \mathfrak{L} \rangle \cong N([\mathfrak{B}, \mathfrak{C}_3])$ , since  $[\mathfrak{B}, \mathfrak{C}_3] = [\mathfrak{B}, O_3(\mathfrak{C})] \neq 1$ . This is impossible, so  $\mathfrak{B} \subset \mathfrak{L}_2$ .

*Case 1a.*  $\mathfrak{C}_2$  is special.

Since  $\mathfrak{B} \cong Z(\mathfrak{C}_2)$ , it follows that  $\mathfrak{C}_3\mathfrak{B}$  is a maximal subgroup of  $\mathfrak{C}$ . Thus,  $O_3(\mathfrak{C})\mathfrak{C}_2/O_3(\mathfrak{C})\mathfrak{B}$  is a chief factor of  $\mathfrak{C}$ . Let  $\mathfrak{B}_0$  be a subgroup of  $\mathfrak{B}$  of order 2 and let  $\mathfrak{C}_2^0 = Z(\mathfrak{C}_2 \text{ mod } \mathfrak{B}_0)$ . Since  $O_3(\mathfrak{S})\mathfrak{B}_0 \triangleleft \mathfrak{C}$ ,  $\mathfrak{C}_3\mathfrak{C}_2^0$  is a group. Hence,  $\mathfrak{C}_2^0 = \mathfrak{B}$  or  $\mathfrak{C}_2^0 = \mathfrak{C}_2$ . If  $\mathfrak{C}_2^0 = \mathfrak{C}_2$ , then  $\mathfrak{C}_2^0 \subseteq \mathfrak{B}_0$ , so that  $\mathfrak{C}_2/\mathfrak{B}_0$  is abelian. This is not the case, since  $\mathfrak{C}_2^0 = \mathfrak{B}$ . Hence,  $\mathfrak{C}_2^0 = \mathfrak{B}$ , so that  $\mathfrak{C}_2/\mathfrak{B}_0$  is extra special of width  $\geq 2$ . It follows from the proof of Lemma 5.52 that  $J(\mathfrak{C}_3) \triangleleft \mathfrak{C}$ . Maximality of  $\mathfrak{C}$  in  $\mathcal{S}$  guarantees that  $\mathfrak{C}_3$  is a  $S_3$ -subgroup of  $\mathfrak{G}$ , against our assumption that  $\mathfrak{C}_3 \subset \mathfrak{B}$ .

*Case 1b.*  $\mathfrak{C}_2$  is the direct product of a quaternion group and a group of order 2.

Since  $\mathfrak{L}_2$  has no noncyclic abelian subgroup of order 8, it follows that  $\mathfrak{B}$  is a self centralizing subgroup of  $\mathfrak{L}_2$ . Hence,  $\mathfrak{L}_2$  is of maximal class. Let  $\tilde{\mathfrak{L}}_2 = \mathfrak{L}_2 \cap O_{3,2}(\mathfrak{L})$ . Thus,  $\tilde{\mathfrak{L}}_2$  has an automorphism of order 3. Being a subgroup of a group of maximal class,  $\tilde{\mathfrak{L}}_2$  is either a quaternion group or a four-group.

Suppose  $\tilde{\mathfrak{L}}_2$  is a quaternion group. In this case,  $\mathfrak{L}/O_3(\mathfrak{L}) \cong GL(2, 3)$  and  $\mathfrak{B}$  normalizes a  $S_3$ -subgroup of  $\mathfrak{L}$ , against our previous argument. we conclude that  $\tilde{\mathfrak{L}}_2$  is a four-group.

If  $\tilde{\mathfrak{L}}_2 = \mathfrak{B}$ , then  $[O_3(\mathfrak{L}), \mathfrak{B}] \triangleleft \mathfrak{L}$ . But  $O_3(\mathfrak{L}) = \mathfrak{C}_3$  and  $[\mathfrak{C}_3, \mathfrak{B}] = [O_3(\mathfrak{C}), \mathfrak{B}] \triangleleft \mathfrak{C}$ . Hence,  $[O_3(\mathfrak{L}), \mathfrak{B}] \triangleleft \langle \mathfrak{L}, \mathfrak{C} \rangle$  against the maximality of  $\mathfrak{C}$  in  $\mathcal{S}$ . We conclude that  $\tilde{\mathfrak{L}}_2 \neq \mathfrak{B}$ , so that  $\mathfrak{L}_2$  is a dihedral group of order 8 whose two four-subgroups are  $\mathfrak{B}$  and  $\tilde{\mathfrak{L}}_2$ .

Since  $\tilde{\mathfrak{L}}_2$  does not centralize  $Z(O_3(\mathfrak{L}))$ , it follows that  $J(\mathfrak{L}_3) \triangleleft \mathfrak{L}_0$ . Hence,  $J(\mathfrak{L}_3) = J(\mathfrak{C}_3)$ , so by construction of  $\mathfrak{L}$ , we conclude that  $\mathfrak{L}_3$  is a  $S_3$ -subgroup of  $\mathfrak{G}$ . We may therefore assume without loss of generality that  $\mathfrak{B} = \mathfrak{L}_3$ . Hence,  $|\mathfrak{B} : \mathfrak{C}_3| = 3$ .

Let  $\mathfrak{X} = \Omega_1(Z(\mathfrak{C}_3))$ . Recalling that  $\mathfrak{C}_3 = O_3(\mathfrak{L})$ , we get  $\mathfrak{X} \triangleleft \mathfrak{L}$ . Since  $\tilde{\mathfrak{L}}_2$  does not centralize  $\mathfrak{B}$ , we get  $\mathfrak{B} \subset \mathfrak{X}$ . Since  $Z(\mathfrak{B})$  is cyclic and  $|\mathfrak{B} : \mathfrak{C}_3| = 3$ , we get  $|\mathfrak{X}| \leq 27$ . Hence,  $|\mathfrak{X}| = 27$  and  $\mathfrak{X}$  is the only minimal normal subgroup of  $\mathfrak{L}$ .

Since  $J(\mathfrak{B}) \triangleleft \mathfrak{L}$ , we get  $\mathfrak{X}_0 \triangleleft \mathfrak{L}$ , where  $\mathfrak{X}_0 = \Omega_1(Z(J(\mathfrak{B})) \cap D(J(\mathfrak{B})))$ , if

$D(J(\mathfrak{P})) \neq 1$ , and  $\mathfrak{x}_0 = J(\mathfrak{P})$  if  $D(J(\mathfrak{P})) = 1$ . If  $|\mathfrak{x}_0| > 27$ , then some element of  $\mathfrak{B}^\#$  centralizes a noncyclic subgroup of  $\mathfrak{x}_0$ . This was shown to be impossible in the proof of Lemma 9.14. Hence,  $|\mathfrak{x}_0| \leq 27$ . This implies that  $\mathfrak{x}_0 = \mathfrak{x}$ .

Suppose  $A, B \in \mathfrak{x}$ , and  $A = B^B$  for some  $G$  in  $\mathfrak{G}$ . Thus,  $\langle J(\mathfrak{P}), J(\mathfrak{P}^{G^{-1}}) \rangle \subseteq C(B)$ , and we can choose  $C$  in  $C(B)$  such that  $J(\mathfrak{P})^C = J(\mathfrak{P}^{G^{-1}})$ . Hence,  $CG = N \in N(J(\mathfrak{P}))$  and  $A = B^G = B^{G^{-1}N} = B^N$ . Thus, elements of  $\mathfrak{x}$  are  $\mathfrak{G}$ -conjugate only if they are  $N(\mathfrak{x})$ -conjugate.

Let  $\mathfrak{x} = \mathfrak{x}_1 \times \mathfrak{x}_2 \times \mathfrak{x}_3$ , where  $|\mathfrak{x}_i| = 3$  and where  $\mathfrak{x}_i$  admits  $\mathfrak{B}$ ,  $i = 1, 2, 3$ . Let  $\mathfrak{V}_i = C(\mathfrak{x}_i) \cap \mathfrak{B} = \langle V_i \rangle$ . Thus,  $\mathfrak{x}_1, \mathfrak{x}_2, \mathfrak{x}_3$  are the only subgroups of  $\mathfrak{x}$  of order 3 which admit  $\mathfrak{B}$ . Let  $Z$  be a generator for  $\mathfrak{B}$ . Then  $Z = X_1 X_2 X_3$  with  $X_i \in \mathfrak{x}_i$ .

We argue that  $\mathfrak{B} \not\sim \mathfrak{x}_i$  for  $i = 1, 2, 3$ . Namely, if  $\mathfrak{B} \sim \mathfrak{x}_i$ , there is  $N \in N(\mathfrak{x})$  such that  $\mathfrak{x}_i = \mathfrak{B}^N$ . Let  $\mathfrak{A} = A_{\mathfrak{G}}(\mathfrak{x})$ . Thus,  $|\mathfrak{A}|_3 = 3$ ,  $\mathfrak{A}$  is solvable, and  $\mathfrak{A} \supseteq \mathfrak{B} = A_{\mathfrak{Q}}(\mathfrak{x}) \cong \Sigma_4$ . So  $\mathfrak{A} = \mathfrak{B}$  or  $\mathfrak{A} = \mathfrak{B} \times \mathfrak{A}_0$ , where  $\mathfrak{A}_0 = \langle A \rangle$  and  $A$  inverts  $\mathfrak{x}$ . In neither case are  $\mathfrak{B}$  and  $\mathfrak{x}_i$  in the same orbit under  $\mathfrak{A}$ .

We now return to our study of  $\mathfrak{S}$ . Let  $\mathfrak{S}_2 = \mathfrak{Q} \times \langle V \rangle$ , where  $\mathfrak{Q}$  is a quaternion group and  $V \in \mathfrak{B}$ . Let  $\langle V_0 \rangle = \mathfrak{Q}'$ . Since  $V_0$  does not centralize  $Z(O_3(\mathfrak{S}))$ , there is a minimal normal subgroup  $\mathfrak{Y}$  of  $\mathfrak{S}$  such that  $\mathfrak{Q}$  is represented faithfully on  $\mathfrak{Y}$ . Let  $\mathfrak{B}^* = C_{\mathfrak{B}}(\mathfrak{Y})$  so that  $|\mathfrak{B}^*| = 2$ ,  $\mathfrak{S}_2 = \mathfrak{Q} \times \mathfrak{B}^*$ . We see that  $|\mathfrak{Y}| = 9$  and that  $\mathfrak{Y}_0 = \mathfrak{Y} \cap Z(\mathfrak{S}_2)$  is of order 3 and admits  $\mathfrak{B}$ . Thus,  $\mathfrak{Y}_0 \subset \mathfrak{x}$ , so  $\mathfrak{Y}_0 = \mathfrak{x}_i$  for some  $i = 1, 2, 3$ . Since  $\mathfrak{x}_i \not\sim \mathfrak{B}$ , it follows that  $\mathfrak{S}_3$  is a  $S_3$ -subgroup of  $C(\mathfrak{Y}_0)$ . Since  $[\mathfrak{Y}, \mathfrak{S}_3] \subseteq \mathfrak{Y}_0$ , it follows that  $\mathfrak{Y} \subseteq O_3(C(\mathfrak{Y}_0))$ . Since  $\mathfrak{Q}$  permutes transitively the subgroups of  $\mathfrak{Y}$  of order 3, it follows that  $\mathfrak{Y} \subseteq O_3(C(\mathfrak{Y}^*))$  for every subgroup  $\mathfrak{Y}^*$  of  $\mathfrak{Y}$  of order 3. This implies that  $\mathfrak{Y} \in \mathcal{E}(3)$ . Now  $C(\mathfrak{B}^*)$  contains  $\mathfrak{Y}$  and also contains an element of  $\mathcal{Z}(2)$ , so  $C(\mathfrak{B}^*)$  is nonsolvable. This contradiction shows that this case does not arise.

*Case 2.*  $O_3(\mathfrak{S})\mathfrak{B}/O_3(\mathfrak{S}) \not\subseteq Z(\mathfrak{S}/O_3(\mathfrak{S}))$ .

We conclude that  $\mathfrak{S}_2$  is special and that  $\mathfrak{B} = \mathfrak{S}_2$ . Since  $\mathfrak{B} = Z(\mathfrak{S}_2)$ , we get that  $\mathfrak{S}_3\mathfrak{B}$  is a maximal subgroup of  $\mathfrak{S}$ . That is,  $O_3(\mathfrak{S})\mathfrak{S}_2/O_3(\mathfrak{S})\mathfrak{B}$  is a chief factor of  $\mathfrak{S}$ .

Let  $\mathfrak{P}_0$  be a maximal element of  $\mathfrak{N}_{\mathfrak{S}}(\mathfrak{B}; 3)$  with  $\mathfrak{P}_0 \subset \mathfrak{S}_3$ . Hence,  $\mathfrak{P}_0$  is of index 3 in  $\mathfrak{S}_3$ , and all involutions of  $\mathfrak{B}$  are fused in  $\mathfrak{S}_3\mathfrak{B}$ . Also,  $[\mathfrak{B}_0, \mathfrak{P}_0] = [O_3(\mathfrak{S}), \mathfrak{B}_0]$  for every subgroup  $\mathfrak{B}_0$  of  $\mathfrak{B}$ .

Suppose  $\mathfrak{P}_0$  is not a maximal element of  $\mathfrak{N}_{\mathfrak{G}}(\mathfrak{B}; 3)$ . Choose  $\mathfrak{P}_1$  in  $\mathfrak{N}_{\mathfrak{G}}(\mathfrak{B}; 3)$  so that  $|\mathfrak{P}_1 : \mathfrak{P}_0| = 3$ , and let  $\mathfrak{B}_0 = C_{\mathfrak{B}}(\mathfrak{P}_1/\mathfrak{P}_0)$ . Then  $\mathfrak{P}_1$  and  $\mathfrak{S}_2$  both normalize  $[\mathfrak{B}_0, \mathfrak{P}_1]$ . Let  $\mathfrak{B}^*$  be a  $S_{2,3}$ -subgroup of  $N([\mathfrak{B}_0, \mathfrak{P}_1])$  which contains  $\mathfrak{B}\mathfrak{P}_1$ , and let  $\mathfrak{B}_p^*$  be a  $S_p$ -subgroup of  $\mathfrak{B}^*$  with  $\mathfrak{P}_1 \subseteq \mathfrak{B}_p^*$ ,  $\mathfrak{B} \subseteq \mathfrak{B}_p^*$ . Note that  $\mathfrak{B}^*$  contains a conjugate of  $\mathfrak{S}_2$ .

By maximality,  $\mathfrak{S}_3 = N_{\mathfrak{B}}(\Omega_1(Z(O_3(\mathfrak{S}))))$ . Hence,  $\mathfrak{B} \subset \mathfrak{S}_3$ , and so  $\mathfrak{B} \subseteq \Omega_1(Z(O_3(\mathfrak{S})))$ , since  $O_2(\mathfrak{S}) = 1$ . If  $u \in \mathcal{Z}(\mathfrak{B})$ , then  $[\mathfrak{B}, u] \subseteq \mathfrak{B}$ , and so

$\mathbb{U} \subseteq \mathfrak{C}_3$ . Since  $\mathfrak{B}$  is a 4-group and does not centralize  $\Omega_1(\mathbf{Z}(\mathfrak{O}_3(\mathfrak{C})))$ , we conclude from  $[\mathfrak{B}, \mathbb{U}, \mathbb{U}] = 1$  that  $\mathbb{U}$  centralizes  $\mathfrak{B}\mathfrak{O}_3(\mathfrak{C})/\mathfrak{O}_3(\mathfrak{C})$ . That is,  $\mathbb{U} \subseteq \mathfrak{F}_0 \subseteq \mathfrak{F}^*$ . Hence,  $\mathfrak{F}^*$  contains an element of  $\mathcal{S}$  so maximality of  $\mathfrak{C}$  guarantees that  $\mathfrak{F}_1 = \mathfrak{F}_3^*$ , since  $\mathfrak{F}_1$  and  $\mathfrak{C}_3$  are of the same order.

Let  $\mathfrak{F}_2^{**} = \mathfrak{F}_2^* \cap \mathfrak{O}_{3,2}(\mathfrak{F}^*)$ . Thus,  $\mathfrak{F}_2^{**}$  contains a noncyclic abelian subgroup of order 8, since  $\mathfrak{F}^*$  contains a conjugate of  $\mathfrak{C}_2$ .

Suppose every subgroup of  $\mathfrak{F}_2^{**}$  which is characteristic and abelian is also cyclic. Let  $\mathfrak{F}_2^{***}$  be a subgroup of  $\mathfrak{F}_2^{**}$  which is minimal subject to (a) containing a noncyclic abelian subgroup of order 8 and (b) being permutable with  $\mathfrak{F}_3^*$ . Then since  $D(\mathfrak{F}_2^{***}) \subseteq D(\mathfrak{F}_2^{**})$ , it follows that  $\mathfrak{F}_2^{***}$  is not a special group with center of order 4. Since  $\mathfrak{F}_3^*\mathfrak{F}_2^{***} \in \mathcal{S}$ , our previous reduction excludes this possibility.

Let  $\tilde{\mathfrak{F}}_2$  be a noncyclic characteristic abelian subgroup of  $\mathfrak{F}_2^{**}$ . If  $|\tilde{\mathfrak{F}}_2| > 4$ , then  $\mathfrak{F}_1\tilde{\mathfrak{F}}_2$  contains an element of  $\mathcal{S}$ , against our previous reduction. We may assume that  $\tilde{\mathfrak{F}}_2$  is a four-group. If  $\tilde{\mathfrak{F}}_2 \cap \mathfrak{B} \neq 1$ , then  $\mathfrak{O}_3(\mathfrak{F}^*)\tilde{\mathfrak{F}}_2/\mathfrak{O}_3(\mathfrak{F}^*)$  is centralized by  $\mathfrak{F}_1$ , since  $\mathfrak{B}$  normalizes  $\mathfrak{F}_1$ . But in this case, there are maximal elements of  $\mathcal{S}$  which do not satisfy our previous reduction. If  $\tilde{\mathfrak{F}}_2 \cap \mathfrak{B} = 1$ , then  $\mathfrak{F}_1\mathfrak{B}\tilde{\mathfrak{F}}_2$  contains an element of  $\mathfrak{C}$  which also violates our previous reduction. Hence,  $\mathfrak{F}_0$  is a maximal element of  $\mathfrak{N}_2(\mathfrak{B}; 3)$ .

Since  $\mathfrak{O}_3(\mathfrak{B}) \in \mathfrak{N}_2(\mathfrak{B}; 3)$ , we have  $\mathfrak{O}_3(\mathfrak{B}) \subseteq \mathfrak{F}_0$ . Maximality of  $\mathfrak{C}$  in  $\mathcal{S}$  implies that  $\mathfrak{B}$  contains no noncyclic abelian subgroup of order 8. Since the involutions of  $\mathfrak{B}$  are fused in  $\mathfrak{C}_3\mathfrak{B}$ , it follows that  $\mathfrak{O}_{3,2}(\mathfrak{B}) = \mathfrak{O}_3(\mathfrak{B})\mathfrak{B}$ . Hence,  $\mathfrak{F}_0 = \mathfrak{O}_3(\mathfrak{B})$  is of index 3 in  $\mathfrak{F}_3$ . This violates  $|\mathfrak{F}_3| > |\mathfrak{C}_3| = 3|\mathfrak{F}_0|$ .

Thus, in all cases, we have shown that  $\mathfrak{C}_3 = \mathfrak{F}$ .

Suppose that  $\mathfrak{B}$  normalizes  $\mathfrak{C}_3$ . Let  $\mathfrak{W}$  be a minimal normal subgroup of  $\mathfrak{C}$ . Clearly,  $\mathfrak{W} \supset \mathfrak{B}$ . Since  $\mathfrak{B}\mathfrak{O}_3(\mathfrak{C})/\mathfrak{O}_3(\mathfrak{C})$  is a central factor of  $\mathfrak{C}$ , some involution of  $\mathfrak{B}$  centralizes  $\mathfrak{W}$ . But  $\mathfrak{W}$  contains an element of  $\mathcal{Z}(\mathfrak{B})$ , so Lemmas 7.4 and 5.38 imply that  $C(V)$  is nonsolvable for some involution  $V$  of  $\mathfrak{B}$ . Thus,  $\mathfrak{B}$  does not normalize  $\mathfrak{F}$ . In particular,  $\mathfrak{C}_2$  is special.

Let  $\mathfrak{F}_0$  be the largest subgroup of  $\mathfrak{F}$  normalized by  $\mathfrak{B}$ . Thus,  $|\mathfrak{F}:\mathfrak{F}_0| = 3$  and  $N_{\mathfrak{F}}(\mathfrak{C}_2)$  permutes transitively the involutions of  $\mathfrak{B}$ .

Let  $\mathfrak{W}$  be a minimal normal subgroup of  $\mathfrak{C}$ . Clearly,  $\mathfrak{W} \supset \mathfrak{B}$ , so  $\mathfrak{B}$  is faithfully represented on  $\mathfrak{W}$ . Hence,  $\mathfrak{C}_2$  is faithfully represented on  $\mathfrak{W}$ , so  $C_{\mathfrak{C}}(\mathfrak{W}) = \mathfrak{O}_3(\mathfrak{W})$ . Let  $\mathfrak{W} = \mathfrak{W}_1 \times \mathfrak{W}_2 \times \mathfrak{W}_3$ , where  $\mathfrak{W}_i = C_{\mathfrak{W}}(V_i)$  and  $V_1, V_2, V_3$  are the involutions of  $\mathfrak{B}$ . Thus,  $\mathfrak{C}_2\mathfrak{F}_0$  normalizes each  $\mathfrak{W}_i$ , and  $\mathfrak{C}$  permutes  $\mathfrak{W}_1, \mathfrak{W}_2, \mathfrak{W}_3$  transitively. Obviously each  $\mathfrak{W}_i$  is an irreducible  $\mathfrak{F}_0\mathfrak{C}_2$ -module.

Let  $\mathfrak{F}_i = C_{\mathfrak{F}_0\mathfrak{C}_2}(\mathfrak{W}_i)$ , and  $\mathfrak{C}_{2i} = \{S \in \mathfrak{C}_2 \mid [S, \mathfrak{C}_2] \subseteq \langle V_i \rangle\}$ , for  $i = 1, 2, 3$ . Then  $N_{\mathfrak{C}}(\mathfrak{C}_2)$  permutes  $\{\mathfrak{C}_{21}, \mathfrak{C}_{22}, \mathfrak{C}_{23}\}$  transitively, and so one of the following holds:

- (a)  $\mathfrak{C}_{2i} = \mathfrak{B}, i = 1, 2, 3$ ,
- (b)  $\mathfrak{C}_{2i} \supset \mathfrak{B}, i = 1, 2, 3$ .

If (a) holds, then  $\mathfrak{S}_2/\langle V_i \rangle$  is extra special, and since  $V_j$  inverts  $\mathfrak{B}_i$  for  $j \neq i$ , it follows that  $\mathfrak{R}_i \cap \mathfrak{S}_2 = \langle V_i \rangle$ , whence  $\mathfrak{R}_i = O_3(\mathfrak{S})\langle V_i \rangle$ . The proof of Lemma 5.52 now shows that  $O_3(\mathfrak{S}) \cong J(\mathfrak{S}_3)$ , the desired contradiction.

Suppose (b) holds. The group  $O_3(\mathfrak{S})\mathfrak{S}_{21}\mathfrak{S}_{22}\mathfrak{S}_{23}$  is clearly  $\mathfrak{S}_3$ -invariant, and since  $O_3(\mathfrak{S})\mathfrak{S}_2/O_3(\mathfrak{S})\mathfrak{B}$  is a chief factor of  $\mathfrak{S}$ , we have  $\mathfrak{S}_2 = \mathfrak{S}_{21}\mathfrak{S}_{22}\mathfrak{S}_{23}$ . Obviously,  $[\mathfrak{S}_{2i}, \mathfrak{S}_{2j}] \cong \langle V_i \rangle \cap \langle V_j \rangle = 1$  for  $i \neq j$ . Because  $\mathfrak{S}_2$  is special, we conclude that  $\mathfrak{S}_{2i'} = \langle V_i \rangle$  and hence that  $\mathfrak{S}_{2j}/\langle V_i \rangle$  is extra special for  $i \neq j$ . If the width of  $\mathfrak{S}_{2j}/\langle V_i \rangle$  is greater than 1, then since  $J(\mathfrak{S}_3) \not\subseteq O_3(\mathfrak{S})$ , and since  $\mathfrak{S}_{2j}/\langle V_i \rangle$  acts faithfully on  $\mathfrak{B}_i$ , it follows that  $J(\mathfrak{S}_3)$  centralizes  $O_3(\mathfrak{S})\mathfrak{S}_{2j}/O_3(\mathfrak{S})\langle V_i \rangle$ . But  $J(\mathfrak{S}_3) \triangleleft \mathfrak{S}_3$ , and so  $J(\mathfrak{S}_3)$  centralizes  $O_3(\mathfrak{S})\mathfrak{S}_2/O_3(\mathfrak{S})\mathfrak{B}$ , that is,  $J(\mathfrak{S}_3) \cong O_3(\mathfrak{S})$ . We may assume that if  $i \neq j$ , then  $\mathfrak{S}_{2i}/\langle V_i \rangle$  is of width 1. But then  $\mathfrak{S}_{21}\mathfrak{S}_{22}/\langle V_3 \rangle$  is the central product of  $\mathfrak{S}_{21}/\langle V_3 \rangle$  and  $\mathfrak{S}_{22}/\langle V_3 \rangle$ , so is extra special of width 2, acts faithfully on  $\mathfrak{B}_3$ , and  $O_3(\mathfrak{S})\mathfrak{S}_{21}\mathfrak{S}_{22}$  admits  $J(\mathfrak{S}_3)$ . By Lemma 5.52,  $J(\mathfrak{S}_3)$  centralizes  $O_3(\mathfrak{S})\mathfrak{S}_{21}\mathfrak{S}_{22}/O_3(\mathfrak{S})$ , so again we get the contradiction  $J(\mathfrak{S}_3) \cong O_3(\mathfrak{S})$ . The proof is complete.

LEMMA 9.16. *If  $\mathfrak{A}$  is a subgroup of  $\mathfrak{G}$  of type (3, 3) and each element of  $\mathfrak{A}$  centralizes an element of  $\mathcal{Z}(3)$ , then*

- (i)  $\mathfrak{A} \in \mathcal{S}$ .
- (ii)  $4 \nmid |C(\mathfrak{A})|$ .

*Proof.* (i) Suppose false, and  $\mathfrak{X}$  is a four-group normalized but not centralized by  $\mathfrak{A}$ . Let  $\mathfrak{A}_0 = C_{\mathfrak{A}}(\mathfrak{X})$ , so that  $|\mathfrak{A}_0| = 3$ . Let  $\mathfrak{Q}_0$  be a maximal 2, 3-subgroup of  $\mathfrak{G}$  which contains a  $S_{2,3}$ -subgroup of  $C(\mathfrak{A}_0)$  containing  $\mathfrak{A}\mathfrak{X}$ . Let  $\mathfrak{Q} = O^{\mathfrak{X}}(\mathfrak{Q}_0)$ . Since  $\mathfrak{Q}$  contains an element of  $\mathcal{Z}(3)$ , Lemma 9.15 implies that  $\mathfrak{Q}$  contains no noncyclic abelian subgroup of order 8. Hence,  $\mathfrak{X}$  is a  $S_2$ -subgroup of  $\mathfrak{Q}$  and  $\mathfrak{Q}/O_3(\mathfrak{Q}) \cong A_4$ . Let  $\mathfrak{Q}_3$  be a  $S_3$ -subgroup of  $\mathfrak{Q}$ . Since  $\mathfrak{X}$  does not centralize  $Z(O_3(\mathfrak{Q}))$ , it follows that  $J(\mathfrak{Q}_3) \triangleleft \mathfrak{Q}$ . Hence,  $\mathfrak{Q}_3$  is a  $S_3$ -subgroup of  $\mathfrak{G}$ , and we may assume that  $\mathfrak{Q}_3 = \mathfrak{B}$ .

Let  $\mathfrak{X}$  be a minimal normal subgroup of  $\mathfrak{Q}$  with  $\mathfrak{X} \cong Z(J(\mathfrak{B}))$ . Thus,  $\mathfrak{X}$  is elementary of order 27 and  $C_{\mathfrak{X}}(\mathfrak{X}) = 1$ . Choose  $T$  in  $\mathfrak{X}^{\mathfrak{X}}$  so that  $\mathfrak{X}_1 = C_{\mathfrak{X}}(T)$  is of order 3 and is inverted by the generator of  $\mathfrak{X}/\langle T \rangle$ . Hence,  $\langle \mathfrak{A}_0, \mathfrak{X}_1 \rangle = \mathfrak{A}^*$  is elementary of order 9 and is normalized by  $\mathfrak{X}$ , and every element of  $\mathfrak{A}^*$  centralizes an element of  $\mathcal{Z}(\mathfrak{B})$ .

Let  $\mathfrak{C}$  be a  $S_{2,3}$ -subgroup of  $C(T)$  which contains  $\mathfrak{A}^*\mathfrak{X}$ . By Lemma 5.38,  $\mathfrak{C}$  contains an element of  $\mathcal{Z}(2)$ , so  $|O_3(\mathfrak{C})| \leq 3$ , and  $O_3(\mathfrak{C}) \cap \mathfrak{A}^* = 1$ . Hence,  $\mathfrak{A}^*$  is faithfully represented on  $O_2(\mathfrak{C})$ . Let  $\mathfrak{C}_0$  be a characteristic abelian subgroup of  $O_2(\mathfrak{C})$ . Suppose  $\mathfrak{A}^*$  does not centralize  $\mathfrak{C}_0$ . Hence, there is an element  $A$  in  $\mathfrak{A}^{**}$  such that  $C(A)$  contains an elementary subgroup of order 8. This is not the case, so  $\mathfrak{A}^*$  centralizes  $\mathfrak{C}_0$ . If  $|\mathfrak{C}_0| > 2$ , then some element  $A$  of  $\mathfrak{A}^{**}$  centralizes a noncyclic



abelian subgroup of  $O_2(\mathbb{C})$  of order 8. This is not the case, by Lemma 9.15. Hence,  $O_2(\mathbb{C})$  is extra special of width at least 2 and  $\langle T \rangle = O_2(\mathbb{C})'$ . Hence,  $\mathbb{C}$  contains a  $S_2$ -subgroup of  $\mathbb{G}$ .

By Lemma 9.15, no element of  $\mathfrak{A}^{**}$  centralizes any noncyclic abelian subgroup of order 8. Hence,  $\langle T \rangle = O_2(\mathbb{C}) \cap C(\mathfrak{A}^*)$ . For each  $A$  in  $\mathfrak{A}^{**}$ ,  $O_2(\mathbb{C}) \cap C(A)$  is either  $\langle T \rangle$  or is extra special. Thus,  $O_2(\mathbb{C}) \cap C(A)$  is either  $\langle T \rangle$  or is a quaternion group, so no element of  $\mathfrak{A}^{**}$  centralizes any four-subgroup of  $O_2(\mathbb{C})$ . Thus, the width of  $O_2(\mathbb{C})$  is at most 4. Since  $O_2(\mathbb{C}) \cap C(\mathfrak{A}_0)$  is centralized by  $\mathfrak{A}^*$ , it follows that  $O_2(\mathbb{C}) \cap C(\mathfrak{A}_0) = \langle T \rangle$ , and so the width of  $O_2(\mathbb{C})$  is at most 3.

Consider  $C^*(\mathfrak{x}_1) = \{G \in \mathbb{G}, G \text{ either centralizes or inverts } \mathfrak{x}_1\}$ . By construction,  $|C_2^*(\mathfrak{x}_1)|_3 = |\mathfrak{P}|/3$ . Also,  $\mathfrak{X} \subseteq C^*(\mathfrak{x}_1)$ , and  $C^*(\mathfrak{x}_1)$  contains no noncyclic abelian subgroup of order 8. Suppose  $\mathfrak{A}_0 \not\subseteq O_3(\tilde{\mathfrak{X}})$ , where  $\tilde{\mathfrak{X}}$  is a  $S_{2,3}$ -subgroup of  $C^*(\mathfrak{x}_1)$  which contains  $C_2^*(\mathfrak{x}_1)$ . Then  $\tilde{\mathfrak{X}}/O_3(\tilde{\mathfrak{X}})$  contains a subgroup isomorphic to  $\mathfrak{A}_0 \times \mathfrak{X}$ . This is obviously impossible, since  $S_2$ -subgroups of  $\tilde{\mathfrak{X}}/O_3(\tilde{\mathfrak{X}})$  are of maximal class. Hence,  $\mathfrak{A}_0 \subseteq O_3(\tilde{\mathfrak{X}})$ . This implies that  $\mathfrak{A}_0$  centralizes  $O_2(\mathbb{C}) \cap C(\mathfrak{x}_1)$ , so the width of  $O_2(\mathbb{C})$  is 2. Hence,  $\mathfrak{x}_1 \times \mathfrak{A}_0$  is a  $S_3$ -subgroup of  $C(T)$ , so  $\mathfrak{x}_1 \times \mathfrak{A}_0$  is a  $S_3$ -subgroup of  $C_3(T)$ . By a formula of Wielandt [40],

$$|O_3(\mathfrak{L})| = |O_3(\mathfrak{L}) \cap C(T)|^3 / |O_3(\mathfrak{L}) \cap C(\mathfrak{X})|^2.$$

Hence,  $|O_3(\mathfrak{L})| = 3^6/3^2 = 3^4$ , so that  $O_3(L) = \mathfrak{A}_0 \times \mathfrak{x}$ . This implies that  $Z(\mathfrak{P})$  is noncyclic, since  $|\mathfrak{P}:O_3(\mathfrak{L})| = 3$ . The proof of (i) is complete.

As for (ii), suppose  $\mathfrak{X}$  is a subgroup of  $C(\mathfrak{A})$  of order 4. Then  $C(T)$  contains an element of  $\mathcal{Z}(2)$ ,  $T$  being an involution of  $\mathfrak{X}$ . Thus, by Lemma 7.5, there is a subgroup  $\mathfrak{M}$  in  $\mathcal{MS}(\mathbb{G})$  which contains  $\mathfrak{A}\mathfrak{X}$  and satisfies  $O_2(\mathfrak{M}) = 1$ , while  $O_2(\mathfrak{M})$  is of symplectic type. Since  $\mathfrak{A}$  acts faithfully on  $O_2(\mathfrak{M}) \cap C(\mathfrak{X})$ , we can therefore choose  $A$  in  $\mathfrak{A}^\#$  such that  $\mathfrak{A}$  does not centralize  $O_2(\mathbb{C}) \cap C(\mathfrak{X}) \cap C(\mathfrak{A})$ . Thus,  $C(A)$  contains a noncyclic abelian subgroup of order 8, against Lemma 9.15. The proof of (ii) is complete.

LEMMA 9.17. *Suppose*

- (a)  $\mathfrak{R}$  is a maximal 2, 3-subgroup of  $\mathbb{G}$ .
- (b)  $\mathfrak{R}$  contains an element of  $\mathcal{D}$ .
- (c)  $\mathfrak{R}$  contains a noncyclic abelian subgroup of order 8.

Then  $O_2(\mathfrak{R}) \neq 1$ .

*Proof.* Let  $\mathfrak{R}_p$  be a  $S_p$ -subgroup of  $\mathfrak{R}$ ,  $p = 2, 3$ . We assume without loss of generality that  $\mathfrak{R}_3 \subseteq \mathfrak{P}$ . Suppose by way of contradiction that  $O_2(\mathfrak{R}) = 1$ . Then  $O_3(\mathfrak{R}) \neq 1$ , so by maximality of  $\mathfrak{R}$ ,  $\mathfrak{R}_3 = N_{\mathfrak{P}}(O_3(\mathfrak{R}))$ . Hence,  $\mathfrak{P} \subseteq \mathfrak{R}_3$ . Since  $O_2(\mathfrak{R}) = 1$ , we get  $\mathfrak{P} \subseteq Z(O_3(\mathfrak{R}))$ . Hence,  $\mathfrak{R}_3$  contains every element of  $\mathcal{Z}(\mathfrak{P})$ . This contradicts Lemma 9.15 and completes the proof.

We now begin the construction of the final configuration.

By hypothesis,  $2 \sim 3$ . Let  $\mathfrak{A}$  be a noncyclic abelian subgroup of  $\mathfrak{G}$  of order 8 and let  $\mathfrak{B}$  be an elementary subgroup of order 9 each of whose elements centralizes an element of  $\mathcal{U}(3)$ , chosen so that  $\langle \mathfrak{A}, \mathfrak{B} \rangle$  is solvable. We assume without loss of generality that  $\langle \mathfrak{A}, \mathfrak{B} \rangle$  is a 2, 3-group. Let  $\mathfrak{R}$  be a maximal 2, 3-subgroup of  $\mathfrak{G}$  which contains  $\langle \mathfrak{A}, \mathfrak{B} \rangle$ .

Let  $\mathfrak{R}_p$  be a  $S_p$ -subgroup of  $\mathfrak{R}$ ,  $p = 2, 3$ , with  $\mathfrak{B} \subseteq \mathfrak{R}_3$ . By Lemma 9.17,  $O_2(\mathfrak{R}) \neq 1$ .

Let  $J$  be an involution in  $Z(\mathfrak{R}_2) \cap O_2(\mathfrak{R})$ . Since  $\mathfrak{B} \in \mathcal{D}$ , by Lemma 9.16,  $\mathfrak{B}$  centralizes  $Z(O_2(\mathfrak{R}))$ . Hence,  $C(J)$  is a solvable subgroup containing  $\mathfrak{B}, \mathfrak{R}_2$ , and an element of  $\mathcal{U}(2)$ .

Since  $\mathfrak{B} \in \mathcal{D}$ , we may apply Lemma 7.5. Let  $\mathfrak{M}$  be an element of  $\mathcal{MS}(\mathfrak{G})$  which contains  $\mathfrak{B}$  and  $\mathfrak{R}_2$  and which satisfies all the conclusions of Lemma 7.5. Let  $\mathfrak{R}$  be a  $S_{2,3}$ -subgroup of  $\mathfrak{M}$  and let  $\mathfrak{R}_0 = O_2(\mathfrak{R})$ . Since  $O_2(\mathfrak{M}) = 1$ , so also  $O_3(\mathfrak{R}) = 1$ . Since no element of  $\mathfrak{B}^*$  centralizes any noncyclic abelian subgroup of order 8, it follows that  $\mathfrak{R}_0$  is extra special of width 2, 3 or 4, and  $C_{\mathfrak{R}_0}(\mathfrak{B}) = \mathfrak{R}'_0 = \langle I \rangle$ , the last equality serving to define  $I$ . Hence,  $\mathfrak{R}_0 = O_2(\mathfrak{M})$ . Let  $\mathfrak{R}_p$  be a  $S_p$ -subgroup of  $\mathfrak{R}$ ,  $p = 2, 3$ , with  $\mathfrak{B} \subseteq \mathfrak{R}_3$ . Let  $\mathfrak{R}_3^* = \mathfrak{R}_3 \cap O_{2,3}(\mathfrak{R})$ ,  $\mathfrak{R}^* = N_{\mathfrak{R}}(\mathfrak{R}_3^*)$ . Thus,  $\mathfrak{R} = \mathfrak{R}_0\mathfrak{R}^*$  and  $\mathfrak{R}_0 \cap \mathfrak{R}^* = C_{\mathfrak{R}_0}(\mathfrak{R}_3^*)$ . Let  $\mathfrak{R}_2^* = \mathfrak{R}^* \cap \mathfrak{R}_2$  so that  $\mathfrak{R}^* = \mathfrak{R}_3\mathfrak{R}_2^*$ . We assume without loss of generality that  $\mathfrak{R}_3 \subseteq \mathfrak{R}_3^*$ .

We argue that

$$(9.25) \quad \mathfrak{R}_0 \cap \mathfrak{R}^* = \langle I \rangle .$$

Namely, choose  $\mathfrak{U}$  in  $\mathcal{U}(\mathfrak{B})$  and suppose  $C(\mathfrak{U}) \cap \mathfrak{R}_3^*$  is noncyclic. Then by Lemma 9.16 (ii), no noncyclic abelian subgroup of  $C(\mathfrak{U}) \cap \mathfrak{R}_3^*$  centralizes any subgroup of order 4, so (9.25) is clear. Suppose  $C(\mathfrak{U}) \cap \mathfrak{R}_3^*$  is cyclic. Hence,  $\mathfrak{R}_3^*$  has a cyclic subgroup of index 3. Assume that (9.25) does not hold. Then  $\mathfrak{B} \not\subseteq \mathfrak{R}_3^*$ , so the 3-length of  $\mathfrak{R}$  is at least 2. Hence,  $\mathfrak{R}_3^*$  is elementary of order 9 and all elements of  $\mathfrak{R}_3^{*\#}$  are fused in  $\mathfrak{R}$ . But then every element of  $\mathfrak{R}_3^*$  centralizes an element of  $\mathcal{U}(3)$ , so again (9.25) holds. Thus, (9.25) holds.

LEMMA 9.18. *If  $\mathfrak{R}$  is any 2, 3-subgroup of  $\mathfrak{G}$  which contains  $\mathfrak{B}, I$ , and also contains a noncyclic abelian subgroup of order 8, then  $\mathfrak{R} \subseteq \mathfrak{M}$ .*

*Proof.* We may assume that  $\mathfrak{R}$  is a maximal 2, 3-subgroup of  $\mathfrak{G}$ . By Lemma 9.17, we have  $O_2(\mathfrak{R}) \neq 1$ . Since  $\mathfrak{B} \in \mathcal{D}$ ,  $\mathfrak{B}$  centralizes  $Z(O_2(\mathfrak{R}))$ . Since  $\langle I \rangle$  is a  $S_2$ -subgroup of  $C(\mathfrak{B})$ , by Lemma 9.16 (ii), it follows that  $\langle I \rangle = Z(O_2(\mathfrak{R}))$ , so  $\mathfrak{R} \subseteq C(I) = \mathfrak{M}$ .

LEMMA 9.19.  *$\mathfrak{R}_2^*$  contains no noncyclic abelian subgroup of order 8.*

*Proof.* Suppose false. In this case,  $\mathfrak{R}^*$  is a  $S_{2,3}$ -subgroup of  $N(\mathfrak{R}_3^*)$ , by the preceding paragraph. Hence, the 3-length of  $\mathfrak{R}^*$  is at least 2. But in this case,  $\mathfrak{B} \subset \mathfrak{R}_3^*$ , so  $\mathfrak{R}_3$  contains every element of  $\mathcal{Z}(\mathfrak{B})$ , against Lemma 7.4. The proof is complete.

LEMMA 9.20. *If  $\tilde{\mathfrak{R}}_3$  is a  $S_3$ -subgroup of  $N(\mathfrak{R}_3)$ , then*

$$\tilde{\mathfrak{R}}_3 = \mathfrak{R}_3 \cdot C_{\tilde{\mathfrak{R}}_3}(\mathfrak{R}_3).$$

*Proof.* Let  $\mathfrak{C}_1 = C(\mathfrak{R}_2)\mathfrak{R}_3$ ,  $\mathfrak{N}_1 = N(\mathfrak{R}_3)$ . Since  $\mathfrak{B} \subseteq \mathfrak{R}_3$ , it follows that  $\langle I \rangle$  is a  $S_2$ -subgroup of  $\mathfrak{C}_1$ , by Lemma 9.16 (ii). Hence,  $\mathfrak{M}$  covers  $\mathfrak{N}_1/\mathfrak{C}_1$ , so  $\mathfrak{C}_1$  contains  $\mathfrak{R}_3$ , which is equivalent to our assertion.

LEMMA 9.21.

(a) *If  $\tilde{\mathfrak{F}}$  is any 3-subgroup of  $\mathfrak{M}$ , then no  $S_3$ -subgroup of  $N(\tilde{\mathfrak{F}})$  is contained in any conjugate of  $\mathfrak{M}$ .*

(b) *If  $P$  is an element of  $\mathfrak{M}$  of order 3, then  $C(P)$  contains a subgroup  $\mathfrak{A}^*$  of type (3, 3) such that  $C(A)$  contains an element of  $\mathcal{Z}(3)$  for each  $A$  in  $\mathfrak{A}^*$ .*

(c) *If  $\tilde{\mathfrak{F}}$  is any nonidentity 3-subgroup of  $\mathfrak{M}$ , then  $N(\tilde{\mathfrak{F}})$  contains no noncyclic abelian group of order 8.*

(d)  $\mathfrak{R}_3$  contains no abelian subgroup of order 27.

(e)  $\mathfrak{R}_3$  is isomorphic to one of the following groups:

(i) an elementary group of order 9.

(ii) a nonabelian group of order 27.

*Proof.* Let  $\mathfrak{F}^*$  be a  $S_3$ -subgroup of  $N(\tilde{\mathfrak{F}})$ . Suppose  $\mathfrak{F}^* \subseteq \mathfrak{M}^G$ . Since  $\tilde{\mathfrak{F}} \subseteq \mathfrak{F}^*$ , we get  $\tilde{\mathfrak{F}} \subseteq \mathfrak{M}^G$ . Let  $\mathfrak{F}_0 = \tilde{\mathfrak{F}}^{G^{-1}}$ ,  $\mathfrak{F}_1 = \mathfrak{F}^{*G^{-1}}$ . Then  $\mathfrak{F}_0$  is a 3-subgroup of  $\mathfrak{M}$  and  $\mathfrak{F}_1$  is a  $S_3$ -subgroup of  $N(\mathfrak{F}_0)$  which is contained in  $\mathfrak{M}$ . This violates Lemma 9.20, since  $\mathfrak{R}_3 \subset \tilde{\mathfrak{R}}_3$ . Hence, (a) holds.

Since  $\mathfrak{B} \subseteq \mathfrak{R}_3$ , it follows from Lemma 9.16 (ii) that  $\langle I \rangle$  is a  $S_2$ -subgroup of  $\mathfrak{R}_3 C(\mathfrak{R}_3)$ . Thus,  $\mathfrak{R}_3 C(\mathfrak{R}_3)$  has a normal 2-complement. We assume without loss of generality that  $I$  normalizes  $\tilde{\mathfrak{R}}_3$ . Let  $\hat{\mathfrak{R}}_3/\mathfrak{R}_3$  be a chief factor of  $\tilde{\mathfrak{R}}_3 \langle I \rangle$ . Hence,  $\hat{\mathfrak{R}}_3 = \mathfrak{R}_3 \times \bar{\mathfrak{R}}_3$ , where  $|\bar{\mathfrak{R}}_3| = 3$ . This implies that  $C_{\hat{\mathfrak{R}}_3}(P)$  contains an elementary subgroup of order 27. Let  $\mathfrak{F}^G$  be a  $S_3$ -subgroup of  $\mathfrak{G}$  containing  $\tilde{\mathfrak{R}}_3$ , and let  $\mathfrak{U} \in \mathcal{Z}(\mathfrak{F}^G)$ . Then  $C(\mathfrak{U}) \cap C_{\hat{\mathfrak{R}}_3}(P)$  is noncyclic, and any noncyclic subgroup of  $C(\mathfrak{U}) \cap C_{\hat{\mathfrak{R}}_3}(P)$  of order 9 may play the role of  $\mathfrak{A}^*$  in (b).

Let  $\mathfrak{R}$  be a maximal 2, 3-subgroup of  $\mathfrak{G}$  which contains a  $S_{2,3}$ -subgroup of  $N(\tilde{\mathfrak{F}})$ . By (b),  $\mathfrak{R}$  contains an element of  $\mathcal{S}$ . Assume that  $\mathfrak{R}$  contains a noncyclic abelian subgroup of order 8. Then by Lemma 9.17,  $O_2(\mathfrak{R}) \neq 1$ . By Lemma 9.16 (ii), we get  $|Z(O_2(\mathfrak{R}))| = 2$ . Clearly,  $\mathfrak{R}$  is a  $S_{2,3}$ -subgroup of  $C(Z(O_2(\mathfrak{R})))$ , and so it contains an

element  $\mathfrak{D}$  of  $\mathcal{S}$  and one of  $\mathcal{U}(2)$ . Applying Lemma 7.5, we get a conjugate  $\mathfrak{M}^g$  of  $\mathfrak{M}$  containing  $\mathfrak{D}$  and a  $S_2$ -subgroup of  $\mathfrak{R}$ . By Lemma 7.5 (f),  $Z(\mathcal{O}_2(\mathfrak{R})) \subseteq C(\mathfrak{D}) \cap \mathcal{O}_2(\mathfrak{M}^g)$ . By Lemma 9.16 (ii), the last group is of order 2, and so equals  $Z(\mathcal{O}_2(\mathfrak{R}))$ . Hence,  $Z(\mathcal{O}_2(\mathfrak{R})) = Z(\mathcal{O}_2(\mathfrak{M}^g))$ , and so  $\mathfrak{R} \subseteq \mathfrak{M}^g$ . This violates (a).

Suppose  $\mathfrak{C}$  is an abelian subgroup of  $\mathfrak{R}_3$  of order 27. Then there is an element  $E$  in  $\mathfrak{C}^*$  of order 3 such that  $C(E) \cap \mathcal{O}_2(\mathfrak{R})$  contains a noncyclic abelian subgroup of order 8. This violates (c) and establishes (d).

(e) is an immediate consequence of (d).

LEMMA 9.22.  $\mathfrak{R}_2 - \mathfrak{R}_0$  contains an involution.

*Proof.* Suppose false. By a result of Glauberman [16],  $\mathfrak{R}_2$  contains an involution  $J$  such that  $J = I^g \neq I$ . Since the lemma is false,  $J \in \mathfrak{R}_0$ . Let  $\mathfrak{X} = C_{\mathfrak{R}_0}(J)$ . Then  $\mathfrak{X}$  is generated by involutions, and  $\mathfrak{X} \subseteq \mathfrak{M}^g = C(J)$ . Since the lemma is false,  $\mathfrak{X} \subseteq \mathfrak{R}_0^g$ . In particular,  $I \in (\mathfrak{R}_0^g)'$ . Hence,  $(\mathfrak{R}_0^g)' = \langle J \rangle = \langle I \rangle$ , a contradiction.

LEMMA 9.23. The 3-length of  $\mathfrak{R}$  is 1.

*Proof.* Suppose false. By Lemma 9.21 (e), it follows that  $\mathfrak{R}_3^*$  is elementary of order 9. Consider  $\mathfrak{R}^*/\langle I \rangle$ . Since  $\mathfrak{R}^* \cap \mathfrak{R}_0 = \langle I \rangle$  by (9.25), it follows that  $\mathfrak{R}_3^* \langle I \rangle / \langle I \rangle = F(\mathfrak{R}^* / \langle I \rangle)$ . This implies that  $\mathfrak{R}^* / \langle I \rangle$  contains a quaternion subgroup  $\mathfrak{Q} / \langle I \rangle$ . Thus,  $\mathfrak{Q}$  is not of maximal class, since no group of maximal class and order 16 has a quaternion factor group. Hence,  $\mathfrak{Q}$  contains a noncyclic abelian subgroup of order 8. This violates Lemma 9.21 (c) with  $\mathfrak{R}_3^*$  in the role of  $\tilde{\mathfrak{F}}$ .

LEMMA 9.24. Each involution  $J$  of  $\mathfrak{R}_2 - \mathfrak{R}_0$  normalizes a  $S_3$ -subgroup of  $\mathfrak{R}$ .

*Proof.* Since  $J \in \mathfrak{R}_0$ , Lemma 5.36 implies that  $J$  inverts an element  $P$  of  $\mathfrak{R}$  of order 3. Let  $\mathfrak{C} = C_{\mathfrak{R}}(P)$ . Suppose  $\mathfrak{C} \cap \mathfrak{R}_0 = \langle I \rangle$ . Then since  $\mathfrak{R}_3 \mathfrak{R}_0 \triangleleft \mathfrak{R}$ , it follows that  $\mathfrak{C}$  is 3-closed. Let  $\mathfrak{C}_3$  be the  $S_3$ -subgroup of  $\mathfrak{C}$ . Thus,  $\mathfrak{C}_3$  is noncyclic. Since  $N(\mathfrak{C}_3) \cap \mathfrak{R}_0 = \langle I \rangle$ ,  $N_{\mathfrak{R}}(\mathfrak{C}_3)$  contains a  $S_3$ -subgroup of  $\mathfrak{R}$  as a normal subgroup. Since  $J \in N(\mathfrak{C}_3)$ , we are done.

We may assume that  $\mathfrak{C} \cap \mathfrak{R}_0 \supset \langle I \rangle$ . Hence,  $\mathfrak{C} \cap \mathfrak{R}_0 = \mathfrak{Q}$  is a quaternion group. Let  $\tilde{\mathfrak{F}}$  be a  $S_3$ -subgroup of  $\mathfrak{C}$ . Thus,  $\tilde{\mathfrak{F}}\mathfrak{Q} = \mathcal{O}^3(\mathfrak{C})$  char  $\mathfrak{C}$ , so  $\langle J \rangle \tilde{\mathfrak{F}}\mathfrak{Q}$  is a group. Let  $\tilde{\mathfrak{F}}_0 = \mathcal{O}_3(\tilde{\mathfrak{F}}\mathfrak{Q})$ . Thus,  $\tilde{\mathfrak{F}}\mathfrak{Q} / \tilde{\mathfrak{F}}_0 \cong SL(2, 3)$ , and  $J$  stabilizes  $\tilde{\mathfrak{F}}\mathfrak{Q} / \tilde{\mathfrak{F}}_0$ . If  $J$  does not centralize  $\mathfrak{Q}\tilde{\mathfrak{F}} / \mathfrak{Q}\tilde{\mathfrak{F}}_0$ , it follows from Lemma 5.36 that  $J$  normalizes a  $S_3$ -subgroup of  $\mathfrak{Q}\tilde{\mathfrak{F}}$ . Suppose  $J$  centralizes  $\tilde{\mathfrak{F}}\mathfrak{Q} / \mathfrak{Q}\tilde{\mathfrak{F}}_0$ . Then  $\langle J \rangle \tilde{\mathfrak{F}}\mathfrak{Q} / \mathfrak{Q}\tilde{\mathfrak{F}}_0$  is a cyclic group of order 6, so  $J$  centralizes  $\mathfrak{Q} / \mathfrak{Q}'$ . This implies that  $|C(J) \cap \mathfrak{Q}| \geq 4$ , so that  $\langle J, \mathfrak{Q} \rangle$  contains a noncyclic abelian subgroup of order 8. Since  $\langle J, \mathfrak{Q} \rangle \subseteq N(\langle P \rangle)$ , Lemma 9.21 (c) is violated. We conclude that  $J$

normalizes a  $S_3$ -subgroup of  $\tilde{\mathfrak{B}}\Omega$ . We assume without loss of generality that  $J$  normalizes  $\tilde{\mathfrak{B}}$ . Since  $N(\tilde{\mathfrak{B}}) \cap \mathfrak{R}_0 = \langle I \rangle$ , it follows that  $N_{\mathfrak{R}}(\tilde{\mathfrak{B}})$  contains a  $S_3$ -subgroup of  $\mathfrak{R}$  as a normal subgroup. The proof is complete.

LEMMA 9.25. (a) *If  $T$  is an involution of  $\mathfrak{G}$ ,  $C(T)$  contains a noncyclic abelian subgroup of order 8.*

(b) *If the width of  $\mathfrak{R}_0$  is 2, then for each involution  $T$  of  $\mathfrak{G}$ ,  $|C(T)|_3 \leq 9$ .*

*Proof.* (a) By Lemma 5.38,  $C(T)$  contains an element  $u$  of  $\mathcal{Z}(2)$ . If  $T \notin u$ , then  $\langle u, T \rangle$  is a noncyclic abelian subgroup of order 8 which is contained in  $C(T)$ . Suppose  $T \in u$ . Since  $\mathcal{S}_{ev_3}(2) \neq \emptyset$ ,  $C(T)$  contains an element of  $\mathcal{S}_{ev_3}(2)$  by Lemma 0.8.9.

Suppose (b) is false, and  $T$  is an involution of  $\mathfrak{G}$  with  $|C(T)|_3 \geq 27$ . Let  $\mathfrak{C}$  be a maximal 2, 3-subgroup of  $\mathfrak{G}$  which contains a  $S_{2,3}$ -subgroup of  $C(T)$ . By Lemma 5.38,  $\mathfrak{C}$  contains an element  $u$  of  $\mathcal{Z}(2)$ . Let  $\mathfrak{C}_p$  be a  $S_p$ -subgroup of  $\mathfrak{C}$ ,  $p = 2, 3$ . We assume without loss of generality that  $\mathfrak{C}_2 \subseteq \mathfrak{R}_2$ .

Case 1.  $O_3(\mathfrak{C}) \neq 1$ .

Since  $u \in \mathcal{Z}(2)$ ,  $u$  centralizes  $O_3(\mathfrak{C})$ . Since  $u$  contains a conjugate of  $I$ , it follows that  $|O_3(\mathfrak{C})| \leq 9$ . Suppose  $|O_3(\mathfrak{C})| = 9$ . Then  $O_3(\mathfrak{C})$  is conjugate to  $\mathfrak{B}$ , since  $\mathfrak{B}$  is a  $S_3$ -subgroup of  $\mathfrak{M}$ . But then Lemma 9.16 (ii) is violated. Hence,  $|O_3(\mathfrak{C})| = 3$ .

Since  $u$  centralizes  $O_3(\mathfrak{C})$ ,  $O_3(\mathfrak{C})$  is conjugate to a subgroup of  $\mathfrak{B}$ . By Lemma 9.21 (b),  $C(O_3(\mathfrak{C}))$  contains an elementary subgroup  $\mathfrak{A}^*$  such that  $C(A)$  contains an element of  $\mathcal{Z}(3)$  for each  $A$  in  $\mathfrak{A}^*$ . Since  $\mathfrak{C}$  is a  $S_{2,3}$ -subgroup of  $N(O_3(\mathfrak{C}))$ , we assume without loss of generality that  $\mathfrak{A}^* \subseteq \mathfrak{C}$ . By Lemma 9.16 (i),  $\mathfrak{A}^* \in \mathcal{S}$ . Now Lemma 9.17 yields  $O_2(\mathfrak{C}) \neq 1$ . Since  $\mathfrak{A}^* \subseteq \mathfrak{C}$ , Lemma 9.16 (ii) forces  $|Z(O_2(\mathfrak{C}))| = 2$ , and forces  $Z(O_2(\mathfrak{C}))$  to be a maximal characteristic abelian subgroup of  $O_2(\mathfrak{C})$ . Since  $|O_3(\mathfrak{C})| = 3$ , it follows that  $|O_2(\mathfrak{C})| > 2$ . Hence,  $O_2(\mathfrak{C})$  is extra special. Thus,  $O_2(\mathfrak{C})'$  is of order 2 and is normalized by every element of  $\mathcal{Z}(\mathfrak{R}_2)$ . Hence, every element of  $\mathcal{Z}(\mathfrak{R}_2)$  is contained in  $\mathfrak{C}_2$ . Thus,  $I$  centralizes  $O_3(\mathfrak{C})$ . Since  $I \in Z(\mathfrak{C}_2)$ , we get  $I \in O_2(\mathfrak{C})$ , so that  $\langle I \rangle = O_2(\mathfrak{C})'$ . Hence,  $\mathfrak{C} \subseteq \mathfrak{M}$ , against  $|\mathfrak{C}|_3 \geq 27$  and  $|\mathfrak{M}|_3 = 9$ .

Case 2.  $O_3(\mathfrak{C}) = 1$ .

Since  $|\mathfrak{C}|_3 \geq 27$ , it follows that  $m(O_2(\mathfrak{C})) \geq 6$ . Since the width of  $\mathfrak{R}_0$  is 2, it follows that  $\mathfrak{R}_2$  has no elementary subgroup of order  $2^6$ . Thus,  $O_2(\mathfrak{C})$  is not elementary.

Now  $\mathfrak{C}$  is clearly not contained in any conjugate of  $\mathfrak{M}$ , since  $|\mathfrak{C}|_3 > |\mathfrak{M}|_3$ . Since  $\langle I \rangle = Z(\mathfrak{R}_2)$ , it follows that  $\mathfrak{C}$  is not 2-closed.

Since  $|\mathfrak{R}_2| \leq 2^8$ , we get  $|O_2(\mathfrak{S})| = 2^7$ . Hence,  $|O_2(\mathfrak{S})| = 2^7$ , so that  $D(O_2(\mathfrak{S}))$  is a subgroup of order 2 and  $\mathfrak{S}_2$  is of order  $2^8$ . Hence,  $\mathfrak{S}_2 = \mathfrak{R}_2$  and  $D(O_2(\mathfrak{S})) = \langle I \rangle$ . This shows that  $\mathfrak{S} \subseteq \mathfrak{M}$ . This contradiction completes the proof.

- LEMMA 9.26. *If  $\tilde{\mathfrak{F}}$  is any subgroup of  $\mathfrak{R}_3$  of order 3, then*
- (a)  $\mathfrak{R}_0 \cap C(\tilde{\mathfrak{F}})$  is either  $\langle I \rangle$  or a quaternion group;
  - (b) if  $\tilde{\mathfrak{F}} \not\subseteq Z(\mathfrak{R}_3)$ , then  $\mathfrak{R}_0 \cap C(\tilde{\mathfrak{F}})$  is quaternion.

*Proof.* (a) Suppose  $\mathfrak{R}_0 \cap C(\tilde{\mathfrak{F}}) \supset \langle I \rangle$ . Then  $\mathfrak{R}_0 \cap C(\tilde{\mathfrak{F}})$  is extra special and does not contain a noncyclic abelian subgroup of order 8. Thus,  $\mathfrak{R}_0 \cap C(\tilde{\mathfrak{F}})$  is either dihedral or quaternion. Now  $C_{\mathfrak{R}_3}(\tilde{\mathfrak{F}})$  contains an elementary subgroup  $\mathfrak{E}$  of order 9 with  $\tilde{\mathfrak{F}} \subset \mathfrak{E}$ . Hence,  $\mathfrak{R}_0 \cap C(\tilde{\mathfrak{F}})$  admits  $\mathfrak{E}$ . Since no element of  $\mathfrak{E}^\#$  centralizes a noncyclic abelian subgroup of  $\mathfrak{R}_0$  of order 8,  $\mathfrak{R}_0 \cap C(\tilde{\mathfrak{F}})$  is quaternion.

(b) Let  $\mathfrak{E} = \langle \tilde{\mathfrak{F}}, Z(\mathfrak{R}_3) \rangle$ , so that by Lemma 9.21 (e),  $\mathfrak{E}$  is elementary of order 9 and  $\mathfrak{E} \triangleleft \mathfrak{R}_3$ . It follows that the three subgroups of  $\mathfrak{E}$  of order 3 which are distinct from  $Z(\mathfrak{R}_3)$  are conjugate in  $\mathfrak{R}_3$ . We can choose  $E$  in  $\mathfrak{E}^\#$  such that  $\mathfrak{R}_0 \cap C(E)$  is not centralized by  $Z(\mathfrak{R}_3)$ . By (a),  $\mathfrak{R}_0 \cap C(E)$  is a quaternion group; so (b) holds.

LEMMA 9.27.  $\mathfrak{R}_2^*$  is a four-group.

*Proof.* Suppose false. By Lemma 9.22,  $\mathfrak{R}_2 - \mathfrak{R}_0$  contains an involution  $J$ . By Lemma 9.24,  $J$  normalizes a  $S_3$ -subgroup of  $\mathfrak{R}$ . Thus, we can choose  $M$  in  $\mathfrak{M}$  such that  $J^M = T_0$  normalizes  $\mathfrak{R}_3$ . Since  $\mathfrak{R}_3$  permutes transitively by conjugation the  $S_2$ -subgroups of  $\mathfrak{R}$ , we may choose  $K$  in  $\mathfrak{R}_3$  such that  $T = T_0^K$  lies in  $\mathfrak{R}_2$ . Thus,  $T \in N(\mathfrak{R}_3) \cap \mathfrak{R}_2$ .

By Lemma 9.23,  $\mathfrak{R}_3^* = \mathfrak{R}_3$ . Thus,  $T \in \mathfrak{R}_2^*$ . If  $\langle T, I \rangle = \mathfrak{R}_2^*$ , we are done, so suppose  $\langle T, I \rangle \subset \mathfrak{R}_2^*$ . Let  $\mathfrak{F}$  be a subgroup of  $\mathfrak{R}_2^*$  of order 8 which contains  $\langle T, I \rangle$ . By Lemma 9.19,  $\mathfrak{F}$  is dihedral of order 8. Let  $\mathfrak{F}_0, \mathfrak{F}_1$  be the four-subgroups in  $\mathfrak{F}$ .

Suppose  $\mathfrak{X}$  is a subgroup of  $\mathfrak{R}_3$  of order 3 which admits  $\mathfrak{F}$  and that  $C(\mathfrak{X}) \cap \mathfrak{R}_0$  is a quaternion group. Hence,  $C_{\mathfrak{M}}(\mathfrak{X})$  contains a normal quaternion subgroup and  $S_2$ -subgroups of  $N_{\mathfrak{M}}(\mathfrak{X})$  are of order at least  $2^5$ . Thus,  $N_{\mathfrak{M}}(\mathfrak{X})$  contains a noncyclic abelian subgroup of order 8. This is impossible, by Lemma 9.21 (c). Hence,  $C_{\mathfrak{R}_0}(\mathfrak{X}) = \langle I \rangle$ , by Lemma 9.26 (a).

By Lemma 9.26 (b) and the preceding paragraph, it follows that  $\mathfrak{F}$  normalizes no noncentral subgroup of  $\mathfrak{R}_3$  of order 3.

Suppose  $\mathfrak{R}_3$  is nonabelian. Then  $\mathfrak{F}$  normalizes  $\Omega_1(\mathfrak{R}_3)$ , a group of exponent 3. Since  $\langle I \rangle$  centralizes  $\mathfrak{R}_3$ , it follows that  $\Omega_1(\mathfrak{R}_3)\mathfrak{F}$  is supersolvable. Thus,  $\Omega_1(\mathfrak{R}_3)\mathfrak{F}$  contains a normal subgroup of order 9, so  $\mathfrak{F}$  normalizes a noncentral subgroup of  $\mathfrak{R}_3$  of order 3. This contradicts the preceding paragraph, so we conclude that  $\mathfrak{R}_3$  is abelian,  $\mathfrak{R}_3 = \mathfrak{B}$ .

Let  $\mathfrak{F}_i = \langle J_i, I \rangle, i = 0, 1$ . If both  $J_0$  and  $J_1$  invert  $\mathfrak{R}_3$ , then  $J_0 J_1$  centralizes  $\mathfrak{R}_3$ , so  $J_0 J_1 \in \langle I \rangle$ . This is not the case, since  $\mathfrak{G}_0 \mathfrak{G}_1$  is of order 8. Thus, we may assume notation is chosen so that  $J_0$  centralizes  $\mathfrak{x}_0$  and inverts  $\mathfrak{x}_1$ . Here,  $|\mathfrak{x}_i| = 3$ , and  $\mathfrak{R}_3 = \mathfrak{x}_0 \times \mathfrak{x}_1$ . Since  $\mathfrak{R}_0 \cap C(\mathfrak{x}_i) = \langle I \rangle$ , for  $i = 0, 1$ , the width of  $\mathfrak{R}_0$  is 2.

Let  $\mathfrak{G}$  be a  $S_{2,3}$ -subgroup of  $N(\mathfrak{R}_3)$  which contains  $\mathfrak{R}_2^*$ . Since  $\mathfrak{R}_3 \subset \mathfrak{B}$ , we get  $|\mathfrak{R}_3| = 9 < |\mathfrak{G}|_3$ . By Lemma 9.16 (ii),  $|C(\mathfrak{R}_3)|_2 = 2$ . By Lemma 9.20, we get that  $\mathfrak{G}$  is 3-closed. Let  $\mathfrak{G}_3 = O_3(\mathfrak{G}) \supset \mathfrak{R}_3$ .

Let  $F_0, F_1, F_2$  be the three involutions of  $\mathfrak{F}_0$ , and set

$$3^{f_i} = |\mathfrak{G}_3 \cap C(F_i)|, i = 0, 1, 2.$$

By Lemma 9.25 (b), we have  $f_i \leq 2$ . Since  $\mathfrak{G}_3 \cap C(\mathfrak{F}_0) = \mathfrak{x}_0$ , a formula of Wielandt [44] yields

$$|\mathfrak{G}_3| = 3^{f_0+f_1+f_2-2} \leq 3^4.$$

Since the dihedral group  $\mathfrak{F}$  is faithfully represented on  $\mathfrak{G}_3/\mathfrak{R}_3$ , it follows that  $|\mathfrak{G}_3| = 3^4$ .

Let  $\mathfrak{D}$  be a  $S_{2,3}$ -subgroup of  $N(\mathfrak{G}_3)$ . Let  $\mathfrak{D}_p$  be a  $S_p$ -subgroup of  $\mathfrak{D}$ , with  $\mathfrak{R}_2^* \subseteq \mathfrak{D}_2$ . By the formula of Wielandt [44] applied to  $\mathfrak{F}_0$  acting on  $O_3(\mathfrak{D})$ , we get  $O_3(\mathfrak{D}) = \mathfrak{G}_3$ . If  $\mathfrak{D}_3 = \mathfrak{G}_3$ , then  $\mathfrak{G}_3$  is a  $S_3$ -subgroup of  $\mathfrak{G}$ . But the center of  $\mathfrak{G}_3$  contains  $\mathfrak{R}_3 = \mathfrak{B}$  by Lemma 9.20, and hence is noncyclic. This contradicts hypothesis (iii) of Theorem 9.1. Therefore  $\mathfrak{D}_3 \supset \mathfrak{G}_3$  and  $\mathfrak{D}$  is not 3-closed.

By Lemma 9.25 (b), we get  $O_2(\mathfrak{D}) = 1$ . Since  $\mathfrak{D}$  is quite obviously contained in no conjugate of  $\mathfrak{M}$ , Lemma 9.18 implies that  $\mathfrak{D}$  contains no noncyclic abelian subgroups of order 8. Thus,  $\mathfrak{D}_2$  is of maximal class. Hence,  $\langle I \rangle = Z(\mathfrak{D}_2)$ , so  $\mathfrak{D}_2 = \mathfrak{R}_2^*$  is of order at most 16. Suppose  $|\mathfrak{D}_2| = 16$ . Since  $O_3(\mathfrak{D}) = 1$ , and  $\mathfrak{D}$  is not 3-closed, it follows that  $O_{3,2}(\mathfrak{D}) \cap \mathfrak{D}_2$  is a quaternion group. But then  $\mathfrak{M}$  covers  $\mathfrak{D}/O_3(\mathfrak{D})$ . This is not the case, since  $O_3(\mathfrak{D})$  contains a  $S_3$ -subgroup of  $\mathfrak{M}$ , and since  $3 \mid |\mathfrak{D}:O_3(\mathfrak{D})|$ . Hence,  $\mathfrak{D}_2 = \mathfrak{F}$  is dihedral of order 8. Let  $\tilde{\mathfrak{D}}_2 = O_{3,2}(\mathfrak{D}) \cap \mathfrak{D}_2$ . Thus,  $\tilde{\mathfrak{D}}_2$  is a four-group and  $\mathfrak{D}/O_3(\mathfrak{D}) \cong \Sigma_4$ .

Since  $\mathfrak{D}/O_3(\mathfrak{D}) \cong \Sigma_4$ , some chief factor of  $\mathfrak{D}$  is of order  $3^3$ . Thus,  $O_3(\mathfrak{D})$  is necessarily elementary, and elements of  $\mathfrak{D}_3 - O_3(\mathfrak{D})$  induce automorphisms of  $O_3(\mathfrak{D})$  with minimal polynomial  $(x - 1)^3$ . Hence,  $O_3(\mathfrak{D}) = J(\mathfrak{D}_3) O_3(\mathfrak{D}) = J(\mathfrak{D}_3)$  char  $\mathfrak{D}_3$ , so  $\mathfrak{D}_3$  is a  $S_3$ -subgroup of  $\mathfrak{G}$ . This is not the case, since  $Z(\mathfrak{D}_3)$  is noncyclic. The proof is complete.

LEMMA 9.28. *If the width of  $\mathfrak{R}_0$  exceeds 2, then  $I$  is the only conjugate of  $I$  in  $\mathfrak{R}_0$ .*

*Proof.* Suppose  $T = I^g \neq I, T \in \mathfrak{R}_0$ . Then  $C(T) \cap \mathfrak{R}_0 \subseteq C(T) = \mathfrak{M}^g$ . By Lemma 9.27,  $C(T) \cap \mathfrak{R}_0 \cap \mathfrak{R}_0^g$  is of index at most 2 in  $C(T) \cap \mathfrak{R}_0$ .

Since  $C_{\mathfrak{R}_0}(T)$  is of index 2 in  $\mathfrak{R}_0$ , we get  $|\mathfrak{R}_0 : C(T) \cap \mathfrak{R}_0 \cap \mathfrak{R}_0^g| \leq 4$ . Since the width of  $\mathfrak{R}_0$  is at least 3, it follows that  $C(T) \cap \mathfrak{R}_0 \cap \mathfrak{R}_0^g$  is nonabelian. Hence,  $\langle I \rangle = (C(T) \cap \mathfrak{R}_0 \cap \mathfrak{R}_0^g)' = \langle T \rangle$ . This contradiction completes the proof.

LEMMA 9.29.  $\mathfrak{R}_3 = \mathfrak{B}$  is of order 9.

*Proof.* Suppose false. By Lemma 9.21 (e),  $\mathfrak{R}_3$  is nonabelian of order 27. Since  $\mathfrak{R}_3$  is faithfully represented on  $\mathfrak{R}_0$ , the width of  $\mathfrak{R}_0$  is at least 3. By a result of Glauberman [14],  $\mathfrak{R}_2$  contains a conjugate  $T$  of  $I$  distinct from  $I$ ,  $T = I^g \neq I$ . By Lemma 9.28,  $T \in \mathfrak{R}_2 - \mathfrak{R}_0$ , so by Lemma 9.24, we may assume that  $T \in \mathfrak{R}_2^*$ . Thus, by Lemma 9.27,  $\mathfrak{R}_2^* = \langle I, T \rangle$ .

Since  $\mathfrak{R}_3$  is nonabelian, it follows that  $\mathfrak{X}_1 = \mathfrak{R}_3 \cap C(T)$  is of order 3. By Lemma 1.3 of [17],  $\mathfrak{R}_3$  has a subgroup  $\mathfrak{X}_0$  of order 3 which centralizes  $\mathfrak{X}_1$  and is inverted by  $T$ . Let  $\mathfrak{X} = \mathfrak{X}_0 \times \mathfrak{X}_1$  and let  $\mathfrak{X}_2, \mathfrak{X}_3$  be the remaining subgroups of  $\mathfrak{X}$  of order 3.

Suppose  $C(\mathfrak{X}_0) \cap \mathfrak{R}_0 \supset \langle I \rangle$ . By Lemma 9.26 (a),  $C(\mathfrak{X}_0) \cap \mathfrak{R}_0 = \mathfrak{Q}$  is a quaternion group. Since  $C(\mathfrak{X}) \cap \mathfrak{R}_0 = \langle I \rangle$ , it follows that  $\mathfrak{X}_1$  is faithfully represented on  $\mathfrak{Q}$ . Since  $\text{Aut}(\mathfrak{Q})$  has no element of order 6,  $T$  centralizes  $\mathfrak{Q}$ . But then  $\langle \mathfrak{Q}, T \rangle = \mathfrak{Q} \times \langle T \rangle$  contains a noncyclic abelian subgroup of order 8. This violates Lemma 9.21 (c) with  $\mathfrak{X}_0$  in the role of  $\mathfrak{P}$ . Hence,  $C(\mathfrak{X}_0) \cap \mathfrak{R}_0 = \langle I \rangle$ .

Since  $C(\mathfrak{X}_0) \cap \mathfrak{R}_0 = \langle I \rangle$ , the width of  $\mathfrak{R}_0$  is at most 3. By Lemma 9.26 (b), we get  $\mathfrak{X}_0 = Z(\mathfrak{R}_3)$ . Thus, if we set  $\mathfrak{Q}_i = \mathfrak{R}_0 \cap C(\mathfrak{X}_i)$ ,  $i = 1, 2, 3$ , then by Lemma 9.26, it follows that each  $\mathfrak{Q}_i$  is quaternion. Hence,  $\mathfrak{R}_0$  is the central product of  $\mathfrak{Q}_1, \mathfrak{Q}_2, \mathfrak{Q}_3$ . Since  $T$  centralizes  $\mathfrak{X}_1$  and interchanges  $\mathfrak{X}_2$  and  $\mathfrak{X}_3$ , it follows that  $T$  normalizes  $\mathfrak{Q}_1$  and interchanges  $\mathfrak{Q}_2$  and  $\mathfrak{Q}_3$ . Since  $\mathfrak{X}_0$  is faithfully represented on  $\mathfrak{Q}_1$ , it follows that  $\mathfrak{Q}_1 \mathfrak{X}_0 \langle T \rangle \cong GL(2, 3)$ . Thus, we can choose generators  $A_i, B_i$  for  $\mathfrak{Q}_i$  such that  $A_1^T = B_1, A_2^T = A_3, B_2^T = B_3$ . It follows that  $C(T) \cap \mathfrak{R}_0 = \langle A_2 A_3, B_2 B_3, I \rangle$ , an elementary group of order 8. Let  $\mathfrak{F} = C(T) \cap \mathfrak{R}_2 = \langle T \rangle \times C(T) \cap \mathfrak{R}_0$ . It now follows that  $\tilde{\mathfrak{R}}_2 = N_{\mathfrak{R}_2}(\mathfrak{F}) = \langle \mathfrak{Q}_2, \mathfrak{Q}_3, A_1 B_1, T \rangle$ , a group of index 2 in  $\mathfrak{R}_2$ . Since  $Z(\tilde{\mathfrak{R}}_2) = \langle I \rangle$ , it follows that  $\tilde{\mathfrak{R}}_2$  is a  $S_3$ -subgroup of  $N(\mathfrak{F})$ .

Now  $T = I^g$ , so  $\mathfrak{F} \cong \mathfrak{M}^g$ . By symmetry,  $N(\mathfrak{F}) \cap \mathfrak{M}^g$  contains a  $S_2$ -subgroup of  $N(\mathfrak{F})$ . This implies that  $O_2(N(\mathfrak{F}))$  centralizes both  $T$  and  $I$ . Hence,  $O_2(N(\mathfrak{F})) = \mathfrak{F}$ .

Now  $\tilde{\mathfrak{R}}_2$  permutes transitively the elements of  $(\mathfrak{F} \cap \mathfrak{R}_0)T$ , so  $\mathfrak{S} = \{(\mathfrak{F} \cap \mathfrak{R}_0)T, I\}$  is the set of all the elements of  $\mathfrak{F}$  which are conjugate to  $I$  in  $\mathfrak{G}$ . Since  $N(\mathfrak{F}) \cap \mathfrak{M}^g$  normalizes  $\mathfrak{S}$  but does not centralize  $I$ , it follows that  $N(\mathfrak{F})$  permutes  $\mathfrak{S}$  transitively.

Since  $N(\mathfrak{F})$  is transitive on  $\mathfrak{S}$ , it follows that  $N(\mathfrak{F}) = 9 \cdot |N(\mathfrak{F}) \cap \mathfrak{M}^g|$ . Since  $C(\mathfrak{F}) = C_{\mathfrak{M}}(\mathfrak{F})$ , it follows that  $\mathfrak{F} = C(\mathfrak{F})$ . Since  $T$  centralizes  $\mathfrak{X}_1$ ,



it follows that  $\mathfrak{X}_1$  normalizes  $\mathfrak{R}_0 \cap C(T)$ , so normalizes  $\mathfrak{F} = \langle T \rangle \times \mathfrak{R}_0 \cap C(T)$ . But it now follows that  $27 \mid |N(\mathfrak{F})|$ . Since  $\mathfrak{F}$  is elementary of order  $2^4$ ,  $\text{Aut}(\mathfrak{F})$  has no subgroup of order 27. This violates the equality  $\mathfrak{F} = C(\mathfrak{F})$ , and the proof is complete.

LEMMA 9.30. *If  $T$  is any involution of  $\mathfrak{G}$ , then  $|C(T)|_3 \leq 9$ .*

*Proof.* Since  $|\mathfrak{R}_2: \mathfrak{R}_0| = 2$ , Lemma 5.38 implies that every involution of  $\mathfrak{G}$  is conjugate to an involution of  $\mathfrak{R}_0$ . Thus, we may assume that  $T \in \mathfrak{R}_0$ . If  $T \sim I$ , we are done by Lemma 9.29, so from now on we suppose  $T \not\sim I$ .

Let  $\mathfrak{U} \in \mathcal{Z}(\mathfrak{R}_2)$ , and let  $\tilde{\mathfrak{R}}_0 = \mathfrak{R}_0 \cap C(\mathfrak{U})$ . Thus,  $\tilde{\mathfrak{R}}_0$  is of index 2 in  $\mathfrak{R}_0$ . Since  $\mathfrak{R}_3$  has no fixed points on  $\mathfrak{R}_0/\langle I \rangle$ , Lemma 5.38 implies that for some  $X$  in  $\mathfrak{R}_3$ ,  $T^X \in \tilde{\mathfrak{R}}_0$ . Thus, we assume without loss of generality that  $T \in \tilde{\mathfrak{R}}_0$ .

We argue that  $C_{\mathfrak{M}}(T)$  contains a  $S_2$ -subgroup of  $C(T)$ . This is clear if  $|\mathfrak{M}: C_{\mathfrak{M}}(T)|_2 = 2$ , since  $T \not\sim I$ . So suppose  $|\mathfrak{M}: C_{\mathfrak{M}}(T)|_2 = 4$ . In this case,  $C_{\mathfrak{R}_0}(T)$  is a  $S_2$ -subgroup of  $C_{\mathfrak{M}}(T)$ . Since  $\langle I \rangle = C_{\mathfrak{R}_0}(T)'$  char  $C_{\mathfrak{R}_0}(T)$ , it follows that  $C_{\mathfrak{R}_0}(T)$  is a  $S_2$ -subgroup of  $C(T)$ .

Let  $\mathfrak{R}$  be a  $S_{2,3}$ -subgroup of  $C(T)$  which contains  $C_{\mathfrak{R}}(T)$ . Suppose  $O_3(\mathfrak{R}) \neq 1$ . Since  $\mathfrak{U} \subseteq \mathfrak{R}$ ,  $\mathfrak{U}$  centralizes  $O_3(\mathfrak{R})$ , so  $O_3(\mathfrak{R}) \subseteq \mathfrak{M}$ . Since no element of  $\mathfrak{R}_3^*$  centralizes a four-subgroup of  $\mathfrak{R}_0$  by Lemma 9.26 (a), we conclude that  $O_3(\mathfrak{R}) = 1$ .

Since  $O_3(\mathfrak{R}) = 1$  and since  $\mathfrak{R} \cap \mathfrak{M}$  contains a  $S_2$ -subgroup of  $C(T)$ , it follows that  $I \in Z(O_2(\mathfrak{R}))$ . Suppose  $X$  is a 3-element of  $\mathfrak{R}$  and  $\mathfrak{X}$  centralizes  $I$ . Then  $X \in C(\langle T, I \rangle)$ , so  $X = 1$  by Lemma 9.26 (a). Thus, a  $S_3$ -subgroup  $\mathfrak{R}_3$  of  $\mathfrak{R}$  is faithfully represented on  $Z(O_2(\mathfrak{R}))$ .

Let  $\Omega_1(Z(O_2(\mathfrak{R}))) = \mathfrak{Y}_1 \times \mathfrak{Y}_2$ , where  $\mathfrak{Y}_1 = \Omega_1(Z(O_2(\mathfrak{R}))) \cap C(\mathfrak{R}_3)$ , and  $\mathfrak{Y}_2 = [\Omega_1(Z(O_2(\mathfrak{R}))), \mathfrak{R}_3]$ . Thus,  $T \in \mathfrak{Y}_1$  and  $\mathfrak{R}_3$  is faithfully represented on  $\mathfrak{Y}_2$ . Hence,  $m(Z(O_2(\mathfrak{R}))) = m(\mathfrak{Y}_1) + m(\mathfrak{Y}_2) \geq 7$ . Thus,  $\mathfrak{R}_0$  has an elementary subgroup of order  $2^6$ , by Lemma 9.27. This is impossible, since the width of  $\mathfrak{R}_0$  is at most 4. The proof is complete.

LEMMA 9.31.  $|\mathfrak{G}|_3 > 3^4$ .

*Proof.* Let  $\mathfrak{X}$  be a subgroup of  $\mathfrak{R}_3$  of order 3 such that  $\mathfrak{R}_0 \cap C(\mathfrak{X}) = \mathfrak{Q}$  is quaternion. Let  $\mathfrak{C}$  be a  $S_{2,3}$ -subgroup of  $C(\mathfrak{X})$  which contains  $\mathfrak{R}_3\mathfrak{Q}$ . Since  $\mathfrak{R}_3 = \mathfrak{B} \in \mathcal{S}$ , it follows that  $\mathfrak{R}_3$  centralizes  $Z(O_2(\mathfrak{C}))$ . Since  $\langle I \rangle$  is a  $S_2$ -subgroup of  $C(\mathfrak{R}_3)$ , it follows that  $O_2(\mathfrak{C}) = 1$ , by Lemma 9.21(a).

Since  $O_2(\mathfrak{C}) = 1$ ,  $\mathfrak{Q}$  is faithfully represented on  $O_3(\mathfrak{C})$ , so is faithfully represented on  $O_3(\mathfrak{C})/\mathfrak{X}$ . Hence,  $|O_3(\mathfrak{C}): \mathfrak{X}| \geq 9$ . Since  $[\mathfrak{R}_3 \cap O_3(\mathfrak{C}), \mathfrak{Q}] \subseteq O_3(\mathfrak{C}) \cap \mathfrak{Q} = 1$ , it follows that  $\mathfrak{R}_3 \cap O_3(\mathfrak{C}) = \mathfrak{X}$ . Hence,  $|\mathfrak{C}|_3 \geq 3^4$ . Suppose the lemma is false. Then  $\mathfrak{C}$  contains a  $S_3$ -subgroup of  $\mathfrak{G}$ , and  $O_3(\mathfrak{C})$  is of order  $3^3$ , while  $\mathfrak{X} \sim \mathfrak{Z}$ . If  $O_3(\mathfrak{C})$  is nonabelian, then Hypothesis 9.1 is satisfied. This is not the case, so  $O_3(\mathfrak{C})$  is elementary. Hence,  $O_3(\mathfrak{C}) = \mathfrak{X} \times [O_3(\mathfrak{C}), \mathfrak{Q}]$ . Hence, the center of a  $S_3$ -subgroup of

$\mathfrak{G}$  is noncyclic. This is not the case. The proof is complete.

LEMMA 9.32. *Choose  $J$  in  $\mathfrak{R}_2^* - \langle I \rangle$ . If  $J$  inverts  $\mathfrak{R}_3$ , then  $A_{\mathfrak{G}}(\mathfrak{R}_2^*) = \text{Aut}(\mathfrak{R}_2^*)$ .*

*Proof.* Let  $\mathfrak{x}$  be any four-subgroup of  $\mathfrak{M}$  which contains  $I$ . We will show that

$$(9.25) \quad |A_{\mathfrak{M}}(\mathfrak{x})| = 2.$$

This is clear if  $\mathfrak{x} \subseteq \mathfrak{R}_0$ . If  $\mathfrak{x} \not\subseteq \mathfrak{R}_0$ , then by Lemmas 9.27 and 9.24, we see that  $\mathfrak{x}$  is conjugate to  $\mathfrak{R}_2^*$  in  $\mathfrak{M}$ . Let  $\mathfrak{Y}$  be a subgroup of  $\mathfrak{R}_3$  such that  $\mathfrak{Q} = \mathfrak{R}_0 \cap C(\mathfrak{Y})$  is quaternion. Since  $J$  inverts  $\mathfrak{R}_3$ ,  $\mathfrak{Q}$  admits  $\langle J \rangle \mathfrak{R}_3 / \mathfrak{Y}$  as a group of automorphisms. Hence,  $J$  inverts an element  $Q$  of  $\mathfrak{Q}$  of order 4. Then  $JQJ = Q^{-1}$ , that is,  $Q^{-1}JQ = JI$ , so  $Q \in N_{\mathfrak{M}}(\mathfrak{R}_2^*)$ . Thus, (9.25) holds.

Suppose that  $\mathfrak{R}_2^* - \langle I \rangle$  contains a conjugate  $J$  of  $I$ . By (9.25), we can choose  $M$  in  $\mathfrak{M} \cap N(\mathfrak{R}_2^*)$  such that  $M^{-1}JM = JI$ . By (9.25) again, this time applied to the group  $C(J)$ , we can choose  $M_0$  in  $C(J)$  with  $M_0^{-1}IM_0 = IJ$ . Thus, the lemma follows in this case.

We may now assume that

$$(9.26) \quad I \text{ is the only conjugate of } I \text{ in } \mathfrak{R}_2^* .$$

By a result of Glauberman [16],  $\mathfrak{R}_2$  contains a conjugate  $T$  of  $I$  with  $T \neq I$ . If the width of  $\mathfrak{R}_0$  exceeds 2, then by Lemma 9.28,  $T \in \mathfrak{R}_0$ , so by Lemma 9.24, (9.26) is violated. So suppose the width of  $\mathfrak{R}_0$  is 2. In this case,  $\mathfrak{R}_0$  has exactly 18 noncentral involutions and they are permuted transitively in  $\mathfrak{M}$ . Since  $T$  lies in no  $\mathfrak{M}$ -conjugate of  $\mathfrak{R}_2^*$ , Lemma 9.24 implies that  $T \in \mathfrak{R}_0$ . Thus, every involution of  $\mathfrak{R}_0$  is conjugate to  $I$  in  $\mathfrak{G}$ . But by Lemma 5.38, every involution of  $\mathfrak{G}$  is conjugate to an element of  $\mathfrak{R}_0$ . The proof is complete.

LEMMA 9.33. *There is a  $S_3$ -subgroup of  $\mathfrak{G}$  which contains  $\mathfrak{R}_3$  and is normalized by  $\mathfrak{R}_2^*$ .*

*Proof.* Let  $\tilde{\mathfrak{F}}$  be a maximal element of  $N(\mathfrak{R}_2^*; 3)$  which contains  $\mathfrak{R}_3$ . Suppose by way of contradiction that  $|\tilde{\mathfrak{F}}| < |\mathfrak{G}|_3$ . Let  $\mathfrak{C}$  be a  $S_{2,3}$ -subgroup of  $N(\tilde{\mathfrak{F}})$  which contains  $\mathfrak{R}_2^*$ . Let  $\mathfrak{C}_p$  be a  $S_p$ -subgroup of  $\mathfrak{C}$ ,  $p = 2, 3$ , with  $\mathfrak{R}_2^* \subseteq \mathfrak{C}_2$ .

Suppose  $O_2(\mathfrak{C}) \neq 1$ . Then since  $\mathfrak{R}_3 \in \mathcal{D}$ , we get  $Z(O_2(\mathfrak{C})) \sim \langle I \rangle$ , so  $\mathfrak{C}$  is in a conjugate of  $\mathfrak{M}$ . This is not the case, by Lemma 9.21(a). Clearly, the maximality of  $\tilde{\mathfrak{F}}$  forces  $\tilde{\mathfrak{F}} = O_3(\mathfrak{C})$ . Since  $O_2(\mathfrak{C}) = 1$ , the proof of Lemma 9.17 implies that  $\mathfrak{C}$  has no noncyclic abelian subgroup of order 8. Thus,  $\bar{\mathfrak{C}} = \mathfrak{C}/O_3(\mathfrak{C})$  is a 2, 3-group of order divisible by 3

such that

- (a)  $O_3(\bar{\mathbb{C}}) = 1$ .
- (b)  $\bar{\mathbb{C}}$  contains a four-group.
- (c)  $\bar{\mathbb{C}}$  contains no noncyclic abelian subgroup of order 8.

It is routine to verify that  $\bar{\mathbb{C}} \cong GL(2, 3)$  or  $\bar{\mathbb{C}} \cong \Sigma_4$  or  $\bar{\mathbb{C}} \cong A_4$ . If  $GL(2, 3) \cong \bar{\mathbb{C}}$ , then every four-subgroup of  $\bar{\mathbb{C}}$  normalizes a  $S_3$ -subgroup of  $\bar{\mathbb{C}}$ , against the maximality of  $\bar{\mathfrak{B}}$ . Hence,

$$\mathbb{C}/O_3(\mathbb{C}) \cong \Sigma_4 \text{ or } A_4.$$

Let  $\mathfrak{R}_2^* = \langle I, J \rangle$ .

*Case 1.*  $J$  does not invert  $\mathfrak{R}_3$ . Let  $\mathfrak{X} = C(J) \cap \mathfrak{R}_3$ , so that  $\mathfrak{X} = C(\mathfrak{R}_2^*) \cap \mathfrak{R}_3 = O_3(\mathbb{C}) \cap C(\mathfrak{R}_2^*)$  is of order 3. By a formula of Wielandt [40], together with Lemma 9.30, we get  $|O_3(\mathbb{C})| \leq 3^4$ . Since  $O_3(\mathbb{C}) = F(\mathbb{C})$ , it follows from (B) that  $m(O_3(\mathbb{C})) \geq 3$ . Hence,  $O_3(\mathbb{C})$  is elementary of order  $3^3$  or  $3^4$ . If  $|O_3(\mathbb{C})| = 3^3$ , then  $O_3(\mathbb{C})$  char  $\mathbb{C}_3$ , and so  $\mathbb{C}_3$  is a  $S_3$ -subgroup of  $\mathbb{G}$ , against Lemma 9.31. Hence,  $O_3(\mathbb{C})$  is elementary of order  $3^4$ . This implies that  $O_3(\mathbb{C})$  char  $\mathbb{C}_3$ . Hence,  $\mathbb{C}_3$  is a  $S_3$ -subgroup of  $\mathbb{G}$ . This is not the case, since  $Z(\mathbb{C}_3)$  is noncyclic.

*Case 2.*  $J$  inverts  $\mathfrak{R}_3$  and  $\bar{\mathbb{C}} \cong \Sigma_4$ .

Let  $\mathfrak{B} = \mathbb{C}_2 \cap O_{3,2}(\mathbb{C})$ . Thus,  $\mathfrak{B}$  is a four-group. Suppose  $\mathfrak{B} = \mathfrak{R}_2^*$ . Let  $\mathfrak{S} = \mathfrak{M} \cap \mathbb{C}$ . Then  $|\mathfrak{S}| = 8.9$ , and  $\mathfrak{R}_3 \triangle \mathfrak{S}$ . This is not the case, since  $\mathfrak{R}_2^*$  is a four-group, by Lemma 9.27.

Since  $\mathfrak{B} \neq \mathfrak{R}_2^*$ , it follows that  $\mathfrak{B}$  and  $\mathfrak{R}_2^*$  are the four-subgroups of  $\mathbb{C}_2$ . By Lemma 9.32,  $A_{\mathbb{G}}(\mathfrak{R}_2^*) = \text{Aut}(\mathfrak{R}_2^*)$ . Thus,  $\mathfrak{B} \cap \mathfrak{R}_2^* = \langle V \rangle$  with  $V \sim I$ . Hence, all involutions of  $\mathfrak{B}$  are conjugate to  $I$  in  $\mathbb{G}$ .

Choose  $V$  in  $\mathfrak{B}^*$ . Suppose  $|C(V) \cap O_3(\mathbb{C})| > 3$ . Then  $C(V) \cap O_3(\mathbb{C})$  is a  $S_3$ -subgroup of  $C(V)$ , by Lemma 9.29, together with  $V \sim I$ . Hence,  $|C_{\mathbb{G}}(V)| = 8.9$ , and  $C_{\mathbb{G}}(V)$  is 3-closed. This violates Lemma 9.27 applied to  $C(V)$ . Hence,  $|O_3(\mathbb{C}) \cap (V)| \leq 3$ .

Since  $N(\mathfrak{B}) \cap \mathbb{C}$  permutes transitively the involutions of  $\mathfrak{B}$ , we get  $|C(V) \cap O_3(\mathbb{C})| = 3$  for all  $V$  in  $\mathfrak{B}^*$ . Hence,  $|O_3(\mathfrak{B})| = 27$  and  $O_3(\mathbb{C}) = J(\mathbb{C}_3)$  char  $\mathbb{C}_3$ . But then  $|\mathbb{C}|_3 = |\mathbb{G}|_3$ , against Lemma 9.31.

*Case 3.*  $J$  inverts  $\mathfrak{R}_3$  and  $\bar{\mathbb{C}} \cong A_4$ .

Let  $\mathbb{C}_0 = O_3(\mathbb{C})$ . Then  $\mathfrak{R}_3 = C_{\mathbb{C}_0}(I)$  is elementary of order  $3^3$  and inverts by each element of  $\mathfrak{R}_2^* - \langle I \rangle$ . Since the involutions of  $\mathfrak{R}_2^*$  are fused in  $\mathbb{C}$ , we conclude

- (a) for each  $K \in (\mathfrak{R}_2^*)^*$ , the group  $C_{\mathbb{C}_0}(K)$  is elementary of order  $3^2$  and is inverted by each element of  $\mathfrak{R}_2^* - \langle K \rangle$ .

It follows that

- (b)  $\mathbb{C}_0$  contains two chief factors of  $\mathbb{C}$ , each of order  $3^3$ .

Suppose that  $\mathfrak{C}_0$  is abelian. By (a) and (b), it is elementary of order  $3^6$  and each element of  $\mathfrak{C}_3 - \mathfrak{C}_0$  has minimal polynomial  $(x - 1)^3$  on  $\mathfrak{C}_0$ . Hence,  $\mathfrak{C}_0 \text{ char } \mathfrak{C}_3$ . So  $\mathfrak{C}_3$  is a  $S_3$ -subgroup of  $\mathfrak{G}$ . But  $Z(\mathfrak{C}_3) = [\mathfrak{C}_0, \mathfrak{C}_3, \mathfrak{C}_3]$  is not cyclic, against hypothesis (iii) of Theorem 9.1. Therefore,  $\mathfrak{C}_0$  is not abelian. So (a) and (b) imply:

- (c1)  $\mathfrak{C}_0$  is special of order  $3^6$  and exponent 3,
- (c2)  $D(\mathfrak{C}_0) = Z(\mathfrak{C}_0)$  is a chief factor of  $\mathfrak{G}$  of order  $3^3$ ,
- (c3)  $\mathfrak{C}_0/D(\mathfrak{C}_0)$  is a chief factor of  $\mathfrak{G}$  of order  $3^3$ ,
- (c4) every element of  $\mathfrak{C}_3 - \mathfrak{C}_0$  has minimal polynomial  $(x - 1)^3$  on both  $D(\mathfrak{C}_0)$  and  $\mathfrak{C}_0/D(\mathfrak{C}_0)$ ,
- (c5) if  $P \in \mathfrak{C}_3 - \mathfrak{C}_0$ , then  $|C_{\mathfrak{C}_3}(P)| \leq 3^3$ .

This implies

- (d)  $\mathfrak{C}_0 \text{ char } \mathfrak{C}_3$ .

Indeed, if  $\mathfrak{C}_1$  is any subgroup of index 3 in  $\mathfrak{C}_3$  different from  $\mathfrak{C}_0$ , then  $\mathfrak{C}_1 \cap \mathfrak{C}_0 \cong D(\mathfrak{C}_0)$ . Hence, (c4) implies that the exponent of  $\mathfrak{C}_1$  is 9. This proves (d), and gives

- (e)  $\mathfrak{C}_3$  is a  $S_3$ -subgroup of  $\mathfrak{G}$ .

Now let  $\mathfrak{X}_0$  be a subgroup of  $\mathfrak{R}_3$  of order 3 such that  $C_{\mathfrak{R}_0}(\mathfrak{X}_0) = \mathfrak{Q} \supset \langle I \rangle$ . Let  $\mathfrak{C}_3 = A_0 \times A_1$ . Thus,  $\mathfrak{Q}$  is a quaternion group and  $\mathfrak{Q}\mathfrak{X}_1 \langle J \rangle \cong GL(2, 3)$ . Let  $\mathfrak{Y}$  be a  $S_{2,3}$ -subgroup of  $C_{\mathfrak{G}}^*(\mathfrak{X}_0)$  with  $\mathfrak{Q}\mathfrak{R}_3 \langle J \rangle \subseteq L$ . Let  $\mathfrak{Y}_0 = O_3(\mathfrak{Y})$ . Since  $D(\mathfrak{C}_0)\mathfrak{R}_3$  is elementary of order  $3^4$  and contains an element of  $\mathcal{Z}(3)$ , it follows that  $O_3(\mathfrak{Y}) = 1$ . Since  $\mathfrak{Y}$  contains no noncyclic abelian subgroup of order 8, we get that  $\mathfrak{Q}\mathfrak{X}_1 \langle J \rangle$  is a complement to  $\mathfrak{Y}_0$  in  $\mathfrak{Y}$ . Since  $I$  inverts  $\mathfrak{Y}_0/\mathfrak{X}_0$ , it follows that  $|\mathfrak{Y}_0: \mathfrak{X}_0| = 3^{2d}$  for some integer  $d \geq 1$ . If  $d = 1$ , then  $D(\mathfrak{C}_0)\mathfrak{R}_3$  is a  $S_3$ -subgroup of  $C(\mathfrak{X}_0)$  and so  $S_3$ -subgroups of  $\mathfrak{Y}$  are abelian. This is absurd, so  $d \geq 2$ . Since  $|\mathfrak{G}|_3 = 3^7$  by (e), and since  $|\mathfrak{Y}|_3 = 3^{2d+2}$ , we get  $d = 2$ .

Let  $\mathfrak{Y}_3$  be a  $S_3$ -subgroup of  $\mathfrak{Y}$  containing  $\mathfrak{R}_3$ . Since  $\mathfrak{Y}_3$  is not a  $S_3$ -subgroup of  $\mathfrak{G}$ , and since  $\mathfrak{X}_0 \triangleleft \mathfrak{Y}$ , it follows that  $Z(\mathfrak{Y}_3)$  is noncyclic. In particular,  $Z(\mathfrak{Y}_0) \cong Z(\mathfrak{Y}_3)$ , so that  $Z(\mathfrak{Y}_0)$  is not cyclic. Hence,  $|Z(\mathfrak{Y}_0)| \geq 3^3$ . This implies that if  $L \in \mathfrak{Y}_0$ , then  $|C_{\mathfrak{Y}_0}(L)| \geq 3^4$ . Choose  $G$  in  $\mathfrak{G}$  so that  $\mathfrak{Y}_3 \subseteq \mathfrak{C}_3^G$ , which is possible by (e). By (c5), we get  $\mathfrak{Y}_0 \subseteq \mathfrak{C}_0^G$ . Hence,  $\mathfrak{Y}_0 = \mathfrak{Y}_{00} \times \mathfrak{Y}_{01}$ , where  $\mathfrak{Y}_{00}, \mathfrak{Y}_{01}$  admit  $\mathfrak{Q}$ ,  $\mathfrak{Y}_{00}$  is nonabelian of exponent 3 and order  $3^3$  and  $\mathfrak{Y}_{01}$  is elementary of order  $3^2$ . Now  $\mathfrak{X}_1 \subseteq \mathfrak{Y}_3$  and  $|C_{\mathfrak{Y}_3}(\mathfrak{X}_1)| \geq 3^4$ , so we get that  $\mathfrak{X}_1 \subseteq \mathfrak{C}_0^G$ . Hence,  $\mathfrak{Y}_3 = \mathfrak{Y}_0\mathfrak{X}_1 \subseteq \mathfrak{C}_0^G$ , and so  $\mathfrak{Y}_3 = \mathfrak{C}_0^G$ . This is impossible, since  $|Z(\mathfrak{Y}_3)| = 3^2, |Z(\mathfrak{C}_0)| = 3^3$ .

LEMMA 9.34. *Each involution of  $\mathfrak{R}_2^* - \langle I \rangle$  inverts  $\mathfrak{R}_3$ .*

*Proof.* Let  $\mathfrak{P}^*$  be a  $S_2$ -subgroup of  $\mathfrak{G}$  which contains  $\mathfrak{R}_3$  and is normalized by  $\mathfrak{R}_2^*$ , set  $\mathfrak{X} = \mathfrak{R}_3 \cap C(\mathfrak{R}_2^*)$ . Suppose  $\mathfrak{X} \neq 1$ . Then  $|\mathfrak{X}| = 3$ , so by a formula of Wielandt [44],  $|\mathfrak{P}^*| \leq 3^4$ . This contradicts Lemma 9.31. Hence,  $\mathfrak{X} = 1$ . As  $\mathfrak{R}_2^* = \langle I, J \rangle$  for some involution  $J$ , the proof is complete.

LEMMA 9.35. (a) *If  $\mathfrak{X}$  is a subgroup of  $\mathfrak{R}_3$  of order 3 and  $C(\mathfrak{X}) \cap \mathfrak{R}_0$  is quaternion, then  $|C(\mathfrak{X})|_3 = 3^4$ .*

(b)  $|C(\mathfrak{R})|_3 = 3^3$ .

*Proof.* (a) Set  $\mathfrak{Q} = C(\mathfrak{X}) \cap \mathfrak{R}_0$ , and let  $\mathfrak{Y}$  be a subgroup of  $\mathfrak{R}_3$  of order 3 distinct from  $\mathfrak{X}$ . Let  $J$  be an involution of  $\mathfrak{R}_2^* - \langle I \rangle$ . Thus,  $J$  inverts  $\mathfrak{R}_3$  by Lemma 9.34. Also,  $\langle J \rangle \mathfrak{Y} \mathfrak{Q} \cong GL(2, 3)$ .

Let  $\mathfrak{C}$  be a  $S_{2,3}$ -subgroup of  $N(\mathfrak{X})$  which contains  $\mathfrak{R}_3 \mathfrak{Q} \mathfrak{R}_2^*$ . Thus,  $\mathfrak{Q} \langle J \rangle$  is a  $S_2$ -subgroup of  $\mathfrak{C}$  and  $O_2(\mathfrak{C}) = 1$ . Since

$$[O_3(\mathfrak{C}) \cap \mathfrak{R}_3, \mathfrak{Q}] \subseteq O_3(\mathfrak{C}) \cap \mathfrak{Q} = 1,$$

it follows that  $O_3(\mathfrak{C}) \cap \mathfrak{R}_3 = \mathfrak{X}$ . Hence,  $I$  inverts  $O_3(\mathfrak{C})/\mathfrak{X}$ . Hence,  $O_3(\mathfrak{C})/\mathfrak{X}$  is the direct sum of a certain number, say  $k$ , of modules each isomorphic to the faithful irreducible  $F_3 \mathfrak{Q}$ -module, so that  $|O_3(\mathfrak{C})/\mathfrak{X}| = 3^{2k}$ . Hence,  $|\mathfrak{C}|_3 = |C(\mathfrak{R})|_3 = 3^{2(k+1)}$ . Suppose  $k \geq 2$ . Then by Lemma 9.33, we get  $|\mathfrak{C}|_3 = |\mathfrak{G}|_3 = 3^6$ .

We argue that  $Z(O_3(\mathfrak{C})) = \mathfrak{X}$ . Suppose false. We get  $Z(O_3(\mathfrak{C})) = (Z(O_3(\mathfrak{C})) \cap C(I)) \times [Z(O_3(\mathfrak{C})), I]$ . Since  $\mathfrak{R}_3 \cap O_3(\mathfrak{C}) = \mathfrak{X}$ , we get

$$Z(O_3(\mathfrak{C})) \cap C(I) = \mathfrak{X};$$

also  $[Z(O_3(\mathfrak{C})), I]$  is normalized by  $\mathfrak{Y}$ , so if  $[Z(O_3(\mathfrak{C})), I] \neq 1$ , then a  $S_3$ -subgroup of  $\mathfrak{G}$  has a noncyclic center. We conclude that  $\mathfrak{X} = Z(O_3(\mathfrak{C}))$ . This implies that  $O_3(\mathfrak{C})$  is extra special of width 2. Since  $\mathfrak{Q} \langle J \rangle$  is a  $S_2$ -subgroup of  $N(\mathfrak{X})$ , it follows that  $O_3(\mathfrak{C}) = O_3(N(\mathfrak{X}))$ . Thus, Hypothesis 9.2 is satisfied. Since this is not the case, we get  $k = 1$ . Thus (a) holds.

By Lemma 9.20, we have  $|C(\mathfrak{R}_3)|_3 \geq 27$ . Since  $\mathfrak{R}_3$  is not central in a  $S_3$ -subgroup of  $\mathfrak{C}$ , (b) follows.

LEMMA 9.36. *Let  $\mathfrak{P}$  be a  $S_3$ -subgroup of  $\mathfrak{G}$ . Then*

(a)  $|\mathfrak{P}| = 3^5$ .

(b)  $\mathfrak{P}/Z(\mathfrak{P})$  is of maximal class and order  $3^4$ .

*Proof.* By Lemma 9.33, there is a conjugate  $\mathfrak{B}$  of  $\mathfrak{R}_2^*$  which normalizes  $\mathfrak{P}$ . By Lemma 9.32, all involutions of  $\mathfrak{B}$  are conjugate to  $I$ . Let  $V_1, V_2, V_3$  be the involutions of  $\mathfrak{B}$ . By Lemma 9.34,  $C_{\mathfrak{P}}(\mathfrak{B}) = 1$ . By Lemma 9.29,  $|C_{\mathfrak{P}}(V_i)| \leq 9$  for  $i = 1, 2, 3$ . Then by Wielandt [44],  $|\mathfrak{P}| \leq 3^6$ .

Set  $\mathfrak{Z} = Z(\mathfrak{P})$ . Since  $\mathfrak{Z}$  is cyclic,  $C(\mathfrak{Z}) \cap \mathfrak{P} \neq 1$ . We may assume notation is chosen so that  $V_1$  is a generator for  $C(\mathfrak{Z}) \cap \mathfrak{P}$ . Thus,  $|\mathfrak{Z}| = 3$ . Suppose  $V_1$  inverts  $\mathfrak{P}/\mathfrak{Z}$ . Then,  $|\mathfrak{P}| \leq 3^5$ , so by Lemma 9.31,  $|\mathfrak{P}| = 3^5$ . In this case, since  $\mathfrak{P}$  is generated by elements of order 3, we get that  $\mathfrak{Z} = \mathfrak{P}' = D(\mathfrak{P})$ . Since  $O_3(N(\mathfrak{Z})) = 1$ , so also  $O_3(\mathfrak{Z})/\mathfrak{Z} = 1$ .

Hence,  $\mathfrak{P} \triangleleft N(\mathfrak{Z})$ . Thus, Hypothesis 9.2 is satisfied. Since this is not the case, we conclude that  $V_1$  does not invert  $\mathfrak{P}/\mathfrak{Z}$ .

Let  $\mathfrak{U}$  be a subgroup of  $C_{\mathfrak{P}}(V_1)$  of order 3 distinct from  $\mathfrak{Z}$ . Thus,  $C_{\mathfrak{P}}(V_1) = \mathfrak{Z}\mathfrak{U}$ .

By Lemma 9.35(b), we get  $|C_{\mathfrak{P}}(\mathfrak{Z}\mathfrak{U})| \leq 27$ . Since  $N_{\mathfrak{P}}(\mathfrak{Z}\mathfrak{U}) = C_{\mathfrak{P}}(\mathfrak{Z}\mathfrak{U})$ , we have  $|C_{\mathfrak{P}}(\mathfrak{Z}\mathfrak{U})| = 27$ . Again, since  $N_{\mathfrak{P}}(\mathfrak{Z}\mathfrak{U}) = C_{\mathfrak{P}}(\mathfrak{Z}\mathfrak{U})$ , it follows that  $|N_{\mathfrak{P}/\mathfrak{Z}}(\mathfrak{Z}\mathfrak{U}/\mathfrak{Z})| = 9$ . Thus,  $\mathfrak{P}/\mathfrak{Z}$  is of maximal class, and  $\mathfrak{U}\mathfrak{Z}/\mathfrak{Z} \not\subseteq (\mathfrak{P}/\mathfrak{Z})'$ . Since  $\mathfrak{U}\mathfrak{Z}/\mathfrak{Z}$  is the set of fixed points of  $V_1$  on  $\mathfrak{P}/\mathfrak{Z}$ , it follows that  $\mathfrak{P}/\mathfrak{Z}$  has a subgroup  $\mathfrak{P}_0/\mathfrak{Z}$  of index 3 which is inverted by  $\mathfrak{V}_1$ . Since  $\mathfrak{P}_0/\mathfrak{Z}$  is generated by elements of order 3,  $\mathfrak{P}_0/\mathfrak{Z}$  is elementary. If  $|\mathfrak{P}_0/\mathfrak{Z}| \geq 3^4$ , then  $\mathfrak{P}/\mathfrak{Z}$  is not of maximal class. Hence,  $|\mathfrak{P}_0/\mathfrak{Z}| \leq 27$ , so by Lemma 9.31, we have  $|\mathfrak{P}_0:\mathfrak{Z}| = 27$ . This establishes both (a) and (b).

We may now complete the proof of Theorem 9.1. Let  $\mathfrak{P}, \mathfrak{P}_0, \mathfrak{Z}, \mathfrak{U}, \mathfrak{V}, V_1$  be as above. Thus,  $|\mathfrak{P}_0| = 3^4$ ,  $\mathfrak{P}_0/\mathfrak{Z}$  is elementary of order 27 and is inverted by  $V_1$ . Being generated by elements of order 3,  $\mathfrak{P}_0$  is of exponent 3. It follows that  $Z(\mathfrak{P}_0)$  is not cyclic. Hence, we can choose a subgroup  $\mathfrak{W}$  of  $Z(\mathfrak{P}_0)$  of order 9 which is normal in  $\mathfrak{P}\mathfrak{P}$ . Set  $\mathfrak{Y} = \mathfrak{W}\mathfrak{U}$ . Thus,  $\mathfrak{Y}$  is of order 27 and  $\mathfrak{Y}$  admits  $V_1$ . Thus,  $\mathfrak{Z}\mathfrak{U} \triangleleft \mathfrak{Y}$ , so  $\mathfrak{Y}$  is abelian, since  $N_{\mathfrak{P}}(\mathfrak{Z}\mathfrak{U}) = C_{\mathfrak{P}}(\mathfrak{Z}\mathfrak{U})$ . This implies that  $\mathfrak{W} \subset Z(\mathfrak{P})$ , since  $\mathfrak{P} = \mathfrak{P}_0\mathfrak{U}$ . This contradiction completes the proof of Theorem 9.1.

Theorems 8.1 and 9.1 provide a proof of Theorem ES.

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